

A Bijection Between Evil-Avoiding and Rectangular Permutations

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Abstract

Evil-avoiding permutations, introduced by Kim and Williams in 2022, arise in the study of the inhomogeneous totally asymmetric simple exclusion process. Rectangular permutations, introduced by Chirivì, Fang, and Fourier in 2021, arise in the study of Schubert varieties and Demazure modules. Taking a suggestion of Kim and Williams, we supply a bijection between evil-avoiding and rectangular permutations in S_n that preserves the number of recoils. We encode these classes of permutations as regular languages and construct a length-preserving bijection between words in these regular languages. We extend the bijection to another Wilf-equivalent class of permutations, namely the 1-almost-increasing permutations, and exhibit a bijection between rectangular permutations and walks of length $2n - 2$ in a path of seven vertices starting and ending at the middle vertex.

Mathematics Subject Classifications: 05A05, 05A15, 68Q45

1 Introduction

A permutation π *contains* a permutation σ as a pattern if some subsequence of the values of π has the same relative order as all of the values of σ . Otherwise, π *avoids* σ . Two classes of pattern-avoiding permutations, called *evil-avoiding* and *rectangular* permutations, have been of recent interest due to their algebraic significance. Kim and Williams gave the following definition of evil-avoiding permutations.

Definition 1. [8] A permutation that avoids the patterns 2413, 4132, 4213 and 3214 is called *evil-avoiding*.¹

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¹Kim and Williams called these permutations *evil-avoiding* because if we replace I by 1, E by 2, L by 3, and V by 4, then EVIL and its anagrams VILE, VEIL and LEIV become the four patterns 2413, 4132, 4213 and 3214 [8]. (Leiv is a name of Norwegian origin meaning “loaf”. Even though LIVE is an anagram of EVIL, an evil-avoiding permutation does not necessarily avoid the pattern 3142.)

For example, the permutation 1674523 is evil-avoiding, but the permutation 562314 is not evil-avoiding since, for instance, the subsequence 5214 reduces to the forbidden pattern 4213. Chirivì, Fang, and Fourier gave the following definition of rectangular permutations.

Definition 2. [6] A permutation that avoids the patterns 2413, 2431, 4213, and 4231 is called *rectangular*.

Both evil-avoiding permutations and rectangular permutations are enumerated by the same sequence, which begins 1, 2, 6, 20, 68, 232, ... and appears in the OEIS as sequence A006012 [11]. The enumeration of evil-avoiding permutations is done by Kim and Williams in [8], and the enumeration of rectangular permutations is done by Biers-Ariel and Chirivì, Fang, and Fourier in [4] and [6].

Theorem 3. [8, Proposition 1.14] *The number of evil-avoiding permutations in S_n satisfies the recurrence $e(1) = 1, e(2) = 2, e(n) = 4e(n - 1) + 2e(n - 2)$.*

Theorem 4. [4, Theorem 8], [6, Corollary 8] *The number of rectangular permutations in S_n satisfies the same recurrence $r(1) = 1, r(2) = 2, r(n) = 4r(n - 1) + 2r(n - 2)$.*

Definition 5. Two families U and V of pattern-avoiding permutations are called *Wilf-equivalent* if for all $n, |S_n \cap U| = |S_n \cap V|$. This is a *trivial Wilf-equivalence* if U avoids patterns in P_U and V avoids patterns in P_V , where for some symmetry (i.e., rotation or reflection) ρ of the square, $P_U = \rho(P_V)$.

Theorems 3 and 4 imply that evil-avoiding and rectangular permutations are Wilf-equivalent. Since no symmetry takes the set of patterns $\{2413, 4132, 4213, 3214\}$ to the set of patterns $\{2413, 2431, 4213, 4231\}$, this Wilf-equivalence is nontrivial. Thus, the following question is natural.

Suggestion 1. [14] Construct a bijection between evil-avoiding and rectangular permutations in S_n .

In this paper, we exhibit such a bijection. In fact, our bijection preserves not only the size of a permutation but also the number of recoils, as motivated in [8] and explained below.

Definition 6. A *recoil* of a permutation $\pi \in S_n$ is a value $i \in \{1, 2, \dots, n - 1\}$ so that i occurs after $i + 1$ in π , which means $\pi_i^{-1} > \pi_{i+1}^{-1}$. Equivalently, i is a recoil of π if i is a descent of π^{-1} .

Kim and Williams [8] enumerated evil-avoiding permutations of size n with k recoils. A computational check shows that for small n , there are the same number of evil-avoiding permutations of size n with k recoils as rectangular permutations of size n with k recoils, thus motivating the following notion.

Definition 7. Two classes of pattern-avoiding permutations are *strongly Wilf-equivalent* if for every n and k the classes have the same counts of permutations in S_n with k recoils.

In [8], Kim and Williams observed that there are several permutation classes enumerated by OEIS sequence A006012, including

1. permutations $\pi \in S_n$ for which the pairs (i, π_i) with $i < \pi_i$, considered as closed intervals $[i + 1, \pi_i]$, do not overlap; equivalently, for each $i \in [n]$ there is at most one $j \leq i$ with $\pi_j > i$,
2. permutations on $\{1, 2, \dots, n\}$ with no subsequence $abcd$ such that bc are adjacent in position and $\max(a, c) < \min(b, d)$,
3. rectangular permutations,

hence the following suggestion, which may be seen as a refinement of Suggestion 1.

Suggestion 2. [8] Find a recoil-preserving bijection between evil-avoiding permutations and any of the above sets of permutations in S_n .

We construct such a refined bijection in Section 3. The bijection is more easily described as a map from rectangular to evil-avoiding permutations than the other way around, so we write the bijection in the easier format from here onward.

Theorem 8. *The map β (defined in Section 3.3) from the set of rectangular permutations to the set of evil-avoiding permutations is a bijection preserving length and number of recoils.*

For example, the bijection sends the rectangular permutation in Figure 1 to the evil-avoiding permutation in Figure 2. The rectangular permutation π was drawn uniformly from $\text{Rect}(100)$, the set of rectangular permutations of size 100, and the evil-avoiding permutation was taken to be $\beta(\pi)$. A description of how the rectangular permutation was drawn from $\text{Rect}(100)$ is in Section 3.

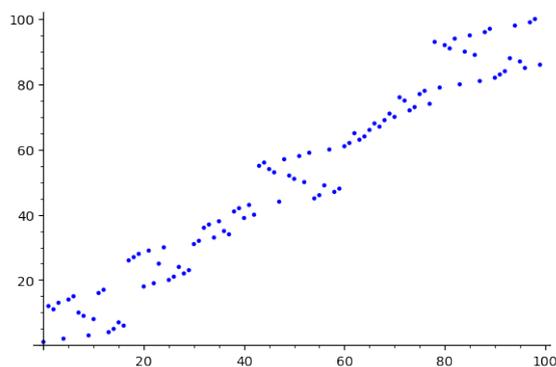


Figure 1: A plot of the rectangular permutation

$\pi = [1, 12, 11, 13, 2, 14, 15, 10, 9, 3, 8, 16, 17, 4, 5, 7, 6, 26, 27, 28, 18, 29, 19, 25, 30, 20, 21, 24, 22, 23, 31, 32, 36, 37, 33, 38, 35, 34, 41, 42, 39, 43, 40, 55, 56, 54, 53, 44, 57, 52, 51, 58, 50, 59, 45, 46, 49, 60, 47, 48, 61, 62, 65, 63, 64, 66, 68, 67, 69, 71, 70, 76, 75, 72, 73, 77, 78, 74, 93, 79, 92, 91, 94, 80, 90, 95, 89, 81, 96, 97, 82, 83, 84, 88, 98, 87, 85, 99, 100, 86]$.

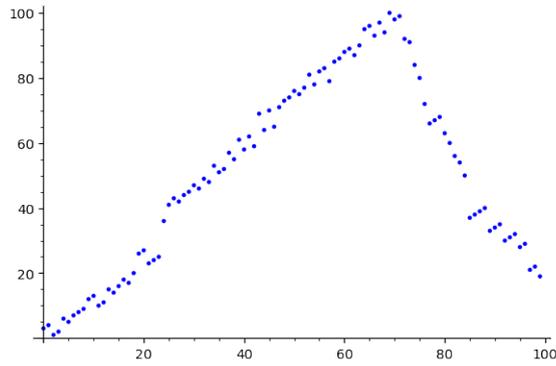


Figure 2: A plot of the evil-avoiding permutation

$\beta(\pi) = [3, 4, 1, 2, 6, 5, 7, 8, 9, 12, 13, 10, 11, 15, 14, 16, 18, 17, 20, 26, 27, 23, 24, 25, 36, 41, 43, 42, 44, 45, 47, 46, 49, 48, 53, 51, 52, 57, 55, 61, 58, 62, 59, 69, 64, 70, 65, 71, 73, 74, 76, 75, 77, 81, 78, 82, 83, 79, 85, 86, 88, 89, 87, 90, 95, 96, 93, 97, 94, 100, 98, 99, 92, 91, 84, 80, 72, 66, 67, 68, 63, 60, 56, 54, 50, 37, 38, 39, 40, 33, 34, 35, 30, 31, 32, 28, 29, 21, 22, 19]$.

The significance of evil-avoiding permutations is rooted in Schubert calculus. There is a type of Markov chain called the *asymmetric simple exclusion process* (ASEP) in which particles hop on a one-dimensional lattice subject to the condition that at most one particle may occupy a given site. The *inhomogeneous totally asymmetric simple exclusion process* (inhomogeneous TASEP) is a type of ASEP where the sites $1, 2, \dots, n$ are arranged in a ring and the hopping rate depends on the weight of the particles. For more details on the inhomogeneous TASEP, see [8], [10], [3], [2], and [5]. An interesting feature about ASEPs is that the expressions for the steady-state probabilities at each site often contain Schubert polynomials, for reasons that are not entirely understood. In the case of the inhomogeneous TASEP, Kim and Williams demonstrate that a specialization of the steady-state probabilities at states corresponding to evil-avoiding permutations can be written as a “trivial factor” times a product of (double) Schubert polynomials [8]. The number of Schubert polynomials in the steady-state formula is equal to the number of recoils in the corresponding evil-avoiding permutation [8].

Rectangular permutations arise in the context of representation theory. The name “rectangular” was coined by Chirivì, Fang, and Fourier in 2021 and we briefly summarize its origin here; we direct the reader to their paper [6] for more details.

Let Φ denote the root system of the Lie algebra \mathfrak{sl}_{n+1} , let $\alpha_1, \dots, \alpha_n$ denote the simple roots of \mathfrak{sl}_{n+1} , and let

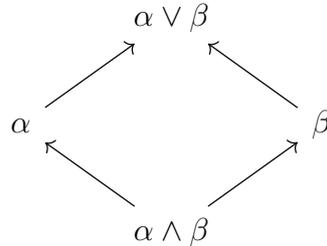
$$\Phi^+ = \{\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j \mid 1 \leq i \leq j \leq n\}$$

denote the set of positive roots. The *support* $\text{supp}(\alpha_{i,j})$ of the root $\alpha_{i,j}$ is defined to be the subset $\{i, i+1, \dots, j\}$ of $\{1, 2, \dots, n\}$. The set Φ^+ admits a poset structure with partial order relation given by $\alpha_{k,\ell} \leq \alpha_{i,j}$ when $k \geq i$ and $j \geq \ell$. The meet $\alpha_{i,j} \wedge \alpha_{k,\ell}$ is defined by $\alpha_{\max(i,k), \min(j,\ell)}$ when it exists and dually, the join $\alpha_{i,j} \vee \alpha_{k,\ell}$ is defined by $\alpha_{\min(i,k), \max(j,\ell)}$.

Definition 9. [6] A subset $A \subset \Phi^+$ is *rectangular* if:

1. it is triangular, i.e., for any $\alpha, \beta \in A$ such that $\text{supp}(\alpha) \cup \text{supp}(\beta)$ is a connected subset of $\{1, 2, \dots, n\}$, we have $\alpha \vee \beta \in A$; moreover, $\alpha \wedge \beta \in A$ if it exists,
2. if $\alpha, \beta \in \Phi^+$ and $\alpha \wedge \beta, \alpha \vee \beta \in A$, then $\alpha, \beta \in A$

so that in the Hasse diagram for the subposet (A, \leq) , we get the following “rectangle”:



The aforementioned rectangular condition translates nicely into a condition on permutations.

