

Integer decomposition property of polytopes

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Abstract

We study the integer decomposition property of lattice polytopes associated to n -dimensional smooth complete fans with at most $n + 3$ rays. Using the classification of smooth complete fans by Kleinschmidt and Batyrev and a reduction to lower dimensional polytopes, we prove the integer decomposition property for lattice polytopes in this setting.

Mathematics Subject Classifications: 52B20, 14M25

1 Introduction

A lattice polytope in \mathbb{R}^n is the convex hull of a finite number of points in the lattice \mathbb{Z}^n . A lattice polytope P is said to satisfy the **integer decomposition property** (or the **IDP** for short) if for all integers $k \geq 1$ and every $x \in kP \cap \mathbb{Z}^n$ there exist $x_1, \dots, x_k \in P \cap \mathbb{Z}^n$ such that $x = x_1 + \dots + x_k$. Following [HH17], we say a pair of lattice polytopes (P, Q) has the **IDP** if

$$(P \cap \mathbb{Z}^n) + (Q \cap \mathbb{Z}^n) = (P + Q) \cap \mathbb{Z}^n,$$

where the addition is the Minkowski sum. We can see that P has the IDP if and only if every pair (P, kP) has the IDP for all integers $k \geq 1$.

Since Oda [Oda08] introduced the question of identifying the polytopes with the IDP, these polytopes have captured the attention of researchers in integer programming, graph theory, commutative algebra, and toric geometry. The IDP does not hold for any general pair of lattice polytopes P and Q , even in the special case when $P = Q$ (see [CLS11, Example 2.2.11]). However, we know that the IDP holds in the following situations:

- Let R be a n -dimensional lattice polytope. Let $P = kdR, Q = ldR$ for $k, l \in \mathbb{N}$ and $d \geq n - 1$ [EW91] (see also [LTZ93], [BGT97, Theorem 1.3.3]).
- Let R be a unimodular simplex or parallelepiped or zonotope or a centrally symmetric 3-dimensional smooth polytope. Let $P = kR, Q = lR$ for $k, l \in \mathbb{N}$ [BHH⁺19, Proposition 1.2, Theorem 1.4].

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- Let R be a lattice polytope of dimension n and suppose every edge has lattice length $2n(n+1)$. Let $P = kR, Q = lR$ for $k, l \in \mathbb{N}$ [HH17, Corollary 6]. We should point out that this bound $2n(n+1)$ is an improvement over the prior one $4n(n+1)$ in [Gub12, Theorem 1.3].

The above results suggest that if P, Q are a big enough dilation of a lattice polytope R , then the pair (P, Q) has the IDP. Apart from the above cases, the following conjecture has received special interest.

Conjecture 1 (Oda’s Conjecture). [Oda08, Problem 3] Let $P, Q \subseteq \mathbb{R}^n$ be two lattice polytopes. Suppose that P is smooth and its normal fan refines the one of Q . Then the pair (P, Q) has the IDP.

Fakhruddin [Fak02] and Ogata [Oga06] give two independent proofs of IDP, for the condition when P is a smooth lattice polygon and the normal fan of P is a refinement of the normal fan of Q . Even if we drop the assumption that P is smooth, IDP holds [HNPS08]. But if we drop the assumption on the normal fan, the IDP does not hold in general (see Figure 1). In Figure 2, we draw the normal fan of $P + Q$ and a minimal smooth refinement. We can see that it has 8 rays. This motivates our research question to study the lowest possible cases that fail to satisfy the IDP. Notice that in 2 dimensions one has a unique minimal smooth refinement, however in higher dimensions this is no longer true.

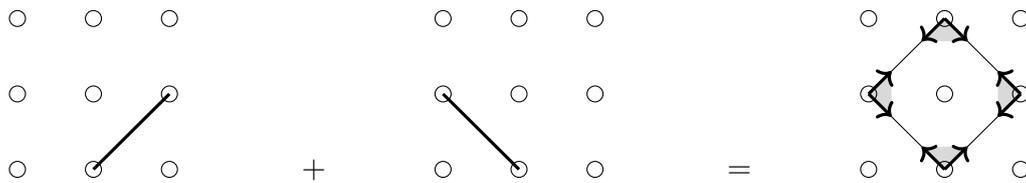


Figure 1: An example of polytopes that do not have IDP

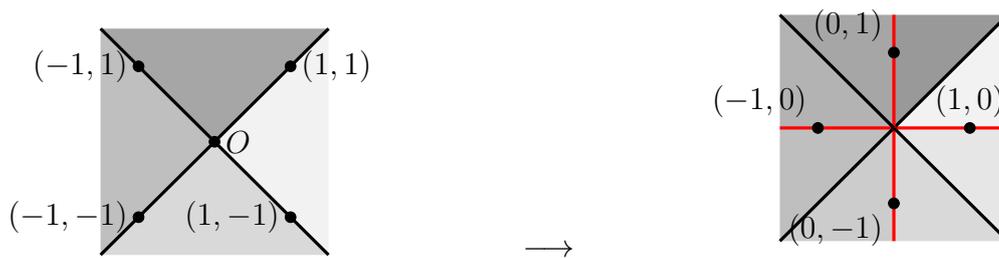


Figure 2: Refining the normal fan into a smooth fan

In general, from a given complete normal fan in \mathbb{R}^n we can construct a simplicial fan by triangulating the maximal cones. By subdividing those cones further, we can construct a smooth complete fan with $n+k$ rays. We can then ask the following question:

Question 2. Let P, Q be two lattice polytopes and Σ be the normal fan of $P+Q$. Suppose we refined Σ to a minimal smooth complete fan Σ' . Can we find a bound on k so that if Σ' has only $n+k$ rays then the pair (P, Q) always has the IDP?

Alternatively, we can start from a smooth complete fan Σ in \mathbb{R}^n with $n+k$ rays and study all lattice polytopes cut out by half spaces from all or some of these rays. All of these lattice polytopes can be studied in terms of convex support functions on Σ . See Section 2 for details. Hence, we can rephrase Question 2 as follows.

Question 3. Let Σ be a smooth complete fan in \mathbb{R}^n with $n+k$ rays and let P, Q be two lattice polytopes associated with two convex support functions for Σ . Does the pair (P, Q) have the IDP?

The answer to Question 3 is known to be positive when $k=1$ or 2 , see Theorem 12. In this paper, we will prove that the IDP holds for the case $k=3$.

Theorem 4. *If P, Q are two lattice polytopes whose coarsest common refinement of normal fans has at most $n+3$ rays in \mathbb{R}^n , then the pair (P, Q) has the IDP.*

It is worth noting that normal fan of a smooth lattice polytope in \mathbb{R}^n with $n+k$ facets is a smooth complete fan in \mathbb{R}^n with $n+k$ rays. Hence, as a corollary to Theorem 4, we observe that Conjecture 1 is true if P is a smooth lattice polytope in \mathbb{R}^n with at most $n+3$ facets. In particular, this implies that every smooth lattice polytope with at most $n+3$ facets has the IDP. An application of the result can be seen in [Rob22, Theorem 2.1].

On the other hand, one can ask what the answer for Question 1.3 is in the particular situation where $n=2$. Our result shows that as long as Σ has 5 or fewer rays IDP holds. However, the example in Figure 1 shows that there exists a smooth complete fan in \mathbb{R}^2 with 8 rays which has a negative answer to Question 3. We ask the following question:

Question 5. Let P, Q be two lattice polygons and suppose that the normal fan of $P+Q$ can be refined to a smooth fan in \mathbb{R}^2 of 6 or 7 rays. Does the pair (P, Q) have the IDP?

We believe the answer is yes, but do not have a proof. We intend to address this question in future research.

