Topology of complements of skeletons

Rowan Rowlands

Abstract

Given a polytopal complex X, we examine the topological complement of its k-skeleton. We construct a long exact sequence relating the homologies of the skeleton complements and links of faces in X, and using this long exact sequence, we obtain characterisations of Cohen–Macaulay and Leray complexes, stacked balls, and neighbourly spheres in terms of their skeleton complements. We also apply these results to CAT(0) cubical complexes, and find new similarities between such a complex and an associated simplicial complex, the crossing complex.

Mathematics Subject Classifications: 57Q05, 05E45, 52B05

1 Introduction

Polytopal complexes are important objects in topology and combinatorics, which include simplicial complexes, cubical complexes and polytopes. Their geometric realisations provide examples of a wide array of topological spaces, and much research has gone into studying the interplay between their combinatorial and topological aspects.

One important feature of a polytopal complex is its *k-skeleton*, the set of faces of the complex of dimension less than or equal to k. Skeletons act as a framework which the high-dimensional faces are attached to, so studying the structure of a skeleton can reveal a lot about a complex. For example, [4] surveys many results about objects that can be reconstructed from their k-skeletons for certain values of k. Skeletons also play an important role in defining cellular homology, and in important topological results such as Poincaré duality.

In this paper, however, we aim to approach this topic from the other direction, starting from the higher-dimensional faces instead of the low-dimensional ones. We define the *kth co-skeleton* of a complex to be the set of faces of dimension *higher* than k. We discover that there is a strong relationship between co-skeletons and *links*: a link of a face in a polytopal complex captures the local structure of the complex around that face, so in a sense, the co-skeletons give us a "global" summary of the "local" information of the complex.

University of Washington (rowanr@uw.edu).
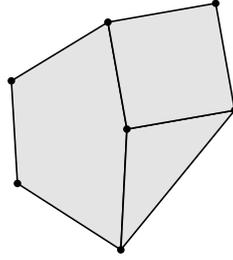
Figure 1: An example of a polytopal complex

Section 2 of this paper sets out the basic definitions we will use. In Section 3, we examine some initial facts about co-skeletons of arbitrary complexes, and their connections with various forms of duality (see Proposition 6 and Corollary 8). In Section 4, we construct a long exact sequence relating the kth and $(k-1)$th co-skeletons and the links of k-dimensional faces (Theorem 11). In Section 5, we use this long exact sequence to study some families of complexes defined by properties of links — specifically, Cohen–Macaulay complexes, Leray complexes, and stacked balls — and give characterisations of these families in terms of the homology of their co-skeletons (Theorems 14, 21 and 22). Finally, Section 6 examines cubical complexes, particularly CAT(0) cubical complexes: we examine the "crossing complex" defined in [19], and show that a CAT(0) cubical complex has one of the properties discussed in Section 5 if and only if its crossing complex shares that property (Theorem 33).

2 Preliminaries

We begin with some definitions. Readers familiar with polytopal complexes and topology may skip most of this section, but beware that we give slightly non-standard definitions in a couple of places, specifically for geometric realisations and links.

2.1 Polytopal complexes

A *polytopal complex* X is a collection of polytopes in Euclidean space $\mathbb{R}^N$ with the following properties:

- If σ is in X and τ is a face of σ, then τ is in X, and

- If σ and σ' are polytopes in X, then $\sigma \cap \sigma'$ is a face of each (possibly the empty face).

See Fig. 1 for a small example. In this paper, we will only consider finite polytopal complexes.

If every polytope in X is a simplex, then we say that $\Delta \coloneqq X$ is a *(geometric) simplicial complex*. If every polytope is a cube (that is, a polytope combinatorially equivalent to $[0,1]^i$ for some dimension $i \geqslant 0$), then $\square \coloneqq X$ is a *cubical complex*.