Definition 10. [6] A permutation σ is *rectangular* if for any $1 \leq i < k < j < \ell \leq n$, the following conditions are equivalent:

1. $\sigma_i > \sigma_j$ and $\sigma_k > \sigma_\ell$,
2. $\sigma_i > \sigma_\ell$ and $\sigma_k > \sigma_j$.

As justified in [6], an equivalent definition is that a permutation is rectangular if it avoids the patterns in $\{2413, 2431, 4213, 4231\}$.

The rest of the paper is organized as follows. Section 2 gives preliminary definitions and constructions, and Section 3 introduces operators for constructing rectangular and evil-avoiding permutations and explicitly states the bijection between the languages L_{Rect} and L_{Evil} . Sections 4 and 5 establish the bijection between L_{Rect} and rectangular permutations and between L_{Evil} and evil-avoiding permutations. Section 6 illustrates the bijection for a number of small permutations. Section 7 discusses 1-almost-increasing permutations and bijects them with rectangular permutations. Section 8 describes insertion encodings for rectangular, evil-avoiding, and 1-almost-increasing permutations, as well as all of their inverses. Section 9 provides a bijection between a family of paths and rectangular permutations. Finally, Section 10 discusses possible future directions, algebraic and enumerative.

2 Preliminaries

Let S_n be the symmetric group on the set $\{1, \dots, n\}$. We say that a permutation $\pi \in S_n$ has *size* n and write π in tabular form as $[\pi_1 \ \pi_2 \ \dots \ \pi_n]$. For $n \geq 0$, we denote by e_n the identity in S_n .

We may grade the rectangular permutations by the size of the permutation, or double grade these permutations by the size and the number of recoils. Let Rect denote the set

of all rectangular permutations, let $\text{Rect}(n)$ denote the set of rectangular permutations in S_n , and let $\text{Rect}(n, k)$ denote the set of rectangular permutations in S_n with k recoils. The evil-avoiding permutations can be singly or doubly graded in the same way, and we define Evil , $\text{Evil}(n)$, and $\text{Evil}(n, k)$ analogously.

Since our bijection involves regular expressions (regexes), we will define them here informally. For a formal definition, see [12].

Definition 11. A *regular expression* (regex) is a formula that describes a language (set of words) over an alphabet Σ . Languages describable by such a formula are called *regular*. A regex is either \emptyset or it is built recursively from individual elements of $\Sigma \cup \{\varepsilon\}$ using the operations of

1. concatenation $(\cdot)(\cdot)$,
2. OR $(\cdot|\cdot)$ (an alternative between two simpler patterns), or
3. the unary Kleene star $(\cdot)^*$ (R^* allows 0 or more repetitions of the pattern R).

The symbol ε represents the empty string. For example, $((a|\varepsilon)b)^*(a|\varepsilon)$ generates words in $\{a, b\}^*$ with no adjacent a 's. For clarity, we can also use the unary Kleene plus $(\cdot)^+$ (R^+ allows 1 or more repetitions of the pattern R). For example, $(a|b)^+$ describes any positive-length string of a 's and b 's.

It is equivalent to be able to describe a language (a subset of Σ^*) as the set of words generated by a regular expression and to say that there is a deterministic finite automaton (DFA) accepting the words [12].

Definition 12. A *deterministic finite automaton* (DFA) M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

1. Q is a finite set of states,
2. Σ is an alphabet consisting of a finite set of input symbols,
3. δ is a transition function from $Q \times \Sigma$ to Q ,
4. $q_0 \in Q$ is an initial state,
5. $F \subset Q$ is a set of accepting states.

If we let $w = a_1a_2 \cdots a_n$ be a string with characters in Σ , then M accepts w if there is a sequence of states $r_0, r_1, \dots, r_n \in Q$ such that

1. $r_0 = q_0$,
2. $r_{i+1} = \delta(r_i, a_{i+1})$ for $i = 0, 1, \dots, n - 1$,
3. $r_n \in F$.

We will be interested in length-preserving bijections between regular languages encoding families of permutations. Some of these come from substitutions.

Definition 13. A *homomorphism* f from language L_1 to language L_2 over alphabets Σ_1 and Σ_2 is a restriction to L_1 of a map $f : \Sigma_1^* \rightarrow \Sigma_2^*$ so that

1. $f(w_1w_2) = f(w_1)f(w_2)$,
2. $f(L_1) \subseteq L_2$.

A homomorphism is determined by its values on Σ_1 . An *isomorphism* between languages is a homomorphism with an inverse homomorphism.

We will consider many lengthening operators (denoted $\rho_{i,j}$ or $\gamma_{a,b}$) on permutations. These operators have domains that are subsets of $\bigcup_{n=0}^{\infty} S_n$. Applying a lengthening operator to a permutation in S_n in the operator's domain produces a permutation in S_{n+1} .

Definition 14. The *insertion operator* $\rho_{i,j}$ takes in a permutation π , inserts the value i at index j , and increments by 1 all values in π greater than or equal to i . Equivalently, the permutation matrix for the result has the original permutation matrix as the (i, j) minor formed by deleting the i th row and j th column, which are both all 0s except for a 1 in the (i, j) position. If $\pi \in S_n$ then $\rho_{i,j}(\pi)$ is defined when $1 \leq i \leq n+1$ and $1 \leq j \leq n+1$. In tabular form, all entries at least i are increased by 1, then an i is inserted between positions $j-1$ and j . If $\sigma = \rho_{i,j}(\pi)$, then for $1 \leq k \leq n+1$,

$$\sigma_k = \begin{cases} i & k = j \\ \pi_k & k < j \text{ and } \pi_k < i \\ \pi_k + 1 & k < j \text{ and } \pi_k \geq i \\ \pi_{k-1} & k > j \text{ and } \pi_{k-1} < i \\ \pi_{k-1} + 1 & k > j \text{ and } \pi_{k-1} \geq i. \end{cases}$$

Example 15. $\rho_{3,4}[5674321] = [67835421]$ because we increase all values in $[5674321]$ greater than or equal to 3 to get $[6785421]$ and then we insert 3 into position 4 to get $[67835421]$.

We also define an operator $\gamma_{a,b}$ which does not increase the length of a permutation. It will be used to define one of the operators on evil-avoiding permutations.

Definition 16. The *shifting operator* $\gamma_{a,b}$ takes in a permutation $\pi \in S_n$, removes the value at index a , and inserts it at index b for $a > b$. More formally, if $\tau = \gamma_{a,b}(\pi)$, then for $1 \leq k \leq n$,

$$\tau_k = \begin{cases} \pi_k & k < b \\ \pi_a & k = b \\ \pi_{k-1} & b < k \leq a \\ \pi_k & k > a. \end{cases}$$

Example 17. $\gamma_{5,2}[5674321] = [5367421]$.

Additionally, we will make use of indicator variables to describe the actions of our operators on individual values in permutations.

Definition 18. For a statement S , the *indicator variable* $\mathbb{1}_S$ is defined as follows:

$$\mathbb{1}_S = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Following Albert, Linton, and Ruškuc [1], we can encode any permutation in S_n as a word of length n in $\bigcup_{i \in \mathbb{N}} \{\ell_i, r_i, m_i, f_i\}$ by observing where each element k from 1 to n is placed relative to *slots*, which are blocks of numbers greater than or equal to k . We represent such a slot using the symbol \diamond .

Definition 19. The k th *insertion pattern* of permutation π is the string produced from the table form for π where each maximal contiguous subsequence with values at least k is replaced by a \diamond .

Example 20. The permutation 4215736 has the following sequence of insertion patterns

$$\begin{array}{c} \diamond \\ \diamond 1 \diamond \\ \diamond 2 1 \diamond \\ \diamond 2 1 \diamond 3 \diamond \\ 4 2 1 \diamond 3 \diamond \\ 4 2 1 5 \diamond 3 \diamond \\ 4 2 1 5 \diamond 3 6 \\ 4 2 1 5 7 3 6 \end{array}$$

To represent the relevant aspects of an insertion pattern, we can use a string of symbols in $\{+, *, \diamond\}$ where $+$ represents one or more numbers in a row and $*$ represents 0 or more numbers in a row. For example, the pattern $\diamond 2 1 \diamond 3 \diamond$ is of type $\diamond + \diamond + \diamond$.

Each insertion fills a slot, is on the left of a slot, is on the right of a slot, or breaks a slot into two. The first type of insertion is called f_i , the second ℓ_i , the third r_i , and the fourth m_i . The subscript indicates which slot contains the number inserted.

Definition 21. The *insertion encoding* of a permutation π is the sequence $x_1 x_2 \cdots x_n$ of elements of

$$\bigcup_{i \in \mathbb{N}} \{\ell_i, r_i, m_i, f_i\}$$

so that x_k indicates the position of k relative to the k th insertion pattern of π .

Example 22. The insertion encoding of 4217536 is $m_1 r_1 m_2 f_1 \ell_1 f_2 f_1$.

Inserting 1 into \diamond split the only slot into two, so this insertion is denoted m_1 . Inserting 2 into $\diamond 1 \diamond$ put 2 on the right side of the first slot, so it is r_1 . Inserting 3 split the second slot, so it is m_2 . Inserting 4 filled and eliminated the first slot, so it is f_1 . Inserting 5 added to the left side of the first slot, ℓ_1 . Then 6 filled the second slot, f_2 , and 7 filled the first slot, f_1 .

3 The bijection

In this section, we construct two regular languages of operators on permutations based on rectangular and evil-avoiding permutations, L_{Rect} and L_{Evil} . We find regular expressions for these languages and construct a length-preserving bijection between them. In Sections 4 and 5, we will establish that these languages are encodings of rectangular and evil-avoiding permutations respectively.

3.1 Rectangular permutations

We define four operators on rectangular permutations: ψ_1, ψ_2, ψ_u , and ψ_d . Each increases the size of the permutation by one, and we restrict the domains of some of the operators. Similar operators were introduced by Biers-Ariel to count permutations avoiding the patterns 1324, 1423, 2314, and 2413 [4], which are trivially Wilf-equivalent to rectangular permutations.

The map ψ_1 is the insertion operator $\rho_{1,1}$. It increments by 1 all values of the permutation, then inserts a 1 at the beginning. The domain of ψ_1 is all rectangular permutations of size at least 0. Applying ψ_1 does not change the number of recoils in a permutation.

The map ψ_2 is the insertion operator $\rho_{1,2}$ with a restricted domain. We restrict ψ_2 to rectangular permutations with a first element greater than 1, that is, rectangular permutations of size at least 2 that could not be in the image of ψ_1 . (We are restricting the domain of ψ_2 to ensure that ψ_2 and ψ_d , defined later, have disjoint images, which avoids redundant encodings.) This restriction means that $\psi_2(\pi)$ has the same number of recoils as π .

The map ψ_u applies $\rho_{\pi_1,1}$ to permutation π . To ensure that ψ_u and ψ_1 have disjoint images, we restrict ψ_u to rectangular permutations of size at least 2 whose first element is not 1. Applying ψ_u does not change the number of recoils of a permutation.

The map ψ_d applies $\rho_{\pi_1+1,1}$ to permutation π . This operator is defined on all rectangular permutations of size at least 1. Applying ψ_d always increases the number of recoils by one.

Here are some examples of the four operators. For more such examples, see Table 1 in Section 6.

Example 23. The permutation $\sigma = [3214]$ does not start with 1 and is in the domain of all four of these operators, and the images are $\psi_1(\sigma) = [14325]$, $\psi_2(\sigma) = [41325]$, $\psi_u(\sigma) = [34215]$, and $\psi_d(\sigma) = [43215]$.

Example 24. The permutation $\tau = [126354]$ starts with a 1, so it is not in the domain of ψ_2 or ψ_u as these operations would duplicate the values of $\psi_d(\tau) = [2137465]$ and $\psi_1(\tau) = [1237465]$, respectively.