This paper is organized as follows. In Section 2, we give the preliminary details about fans and lattice polytopes. We also recall the classification of smooth complete fans in \mathbb{R}^n with $n+3$ rays. We prove the Theorem 4 in Section 3.

2 Preliminaries

In this section, we fix notation and recall some basic facts about fans. For details see [CLS11, Chapter 3]. We also recall the classification of smooth complete fans in \mathbb{R}^n with $n+3$ rays by Batyrev [Bat91].

All cones in this work are assumed to be rational polyhedral, i.e., subsets of \mathbb{R}^n of the form $\sigma = \{\sum_{i=1}^m \lambda_i u_i : \lambda_i \geq 0\}$ for some $u_1, \dots, u_m \in \mathbb{Z}^n$. A fan Σ in \mathbb{R}^n is said to be

complete if the union of all cones in Σ is \mathbb{R}^n . Moreover, Σ is said to be **smooth** if every cone $\sigma \in \Sigma$ is generated by part of a basis of \mathbb{Z}^n . Let Σ be a smooth complete fan in \mathbb{R}^n with $n + m$ rays. We denote the rays of Σ by $\Sigma(1)$. Given a ray $\rho \in \Sigma(1)$, we denote by u_ρ the **primitive generator** in \mathbb{Z}^n of ρ .

Two fans Σ_1 and Σ_2 are said to be **unimodular equivalent** if there exists a unimodular transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserves \mathbb{Z}^n and maps the cones of Σ_1 bijectively to the cones of Σ_2 . The unimodularity condition is motivated by algebraic geometry: the toric varieties associated with the unimodularly equivalent fans are isomorphic. We will consider fans up to unimodular equivalence. For every smooth complete fan, there is a unimodularly equivalent fan with one of the maximal cones generated by the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n .

Let Σ be a smooth complete fan in \mathbb{R}^n with $n + m$ rays. Let A be the $(n + m) \times n$ matrix whose rows represent the primitive ray generators of Σ in terms of the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . If necessary by rearranging the ordering of rays, A has the form $\begin{bmatrix} I \\ B \end{bmatrix}$ where I is the identity matrix and B is an integer matrix. We denote by $\{f_1, \dots, f_{n+m}\}$ the standard basis of \mathbb{R}^{n+m} . If we are not specifying the ordering of the rays, then we alternatively denote the standard basis of \mathbb{R}^{n+m} by $\{f_\rho : \rho \in \Sigma(1)\}$. Then it is easy to see that $\{Ae_1, \dots, Ae_n\}$ can be extended to a \mathbb{Z} -basis of \mathbb{Z}^{n+m} by adding the vectors f_{n+1}, \dots, f_{n+m} . Therefore, we can identify $\mathbb{Z}^{n+m}/A\mathbb{Z}^n$ with \mathbb{Z}^m , and we get the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n+m} \longrightarrow \mathbb{Z}^m \longrightarrow 0. \quad (1)$$

To any $h \in \mathbb{Z}^{n+m}$, we can associate a (possibly empty) polytope $P(A, h) \subseteq \mathbb{R}^n$ defined by

$$P(A, h) = \{x \in \mathbb{R}^n : Ax \geq -h\}.$$

If $h - h' = Ax$ for some $x \in \mathbb{Z}^n$, then $P(A, h)$ is a translation of $P(A, h')$ by an integral vector. Since translations by integral vectors preserve the IDP, it is enough to study the polytopes up to the image of $A\mathbb{Z}^n$.

A **support function** on Σ is a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ that is linear on each cone of Σ and is \mathbb{Z} -valued on \mathbb{Z}^n . To any $h \in \mathbb{Z}^{n+m}$, we define a function φ_h such that

$$\varphi_h(u_\rho) = -h_\rho \text{ for all } \rho \in \Sigma(1), \quad (2)$$

where h_ρ is the ρ^{th} coordinate of h . Since Σ is smooth, φ_h can be extended uniquely to a support function of Σ . We say φ_h is convex if

$$\varphi(x + y) \geq \varphi(x) + \varphi(y)$$

for all $x, y \in \mathbb{R}^n$. Moreover, we say φ_h is strictly convex if

$$\varphi(x + y) > \varphi(x) + \varphi(y)$$

for all $x, y \in \mathbb{R}^n$ not lying on the same maximal cone of Σ . If $h \in \mathbb{Z}^{n+m}$ defines a convex support function φ_h , then $P(A, h)$ is a lattice polytope. Conversely, if Q is a lattice

polytope whose normal fan can be refined into a smooth complete fan Σ , then there exists a convex support function φ_h on Σ such that $P(A, h)$ is equal to Q . If $\varphi_h, \varphi_{h'}$ are two convex support functions on Σ , then

$$P(A, h + h') = P(A, h) + P(A, h'), \quad (3)$$

where the addition of polytopes is the Minkowski sum.

In terms of support functions, we can state Conjecture 1 as follows.

Conjecture 6. Let Σ be a smooth projective fan, and A be the matrix representation of Σ . Suppose that φ_h is strictly convex, and $\varphi_{h'}$ is convex on Σ . Then the pair $(P(A, h), P(A, h'))$ has the IDP.

Now we will discuss the definition of primitive collections introduced by Batyrev, which makes the classifications and computations easier.

Definition 7. Let Σ be a fan. A subset $\mathcal{P} = \{\rho_1, \rho_2, \dots, \rho_k\} \subset \Sigma(1)$ is called a primitive collection if \mathcal{P} is not contained in a single cone of Σ , but every proper subset is. Let \mathcal{P} be a primitive collection and $\sigma \in \Sigma$ be the cone of the smallest dimension containing $u_{\rho_1} + \dots + u_{\rho_k}$. Then there exists a unique expression

$$u_{\rho_1} + \dots + u_{\rho_k} = \sum_{\rho \in \sigma(1)} c_\rho u_\rho, \quad c_\rho \in \mathbb{Z}_{>0}. \quad (4)$$

The equation (4) is called the primitive relation of \mathcal{P} .

Definition 8. A fan is called a splitting fan if there is no intersection between any two primitive collections.

It is easy to check whether φ_h is convex or not using primitive collections.

Theorem 9 ([Bat91, Theorem 2.15]). *Let Σ be a smooth projective fan in \mathbb{R}^n and $h \in \mathbb{Z}^{(n+m)}$. Then φ_h is convex if and only if it satisfies*

$$\varphi_h(u_{\rho_1} + \dots + u_{\rho_k}) \geq \varphi_h(u_{\rho_1}) + \dots + \varphi_h(u_{\rho_k})$$

for all primitive collections $\mathcal{P} = \{\rho_1, \dots, \rho_k\}$ of Σ . Similarly, φ_h is strictly convex if and only if it satisfies

$$\varphi_h(u_{\rho_1} + \dots + u_{\rho_k}) > \varphi_h(u_{\rho_1}) + \dots + \varphi_h(u_{\rho_k})$$

for all primitive collections $\mathcal{P} = \{\rho_1, \dots, \rho_k\}$ of Σ .

We also point out that these results hold in a broader context (see [CvR09, Theorem 1.4] and [CLS11, Theorem 6.4.9]). For the rest of this section we discuss the classification of smooth complete fans in \mathbb{R}^n with at most $n + 3$ rays.