(a) X and Y (b) $|X \setminus Y|$ (c) $|X - Y|$
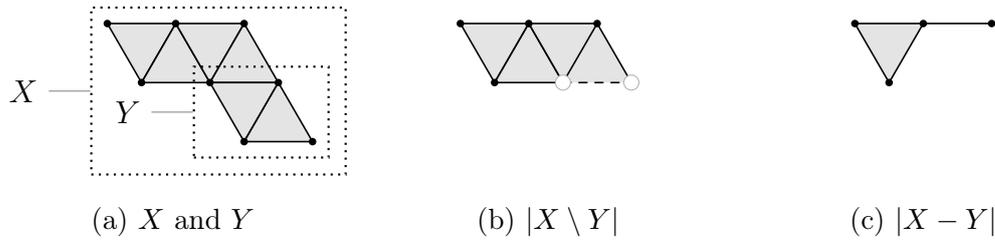
Figure 2: A comparison of the different types of deletion

An *abstract simplicial complex* Δ is a poset isomorphic to the poset of faces of a geometric simplicial complex, ordered by inclusion. Equivalently, it is a collection of subsets of some finite set, with the property that $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$. The dimension of a face σ is $\dim \sigma := \#\sigma - 1$.

In any of these types of complex, faces of dimension 0 and 1 are called *vertices* and *edges* respectively, and a maximal face (by inclusion) is called a *facet*. If all facets of a complex have the same dimension, the complex is *pure*. The dimension of the complex is the maximum dimension of its faces. The number of faces of dimension k in a complex X is denoted $f_k(X)$.

If σ is a polytope, $|\sigma|$ will denote its relative interior. If S is a collection of faces in a polytopal complex, then their *geometric realisation* $|S|$ is the union of the relative interiors of the polytopes in S. This is slightly different from the usual definition of a geometric realisation — if σ is in S, we do not include the boundary of σ in the geometric realisation $|S|$ unless those boundary faces are also part of S, unlike other definitions. We will take care to distinguish between S as a set of polytopes and $|S|$ as a topological space.

Suppose Y is a subset of a polytopal complex X. We write $X \setminus Y$ to denote the difference of sets, so $X \setminus Y$ is the set of faces of X that are not in Y (which is not generally a polytopal complex). If Y is a polytopal subcomplex of X, then $X - Y$ will denote the polytopal complex consisting of all faces of X that do not intersect any faces of Y. We will reserve "$-$" to denote this "combinatorial" deletion, and use "$\setminus$" to denote deletion of sets or topological spaces. Note that $|X \setminus Y| = |X| \setminus |Y|$, but the space $|X - Y|$ is not in general the same; see Fig. 2, for an example. However, these spaces are sometimes related by the following lemma:

Lemma 1 ([15, Lemma 70.1]). *If Δ is a simplicial complex and Λ is an induced subcomplex (in other words, every face of Δ whose vertices are contained in the vertex set of Λ is a face of Λ), then $|\Delta - \Lambda|$ and $|\Delta| \setminus |\Lambda|$ are homotopy equivalent.*

Given a face σ of X, the *(open) star* of σ is the set of faces

$$\operatorname{star}_X \sigma := \{\tau \in X : \sigma \subseteq \tau\}.$$

The geometric realisation of a star is always contractible, if $\sigma \neq \varnothing$. The *link* of σ is the
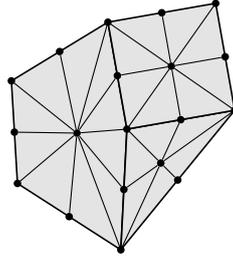
Figure 3: The barycentric subdivision of the polytopal complex in Fig. 1

set

$$\mathrm{link}_X \sigma := \{\tau \in X : \sigma \subset \tau,\ \tau \neq \sigma\}$$
$$= \mathrm{star}_X \sigma \setminus \{\sigma\}.$$

We will sometimes simply write "star σ" and "link σ" if the space X is clear from context. Note that if Δ is an abstract simplicial complex, the usual definition of a link is slightly different: we will refer to the usual definition as the "simplicial link", defined by

$$\mathrm{s\text{-}link}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \varnothing\}.$$

Although our link and the simplicial link are not the same, they are homotopy equivalent.