Operators on Rectangular Permutations			
Name	Domain	Insertion	Image of $\pi \in S_n$
ψ_1	Rect	$\rho_{1,1}$	$[1, \pi_1 + 1, \pi_2 + 1, \dots, \pi_n + 1]$
ψ_2	$\text{Rect} \setminus \psi_1(\text{Rect})$	$\rho_{1,2}$	$[\pi_1 + 1, 1, \pi_2 + 1, \dots, \pi_n + 1]$
ψ_u	$\text{Rect} \setminus \psi_1(\text{Rect})$	$\rho_{\pi_1,1}$	$[\pi_1, \pi_1 + 1, \pi_2 + \mathbb{1}_{\pi_2 > \pi_1}, \dots, \pi_n + \mathbb{1}_{\pi_n > \pi_1}]$
ψ_d	$\text{Rect} \setminus \{e_0\}$	$\rho_{\pi_1+1,1}$	$[\pi_1 + 1, \pi_1, \pi_2 + \mathbb{1}_{\pi_2 > \pi_1}, \dots, \pi_n + \mathbb{1}_{\pi_n > \pi_1}]$

A sequence of compositions of these operators can be abbreviated as a word in the alphabet $A_r = \{1, 2, u, d\}$, e.g., $\psi_u \circ \psi_d \circ \psi_1$ corresponds to the word $ud1$.

We prove in Section 4 that each rectangular permutation can be expressed uniquely as a composition of maps in $A_r^* := \{\psi_1, \psi_2, \psi_u, \psi_d\}$ applied to e_0 . The procedures of ψ_2 and ψ_d could be applied more widely, but since the domains are restricted to ensure uniqueness, they cannot be applied to any permutation in the image of ψ_1 . Since only ψ_1 can be applied to the identity in S_0 , the word encoding a permutation of positive length must end in 1. Words in A_r^* satisfying these restrictions form a regular language.

Lemma 25. *Let L_{Rect} be the language in A_r^* of sequences of compositions of positive length of $\{\psi_1, \psi_2, \psi_u, \psi_d\}$ applied to the permutation in S_0 .*

1. *The words of L_{Rect} are precisely the words in A_r^* that end in 1 with no 21 or u1.*
2. *A regular expression for L_{Rect} is $(1^*(2|u)^*d)^*1^+$.*

Proof. Only ψ_1 can be applied to the element of S_0 , so the first (rightmost) operator must be ψ_1 , and the word must end in 1. The domains of ψ_2 and ψ_u are restricted to exclude precisely the images of ψ_1 , so the substrings 21 and u1 are forbidden. Because the image of each operator consists of rectangular permutations, proven in Lemma 34, there are no other restrictions on the words in L_{Rect} .

One way to produce the regular expression is to consider splitting a word encoding a rectangular permutation by the d 's (if any) into strings in $\{1, 2, u\}^*$. Since there are no 21 or u1 substrings, the 1s must be to the left of the 2 and u symbols (if any), so the strings between the d 's must be of the form $1^*(2|u)^*$. Further, there must be a terminal 1, so the last substring must be of positive length and all 1's. Hence, a regular expression generating L_{Rect} is $(1^*(2|u)^*d)^*1^+$. \square

Understanding the structure of L_{Rect} lets us describe in more detail the permutation selection algorithm in Section 1. Let $L_{\text{Rect}}(n)$ denote the set of length n words in L_{Rect} . In the algorithm, we chose a word uniformly at random from $L_{\text{Rect}}(100)$. Using the results of Section 4, we converted this word to a rectangular permutation illustrated in Figure 1. Lastly, we applied β to get the evil-avoiding permutation illustrated in Figure 2.

The algorithm works by generating some $w \in L_{\text{Rect}}(100)$ character by character. Let us index w from right to left, so that $w = w(100)w(99) \cdots w(2)w(1)$.

1. Set $w(1) = 1$.
2. Let $a_\ell(n)$ be the number of length n strings containing no 21 or $u1$ and ending in ℓ . For $i \in \{2, 3, \dots, 100\}$, choose $w(i)$ according to the following probability distribution. If $w(i-1) = 1$, then $w(i) = 1$ or $w(i) = d$. Choose 1 or d with odds $a_1(101-i) : a_d(101-i)$ in favor of choosing 1. If $w(i-1) \neq 1$, then $w(i)$ could be any character. Choose 1, 2, d , or u with odds

$$a_1(101-i) : a_2(101-i) : a_d(101-i) : a_u(101-i).$$

Theorem 26. *This algorithm selects each word in L_{Rect} with equal probability.*

Proof. Let $w \in L_{\text{Rect}}$. Let A_i denote the event that a word of length 100 has i th letter equal to $w(i)$. Then, according to the algorithm,

$$\begin{aligned} P(w \text{ is selected}) &= P(A_1 \cap A_2 \cap \dots \cap A_{100}) \\ &= P(A_2 \cap \dots \cap A_{100} \mid A_1)P(A_1) \\ &= P(A_{100} \mid A_1, A_2, \dots, A_{99})P(A_{99} \mid A_1, \dots, A_{98}) \dots P(A_2 \mid A_1)P(A_1). \\ &= \frac{a_{w(100)}(1)}{a_{w(99)}(2)} \cdot \frac{a_{w(99)}(2)}{a_{w(98)}(3)} \cdot \dots \cdot \frac{a_{w(3)}(98)}{a_{w(2)}(99)} \frac{a_{w(2)}(99)}{a_{w(1)}(100)} \cdot 1 \end{aligned}$$

We can use the odds ratios above to write each $P(A_i \mid A_1, \dots, A_i)$ as a fraction. Doing so, we get a telescoping product that simplifies to

$$P(w \text{ is selected}) = \frac{1}{a_{w(1)}(100)} = \frac{1}{|L_{\text{Rect}}(100)|}.$$

Thus, the letter-by-letter generation is equivalent to drawing uniformly at random from $L_{\text{Rect}}(100)$, as desired. \square

3.2 Evil-avoiding permutations

We can similarly encode evil-avoiding permutations as compositions of size-increasing operators applied to the permutation e_0 in S_0 . These operators are ψ_p, ψ_q, ψ_r , and ψ_s , each of which has domain equal to a proper subset of evil-avoiding permutations. Compositions of these operators can be abbreviated as words in $A_e = \{p, q, r, s\}$.

The operator ψ_p works the same way as ψ_1 and $\rho_{1,1}$, but is restricted to evil-avoiding permutations with at least one recoil (i.e., to nonidentity permutations).

The operator ψ_q is defined on evil-avoiding permutations with at least one recoil. Let π be a permutation of size n in the domain of ψ_q . Let the least value involved in a recoil be t ; precisely, $\pi_t^{-1} > \pi_{t+1}^{-1}$ but for all $i < t$, we have $\pi_i^{-1} < \pi_{i+1}^{-1}$. Then calculate $\rho_{t+1,1}(\pi)$.

For most π in the domain, we will set $\psi_q(\pi) := \rho_{t+1,1}(\pi)$, but if π takes a specific form, we will need to make a slight modification. We say that π is (a, b) -sandwiched (or *sandwiched* for short) if it begins with a possibly empty increasing run of consecutive values $1, 2, \dots, a$ and ends with an increasing run $a+1, a+2, \dots, a+b$ for some nonnegative

integer a and positive integer b . If π is (a, b) -sandwiched, then we remove the value in $\rho_{t+1,1}(\pi)$ at index $n - b + 2$ and insert it at index $a + 2$ to create a new permutation $\gamma_{n-b+2,a+2} \circ \rho_{t+1,1}(\pi)$. If π has the above form, we set $\psi_q(\pi) := \gamma_{n-b+2,a+2} \circ \rho_{t+1,1}(\pi)$. A straightforward term-by-term evaluation shows this is equivalent to setting

$$\psi_q(\pi) := (a + b + 1, 1, 2, \dots, a + 1, \pi_{a+1} + 1, \pi_{a+2} + 1, \dots, \pi_{n-b} + 1, a + 2, a + 3, \dots, a + b).$$

Otherwise, we set $\psi_q(\pi) = \rho_{t+1,1}(\pi)$.

The operator ψ_r is defined on all evil-avoiding permutations of size at least 1. On an evil-avoiding permutation in S_n , this is $\rho_{1,n+1}$.

The operator ψ_s is defined on evil-avoiding permutations that have e_t as a suffix for some positive integer t . We allow there to be no values before the 1, so that any identity permutation of size at least 0 is included in the domain of ψ_s . This operator is a restriction of $\rho_{t+1,n+1} = \rho_{\pi_n+1,n+1}$.

Here are some examples of the four operators. For more such examples, see Section 6, Table 1.

Example 27. The permutation $\tau = [45123]$ is in the domain of all four operators ψ_p, ψ_q, ψ_r , and ψ_s . The images are $\psi_p(\tau) = [156234]$, $\psi_q(\tau) = [415623]$, $\psi_r(\tau) = [562341]$, and $\psi_s(\tau) = [561234]$.

Example 28. The permutation $\nu = [21453]$ does not end in e_t for $t \geq 1$ so it is not in the domain of ψ_s . The images under the other operators are $\psi_p(\nu) = [132564]$, $\psi_q(\nu) = [231564]$, and $\psi_r(\nu) = [325641]$.

Operators on Evil-Avoiding Permutations			
Name	Domain	Insertion	Image of $\pi \in S_n$
ψ_p	$\text{Evil} \setminus \{e_n n \geq 0\}$	$\rho_{1,1}$	$[1, \pi_1 + 1, \pi_2 + 1, \dots, \pi_n + 1]$
ψ_q	$\text{Evil} \setminus \{e_n n \geq 0\}$	$\rho_{t+1,1}$ if π is not sandwiched $\gamma_{n-b+2,a+2} \circ \rho_{t+1,1}$ otherwise	$[t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_n + \mathbb{1}_{\pi_n > t}]$ OR $[t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_{a+1} + \mathbb{1}_{\pi_{a+1} > t},$ $\pi_{n-b+2} + \mathbb{1}_{\pi_{n-b+2} > t},$ $\pi_{a+2} + \mathbb{1}_{\pi_{a+2} > t}, \dots, \pi_{n-b+1} + \mathbb{1}_{\pi_{n-b+1} > t},$ $\pi_{n-b+3} + \mathbb{1}_{\pi_{n-b+3} > t}, \dots, \pi_n + \mathbb{1}_{\pi_n > t}]$
ψ_r	$\text{Evil} \setminus \{e_0\}$	$\rho_{1,n+1}$	$[\pi_1 + 1, \pi_2 + 1, \dots, \pi_n + 1, 1]$
ψ_s	$\{e_n n \geq 0\} \cup$ $\bigcup_k \psi_s^k \circ \psi_r(\text{Evil} \setminus \{e_0\})$	$\rho_{\pi_n+1,n+1}$	$[\pi_1 + 1, \pi_2 + 1, \dots, 1, 2, \dots, \pi_n, \pi_n + 1]$

In Section 5, we show that every evil-avoiding permutation can be written uniquely as a composition of these operators applied to the permutation of S_0 . Because of domain restrictions, not all words in $A_e^* = \{\psi_p, \psi_q, \psi_r, \psi_s\}$ are valid.

Lemma 29. Let L_{Evil} be the language in A_e^* of compositions of $\{\psi_p, \psi_q, \psi_r, \psi_s\}$ of positive length applied to the permutation in S_0 .

1. The words of L_{Evil} are precisely the words in A_e^* ending in s with no sp or sq and neither p nor q before the terminal string of s 's.

2. A regular expression for L_{Evil} is $((p|q)^*s^*r)^*s^+$.

Proof. The operator ψ_s can be applied to the identity e_0 of S_0 , while ψ_p, ψ_q , and ψ_r cannot, so a word in L_{Evil} must end in s . The domain of ψ_s is restricted so that it can only be applied after ψ_r or ψ_s , so the substrings sp and sq are forbidden. The operators ψ_p and ψ_q cannot be applied to the identity $e_n \leftrightarrow s^n$, so before the terminal string of s 's there can only be an r (or nothing). Because the image of each operator consists of evil-avoiding permutations, proven in Section 5, there are no other restrictions on the words in L_{Evil} .