Example 10. Let Σ be a smooth complete fan in \mathbb{R}^n with $n+1$ rays. It is easy to see that there is only one unimodular equivalent class. The ray generators of Σ can be written in the matrix A as follows:

$$A = \begin{matrix} & \xleftarrow{n \text{ columns}} & \\ \begin{matrix} \rho_1 \\ \vdots \\ \rho_n \\ \rho_{n+1} \end{matrix} & \left(\begin{array}{c} I \\ \hline -1 \cdots -1 \end{array} \right) & \end{matrix} \quad (5)$$

It has only one primitive collection that consists of all rays and the primitive relation is given by

$$u_{\rho_1} + \cdots + u_{\rho_{n+1}} = 0.$$

The support function φ_h is convex if and only if

$$\varphi_h(u_{\rho_1} + \cdots + u_{\rho_{n+1}}) \geq \varphi_h(u_{\rho_1}) + \cdots + \varphi_h(u_{\rho_{n+1}}).$$

Simplifying the inequality, we get

$$\varphi_h(0) \geq - \sum_{\rho \in \Sigma(1)} h_\rho,$$

i.e., h is convex support function if and only if $\sum_{\rho \in \Sigma(1)} h_\rho \geq 0$.

Kleinschmidt classified smooth complete n -dimensional fans with $n+2$ rays. It is known that a smooth complete fan in \mathbb{R}^n of at most $n+2$ rays is a splitting fan.

Theorem 11 ([Kle88, Theorem 1]). *Let Σ be a smooth complete fan \mathbb{R}^n with $n+2$ rays. In our setting, the generators of the smooth complete fan in \mathbb{R}^n with $n+2$ rays can be written in the matrix A as follows:*

$$A = \begin{matrix} & \xleftarrow{n \text{ columns}} & \\ \begin{matrix} \rho_1 \\ \vdots \\ \rho_k \\ \rho_{k+1} \\ \vdots \\ \rho_n \\ \rho_{n+1} \\ \rho_{n+2} \end{matrix} & \left(\begin{array}{cc|cc} I & 0 \\ \hline 0 & I \\ \hline -1 \cdots -1 & 0 \cdots 0 \\ a_1 \cdots a_k & -1 \cdots -1 \end{array} \right) & \end{matrix} \quad (6)$$

where $1 \leq k \leq n-1$ and $a_k \leq \cdots \leq a_1 \leq 0$. The primitive collections are given by $\{\rho_1, \dots, \rho_k, \rho_{n+1}\}$ and $\{\rho_{k+1}, \dots, \rho_n, \rho_{n+2}\}$.

Theorem 12 ([Ike09, Corollary 4.2]). *If Σ is a smooth complete splitting fan and $\varphi_h, \varphi_{h'}$ are convex support functions of Σ , then the pair $(P(A, h), P(A, h'))$ has the IDP. In particular, if Σ is a smooth complete fan in \mathbb{R}^n with at most $n+2$ rays then the statement holds.*

Batyrev classified smooth complete n -dimensional fans with $n+3$ rays in terms of primitive collections. Batyrev showed that the number of primitive collections of its rays is 3 or 5 [Bat91, Theorem 5.7]. The case of 3 primitive collections is a splitting fan. The case of 5 primitive collections is as in the following theorem.

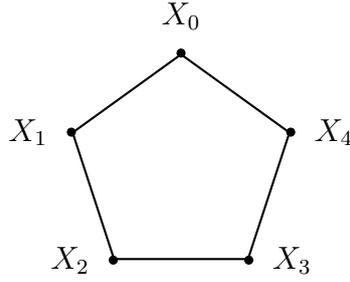


Figure 3

Theorem 13 ([Bat91, Theorem 6.6]). Let $\mathcal{X}_\alpha = X_\alpha \cup X_{\alpha+1}$, where $\alpha \in \mathbb{Z}/5\mathbb{Z}$,

$$\begin{aligned} X_0 &= \{v_1, \dots, v_{p_0}\}, & X_1 &= \{y_1, \dots, y_{p_1}\}, & X_2 &= \{z_1, \dots, z_{p_2}\}, \\ X_3 &= \{t_1, \dots, t_{p_3}\}, & X_4 &= \{u_1, \dots, u_{p_4}\}, \end{aligned}$$

and $p_0 + p_1 + p_2 + p_3 + p_4 = n + 3$. It is convenient to use a picture of a pentagon with the vertices $ie^{i2\pi\alpha/5}$ in the complex plane (see Figure 3). Then any smooth complete n -dimensional fan Σ with the set of generators $\bigcup X_\alpha$ and five primitive collections \mathcal{X}_α can be described up to a symmetry of the pentagon by the following primitive relations with nonnegative integral coefficients $c_2, \dots, c_{p_2}, b_1, \dots, b_{p_3}$:

$$\begin{aligned} v_1 + \dots + v_{p_0} + y_1 + \dots + y_{p_1} &= c_2 z_2 + \dots + c_{p_2} z_{p_2} + (b_1 + 1)t_1 + \dots + (b_{p_3} + 1)t_{p_3}, \\ y_1 + \dots + y_{p_1} + z_1 + \dots + z_{p_2} &= u_1 + \dots + u_{p_4}, \\ z_1 + \dots + z_{p_2} + t_1 + \dots + t_{p_3} &= 0, \\ t_1 + \dots + t_{p_3} + u_1 + \dots + u_{p_4} &= y_1 + \dots + y_{p_1}, \\ u_1 + \dots + u_{p_4} + v_1 + \dots + v_{p_0} &= c_2 z_2 + \dots + c_{p_2} z_{p_2} + b_1 t_1 + \dots + b_{p_3} t_{p_3}. \end{aligned}$$

In Theorem 13 we can identify the set

$$\{v_1, \dots, v_{p_0}, u_2, \dots, u_{p_4}, y_2, \dots, y_{p_1}, t_1, \dots, t_{p_3}, z_2, \dots, z_{p_2}\}$$

with the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{Z}^n . Thus z_1, u_1, y_1 are defined by

$$\begin{aligned} z_1 &= -z_2 - \dots - z_{p_2} - t_1 - \dots - t_{p_3}, \\ u_1 &= -v_1 - \dots - v_{p_0} - u_2 - \dots - u_{p_4} + b_1 t_1 + \dots + b_{p_3} t_{p_3} + c_2 z_2 + \dots + c_{p_2} z_{p_2}, \\ y_1 &= -v_1 - \dots - v_{p_0} - y_2 - \dots - y_{p_1} + (b_1 + 1)t_1 + \dots + (b_{p_3} + 1)t_{p_3} + \\ &\quad c_2 z_2 + \dots + c_{p_2} z_{p_2}. \end{aligned}$$

Let Σ be a fan as in Theorem 13. By arranging the primitive ray generators in rows of a matrix we will get a matrix A of the following form

Recall the construction of φ_h from (2). We use the associated primitive relations in Theorem 13 to obtain a more concrete characterization. It follows that φ_h is convex if and only if

$$\begin{aligned} 0 &\geq -d, \\ -f &\geq -e, \\ 0 &\geq -e, \\ 0 &\geq -f, \\ 0 &\geq -f - d. \end{aligned}$$

That is $d, e, f \geq 0$ and $e \geq f$. Hence, we can consider h of the following form

$$h_\rho = \begin{cases} d, & \text{if } \rho = v_1 \\ f, & \text{if } \rho = u_1 \\ e + f, & \text{if } \rho = z_1 \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

for any $d, e, f \in \mathbb{Z}_{\geq 0}$.

In the following paragraphs, using (5), we identify projections onto dilations of the standard simplex where the IDP is well-known. We can next verify the IDP for a pair of polytopes by carefully examining the fibers of these projections. First, we will discuss some additional notations.