If X is a polytopal complex, its *barycentric subdivision* is the abstract simplicial complex $\mathrm{bary}(X)$ which has one vertex v_σ for each non-empty face σ of X, and a set $\{v_{\sigma_1}, \ldots, v_{\sigma_m}\}$ forms a face of $\mathrm{bary}(X)$ whenever $\{\sigma_1, \ldots, \sigma_m\}$ is a chain in the poset of faces of X ordered by inclusion (that is, $\sigma_1 \subset \cdots \subset \sigma_m$, up to reordering). If each vertex v_σ is positioned at the barycentre of the polytope σ, then the geometric realisation of $\mathrm{bary}(X)$ is exactly the geometric realisation of X, as a topological space. See Fig. 3 for an example.

Given a polytopal complex X, the set of k-dimensional faces of X will be denoted X_k. The *k-skeleton* of X is the subcomplex

$$\mathrm{Skel}_k X := \{\sigma \in X : \dim \sigma \leqslant k\}.$$

The central definition in this paper is the *kth co-skeleton* of X, which is the complement of the the k-skeleton:

$$\mathrm{Skel}_k^c X := X \setminus \mathrm{Skel}_k X$$
$$= \{\sigma \in X : \dim \sigma > k\}.$$

We will mostly be interested in the topological properties of $|\mathrm{Skel}_k^c X| = |X| \setminus |\mathrm{Skel}_k X|$. Note that the co-skeleton is not in general a polytopal complex, except in two special cases: $\mathrm{Skel}_{-1}^c X$ is X itself (modulo the empty face, which makes no difference to the topology), and if X is d-dimensional, $\mathrm{Skel}_d^c X$ is the polytopal complex with no faces.

2.2 Topology

We will use the notation $A \simeq B$ to indicate that the spaces A and B are homotopy equivalent.

We assume that the reader is familiar with homology and cohomology; for background, refer to [11, 15]. While this paper does focus on cell complexes, many spaces we consider are not themselves cell complexes, so $H_i(A)$ will denote singular homology, with coefficients in R where R is a field or $\mathbb{Z}$. Reduced homology is denoted by $\widetilde{H}_i(A)$. We use the convention that $\widetilde{H}_{-1}(\varnothing) = R$. Analogous statements apply to cohomology, denoted $H^i(A)$.

For reference, here are two important theorems from algebraic topology that we will use repeatedly.

Theorem 2 (Mayer–Vietoris, [11, p. 149], [15, Theorem 33.1]). *If A and B are open subsets of a topological space, then there is a long exact sequence:*

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(A \cup B) \longrightarrow H_{i-1}(A \cap B) \longrightarrow \cdots.$$

If $A \cap B \neq \varnothing$, then we may replace these unreduced homology groups with reduced homology throughout.

Theorem 3 (Nerve theorem, [11, Corollay 4G.3], [5, Theorem 10.7]). *Suppose $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a family of open sets whose union is a paracompact space (e.g. any subspace of $\mathbb{R}^n$), or a family of closed sets whose union is a triangulable space. Suppose further that for every index set $I \subseteq \{1, \ldots, n\}$, the intersection $\bigcap_{i \in I} U_i$ is either contractible or empty.*

Construct a simplicial complex $N(\mathcal{U})$ (called the "nerve" of $\mathcal{U}$) where the vertex set is $\{1, \ldots, n\}$ and the set I forms a face whenever $\bigcap_{i \in I} U_i$ is non-empty. Then

$$\bigcup_{i=1}^{n} U_i \simeq |N(\mathcal{U})|.$$

3   First results about co-skeletons

Let us begin to study the co-skeleton $\mathrm{Skel}_k^c X$.

This first lemma tells us that while $\mathrm{Skel}_k^c X$ is not itself a polytopal complex, it is homotopy equivalent to one. See Fig. 4.