To obtain the regular expression for such words, split on the r 's, if any, to produce a sequence of words in $\{p, q, s\}^*$. The last word must be a string of s 's of positive length, so it can be described by s^+ . In each earlier word, the s 's must come after all of the p 's and q 's, so it can be described by $(p|q)^*s^*$. Thus, a regular expression for L_{Evil} is $((p|q)^*s^*r)^*s^+$. \square

3.3 Definition of the bijection

The bijection $\beta : \text{Rect} \rightarrow \text{Evil}$ may be decomposed into three intermediate bijections

$$\text{Rect} \xrightarrow{b_r} L_{\text{Rect}} \xrightarrow{b} L_{\text{Evil}} \xrightarrow{b_e^{-1}} \text{Evil}$$

each of which we explain in this section.

Theorem 30. *There is a bijection $b_r : \text{Rect} \rightarrow L_{\text{Rect}}$ given by the below operations:*

1. *writing a rectangular permutation as a composition of maps in $\{\psi_1, \psi_2, \psi_u, \psi_d\}$,*
2. *abbreviating this composition of maps by substituting $\psi_1 \mapsto 1$, $\psi_2 \mapsto 2$, $\psi_u \mapsto u$, $\psi_d \mapsto d$, and suppressing \circ signs.*

This bijection has inverse $b_r^{-1} : L_{\text{Rect}} \rightarrow \text{Rect}$ given by:

1. *substituting $1 \mapsto \psi_1$, $2 \mapsto \psi_2$, $u \mapsto \psi_u$, $d \mapsto \psi_d$ and inserting \circ signs between each map,*
2. *evaluating the composition of maps.*

Theorem 31. *There is a length-preserving bijection $b : L_{\text{Rect}} \rightarrow L_{\text{Evil}}$ given by the below operations:*

1. *substituting $2 \mapsto p$, $u \mapsto q$, $d \mapsto r$, and $1 \mapsto s$,*
2. *reversing the prefix before the last r , if it exists.*

This bijection has inverse $b^{-1} : L_{\text{Evil}} \rightarrow L_{\text{Rect}}$ given by:

1. *reversing the prefix before the last r , if it exists,*
2. *substituting $p \mapsto 2$, $q \mapsto u$, $r \mapsto d$ and $s \mapsto 1$.*

The fact that b is a bijection and b^{-1} is its inverse follows immediately from the regular expressions $(1^*(u|2)^*d)^*1^+$ and $((p|q)^*s^*r)^*s^+$ for L_{Rect} and L_{Evil} .

Theorem 32. *There is a bijection $b_e : \text{Evil} \rightarrow L_{\text{Evil}}$ given by the below operations:*

1. *writing an evil-avoiding permutation as a composition of maps in $\{\psi_p, \psi_q, \psi_r, \psi_w\}$,*
2. *abbreviating this composition of maps by substituting $\psi_p \mapsto p$, $\psi_q \mapsto q$, $\psi_r \mapsto r$, $\psi_s \mapsto s$, and suppressing \circ signs.*

This bijection has inverse $b_e^{-1} : L_{\text{Evil}} \rightarrow \text{Evil}$ given by:

1. *substituting $p \mapsto \psi_p$, $q \mapsto \psi_q$, $r \mapsto \psi_r$, $s \mapsto \psi_s$ and inserting \circ signs between each map,*
2. *evaluating the composition of maps.*

Example 33. The rectangular permutation

$$[4, 1, 2, 5, 6, 3, 9, 8, 10, 7, 11, 13, 12, 15, 14, 17, 18, 19, 20, 16]$$

may, after applying b_r , be encoded as the word

$$22uud1dud11d1d1uuud1$$

in L_{Rect} . To apply the map b to this word, we substitute $2 \mapsto p$, $u \mapsto q$, $d \mapsto r$, and $1 \mapsto s$ to get

$$ppqqr sr sr qr s sr sr sqqrs$$

and then reverse the prefix before the last r to get

$$qqqsr sr s sr qr sr qpprs$$

in L_{Evil} . After decoding this word using b_e^{-1} , we get the evil-avoiding permutation

$$[3, 4, 5, 1, 12, 11, 18, 19, 15, 16, 17, 20, 13, 14, 8, 9, 10, 6, 7, 2].$$

4 A regular expression for rectangular permutations

We next establish the encoding of rectangular permutations of positive length as words in L_{Rect} , which is generated by the regular expression $(1^*(2|u)^*d)^*1^+$. Throughout the proofs in this section, we say that two values π_i, π_j in a permutation π are *adjacent* if $|\pi_i - \pi_j| = 1$. In particular, we do not require $|i - j| = 1$.

Lemma 34. *If π is rectangular, then any of the permutations $\psi_1(\pi), \psi_2(\pi), \psi_u(\pi)$, and $\psi_d(\pi)$ that are defined are also rectangular.*

Proof. These operators can be viewed as inserting an element into the first position for ψ_1, ψ_u, ψ_d or second position for ψ_2 . We only need to demonstrate that there are no patterns 2413, 2431, 4213, and 4231 using the added element since any patterns without the added element would have been present in π .

The added element of $\psi_1(\pi)$ is a leading 1. Any pattern involving this entry has a leading 1, but none of the forbidden patterns start with a 1, so if π is rectangular, so is $\psi_1(\pi)$.

The added elements of ψ_u and ψ_d are in the first position and are adjacent to the value in the second position. In all of the forbidden patterns, the first two values in the pattern are not adjacent. Thus, any forbidden pattern in $\psi_u(\pi)$ or $\psi_d(\pi)$ using the first value cannot also use the second value. Then, we can replace the first value with the second value to produce the same forbidden pattern, which means π also would have to have that forbidden pattern. Thus, if π is rectangular, so are $\psi_u(\pi)$ and $\psi_d(\pi)$ (if defined).

The added element of $\psi_2(\pi)$ is a 1 in the second position. If the 1 is used in a pattern, the pattern must have a 1 in the first or second position, but none of 2413, 2431, 4213, or 4231 have a 1 in the first or second position. Thus, if π is rectangular, so is $\psi_2(\pi)$ (if defined). \square

Lemma 35. *For $n \geq 1$, every rectangular permutation in $\text{Rect}(n)$ is uniquely expressible as a map in the set $\{\psi_1, \psi_2, \psi_u, \psi_d\}$ applied to an element of $\text{Rect}(n - 1)$.*

Proof. We claim that every rectangular permutation π must have adjacent first two elements or a 1 in one of the first two positions. To see why, assume otherwise. Then the first two elements π_1, π_2 are not adjacent. Let v be a value at least $\min(\pi_1, \pi_2)$ and at most $\max(\pi_1, \pi_2)$. Then the positions $\{1, 2, \pi_1^{-1}, \pi_v^{-1}\}$ form a 2413, 2431, 4213, or 4231, which is a contradiction. Thus, the claim is shown.

If $\pi_1 = 1$, then π is in the image of ψ_1 . If $\pi_2 = 1$, then π is in the image of ψ_2 if $\pi_1 > 2$ or in the image of ψ_d if $\pi_1 = 2$. Otherwise, $|\pi_1 - \pi_2| = 1$. If $\pi_2 = \pi_1 + 1$ with $\pi_1 > 1$ then π is in the image of ψ_u . If $\pi_2 = \pi_1 - 1$ then π is in the image of ψ_d . The restrictions of the domains for the four operators mean there is no ambiguity in deciding which operator produced π . \square

The following corollary follows immediately by induction.

Corollary 36. *Every rectangular permutation can be expressed uniquely as a composition of ψ_1, ψ_2, ψ_u , and ψ_d .*

Further, we can determine the number of recoils in a permutation from the corresponding composition.

Lemma 37. *The operators ψ_1, ψ_2 , and ψ_u do not change the number of recoils of a permutation. The operator ψ_d increases the number of recoils by 1.*

This lemma can be checked in a straightforward manner by examining each recoil π_i in π and the recoil's image under ψ_1, ψ_2, ψ_u , or ψ_d . Under any of the first three maps, this process accounts for all of the recoils in the image permutation, but under the fourth map,

there is one recoil unaccounted for in the image permutation, namely the one formed by π_1 .

The following corollary is then immediate.

Corollary 38. *The permutations in $\text{Rect}(n, k)$ are encoded uniquely by the words in L_{Rect} of length n with k d 's.*

5 A regular expression for evil-avoiding permutations

We now establish the encoding of evil-avoiding permutations of positive length as words in L_{Evil} , which is generated by the regular expression $((p|q)^*s^*r)^*s^+$.

The proof of this bijection involves some machinery in [8] used to analyze and count evil-avoiding permutations. We use slightly different notation than in [8] in some places, which we indicate explicitly.

Definition 39. [8] Let $\text{St}(n, k)$ be the set of permutations $\pi = (\pi_1, \dots, \pi_n)$ so that

1. $\pi_1 = 1$,
2. π is evil-avoiding,
3. π has k recoils.

An element of $\text{St}(n, k)$ is called a *k-Grassmannian permutation*.

Definition 40. A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of a nonnegative integer L is a k -tuple of nonnegative integers so that $\sum_{i=1}^k \lambda_i = L$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$.

Definition 41. [8] A partition λ is *valid* for n if λ is nonempty, it is contained in a $\text{length}(\lambda) \times (n - \text{length}(\lambda))$ rectangle, and it is not the whole rectangle.

This definition does not require λ to be a partition of n .

Definition 42. [8] For $1 \leq k \leq n - 2$, let $\text{ParSeq}(n, k)$ denote the set of all sequences of partitions $(\lambda^1, \dots, \lambda^k)$ such that each λ^i is valid for n , and for all $1 \leq i \leq k - 1$, if ℓ is the smallest part of λ^i , then the first $(n - \ell)$ parts of λ^{i+1} are equal. If $k = 0$, then $\text{ParSeq}(n, k)$ consists of one element, the empty sequence.

The sets $\text{St}(n, k)$ and $\text{ParSeq}(n, k)$ are natural to examine, as we explain. Observe that the sets $\text{Evil}(n - 1, k)$ and $\text{St}(n, k)$ can be bijected by $f : \text{Evil}(n - 1, k) \rightarrow \text{St}(n, k)$ with $f(\pi) = (1, \pi_1 + 1, \dots, \pi_{n-1} + 1)$. The inverse of f is given by $f^{-1}(\pi) = (\pi_2 - 1, \dots, \pi_n - 1)$.

In preparation for the next proposition, we recall that the *Lehmer code* of a permutation π is the tuple $c = (c_1, \dots, c_n)$ where $c_i = |\{j \mid j > i, \pi_j < \pi_i\}|$.

Proposition 43. [8] *The sets $\text{St}(n, k)$ and $\text{ParSeq}(n, k)$ are in bijection. To obtain the forward bijection P ,² we associate to each $\pi \in \text{St}(n, k)$ a sequence of partitions $P(\pi) = (\lambda^1, \dots, \lambda^k)$ as follows. Write the Lehmer code of π^{-1} as $c(\pi^{-1}) = c = (c_1, \dots, c_n)$; since π^{-1} has k descents, c has k descents (i.e. values c_i where $c_i > c_{i+1}$) in positions we denote by a_1, \dots, a_k (where the position of a descent c_i is the index i). Set $a_0 = 0$. For $1 \leq i \leq k$, define $\lambda^i = \underbrace{(n - a_i, \dots, n - a_i)}_{a_i} - \underbrace{(0, \dots, 0)}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \dots, c_{a_i}$ where subtraction of tuples is coordinatewise.*

Example 44. If $\pi \in \text{St}(5, 2)$ is [13254], then the Lehmer code of π^{-1} is $(0, 1, 0, 1, 0)$. This Lehmer code has two descents, one at index 2 and one at index 4, so we set $a_0 = 0, a_1 = 2, a_2 = 4$. Then

$$\lambda^1 = (5 - 2, 5 - 2) - (c_1, c_2) = (3, 3) - (0, 1),$$

which we subtract coordinatewise to get $(3, 2)$. We may also calculate

$$\lambda^2 = \underbrace{(5 - 4, \dots, 5 - 4)}_4 - (0, 0, c_3, c_4) = (1, 1, 1, 1) - (0, 0, 0, 1) = (1, 1, 1, 0).$$

We drop the trailing 0 for brevity. We then say that $P(\pi) = (\lambda^1, \lambda^2) = ((3, 2), (1, 1, 1))$.