Let J, K, S, T be index sets given by

$$\begin{aligned} J &= \{t_1, \dots, t_{p_3}, z_2, \dots, z_{p_2}\}, \\ K &= J \cup \{z_1\}, \\ S &= \{v_1, \dots, v_{p_0}, u_2, \dots, u_{p_4}, y_2, \dots, y_{p_1}\}, \\ T &= S \cup \{u_1, y_1\}. \end{aligned}$$

Now consider the projection maps to respective coordinates given by the index sets as follows:

$$\begin{array}{ccc} \mathbb{R}^{n+3} & & \mathbb{R}^n \\ \swarrow \pi_T & \searrow \pi_K & \swarrow \pi_S \quad \searrow \pi_J \\ \mathbb{R}^{|T|} & \mathbb{R}^{|K|} & \mathbb{R}^{|S|} \quad \mathbb{R}^{|J|} \end{array}$$

Given a lattice polytope $P = P(A, h)$, we can write it as

$$P = \left\{ (x_S, x_J) = x \in \mathbb{R}^n : \left[\begin{array}{c|c} F & G \\ \hline 0 & H \end{array} \right] \cdot \begin{bmatrix} x_S \\ x_J \end{bmatrix} \geq \begin{bmatrix} -h_T \\ -h_K \end{bmatrix} \right\}$$

where $x_S = \pi_S(x)$, $x_J = \pi_J(x)$ and $h_T = \pi_T(h)$, $h_K = \pi_K(h)$. First, consider the polytope P^Δ defined by

$$P^\Delta := P(H, h_K) \subseteq \mathbb{R}^{|J|}.$$

It is worth noting that $P^\Delta = (e + f)\Delta_{|J|}$, where $\Delta_{|J|} = \text{Conv}(0, e_{n+1-|J|}, \dots, e_n)$ is the standard $|J|$ -simplex. We can also look at P^Δ from a geometrical viewpoint: indeed, we can show that $P^\Delta = \pi_J(P)$. It is easy to see that $\pi_J(P) \subseteq P(H, h_K)$. Conversely, if $x_J \in P(H, h_K)$, then $(0, x_J) \in P$.

Let $\alpha \in P \cap \mathbb{Z}^n$. Set $\theta(\alpha) = \pi_T(h) + G\alpha_J$ and define a new polytope

$$\tilde{P}(\alpha) := P(F, \theta(\alpha)) \subseteq \mathbb{R}^{|S|}.$$

By construction, if $x_S \in \tilde{P}(\alpha)$, then $(x_S, \alpha_J) \in P$. Hence, we can also write

$$\tilde{P} = \pi_S(\pi_J^{-1}(\alpha_J) \cap P).$$

Geometrically, $\tilde{P}(\alpha)$ is constructed by intersecting the original polytope P with hyperplane sections defined by $x_j = \alpha_j$ for $j \in J$ and projecting into $\mathbb{R}^{|S|}$. See Figure 4 for the three-dimensional interpretation.

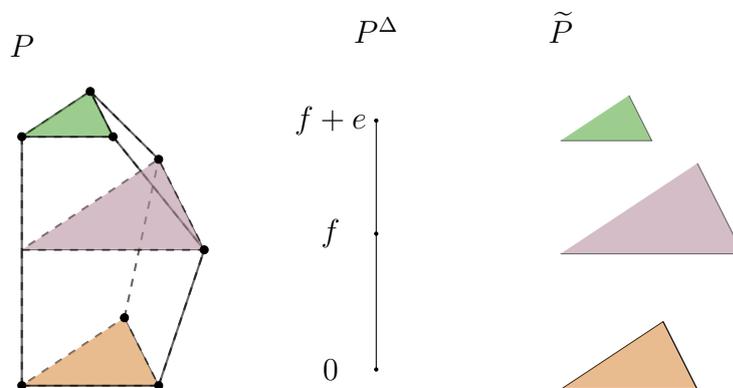


Figure 4: Projection of a polytope

Using the construction of the above polytopes we will now use the following algorithm.

- Given two lattice polytopes $P = P(A, h), Q = P(A, h')$ and their Minkowski sum $(P + Q) = P(A, h + h')$.
- We find projection to $\mathbb{R}^{|J|}$ and construct $P^\Delta, Q^\Delta, (P + Q)^\Delta$. Note that the pair (P^Δ, Q^Δ) has the IDP.
- For every $\alpha_J \in (P + Q) \cap \mathbb{Z}^{|J|}$ find all possible $\beta_J \in P \cap \mathbb{Z}^{|J|}$ and $\gamma_J \in Q \cap \mathbb{Z}^{|J|}$ such that $\beta_J + \gamma_J = \alpha_J$.
- Analyze the fibers of α_J, β_J and γ_J by using the construction $\tilde{P}(\beta_J), \tilde{Q}(\gamma_J)$ and $\widetilde{(P + Q)}(\alpha_J)$.
- For all $(\alpha_S, \alpha_J) \in \pi_J^{-1}(\alpha_J) \cap (P + Q) \cap \mathbb{Z}^n$ find $(\beta_J, \beta_S) \in \pi_J^{-1}(\beta_J) \cap P \cap \mathbb{Z}^n$ and $(\gamma_J, \gamma_S) \in \pi_J^{-1}(\gamma_J) \cap Q \cap \mathbb{Z}^n$ such that $(\beta_S, \beta_J) + (\gamma_S, \gamma_J) = (\alpha_S, \alpha_J)$.

With the following example, we will illustrate how to use the construction of P^Δ and \tilde{P} to prove the IDP.

Example 14. Let

$$A = \begin{matrix} v_1 \\ v_2 \\ u_1 \\ y_1 \\ t_1 \\ z_1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad h' = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix},$$

and $P = P(A, h)$, $Q = P(A, h')$. Then $P + Q = P(A, h + h')$. We will show that $P \cap \mathbb{Z}^3 + Q \cap \mathbb{Z}^3 = (P + Q) \cap \mathbb{Z}^3$. In Figure 5, we have drawn the polytopes P^Δ, Q^Δ and $(P + Q)^\Delta$ and it is easy to see that $P^\Delta + Q^\Delta = (P + Q)^\Delta$ and (P^Δ, Q^Δ) has the IDP. Hence, for every $\alpha_{t_1} \in (P + Q)^\Delta \cap \mathbb{Z}$, we can find $\beta_{t_1} \in P^\Delta \cap \mathbb{Z}, \gamma_{t_1} \in Q^\Delta \cap \mathbb{Z}$ such that $\beta_{t_1} + \gamma_{t_1} = \alpha_{t_1}$.

Let

$$F = \begin{matrix} v_1 \\ v_2 \\ u_1 \\ y_1 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \theta(\beta_{t_1}) = \begin{pmatrix} 0 \\ 0 \\ 3 + \beta_{t_1} \\ 2\beta_{t_1} \end{pmatrix}, \quad \theta(\gamma_{t_1}) = \begin{pmatrix} 2 \\ 0 \\ 1 + \gamma_{t_1} \\ 2\gamma_{t_1} \end{pmatrix}.$$

Then $\tilde{P}(\beta_{t_1}) = P(F, \theta(\beta_{t_1}))$, $\tilde{Q}(\gamma_{t_1}) = P(F, \theta(\gamma_{t_1}))$ and $(\widetilde{P + Q})(\alpha_{t_1}) = P(F, \theta(\alpha_{t_1}))$. Let $(\alpha_{v_1}, \alpha_{v_2}, \alpha_{t_1}) \in (P + Q) \cap \mathbb{Z}^3$. The basic idea is to show that for every $\alpha_{t_1} \in (P + Q)^\Delta \cap \mathbb{Z}$, we can find $\beta_{t_1}, \gamma_{t_1}$ additionally satisfying the following two conditions:

1. $\tilde{P}(\beta_{t_1}) + \tilde{Q}(\gamma_{t_1}) = (\widetilde{P + Q})(\alpha_{t_1})$.
2. $(\tilde{P}(\beta_{t_1}), \tilde{Q}(\gamma_{t_1}))$ has the IDP.