The inverse map $P^{-1} : \text{ParSeq}(n, k) \rightarrow \text{St}(n, k)$ can be described as follows. Let $(\lambda^1, \dots, \lambda^k) \in \text{ParSeq}(n, k)$, and let (f_1, \dots, f_k) be the sequence of first parts of $\lambda^1, \dots, \lambda^k$, i.e., $f_i = \lambda^i_1$. Then

$$(((\underbrace{f_1, \dots, f_1}_{n-f_1}) - \lambda^1) + ((\underbrace{f_2, \dots, f_2}_{n-f_2}) - \lambda^2) + \dots + ((\underbrace{f_k, \dots, f_k}_{n-f_k}) - \lambda^k))$$

is the code of a permutation π^{-1} of $\pi \in \text{St}(n, k)$. We define $P^{-1}(\lambda^1, \dots, \lambda^k) = \pi$.

In [8], Kim and Williams also define the injective maps Ψ_1 , Ψ_2 , and $\Phi_{i,k,n}$, whose images partition the set $\text{ParSeq}(n, k)$ into disjoint parts and make it simpler to count recursively.

Definition 45. [8] For $k \geq 1$, the map $\Psi_1 : \text{ParSeq}(n - 1, k) \rightarrow \text{ParSeq}(n, k)$ takes $(\lambda^1, \dots, \lambda^k)$ to (μ^1, \dots, μ^k) where, for all $i \geq 1$, μ^i is obtained from λ^i by duplicating its first part.

Example 46. $\Psi_1((2), (1, 1)) = ((2, 2), (1, 1, 1))$.

Definition 47. [8] For $k \geq 1$, the map $\Psi_2 : \text{ParSeq}(n - 1, k) \rightarrow \text{ParSeq}(n, k)$ takes $(\lambda^1, \dots, \lambda^k)$ to (μ^1, \dots, μ^k) where, for all $i \geq 2$, μ^i is obtained from λ^i by duplicating its first part. For $i = 1$, we split into cases based on whether all parts of $\lambda^1 = (\lambda^1_1, \dots, \lambda^1_r)$ are equal. If so, then let $\mu^1 = (\lambda^1_1 + 1, \lambda^1_1, \dots, \lambda^1_r)$. Otherwise, we define $\mu^1 = (\lambda^1_1 + 1, \lambda^1_2, \dots, \lambda^1_r)$.

²This operator is denoted Ψ in [8], but we call it P instead to emphasize that it is a bijection and not an operator on permutations or partition sequences.

Example 48. $\Psi_2((2), (1, 1)) = ((3, 2), (1, 1, 1))$ and $\Psi_2((3, 2), (1, 1, 1)) = (4, 2), (1, 1, 1, 1)$.

Definition 49. [8] For $k < i < n$, we define $\Phi_{i,k,n} : \text{ParSeq}(i, k-1) \rightarrow \text{ParSeq}(n, k)$ to be the map which takes $(\lambda^1, \dots, \lambda^{k-1})$ to $((i-1), \mu^1, \dots, \mu^{k-1})$, where μ^j is obtained from λ^j by duplicating the first part of λ^j $n-i$ times.

Example 50. $\Phi_{4,2,5}((2, 1)) = ((3), (2, 2, 1))$.

Proposition 51. [8] *The set $\text{ParSeq}(n, k)$ equals the disjoint union*

$$\Psi_1(\text{ParSeq}(n-1, k)) \cup \Psi_2(\text{ParSeq}(n-1, k)) \cup \prod_{i=k-1}^{n-1} \Phi_{i,k,n}(\text{ParSeq}(i, k-1)).$$

Kim and Williams used this to prove a recurrence for the size of $\text{ParSeq}(n, k)$ in [8] Proposition 3.14. It also implies the following corollary:

Corollary 52. *Every partition sequence in $\text{ParSeq}(n_0, k_0)$ is a composition of maps in $\{\Psi_1, \Psi_2, \Phi_{i,k,n}\}$ applied to an empty sequence in some $\text{ParSeq}(n_1, 0)$.*

Theorem 53. *Every evil-avoiding permutation can be written uniquely as a composition of maps in the set $\{\psi_p, \psi_q, \psi_r, \psi_s\}$ applied to the identity in S_0 .*

Let $\psi_{i,k,n}$ be a map sending $(\pi_1, \dots, \pi_{i-1}) \mapsto (\pi_1 + n - i, \dots, \pi_{i-1} + n - i, 1, 2, \dots, n - i)$. To prove this theorem, we provide alternative descriptions of maps $\psi_p, \psi_q, \psi_{i,k,n}$ which allow us to use a result from [8].

Theorem 54. *The maps $\psi_p, \psi_q, \psi_{i,k,n}$ may be written as*

$$\begin{aligned} \psi_p(\pi) &= f^{-1} \circ P^{-1} \circ \Psi_1 \circ P \circ f(\pi) \\ \psi_q(\pi) &= f^{-1} \circ P^{-1} \circ \Psi_2 \circ P \circ f(\pi) \\ \psi_{i,k,n}(\pi) &= f^{-1} \circ P^{-1} \circ \Phi_{i,k,n} \circ P \circ f(\pi). \end{aligned}$$

Proof. To show that $\psi_p(\pi) = (1, \pi_1 + 1, \dots, \pi_{n-2} + 1) = f^{-1} \circ P^{-1} \circ \Psi_1 \circ P \circ f(\pi)$, we compute $P \circ f(\pi)$ and compare with $P \circ f(1, \pi_1 + 1, \dots, \pi_{n-2} + 1)$. Let $c = (c_1, \dots, c_{n-1})$ be the Lehmer code of the inverse of $f(\pi)$, and let a_1, \dots, a_k be the positions of the descents in c . Then the i th partition in the partition sequence $P \circ f(\pi)$ may be written as

$$\lambda^i = \underbrace{(n-1-a_i, \dots, n-1-a_i)}_{a_i} - \underbrace{(0, \dots, 0, c_{a_{i-1}+1}, \dots, c_{a_i})}_{a_{i-1}}.$$

To get the Lehmer code of the inverse of $f(1, \pi_1 + 1, \dots, \pi_{n-2} + 1)$ from c , we insert a 0 at the start of c because the first entry of the Lehmer code of the inverse of a permutation counts the number of values in the permutation before the 1. Thus, the i th partition in the partition sequence $P \circ f(1, \pi_1 + 1, \dots, \pi_{n-2} + 1)$ is

$$\mu^i = \underbrace{(n-(a_i+1), \dots, n-(a_i+1))}_{a_i+1} - \underbrace{(0, \dots, 0, c_{a_{i-1}+1}, \dots, c_{a_i})}_{a_{i-1}+1}.$$

Comparing λ^i with μ^i , we see that the latter is the same as the former, except with an extra copy of $n - 1 - a_i$, which is the largest part of λ^i . (Observe that in the expression for λ^i , the subtrahend clearly has leading zeroes when $i \geq 2$, and when $i = 1$, we have $c_1 = 0$.) Note also that Ψ_1 acts on a partition sequence by mapping each partition to one with the first part duplicated, so $\Psi_1 \circ P \circ f(\pi) = P \circ f(1, \pi_1 + 1, \dots, \pi_{n-2} + 1)$, as desired.

Let $\pi_{ikn} = (\pi_1 + n - i, \pi_2 + n - i, \dots, \pi_{i-1} + n - i, 1, 2, \dots, n - i)$. Now, we show that

$$\psi_{i,k,n}(\pi) = \pi_{ikn} = f^{-1} \circ P^{-1} \circ \Phi_{i,k,n} \circ P \circ f(\pi).$$

(We consider $\psi_{i,k,n}$ before ψ_q to make the exposition clearer.) Using a similar technique as for ψ_p , we compute $P \circ f(\pi)$ and compare with $P \circ f(\pi_{ikn})$. As computed earlier in this proof, the j th partition in the partition sequence π may be written as

$$\lambda^j = \underbrace{(i - a_j, \dots, i - a_j)}_{a_j} - \underbrace{(0, \dots, 0)}_{a_{j-1}}, c_{a_{j-1}+1}, \dots, c_{a_j}$$

with c defined as above.

To get the Lehmer code of the inverse of $f(\pi_{ikn})$, observe that

$$\begin{aligned} \pi_{ikn} &= (\pi_1 + n - i, \pi_2 + n - i, \dots, \pi_{i-1} + n - i, 1, 2, \dots, n - i), \\ f(\pi_{ikn}) &= (1, \pi_1 + n - i + 1, \pi_2 + n - i + 1, \dots, \pi_{i-1} + n - i + 1, 2, \dots, n - i + 1), \\ &\text{and} \\ (f(\pi_{ikn}))^{-1} &= (1, i + 1, i + 2, \dots, n, \pi_1^{-1} + 1, \dots, \pi_{i-1}^{-1} + 1), \end{aligned}$$

so we have as our Lehmer code $(0, \underbrace{i - 1, \dots, i - 1}_{n-i}, c_2, c_3, \dots, c_i)$.

Thus, the first partition in the partition sequence $P \circ f(\pi_{ikn})$ is

$$\begin{aligned} \mu^1 &= \underbrace{(n - (n - i + 1), \dots, n - (n - i + 1))}_{n-i+1} - \underbrace{(0, i - 1, \dots, i - 1)}_{n-i} \\ &= (i - 1, \underbrace{0, \dots, 0}_{n-i}) \\ &= (i - 1). \end{aligned}$$

The j th partition, for $j > 1$, in the partition sequence $P \circ f(\pi_{ikn})$ is

$$\begin{aligned} \mu^j &= \underbrace{(n - (a_{j-1} + n - i), \dots, n - (a_{j-1} + n - i))}_{a_{j-1}+n-i} - \underbrace{(0, \dots, 0)}_{a_{j-2}+n-i}, c_{a_{j-2}+1}, \dots, c_{a_{j-1}} \\ &= \underbrace{(i - a_{j-1}, \dots, i - a_{j-1})}_{a_{j-1}+n-i} - \underbrace{(0, \dots, 0)}_{a_{j-2}+n-i}, c_{a_{j-2}+1}, \dots, c_{a_{j-1}}. \end{aligned}$$

Comparing λ^j with μ^{j+1} , we observe that the latter is the same as the former, except with $n - i$ leading copies of $i - a_j$. Note that $\Phi_{i,k,n}$ takes a partition sequence

$(\lambda^1, \lambda^2, \dots, \lambda^{k-1})$ to $((i-1), \mu^1, \mu^2, \dots, \mu^{k-1})$, where μ^j is obtained from λ^j by duplicating the first part of λ^j $n-i$ times, so the partition sequences match up as desired, and in fact $\Phi_{i,k,n} \circ P \circ f(\pi) = P \circ f(\pi_{ikn})$.

Finally, we consider ψ_q , and divide into cases based on whether π is (a, b) -sandwiched.

Case 1: π is (a, b) -sandwiched.

Then

$$g(\pi) = (a+b+1, 1, 2, \dots, a+1, \pi_{a+1}+1, \pi_{a+2}+1, \dots, \pi_{n-b-2}+1, a+2, a+3, \dots, a+b).$$

We write the Lehmer code of $f(\pi)^{-1}$ as $c = (c_1, c_2, \dots, c_{n-1})$. where values c_1, \dots, c_{a+1} are equal to 0 and values $c_{a+2}, \dots, c_{a+b+1}$ are equal to $n-2-a-b$. Now observe that $a_1 = a+b+1$ (i.e., the first descent in c is at index $a+b+1$) because the value $a+b+1$ appears at the end of $f(\pi)$ and the value $a+b+2$ appears earlier (i.e., after the first ascending run and before the second one). Thus, the number of values that appear before and exceed $a+b+1$ is greater than the number of values that appear before and exceed $a+b+2$. Equivalently, $c_{a+b+1} > c_{a+b+2}$.

We can compute

$$\begin{aligned} \lambda^1 &= \underbrace{(n-1-a_1, \dots, n-1-a_1)}_{a_1} - (c_1, c_2, \dots, c_{a_1}) \\ &= \underbrace{(n-1-(a+b+1), \dots, n-1-(a+b+1))}_{a+b+1} \\ &\quad - \underbrace{(0, \dots, 0)}_{a+1}, \underbrace{(n-2-a-b, \dots, n-2-a-b)}_b \\ &= \underbrace{(n-2-a-b, \dots, n-2-a-b)}_{a+1}, \end{aligned}$$

i.e., all nonzero parts of λ^1 are equal.

For $i > 1$, we can write

$$\lambda^i = \underbrace{(n-1-a_i, \dots, n-1-a_i)}_{a_i} - \underbrace{(0, \dots, 0)}_{a_{i-1}}, c_{a_{i-1}+1}, \dots, c_{a_i}.$$

Now observe that the Lehmer code for $f(g(\pi))^{-1}$ can be written as

$$(0, \underbrace{1, \dots, 1}_{a+1}, \underbrace{n-1-a-b, \dots, n-1-a-b}_{b-1}, 0, c_{a+b+1}, c_{a+b+2}, \dots, c_n).$$

From this, we can calculate that the first partition in the partition sequence $P \circ f(g(\pi))$ is

$$\begin{aligned} \mu^1 &= \underbrace{(n-a_1, \dots, n-a_1)}_{a_1} - (0, \underbrace{1, \dots, 1}_{a+1}, \underbrace{n-1-a-b, \dots, n-1-a-b}_{b-1}) \\ &= \underbrace{(n-(a+b+1), \dots, n-(a+b+1))}_{a+b+1} \end{aligned}$$

$$\begin{aligned}
& - (0, \underbrace{1, \dots, 1}_{a+1}, \underbrace{n-1-a-b, \dots, n-1-a-b}_{b-1}) \\
& = (n-1-a-b) \underbrace{(n-2-a-b, \dots, n-2-a-b)}_{a+1},
\end{aligned}$$

and the i th partition for $i > 1$ is

$$\begin{aligned}
\mu^i & = \underbrace{(n - (a_i + 1), \dots, n - (a_i + 1))}_{a_i+1} - \underbrace{(0, \dots, 0)}_{a_{i-1}+1}, c_{a_{i-1}+1}, \dots, c_{a_i} \\
& = \underbrace{(n - 1 - a_i, \dots, n - 1 - a_i)}_{a_i+1} - \underbrace{(0, \dots, 0)}_{a_{i-1}+1}, c_{a_{i-1}+1}, \dots, c_{a_i}.
\end{aligned}$$

Note that in the first subcase, when π is (a, b) -sandwiched, Ψ_2 takes $(\lambda^1, \dots, \lambda^k)$ to (μ^1, \dots, μ^k) , where μ^1 is λ^1 with the first part duplicated and then increased by 1, and for $i > 1$, μ^i is λ^i with the first part duplicated. This is precisely what we found, so $\Psi_2 \circ P \circ f(\pi) = P \circ f(g(\pi))$. It is a tedious though straightforward term-by-term arithmetic calculation to verify

$$\begin{aligned}
g(\pi) & = [t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_{a+1} + \mathbb{1}_{\pi_{a+1} > t}, \pi_{n-b+2} + \mathbb{1}_{\pi_{n-b+2} > t}, \\
& \quad \pi_{a+2} + \mathbb{1}_{\pi_{a+2} > t}, \dots, \pi_{n-b+1} + \mathbb{1}_{\pi_{n-b+1} > t}, \pi_{n-b+3} + \mathbb{1}_{\pi_{n-b+3} > t}, \dots, \pi_n + \mathbb{1}_{\pi_n > t}].
\end{aligned}$$

Case 2: π is not (a, b) -sandwiched.

The proof is similar to the one for the previous case. As computed earlier, the i th partition in the partition sequence $P \circ f(\pi)$ may be written as

$$\lambda^i = \underbrace{(n - 1 - a_i, \dots, n - 1 - a_i)}_{a_i} - \underbrace{(0, \dots, 0)}_{a_{i-1}}, c_{a_{i-1}+1}, \dots, c_{a_i}$$

and, in particular,

$$\lambda^1 = \underbrace{(n - 1 - a_1, \dots, n - 1 - a_1)}_{a_1} - (c_1, c_2, \dots, c_{t+1}).$$

Then the Lehmer code of

$$f(t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_{n-2} + \mathbb{1}_{\pi_{n-2} > t})^{-1}$$

is

$$(0, c_2 + 1, c_3 + 1, \dots, c_{t+1} + 1, 0, c_{t+2}, c_{t+3}, \dots, c_n)$$

with $a_1 = t + 1$. We may then write

$$\mu^1 = \underbrace{(n - a_1, \dots, n - a_1)}_{a_1} - (c_1, c_2 + 1, c_3 + 1, \dots, c_{t+1})$$

and then comparing μ^1 with λ^1 part by part, we see that μ^1 is identical to λ^1 except with the first part incremented by 1. Since there is a 0 at index $t + 2$ in the Lehmer code for π , offsetting the Lehmer code for $f(t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_{n-2} + \mathbb{1}_{\pi_{n-2} > t})^{-1}$ back by 1, we get for $i > 1$,

$$\begin{aligned} \mu^i &= \underbrace{(n - (a_i + 1), \dots, n - (a_i + 1))}_{a_i+1} - \underbrace{(0, \dots, 0, c_{a_{i-1}+1}, \dots, c_{a_i})}_{a_{i-1}+1} \\ &= \underbrace{(n - 1 - a_i, \dots, n - 1 - a_i)}_{a_i+1} - \underbrace{(0, \dots, 0, c_{a_{i-1}+1}, \dots, c_{a_i})}_{a_{i-1}+1}. \end{aligned}$$

So in the second subcase, when π is not (a, b) -sandwiched, Ψ_2 takes $(\lambda^1, \dots, \lambda^k)$ to (μ^1, \dots, μ^k) where μ^1 is λ^1 with the first part increased by 1, and for $i > 1$, μ^i is λ^i with the first part duplicated. We find that $\Psi_2 \circ P \circ f(\pi) = P \circ f(t + 1, \pi_1 + \mathbb{1}_{\pi_1 > t}, \dots, \pi_{n-2} + \mathbb{1}_{\pi_{n-2} > t})$. So the forms for $\psi_p, \psi_q, \psi_{i,k,n}$ stated at the start of this proposition are equivalent to the forms provided in Section 3. \square

Proposition 55. *If $k = 0$, then $\text{Evil}(n, k)$ contains only the identity permutation of length n . For $k > 0$,*

$$\text{Evil}(n, k) = \psi_p(\text{Evil}(n - 1, k)) \cup \psi_q(\text{Evil}(n - 1, k)) \cup \prod_{i=k-1}^{n-1} \psi_{i,k,n}(\text{Evil}(i, k - 1))$$

and the terms on the right-hand side are pairwise disjoint.

Proof. The first sentence of the proposition is true because every nonidentity permutation has at least 1 recoil. To see the second part, observe that from the proof of Proposition 3.14 in [8] we have a very similar result, namely

$$\text{ParSeq}(n, k) = \Psi_1(\text{ParSeq}(n - 1, k)) \cup \Psi_2(\text{ParSeq}(n - 1, k)) \cup \prod_{i=k-1}^{n-1} \Phi_{i,k,n}(\text{ParSeq}(i, k - 1)),$$

where the terms on the right-hand side are disjoint.

After conjugating both sides by $P \circ f$, we get the desired result. \square

Proposition 56. *The operators ψ_p, ψ_q , and ψ_s preserve the number of recoils. The operator ψ_r increases the number of recoils by one.*

Proof. Observe that each value in $\psi_v(\pi)$ for any $v \in \{p, q, s\}$ depends on at most one value in π , and no two values in $\psi_v(\pi)$ depend on the same value in π . Throughout the proof, we say that a recoil x at index i in π is mapped to a recoil y at index j in $\sigma = \psi_v(\pi)$ if the expression for σ_j depends on π_i and the evaluation y of σ_j at $x = \pi_i$ is a recoil.

To see that ψ_p preserves the number of recoils, observe that a recoil π_i in π is mapped to recoil $\pi_i + 1$, and this map produces all recoils in $\psi_p(\pi)$. (The value 1 is first in $\psi_p(\pi)$ so cannot be a recoil.)

To see that ψ_q preserves the number of recoils, we casework on whether π is (a, b) -sandwiched. If not, observe that a recoil π_i in π is mapped to recoil $\pi_i + \mathbb{1}_{\pi_i > t}$ in $\psi_q(\pi)$ regardless of whether $\pi_i > t$ or $\pi_i < t$ or $\pi_i = t$, and this map produces all recoils in $\psi_q(\pi)$. If π is (a, b) -sandwiched, then π always contains the recoil $a + b$, and $\psi_q(\pi) = (a + b + 1, 1, 2, \dots, a + 1, \pi_{a+1} + 1, \pi_{a+2} + 1, \dots, \pi_{n-b} + 1, a + 2, a + 3, \dots, a + b)$ necessarily contains $a + b$ as a recoil. Any other recoil in π takes the form π_i for $a + 1 \leq \pi \leq n - b$ and is mapped to $\pi_i + 1$. Now, the values $1, 2, \dots, a + b - 1$ fail to be recoils in π and the values $1, 2, \dots, a + b - 1, a + b + 1$ fail to be recoils in $\psi_q(\pi)$, so we have accounted for all possible recoils in π and its image $\psi_q(\pi)$, and shown these recoils to be in bijection.

To see that ψ_s preserves the number of recoils, observe that the smallest recoil in π is t , and that a recoil π_i in π is mapped to $\pi_i + 1$ in $\psi_s(\pi)$. This map produces all recoils in $\psi_s(\pi)$. (The recoils in $\psi_s(\pi)$ all appear before the 1.)

To see that ψ_r increases the number of recoils by one, observe that a recoil π_i in π is mapped to recoil $\pi_i + 1$ in $\psi_r(\pi)$, and this map produces all recoils in $\psi_r(\pi)$ except for the recoil 1. \square

Now, making the observation that $\psi_{i,k,n} = \psi_s^{n-i-1} \circ \psi_r$, we can describe the set of words in $A_e = \{p, q, r, s\}$ that the set of evil-avoiding permutations is in bijection with. This set was stated in Lemma 29.

Corollary 57. *The words of L_{Evil} encode evil-avoiding permutations uniquely.*

5.1 Other consequences

In a word of L_{Rect} encoding a rectangular permutation, the number of recoils in the permutation equals the number of d 's. The bijection converts d 's to r 's, which count the recoils in the associated evil-avoiding permutation. Since the bijection preserves the number of recoils, the following consequence is immediate.

Corollary 58. *The map β is a bijection between rectangular permutations and evil-avoiding permutations that preserves size and number of recoils.*

Kim and Williams [8] proved a number of enumerative results about evil-avoiding permutations. The bijection between evil-avoiding permutations and L_{Evil} allows us to provide some simpler proofs.

Kim and Williams enumerated $\text{Evil}(n, k)$ using a recurrence and induction, but the structure of L_{Evil} lets us enumerate this directly, which we can also do for $\text{Rect}(n, k)$.

Theorem 59. [8] Corollary 3.15 *For $k > 0$,*

$$|\text{Evil}(n, k)| = \sum_{i=0}^{n-k-1} 2^i \binom{i+k-1}{k-1} \binom{n-i-1}{k}.$$

Proof. Permutations in $\text{Evil}(n, k)$ correspond to words in L_{Evil} of size n with k r 's. The r 's split the word into $k + 1$ pieces, where the last piece is all s 's. Let i be the total number

of p 's and q 's in the word. These i p 's and q 's can be distributed into the first k pieces between the r 's in $\binom{i+k-1}{k-1}$ ways. The remaining $n - k - i$ positions must be s 's, and at least one s must be in the last piece, so there are $n - k - i - 1$ spares. We can allocate these among $k + 1$ pieces in $\binom{n-i-1}{k}$ ways. For each such allocation of the letters to the $k + 1$ pieces, the s 's must come at the end of the pieces, so the remaining choice is which of the i $p|q$ positions are p 's and which are q 's. There are 2^i ways to make this choice. Thus, for each i , there are $2^i \binom{i+k-1}{k-1} \binom{n-i-1}{k}$ words in $\text{Evil}(n, k)$. Therefore, for $k > 0$,

$$|\text{Evil}(n, k)| = \sum_i 2^i \binom{i+k-1}{k-1} \binom{n-i-1}{k}. \quad \square$$

Proposition 55 gives the following recurrence for $|\text{Evil}(n, k)|$ with $k > 0$:

$$|\text{Evil}(n, k)| = 2|\text{Evil}(n-1, k)| + \sum_{i=1} |\text{Evil}(n-i, k-1)|.$$

There is a simpler recurrence, also mentioned in [8] in the proof of their Corollary 3.15.