Suppose such a choice exists. Since $(\alpha_{v_1}, \alpha_{v_2}) \in (\widetilde{P + Q})(\alpha_{t_1})$ we can find $(\beta_{v_1}, \beta_{v_2}) \in \tilde{P}(\beta_{t_1}) \cap \mathbb{Z}^2$ and $(\gamma_{v_1}, \gamma_{v_2}) \in \tilde{Q}(\gamma_{t_1}) \cap \mathbb{Z}^2$ such that

$$(\beta_{v_1}, \beta_{v_2}) + (\gamma_{v_1}, \gamma_{v_2}) = (\alpha_{v_1}, \alpha_{v_2}).$$

Then we conclude the proof in this case by noticing that

$$(\beta_{v_1}, \beta_{v_2}, \beta_{t_1}) + (\gamma_{v_1}, \gamma_{v_2}, \gamma_{t_1}) = (\alpha_{v_1}, \alpha_{v_2}, \alpha_{t_1}).$$

Hence, it is enough to show that choices of $\beta_{t_1}, \gamma_{t_1}$ additionally satisfying the extra two conditions. We proceed by two cases depending on $0 \leq \alpha_{t_1} \leq 4$ or $4 \leq \alpha_{t_1} \leq 7$.

Case 1. If $4 \leq \alpha_{t_1} \leq 7$, then choose $3 \leq \beta_{t_1} \leq 4, \beta_{t_1} \in \mathbb{Z}$ and $1 \leq \gamma_{t_1} \leq 3, \gamma_{t_1} \in \mathbb{Z}$ such that $\alpha_{t_1} = \beta_{t_1} + \gamma_{t_1}$. The case $\alpha_{t_1} = 7$, is shown in Figure 6. Note that the half-space generated by the ray u_1 is not a supporting half-space for $(\widetilde{P + Q})(\alpha_{t_1})$. Our choice

of $\beta_{t_1}, \gamma_{t_1}$ ensure that the half-space generated by the ray u_1 is also not a supporting half-space for $\tilde{P}(\beta_{t_1})$ and $\tilde{Q}(\gamma_{t_1})$. Hence, we have $\tilde{P} = P(\tilde{F}, \tilde{\theta})$, $\tilde{Q} = P(\tilde{F}, \tilde{\theta}')$, where

$$\tilde{F} = \begin{matrix} v_1 \\ v_2 \\ y_1 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \tilde{\theta} = \begin{pmatrix} 0 \\ 0 \\ 2\beta_{t_1} \end{pmatrix}, \quad \tilde{\theta}' = \begin{pmatrix} 2 \\ 0 \\ 2\gamma_{t_1} \end{pmatrix}.$$

Since \tilde{F} corresponds to a smooth complete splitting fan in \mathbb{R}^2 and $\tilde{\theta}, \tilde{\theta}'$ are convex support functions we have $\tilde{P} + \tilde{Q} = \widetilde{P + Q}$ and (\tilde{P}, \tilde{Q}) has IDP.

Case 2. If $0 \leq \alpha_{t_1} \leq 4$, then choose $0 \leq \beta_{t_1} \leq 3, \beta_{t_1} \in \mathbb{Z}$ and $0 \leq \gamma_{t_1} \leq 1, \gamma_{t_1} \in \mathbb{Z}$ such that $\alpha_{t_1} = \beta_{t_1} + \gamma_{t_1}$. The case $\alpha_{t_1} = 4$, is shown in Figure 7 and the case $\alpha_{t_1} = 0$, is shown in Figure 8. Note that the half-space generated by the ray y_1 is not a supporting half-space for $\widetilde{(P + Q)}(\alpha_{t_1})$. Our choice of $\beta_{t_1}, \gamma_{t_1}$ ensure that the half-space generated by the ray y_1 is also not a supporting half-space for $\tilde{P}(\beta_{t_1})$ and $\tilde{Q}(\gamma_{t_1})$. Hence, we have $\tilde{P} = P(\tilde{F}, \tilde{\theta})$, $\tilde{Q} = P(\tilde{F}, \tilde{\theta}')$, where

$$\tilde{F} = \begin{matrix} v_1 \\ v_2 \\ u_1 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \tilde{\theta} = \begin{pmatrix} 0 \\ 0 \\ \beta_{t_1} + 3 \end{pmatrix}, \quad \tilde{\theta}' = \begin{pmatrix} 2 \\ 0 \\ \gamma_{t_1} + 1 \end{pmatrix}.$$

Since \tilde{F} corresponds to a smooth complete splitting fan in \mathbb{R}^2 and $\tilde{\theta}, \tilde{\theta}'$ are convex support functions we have $\tilde{P} + \tilde{Q} = \widetilde{P + Q}$ and (\tilde{P}, \tilde{Q}) has IDP.

Remark 15. If we use a similar type of construction for a general lattice polytope, $\tilde{P}(\alpha)$ does not have to be a lattice polytope. However, in our case, we can associate it with a smooth complete fan and a convex support function (see cases 1,2,3,4 in the Proof of Theorem 1.4). As a result, it is always a lattice polytope.

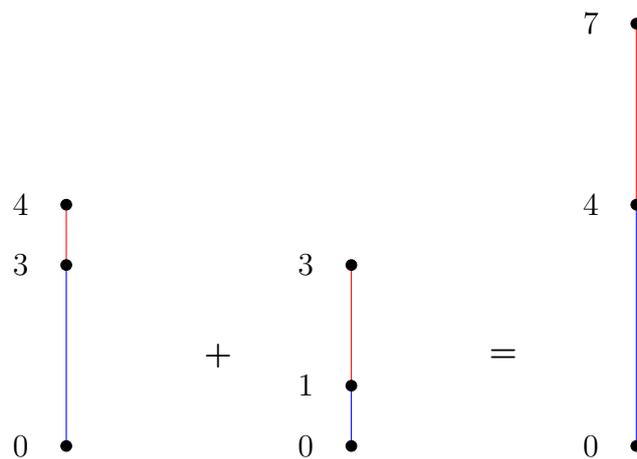


Figure 5: Polytopes $P^\Delta, Q^\Delta, (P + Q)^\Delta$

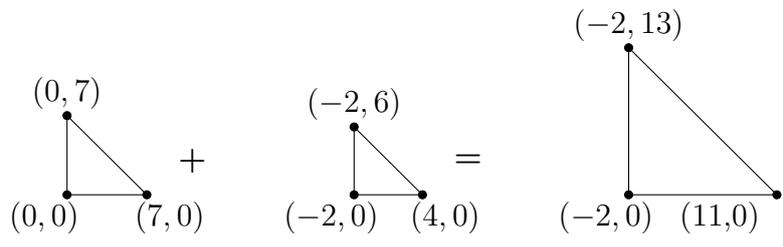


Figure 6: $\widetilde{P}(4), \widetilde{Q}(3), (\widetilde{P+Q})(7)$

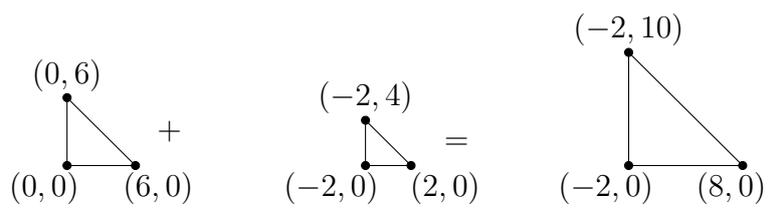


Figure 7: $\widetilde{P}(3), \widetilde{Q}(1), (\widetilde{P+Q})(4)$

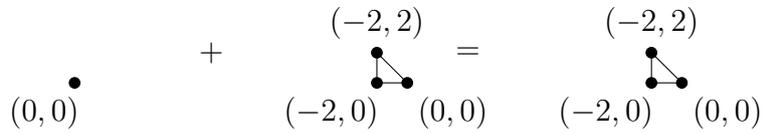


Figure 8: $\widetilde{P}(0), \widetilde{Q}(0), (\widetilde{P+Q})(0)$

Proof of Theorem 4. Let Σ be a smooth complete fan in \mathbb{R}^n . Suppose Σ has at most $n+2$ rays or a splitting fan with $n+3$ rays. Then the result follows from Theorem 12. Hence we can assume $P = P(A, h)$ and $Q = P(A, h')$, where A is a matrix as in (5) and h, h' are convex support functions as in (8). Then we have $P + Q = P(A, h + h')$ by (3). Our goal is to show that the pair (P, Q) has the IDP. Let $\alpha \in (P + Q) \cap \mathbb{Z}^n$. We will find $\beta \in P \cap \mathbb{Z}^n$ and $\gamma \in Q \cap \mathbb{Z}^n$ such that $\beta + \gamma = \alpha$. We will use the projection technique discussed in the beginning of this section.

Let

$$P^\Delta = P(H, h_K), \quad Q^\Delta = P(H, h'_K), \quad (P + Q)^\Delta = P(H, (h + h')_K).$$

Since $P^\Delta = (e + f)\Delta_{|J|}$, $Q^\Delta = (e' + f')\Delta_{|J|}$ and $(P + Q)^\Delta = (e + e' + f + f')\Delta_{|J|}$ we have

$$(P^\Delta \cap \mathbb{Z}^{|J|}) + (Q^\Delta \cap \mathbb{Z}^{|J|}) = (P + Q)^\Delta \cap \mathbb{Z}^{|J|}.$$

Also notice that $\alpha_J \in (P + Q)^\Delta \cap \mathbb{Z}^{|J|}$. Hence, we can find $\beta_J \in P^\Delta \cap \mathbb{Z}^{|J|}$ and $\gamma_J \in Q^\Delta \cap \mathbb{Z}^{|J|}$ such that $\alpha_J = \beta_J + \gamma_J$. Furthermore, we will consider two subcases depending on the value of $\sum_{i \in X_3} \alpha_i$.

- If $0 \leq \sum_{i \in X_3} \alpha_i \leq f + f'$, we can choose β_J, γ_J such that $0 \leq \sum_{i \in X_3} \beta_i \leq f$ and $0 \leq \sum_{i \in X_3} \gamma_i \leq f'$.
- Similarly, if $f + f' \leq \sum_{i \in X_3} \alpha_i \leq f + f' + e + e'$, we can choose β_J, γ_J such that $f \leq \sum_{i \in X_3} \beta_i \leq f + e$ and $f' \leq \sum_{i \in X_3} \gamma_i \leq f' + e'$.

Now consider polytopes

$$\tilde{P} = P(F, \theta), \quad \tilde{Q} = P(F, \theta'), \quad \widetilde{P + Q} = P(F, (\theta + \theta')).$$

Note that F is an $(|S| + 2) \times |S|$ matrix. Depending on the values of p_1 and p_4 , we will consider four cases. In each case, we will consider two subcases depending on the value of $\sum_{i \in X_3} \alpha_j$. Further to that, in every subcase, we can find an alternative description of \tilde{P}, \tilde{Q} and $\widetilde{P + Q}$ as

$$\tilde{P} = P(\tilde{F}, \tilde{\theta}), \quad \tilde{Q} = P(\tilde{F}, \tilde{\theta}'), \quad \widetilde{P + Q} = P(\tilde{F}, \tilde{\theta} + \tilde{\theta}'),$$

where \tilde{F} is associated to a smooth complete splitting fan $\tilde{\Sigma}$ and $\tilde{\theta}, \tilde{\theta}'$ are convex support functions for $\tilde{\Sigma}$. Then by Theorem 12 we have

$$\tilde{P} \cap \mathbb{Z}^{|S|} + \tilde{Q} \cap \mathbb{Z}^{|S|} = \widetilde{P + Q} \cap \mathbb{Z}^{|S|}.$$

Hence, we can find $\beta_L \in \tilde{P} \cap \mathbb{Z}^{|S|}$ and $\gamma_L \in \tilde{Q} \cap \mathbb{Z}^{|S|}$ such that $\alpha_L = \beta_L + \gamma_L$. Finally, we can complete the proof by noticing that $(\beta_L, \beta_J) \in P \cap \mathbb{Z}^n$, $(\gamma_L, \gamma_J) \in Q \cap \mathbb{Z}^n$ and $(\beta_L, \beta_J) + (\gamma_L, \gamma_J) = (\alpha_L, \alpha_J)$.

Here are the four cases.

Case 1. If $p_1 = p_4 = 1$, then we have

$$F = \begin{matrix} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ y_1 \end{matrix} \begin{pmatrix} \xrightarrow{p_0 \text{ columns}} \\ I \\ \hline -1 \dots -1 \\ \hline -1 \dots -1 \end{pmatrix}, \quad \theta + \theta' = \begin{matrix} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ y_1 \end{matrix} \begin{pmatrix} d \\ \vdots \\ 0 \\ f + \sum_{i \in X_3} b_i \alpha_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \alpha_j \\ \sum_{i \in X_3} (b_i + 1) \alpha_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \alpha_j \end{pmatrix},$$

$$\theta = \begin{matrix} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ y_1 \end{matrix} \begin{pmatrix} d \\ \vdots \\ 0 \\ f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ \sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \end{pmatrix}, \quad \theta' = \begin{pmatrix} d' \\ \vdots \\ 0 \\ f' + \sum_{i \in X_3} b_i \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ \sum_{i \in X_3} (b_i + 1) \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \end{pmatrix}.$$

As discussed in the previous paragraph, we need to associate the above data to a smooth complete fan and convex support functions. We first analyze the polytope $(\widetilde{P + Q})(\alpha) = P(F, \theta + \theta')$.

It is clear that depending on the sum $\sum_{i \in X_3} \alpha_i$ one of the equations from the rows u_1, y_1 is redundant.

- If $0 \leq \sum_{i \in X_3} \alpha_i \leq f + f'$, we choose β_J, γ_J such that $0 \leq \sum_{i \in X_3} \beta_i \leq f$ and $0 \leq \sum_{i \in X_3} \gamma_i \leq f'$. In this case, we construct $\widetilde{F}, \widetilde{\theta}, \widetilde{\theta}', \widetilde{\theta + \theta'}$ from $F, \theta, \theta', \theta + \theta'$ by removing the row u_1 . Then we have

$$\widetilde{P}(\beta) = P(\widetilde{F}, \widetilde{\theta}), \quad \widetilde{Q}(\gamma) = P(\widetilde{F}, \widetilde{\theta}'), \quad (\widetilde{P + Q})(\alpha) = P(\widetilde{F}, \widetilde{\theta + \theta'}).$$

- If $f + f' \leq \sum_{i \in X_3} \alpha_i \leq f + e + f' + e'$, we choose β_J, γ_J such that $f \leq \sum_{i \in X_3} \beta_i \leq f + e$ and $f' \leq \sum_{i \in X_3} \gamma_i \leq f' + e'$. In this case, we construct $\widetilde{F}, \widetilde{\theta}, \widetilde{\theta}', \widetilde{\theta + \theta'}$ from $F, \theta, \theta', \theta + \theta'$ by removing the row y_1 . Then we have

$$\widetilde{P}(\beta) = P(\widetilde{F}, \widetilde{\theta}), \quad \widetilde{Q}(\gamma) = P(\widetilde{F}, \widetilde{\theta}'), \quad (\widetilde{P + Q})(\alpha) = P(\widetilde{F}, \widetilde{\theta + \theta'}).$$

Note that in both cases \widetilde{F} is the same. Consider the smooth complete splitting fan Σ in \mathbb{R}^{p_0} with $p_0 + 1$ rays (see Example 10). Then the matrix representing the ray generators of Σ is equal to \widetilde{F} . Also note that entries in $\widetilde{\theta}, \widetilde{\theta}'$ are nonnegative. Hence, it follows that they define convex support functions on Σ (see Example 10).