Theorem 60. For $k > 0$ and $n > 1$,

$$|\text{Evil}(n, k)| = 3|\text{Evil}(n-1, k)| + |\text{Evil}(n-1, k-1)| - 2|\text{Evil}(n-2, k)|.$$

Proof. This follows algebraically from the previous recurrence, but it also admits a relatively short combinatorial proof.

By Propositions 55 and 56, $\text{Evil}(n, k)$ is the disjoint union of the images of ψ_p, ψ_q , and ψ_s on the intersections of their domains with $\text{Evil}(n-1, k)$ together with the image of ψ_r on $\text{Evil}(n-1, k-1)$.

The operators ψ_p and ψ_q are defined on all of $\text{Evil}(n-1, k)$, together contributing $2|\text{Evil}(n-1, k)|$ permutations.

The operator ψ_r is defined on all of $\text{Evil}(n-1, k-1)$ contributing $|\text{Evil}(n-1, k-1)|$ permutations.

The operator ψ_s is only defined on permutations that end with a copy of e_t for some $t \geq 1$. The intersection with $\text{Evil}(n-1, k)$ are those permutations that are not in $\psi_p(\text{Evil}(n-2, k))$ or $\psi_q(\text{Evil}(n-2, k))$. This contributes $|\text{Evil}(n-1, k)| - 2|\text{Evil}(n-2, k)|$ permutations.

This decomposition of $\text{Evil}(n, k)$ combinatorially proves

$$\begin{aligned} |\text{Evil}(n, k)| &= 2|\text{Evil}(n-1, k)| + |\text{Evil}(n-1, k-1)| + (|\text{Evil}(n-1, k)| - 2|\text{Evil}(n-2, k)|) \\ &= 3|\text{Evil}(n-1, k)| + |\text{Evil}(n-1, k-1)| - 2|\text{Evil}(n-2, k)|. \end{aligned} \quad \square$$

6 Table of examples

We give examples of the bijection between rectangular and evil-avoiding permutations below.

Table 1: The bijection for all rectangular permutations in S_1, S_2, S_3, S_4 and three randomly selected permutations in S_5 .

Rect. perm.	Word in L_{Rect}	Word in L_{Evil}	Evil-av. perm.
1	1	<i>s</i>	1
12	11	<i>ss</i>	12
21	<i>d1</i>	<i>rs</i>	21
123	111	<i>sss</i>	123
132	<i>1d1</i>	<i>srs</i>	312
213	<i>d11</i>	<i>rss</i>	231
231	<i>ud1</i>	<i>qrs</i>	213
312	<i>2d1</i>	<i>prs</i>	132
321	<i>dd1</i>	<i>rrs</i>	321
1234	1111	<i>ssss</i>	1234
1243	<i>11d1</i>	<i>ssrs</i>	4123
1324	<i>1d11</i>	<i>srss</i>	3412
1342	<i>1ud1</i>	<i>qsr</i>	3142
1423	<i>12d1</i>	<i>psrs</i>	1423
1432	<i>1dd1</i>	<i>rsrs</i>	4231
2134	<i>d111</i>	<i>rsss</i>	2341
2143	<i>d1d1</i>	<i>srrs</i>	4312
2314	<i>ud11</i>	<i>qrss</i>	2134
2341	<i>uud1</i>	<i>qqr</i>	2314
3124	<i>2d11</i>	<i>prss</i>	1342
3142	<i>2ud1</i>	<i>qprs</i>	3124
3214	<i>dd11</i>	<i>rrss</i>	3421
3241	<i>dud1</i>	<i>qrrs</i>	2143
3412	<i>u2d1</i>	<i>pqr</i>	1324
3421	<i>udd1</i>	<i>rqr</i>	3241
4123	<i>22d1</i>	<i>pprs</i>	1243
4132	<i>2dd1</i>	<i>rprs</i>	2431
4312	<i>d2d1</i>	<i>prrs</i>	1342
4321	<i>ddd1</i>	<i>rrrs</i>	4321
15423	<i>1d2d1</i>	<i>prsr</i>	15342
43125	<i>d2d11</i>	<i>prrss</i>	14532
51432	<i>2ddd1</i>	<i>rrprs</i>	35421

7 The 1-almost-increasing permutations

We consider a class of permutations studied by Knuth [9] and Elizalde [7]. Kim and Williams [8] mentioned these permutations as Wilf-equivalent to evil-avoiding permutations and stated it would be particularly interesting to find a length-preserving bijection.

Definition 61. [7] Let $\mathcal{A}_n^{(1)}$ be the permutations $\pi \in S_n$ so for every $i \in \{1, 2, \dots, n\}$

there is at most one $j \leq i$ with $\pi_j > i$. Equivalently, π avoids 4321, 4312, 3421, and 3412. These permutations π are called *1-almost-increasing*.³

This class of permutations is motivated by sorting algorithms, but it also arises as a property of products of pattern-avoiding permutations.

Proposition 62. *If τ_1 and τ_2 avoid 132 and 123, then $\tau_1 \circ \tau_2$ is 1-almost-increasing.*

Proof. By inspection, this is true in S_4 : The permutations in S_4 avoiding 132 and 123 are $\{3214, 3241, 3412, 3421, 4213, 4231, 4312, 4321\}$. The products of two such permutations cover all permutations in S_4 except 4321, 4312, 3421, and 3412.

Any pattern of $\tau_1 \circ \tau_2$ in positions $\{a, b, c, d\}$ would produce that pattern in the size 4 permutation $\tau'_1 \circ \tau'_2$, where τ'_2 is the restriction of τ_2 to $\{a, b, c, d\}$ and τ'_1 is the restriction of τ_1 to $\tau_2(\{a, b, c, d\})$.

Since the reduction of $\tau'_1 \circ \tau'_2$ cannot be in $\{4321, 4312, 3421, 3412\}$, $\tau_1 \circ \tau_2$ avoids these patterns. \square

These types of product constructions are studied more by the author in [13].

Theorem 63. *The 1-almost-increasing permutations are Wilf-equivalent to evil-avoiding permutations. That is, for all n , $|\mathcal{A}_n^{(1)}| = |\text{Evil}(n)|$.*

Wilf-equivalence follows from the enumerations of Kim and Williams [8] and Knuth [9]. These collections of patterns are not trivially Wilf-equivalent. We prove Wilf-equivalence with an isomorphism between the regular language L_{Rect} and a regular language for 1-almost-increasing permutations.

Theorem 64. *For $n \geq 1$,*

$$\mathcal{A}_{n+1}^{(1)} = \rho_{1,1}(\mathcal{A}_n^{(1)}) \cup \rho_{1,2}(\mathcal{A}_n^{(1)}) \cup \rho_{2,1}(\mathcal{A}_n^{(1)}) \cup \rho_{2,2}(\mathcal{A}_n^{(1)}).$$

Proof. We prove containment both ways.

First, every element π of $\mathcal{A}_{n+1}^{(1)}$ must have $\pi_1 \in \{1, 2\}$ or $\pi_2 \in \{1, 2\}$, since otherwise the values π_1 and π_2 are greater than 1 and 2, so one of the forbidden patterns appears in positions $\{1, 2, \pi_1^{-1}, \pi_2^{-1}\}$. Thus,

$$\mathcal{A}_{n+1}^{(1)} \subset \rho_{1,1}(\mathcal{A}_n^{(1)}) \cup \rho_{1,2}(\mathcal{A}_n^{(1)}) \cup \rho_{2,1}(\mathcal{A}_n^{(1)}) \cup \rho_{2,2}(\mathcal{A}_n^{(1)}).$$

Second, none of the forbidden patterns has a 1 or 2 in the first two positions. Thus, inserting a 1 or 2 in positions 1 or 2 in $\pi \in \mathcal{A}_n^{(1)}$ cannot create a forbidden 4321, 4312, 3421, or 3412. Thus,

$$\rho_{1,1}(\mathcal{A}_n^{(1)}) \cup \rho_{1,2}(\mathcal{A}_n^{(1)}) \cup \rho_{2,1}(\mathcal{A}_n^{(1)}) \cup \rho_{2,2}(\mathcal{A}_n^{(1)}) \subset \mathcal{A}_{n+1}^{(1)}. \quad \square$$

³We suggest the name *bael-avoiding* because if we replace 4 with B, 3 with A, 2 with E, and 1 with L, then the avoided patterns correspond to BAEL, BALE, ABEL, and ABLE. Bael is a name for a demon and for a type of tree native to the Indian subcontinent.

Let the permutation in S_1 be encoded as $\rho_{1,1}$.

Theorem 65. *Each 1-almost-increasing permutation of size $n \geq 1$ can be written uniquely as a string of size n in $\rho_{1,1}, \rho_{1,2}, \rho_{2,1}, \rho_{2,2}$ ending in $\rho_{1,1}$ with no substrings $\rho_{2,1}\rho_{1,1}$ or $\rho_{2,2}\rho_{1,1}$.*

Proof. By the previous theorem, every 1-almost-increasing permutation can be written as a string of lengthening operators in $\{\rho_{1,1}, \rho_{1,2}, \rho_{2,1}, \rho_{2,2}\}$. There are redundant encodings. If $\pi_1 = 1$, then $\rho_{1,1}(\pi) = \rho_{2,2}(\pi)$ and $\rho_{1,2}(\pi) = \rho_{2,1}(\pi)$, which imply $\{\pi_1, \pi_2\} = \{1, 2\}$. Otherwise, $\pi \in \mathcal{A}_{n+1}^{(1)}$ is in the image of precisely one of $\rho_{1,1}, \rho_{1,2}, \rho_{2,1}$, and $\rho_{2,2}$.

Suppose encodings without $\rho_{2,2}\rho_{1,1}$ or $\rho_{2,1}\rho_{1,1}$ are not unique, and that π is a shortest example with two encodings. If π were in the image of just one of $\{\rho_{1,1}, \rho_{1,2}, \rho_{2,1}, \rho_{2,2}\}$, then $\pi = \rho_{i,j}\tau$ for some shorter τ that also doesn't have a unique encoding, contradicting that π is a shortest example.

If $\pi_1 = 2$ and $\pi_2 = 1$, then there is a permutation τ so that $\pi = \rho_{2,1}\rho_{1,1}\tau = \rho_{1,2}\rho_{1,1}\tau$. Since $\rho_{1,1}\tau$ is shorter than π , it must have a unique encoding $\rho_{1,1}w$, and since $\rho_{2,1}\rho_{1,1}$ is forbidden, π has a unique encoding in this language, namely $\rho_{1,2}\rho_{1,1}w$.

If $\pi_1 = 1$ and $\pi_2 = 2$, then π starts with an increasing consecutive sequence. Then since $\rho_{2,2}\rho_{1,1}$ is forbidden, all encodings for π must start with $\rho_{1,1}$. If $\pi = \rho_{1,1}\tau$, there must be multiple encodings for τ which is shorter than π , a contradiction.

Therefore, encodings without $\rho_{2,2}\rho_{1,1}$ or $\rho_{2,1}\rho_{1,1}$ are unique. \square

Let L_{AI} be this language over the alphabet $\{\rho_{1,1}, \rho_{1,2}, \rho_{2,1}, \rho_{2,2}\}$. The languages L_{AI} and L_{Rect} are isomorphic by the length-preserving substitution $\rho_{1,1} \mapsto 1, \rho_{2,1} \mapsto 2, \rho_{2,2} \mapsto u, \rho_{1,2} \mapsto d$. Thus, 1-almost-increasing permutations and evil-avoiding permutations are Wilf-equivalent, and Theorem 63 is proven.