Case 2. If $p_1, p_4 \geq 2$, then we have

$$F = \begin{array}{c} \begin{array}{c} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ u_2 \\ \vdots \\ u_{p_4} \\ y_1 \\ y_2 \\ \vdots \\ y_{p_1} \end{array} \left(\begin{array}{c|c|c} \xleftarrow{p_0+p_1+p_4-2 \text{ columns}} & & \\ \hline I & 0 & 0 \\ \hline -1 \dots -1 & -1 \dots -1 & 0 \dots 0 \\ \hline 0 & I & 0 \\ \hline -1 \dots -1 & 0 \dots 0 & -1 \dots -1 \\ \hline 0 & 0 & I \end{array} \right), \end{array}$$

$$\theta = \begin{array}{c} \begin{array}{c} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ u_2 \\ \vdots \\ u_{p_4} \\ y_1 \\ y_2 \\ \vdots \\ y_{p_1} \end{array} \left(\begin{array}{c} d \\ \vdots \\ 0 \\ f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ 0 \\ \vdots \\ 0 \\ \sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ 0 \\ \vdots \\ 0 \end{array} \right), \quad \theta' = \begin{array}{c} \begin{array}{c} d' \\ \vdots \\ 0 \\ f' + \sum_{i \in X_3} b_i \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ 0 \\ \vdots \\ 0 \\ \sum_{i \in X_3} (b_i + 1) \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ 0 \\ \vdots \\ 0 \end{array} \right).$$

Consider two fans Σ_1, Σ_2 with the rays coming from the rows of F but with different primitive collections. For Σ_1 , we consider the primitive collections given by $\{v_1, \dots, v_{p_0}, u_1, \dots, u_{p_4}\}, \{y_1, \dots, y_{p_1}\}$. The primitive relations are given by

$$\begin{aligned} v_1 + \dots + v_{p_0} + u_1 + \dots + u_{p_4} &= 0 \\ y_1 + \dots + y_{p_1} &= u_1 + \dots + u_{p_4}. \end{aligned}$$

The support function φ_θ is convex with respect to Σ_1 if and only if

$$\begin{aligned} \varphi_\theta(v_1 + \dots + v_{p_0} + u_1 + \dots + u_{p_4}) &\geq \varphi_\theta(v_1) + \dots + \varphi_\theta(v_{p_0}) + \varphi_\theta(u_1) + \dots + \varphi_\theta(u_{p_4}) \\ \varphi_\theta(y_1 + \dots + y_{p_1}) &\geq \varphi_\theta(y_1) + \dots + \varphi_\theta(y_{p_1}). \end{aligned}$$

Simplifying the first inequality gives

$$0 \geq -d - \left(f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right),$$

which holds in this case because d, b_i, c_j, β_j are all non-negative. If we simplify the second inequality, we get

$$-\left(f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right) \geq -\left(\sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right)$$

which holds if and only if $\sum_{i \in X_3} \beta_i \geq f$. Similarly, we get $\varphi_{\theta'}$ is convex if and only if $\sum_{i \in X_3} \gamma_i \geq f'$ and $\varphi_{(\theta+\theta')}$ is convex if and only if $\sum_{i \in X_3} \alpha_i \geq f + f'$. As a result, if

$$f + f' \leq \sum_{i \in X_3} \alpha_i \leq f + e + f' + e',$$

we find $\beta_J, \gamma_J \in \mathbb{Z}^{|J|}$ such that

$$f \leq \sum_{i \in X_3} \beta_i \leq f + e, \quad f' \leq \sum_{i \in X_3} \gamma_i \leq f' + e'.$$

We have seen that θ, θ' are convex in this case.

For the fan Σ_2 , we consider the primitive collections given by $\{v_1, \dots, v_{p_0}, y_1, \dots, y_{p_1}\}, \{u_1, \dots, u_{p_4}\}$. The primitive relations are given by

$$\begin{aligned} v_1 + \dots + v_{p_0} + y_1 + \dots + y_{p_1} &= 0 \\ u_1 + \dots + u_{p_4} &= y_1 + \dots + y_{p_1}. \end{aligned}$$

The support function φ_θ is convex with respect to Σ_1 if and only if

$$\begin{aligned} \varphi_\theta(v_1 + \dots + v_{p_0} + y_1 + \dots + y_{p_1}) &\geq \varphi_\theta(v_1) + \dots + \varphi_\theta(v_{p_0}) + \varphi_\theta(y_1) + \dots + \varphi_\theta(y_{p_1}) \\ \varphi_\theta(u_1 + \dots + u_{p_4}) &\geq \varphi_\theta(u_1) + \dots + \varphi_\theta(u_{p_4}). \end{aligned}$$

Simplifying the first inequality gives

$$0 \geq -d - \left(f + \sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right),$$

which holds in this case because d, b_i, c_j, β_j are all non-negative. If we simplify the second inequality, we get

$$-\left(\sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right) \geq -\left(f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right)$$

which holds if and only if $\sum_{i \in X_3} \beta_i \leq f$. Similarly, we get $\varphi_{\theta'}$ is convex if and only if $\sum_{i \in X_3} \gamma_i \leq f'$ and $\varphi_{(\theta+\theta')}$ is convex if and only if $\sum_{i \in X_3} \alpha_i \leq f + f'$. As a result, if

$$0 \leq \sum_{i \in X_3} \alpha_i \leq f + f',$$

we find $\beta_J, \gamma_J \in \mathbb{Z}^{|J|}$ such that $0 \leq \sum_{i \in X_3} \beta_i \leq f$ and $0 \leq \sum_{i \in X_3} \gamma_i \leq f'$. We have seen that θ, θ' are convex in this case.

Case 3. If $p_1 > p_4 = 1$, then we have

$$F = \begin{array}{c} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ y_1 \\ y_2 \\ \vdots \\ y_{p_1} \end{array} \left(\begin{array}{c|c} \xrightarrow{p_0+p_1-1 \text{ columns}} & \\ \hline I & 0 \\ \hline -1 \dots -1 & 0 \dots 0 \\ -1 \dots -1 & -1 \dots -1 \\ \hline 0 & I \end{array} \right),$$

$$\theta = \begin{array}{c} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ y_1 \\ y_2 \\ \vdots \\ y_{p_1} \end{array} \left(\begin{array}{c} d \\ \vdots \\ 0 \\ f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ \sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ 0 \\ \vdots \\ 0 \end{array} \right), \quad \theta' = \begin{array}{c} d' \\ \vdots \\ 0 \\ f' + \sum_{i \in X_3} b_i \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ \sum_{i \in X_3} (b_i + 1) \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ 0 \\ \vdots \\ 0 \end{array}.$$

In this case, we consider the fan Σ with ray generators given by the rows of F and primitive collections given by $\{v_1, \dots, v_{p_0}, u_1\}, \{y_1, \dots, y_{p_1}\}$. The primitive relations are given by

$$\begin{aligned} v_1 + \dots + v_{p_0} + u_1 &= 0 \\ y_1 + \dots + y_{p_1} &= u_1. \end{aligned}$$