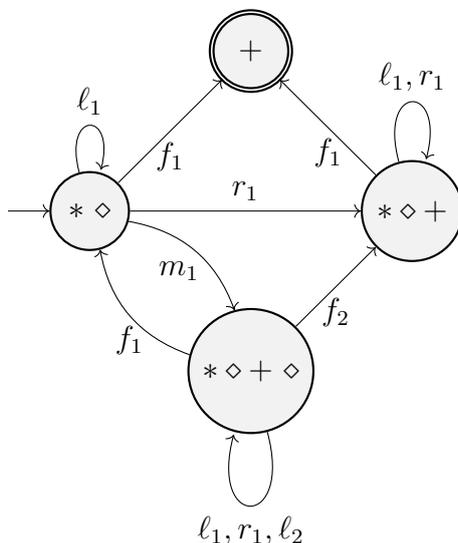
8 Insertion encoding

In Lemma 25, we encode rectangular permutations with a regular language so that a permutation of length n is encoded as a word of length n . The insertion encoding of Albert, Linton, and Ruškuc [1] is another way to encode permutations by words of the same length. When restricted to the inverses of rectangular permutations, this is also a regular language. We thank the referee for suggesting this idea.

Rectangular permutations avoid the patterns $\{2413, 2431, 4213, 4231\}$, so the inverses of rectangular permutations avoid the inverses of those patterns, $\{3142, 4132, 3241, 4231\}$. This set consists of all the length-4 patterns with the property that the 1 and 2 are in positions 2 and 4 in either order, and the 3 and 4 are in positions 1 and 3 in either order. Thus, the insertion pattern $\diamond + \diamond +$ is forbidden. Thus, when inserting elements of an inverse of a rectangular permutation, there can be at most two slots, and if there are two, there is nothing to the right of the second slot, the pattern must be $+ \diamond + \diamond$. Since the number of slots is bounded, by Theorem 8 of [1], the language of possible insertion encodings of inverses of rectangular permutations is regular.

The states of a minimal DFA accepting the insertion encodings of inverses of rectangular permutations correspond to the sets of acceptable suffixes. Since only the pattern

$\diamond + \diamond +$ is forbidden, the suffixes leading to the accepting state (no slots) are determined by the configurations of slots. There is one accepting state when there are no slots. There is one state with two slots: $* \diamond + \diamond$. There are two states with one slot: $* \diamond$ and $* \diamond +$, the initial state with one slot on the right: $* \diamond$, and another state with one slot not on the right: $* \diamond +$. The possible insertion symbols are $\{m_1, f_1, f_2, \ell_1, \ell_2, r_1\}$.



Let L'_{Rect} be the language of insertion encoding operators allowed to act on inverses of rectangular permutations. We can compute a regular expression for the language from the DFA by Arden's method [12].

Theorem 66. *A regular expression for L'_{Rect} is*

$$(\ell_1 | m_1 (\ell_1 | \ell_2 | r_1)_1^*)^* (f_1 | r_1 (\ell_1 | r_1)^* f_1 | m_1 (\ell_1 | \ell_2 | r_1)^* f_2 (\ell_1 | r_1)^* f_1).$$

We now compare the languages L_{Rect} and L'_{Rect} . From a word in either language, we can quickly retrieve the number of recoils in the corresponding rectangular permutation. In L_{Rect} , we count the number of d 's. In L'_{Rect} , we count the number of r 's and f 's. This count gives the number of descents plus 1 in the corresponding inverse rectangular permutation, which gives the number of recoils plus 1 after taking the inverse. Though L'_{Rect} has more letters than L_{Rect} , and the regular expression for L_{Rect} we found is simpler than the one for L'_{Rect} , there are many families of permutations that could admit regular languages in r 's, ℓ 's, and f 's, which we elaborate on in Section 10.

However, not all families of permutations have regular languages. It is natural to ask about the insertion encodings of rectangular permutations, evil-avoiding permutations, inverses of evil-avoiding permutations, and 1-almost-increasing permutations, the last of which are self-inverse.

Lemma 67. *The insertion encodings of rectangular permutations, evil-avoiding permutations, and inverses of evil-avoiding permutations are not regular languages.*

Proof. We construct examples of permutations in these classes with arbitrarily many slots.

We claim the permutation $\pi = [1\ n+1\ 2\ n+2\ \cdots\ 2n-1\ n\ 2n]$ is rectangular. This permutation is a union of two increasing subsequences so avoids 321. Hence, π avoids 2431, 4213, and 4231, each of which contains a 321 subpattern. The last pattern to avoid is 2413. If there were a 2413 pattern in π in positions $a < b < c < d$, we would have one of (1) $\pi_a, \pi_b \leq n$, (2) $\pi_a, \pi_b > n$, or (3) $\pi_a \leq n$ and $\pi_b > n$. Cases (1) and (3) are impossible because they leave no choice for π_c : all values less than π_a appear to its left. Case (2) is impossible because it leaves no choice for π_d : all values between π_a and π_b appear between a and b .

Indeed, π is also evil-avoiding: π avoids 321, so π avoids 4132, 4213, and 3214. The fourth pattern is 2413, which π avoids as explained in the previous paragraph.

We claim the permutation $\sigma = [1\ 3\ 5\ \cdots\ 2n-1\ 2n\ 2n-2\ \cdots\ 4\ 2]$ is evil-avoiding. Suppose one of the patterns 2413, 4213, or 3214 occurs in σ in positions $a < b < c < d$. Since σ_c is the least of $\{\sigma_a, \sigma_b, \sigma_c, \sigma_d\}$, we have that $\sigma_c < \sigma_d$. Thus, the structure of σ gives us that $c \leq n$. The structure of σ also tells us that $\sigma_a < \sigma_c$, contradicting the pattern. By a similar logic, if 4132 occurs in σ , then $\sigma_a < \sigma_b < \sigma_d$, contradicting the pattern.

In both $\pi = [1\ n+1\ 2\ n+2\ \cdots\ 2n-1\ n\ 2n]$ and $\sigma^{-1} = [1\ 2n\ 2\ 2n-1\ \cdots\ n\ n+1]$, the $(n+1)$ st insertion pattern $1 \diamond 2 \diamond \cdots n \diamond$ contains n slots. So, the insertion encodings of rectangular, evil-avoiding, and inverses of evil-avoiding permutations are not regular languages. \square

The 1-almost-increasing permutations forbid a set of patterns $\{4321, 4312, 3421, 3412\}$ preserved under inversion so the inverses of 1-almost-increasing permutations are also 1-almost-increasing.

Lemma 68. *The insertion encodings of 1-almost-increasing permutations are regular.*

Proof. If the pattern $\diamond + \diamond + \diamond +$ occurs when inserting a permutation, values in the last two $+$ symbols are lower than the values going into the first two slots, so one of the patterns in $\{4321, 4312, 3421, 3412\}$ must appear. Although three slots can occur with the pattern $\diamond + \diamond + \diamond$ such as in 142536, four slots can't occur, so the insertion encodings are regular. \square

9 Paths

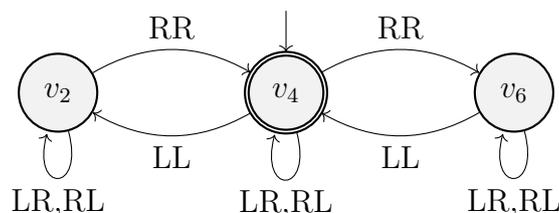
There are several other families of combinatorial objects counted by the same sequence A006012 [11]. One is walks of length $2n-2$ starting and ending in the middle vertex of P_7 , a path graph with 7 vertices. Let the vertices of P_7 be $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

Theorem 69. *There is a bijection between walks of length $2n-2$ in P_7 starting and ending at v_4 and rectangular permutations of length n .*

Elizalde [7] also produced a bijection between walks of length $2n-2$ starting and ending at the middle vertex of P_7 and 1-almost-increasing permutations in S_n by bijecting both with words of length $n-1$ in a regular language on the alphabet $\{E, W, R, L\}$ ending in W or E with no LE or RW . Our proof is different.

Proof. We produce a bijection between these paths and words of length $n - 1$ in a language that is L_{Rect} with the trailing 1 of each word truncated, hence to L_{Rect} and to rectangular permutations. We create a regular language encoding paths using a discrete finite automaton (DFA).

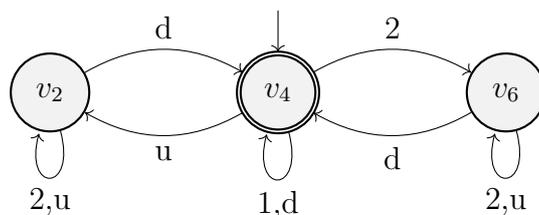
Instead of using all 7 vertices of P_7 as states, we consider taking steps two by two. By parity, after starting at the middle vertex v_4 , a walk must reach a vertex in $\{v_2, v_4, v_6\}$ after an even number of steps. Elements of this set will correspond to the states of a DFA with alphabet $\{\text{LL}, \text{LR}, \text{RL}, \text{RR}\}$.



The extra arrow into v_4 indicates that it is the initial state and the double circle indicates that it is the only accepting state. The edges are labeled by the pair or pairs of steps that move from one state to the next. The character L indicates a leftward step and the character R indicates a rightward step. We can convert this DFA to a regular expression generating the language, producing

$$(\text{LR}|\text{RL}|\text{LL}(\text{LR}|\text{RL})^*\text{RR}|\text{RR}(\text{LR}|\text{RL})^*\text{LL})^*.$$

There is no way of substituting the pairs LL, LR, RL, and RR to $\{1, 2, d, u\}$ to produce a language similar to L_{Rect} . To see why, we observe that the number of LL symbols must be equal to the number of RR symbols, and no analogous restriction occurs in L_{Rect} . Instead, we relabel the state transitions as follows:



If we convert this to a regular expression, we get

$$(1|d|u(2|u)^*d|2(2|u)^*d)^* = (1|(2|u)^*d)^*.$$

The language of this expression is all strings in $\{1, 2, d, u\}$ with no 21 or $u1$ substring that do not end in 2 or u . This is almost the same as L_{Rect} . We can make a bijection by adding a terminal 1. This proves the theorem. \square

10 Future directions

We discuss possible future directions, both algebraic and enumerative.

In the algebraic direction, the bijection between rectangular and evil-avoiding permutations may help us derive the inhomogeneous TASEP steady-state probabilities for new families of permutations. Kim and Williams express the steady-state probabilities corresponding to evil-avoiding permutations as a “trivial factor” times a product of (double) Schubert polynomials (for a more precise statement, see Theorem 1.11 in [8]), so there is hope that other permutation families may give rise to nice steady-state expressions. Perhaps the structure of evil-avoiding permutations and nearly-evil-avoiding permutations can be illuminated better by studying nearly rectangular permutations. Thus, we could get steady-state probabilities for some nearly evil-avoiding permutations.

In an enumerative direction, one could try to biject other pairs of objects counted by A006012, as suggested by Williams in personal communication [14]. Two sets of objects seem likely to shed light on evil-avoiding permutations, either because they are permutation classes or are clearly recursively constructed. The last was included in [8] as particularly interesting.

1. paths of length $2n$ with $n = 0$ starting at the initial node on the path graph with 7 vertices,
2. permutations on $[n]$ with no subsequence $abcd$ such that (i) bc are adjacent in position and (ii) $\max(a, c) < \min(b, d)$.

From another enumerative perspective, we may want to consider which sets of permutations can be bijected using similar techniques as those used here. Permutations avoiding $\{4312, 4231\}$, $\{4312, 4213\}$, $\{4231, 4213\}$, $\{4213, 4132\}$, and $\{4213, 3214\}$ are known to be equinumerous. When they are triply graded by size, recoils, and descents, these five sets of permutations still appear equinumerous. It would be interesting to construct a bijection between any pair of these five sets of permutations, and particularly so if the bijection preserves the number of recoils and/or descents. Evil-avoiding and rectangular permutations also seem to be equinumerous when triply graded by size, recoils, and descents.

Here, we used regular languages to biject two particular families of permutations. This technique may be more widely applicable to other permutation families, but there is a limitation that the generating functions of regular languages must be rational (the sequences satisfy linear recurrence relations). Many families of pattern-avoiding permutations are known not to have rational generating functions, e.g., permutations avoiding any one pattern of size three are counted by Catalan numbers, and the generating function for Catalan numbers is algebraic but not rational. However, some pattern-avoiding permutation families have rational generating functions. For these permutation families, it would be interesting to check if there are natural bijections with regular languages, perhaps using insertion encodings. If so, then the techniques of this paper might be used to establish many other nontrivial Wilf-equivalences.

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