It is a smooth complete fan in $\mathbb{R}^{p_0+p_1-1}$ with $p_0 + p_1 + 1$ rays. Next, we will determine when $\varphi_\theta, \varphi_{\theta'}, \varphi_{(\theta+\theta')}$ are convex with respect to the fan Σ . The support function φ_θ is convex if and only if

$$\begin{aligned}\varphi_\theta(v_1 + \cdots + v_{p_0} + u_1) &\geq \varphi_\theta(v_1) + \cdots + \varphi_\theta(v_{p_0}) + \varphi_\theta(y_1) \\ \varphi_\theta(y_1 + \cdots + y_{p_1}) &\geq \varphi_\theta(y_1) + \cdots + \varphi_\theta(y_{p_1}).\end{aligned}$$

Simplifying the first inequality gives

$$0 \geq -d - \left(\sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right),$$

which holds in this case because d, b_i, c_j, β_j are all non-negative. If we simplify the second inequality, we get

$$-\left(f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right) \geq -\left(\sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right),$$

which holds if and only if $\sum_{i \in X_3} \beta_i \geq f$. Similarly, we get $\varphi_{\theta'}$ is convex if and only if $\sum_{i \in X_3} \gamma_i \geq f'$ and $\varphi_{(\theta+\theta')}$ is convex if and only if $\sum_{i \in X_3} \alpha_i \geq f + f'$. As a result, if

$$f + f' \leq \sum_{i \in X_3} \alpha_i \leq f + e + f' + e',$$

we find $\beta_J, \gamma_J \in \mathbb{Z}^{|J|}$ such that $f \leq \sum_{i \in X_3} \beta_i \leq f + e$ and $f' \leq \sum_{i \in X_3} \gamma_i \leq f' + e'$. We have seen that θ, θ' are convex in this case.

But if $0 \leq \sum_{j \in J} \alpha_j < f + f'$ then $\theta + \theta'$ is not convex. Nonetheless, we can see that the half-space defined by row u_1 is not a supporting half-space for the polytope, so we translate it without altering the polytope and construct a convex support function. Define

$$\begin{aligned}\tilde{\theta}_j &= \begin{cases} \theta_{y_1} & \text{if } j = u_1 \\ \theta_j & \text{otherwise,} \end{cases} \\ \tilde{\theta}'_j &= \begin{cases} \theta'_{y_1} & \text{if } j = u_1 \\ \theta'_j & \text{otherwise,} \end{cases} \\ \widetilde{(\theta + \theta')}_j &= \begin{cases} (\theta + \theta')_{y_1} & \text{if } j = u_1 \\ (\theta + \theta')_j & \text{otherwise.} \end{cases}\end{aligned}$$

It easy to see that $\tilde{\theta}, \tilde{\theta}'$ are convex and

$$\tilde{P} = P(F, \tilde{\theta}), \quad \tilde{Q} = P(F, \tilde{\theta}'), \quad \widetilde{P + Q} = P(F, \tilde{\theta} + \tilde{\theta}').$$

Case 4. If $p_4 > p_1 = 1$, then we have

$$F = \begin{matrix} & \xleftarrow{p_0+p_4-1 \text{ columns}} & \\ \begin{matrix} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ u_2 \\ \vdots \\ u_{p_4} \\ y_1 \end{matrix} & \begin{pmatrix} I & 0 \\ \hline -1 \cdots -1 & -1 \cdots -1 \\ \hline 0 & I \\ \hline -1 \cdots -1 & 0 \cdots 0 \end{pmatrix} & \end{matrix},$$

$$\theta = \begin{matrix} v_1 \\ \vdots \\ v_{p_0} \\ u_1 \\ u_2 \\ \vdots \\ u_{p_4} \\ y_1 \end{matrix} \begin{pmatrix} d \\ \vdots \\ 0 \\ f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \\ 0 \\ \vdots \\ 0 \\ \sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \end{pmatrix}, \quad \theta' = \begin{pmatrix} d' \\ \vdots \\ 0 \\ f' + \sum_{i \in X_3} b_i \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \\ 0 \\ \vdots \\ 0 \\ \sum_{i \in X_3} (b_i + 1) \gamma_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \gamma_j \end{pmatrix}.$$

In this case, we consider the fan Σ with ray generators given by the rows of F and primitive collections given by $\{v_1, \dots, v_{p_0}, y_1\}, \{u_1, \dots, u_{p_4}\}$. The primitive relations are given by

$$\begin{aligned} v_1 + \cdots + v_{p_0} + y_1 &= 0 \\ u_1 + \cdots + u_{p_4} &= y_1. \end{aligned}$$

It is a smooth complete fan in $\mathbb{R}^{p_0+p_4-1}$ with $p_0 + p_4 + 1$ rays. Next, we will determine when $\varphi_\theta, \varphi_{\theta'}, \varphi_{(\theta+\theta')}$ are convex with respect to the fan Σ . The support function φ_θ is convex if and only if

$$\begin{aligned} \varphi_\theta(v_1 + \cdots + v_{p_0} + y_1) &\geq \varphi_\theta(v_1) + \cdots + \varphi_\theta(v_{p_0}) + \varphi_\theta(y_1) \\ \varphi_\theta(u_1 + \cdots + u_{p_4}) &\geq \varphi_\theta(u_1) + \cdots + \varphi_\theta(u_{p_4}). \end{aligned}$$

Simplifying the first inequality gives

$$0 \geq -d - \left(\sum_{i \in X_3} (b_i + 1) \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j \right),$$

which holds in this case because d, b_i, c_j, β_j are all non-negative. If we simplify the second inequality, we get

$$-\left(\sum_{i \in X_3} (b_i + 1)\beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j\right) \geq -\left(f + \sum_{i \in X_3} b_i \beta_i + \sum_{j \in X_2 \setminus \{z_1\}} c_j \beta_j\right),$$

which holds if and only if $\sum_{i \in X_3} \beta_i \leq f$. Similarly, we get $\varphi_{\theta'}$ is convex if and only if $\sum_{i \in X_3} \gamma_i \leq f'$ and $\varphi_{(\theta+\theta')}$ is convex if and only if $\sum_{i \in X_3} \alpha_i \leq f + f'$. As a result, if

$$0 \leq \sum_{i \in X_3} \alpha_i \leq f + f',$$

we find $\beta_J, \gamma_J \in \mathbb{Z}^{|J|}$ such that $0 \leq \sum_{i \in X_3} \beta_i \leq f$ and $0 \leq \sum_{i \in X_3} \gamma_i \leq f'$. We have seen that θ, θ' are convex in this case.

But if $f + f' < \sum_{j \in J} \alpha_j \leq f + e + f' + e'$ then $\theta + \theta'$ is not convex. Nonetheless, we can see that the half-space defined by row y_1 is not a supporting half-space for the polytope, so we translate it without altering the polytope and construct a convex support function. Define

$$\begin{aligned} \tilde{\theta}_j &= \begin{cases} \theta_{u_1} & \text{if } j = y_1 \\ \theta_j & \text{otherwise,} \end{cases} \\ \tilde{\theta}'_j &= \begin{cases} \theta'_{u_1} & \text{if } j = y_1 \\ \theta'_j & \text{otherwise,} \end{cases} \\ \widetilde{(\theta + \theta')}_j &= \begin{cases} (\theta + \theta')_{u_1} & \text{if } j = y_1 \\ (\theta + \theta')_j & \text{otherwise.} \end{cases} \end{aligned}$$

It easy to see that $\tilde{\theta}, \tilde{\theta}'$ are convex and

$$\tilde{P} = P(F, \tilde{\theta}), \quad \tilde{Q} = P(F, \tilde{\theta}'), \quad \widetilde{P + Q} = P(F, \tilde{\theta} + \tilde{\theta}').$$

□

Remark 16. We can use the strategy of elimination variables together with induction to give a different proof for the fact that IDP holds for any smooth complete splitting fan.

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