Lemma 4. $|\mathrm{Skel}_k^c X| \simeq |\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_k X)|.$

Proof. By definition,

$$|\mathrm{Skel}_k^c X| = |X| \setminus |\mathrm{Skel}_k X|$$
$$= |\mathrm{bary}(X)| \setminus |\mathrm{bary}(\mathrm{Skel}_k X)|.$$

(a) $\mathrm{Skel}_0^c X$ (b) $\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_0 X)$
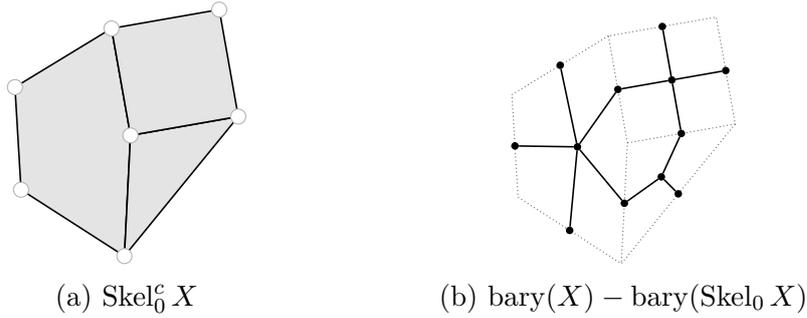
Figure 4: Lemma 4 applied to Fig. 1, with $k = 0$

Now, $\mathrm{bary}(\mathrm{Skel}_k X)$ is an induced subcomplex of the simplicial complex $\mathrm{bary}(X)$: if $v_{\sigma_1}, \ldots, v_{\sigma_m}$ are vertices of $\mathrm{bary}(\mathrm{Skel}_k X)$ that form a face of $\mathrm{bary}(X)$, then the faces $\sigma_1, \ldots, \sigma_m$ form a chain in the face poset of X, so they still form a chain in the face poset of $\mathrm{Skel}_k X$. Therefore, we may invoke Lemma 1:

$$|\mathrm{bary}(X)| \setminus |\mathrm{bary}(\mathrm{Skel}_k X)| \simeq |\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_k X)|. \qquad \square$$

Corollary 5. $H_i(\mathrm{Skel}_k^c X) = 0$ when $i > \dim X - k - 1$.

Proof. The vertices of $\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_k X)$ correspond to faces σ of X of dimensions between $k + 1$ and $\dim X$; therefore, the largest possible face of $\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_k X)$ has $\dim X - k$ vertices, so its dimension is $\dim X - k - 1$. $\qquad \square$

The name "co-skeleton" was chosen to suggest "complement", but also duality. Let us illustrate why.

Proposition 6. *Suppose P is a $(d+1)$-dimensional polytope. Then*

$$|\mathrm{Skel}_k^c \partial P| \simeq |\mathrm{Skel}_{d-k-1} \partial P^*|,$$

where P^* is the polar dual polytope to P.

Proof. By Lemma 4,

$$|\mathrm{Skel}_k^c \partial P| \simeq |\mathrm{bary}(\partial P) - \mathrm{bary}(\mathrm{Skel}_k \partial P)|.$$

The faces of $\mathrm{bary}(\partial P) - \mathrm{bary}(\mathrm{Skel}_k \partial P)$ are the chains in the poset of faces of ∂P of dimension greater than k. But the poset of non-empty faces of ∂P is isomorphic to the poset of non-empty faces of ∂P^*, flipped upside down. Under this flip, the faces of ∂P of dimension greater than k become faces of ∂P^* of dimension less than or equal to $d - k - 1$. Therefore, $\mathrm{bary}(\partial P) - \mathrm{bary}(\mathrm{Skel}_k \partial P) = \mathrm{bary}(\mathrm{Skel}_{d-k-1} \partial P^*)$, so

$$\begin{aligned}
|\mathrm{bary}(\partial P) - \mathrm{bary}(\mathrm{Skel}_k \partial P)| &= |\mathrm{bary}(\mathrm{Skel}_{d-k-1} \partial P^*)| \\
&= |\mathrm{Skel}_{d-k-1} \partial P^*|. \qquad \square
\end{aligned}$$

Note that the assumption that P is a polytope is not essential to this proposition, just that P has an associated "dual cell structure". In fact, if X is a d-dimensional homology manifold, the space $|\mathrm{bary}(X) - \mathrm{bary}(\mathrm{Skel}_k X)|$ is exactly the $(d - k - 1)$-skeleton of the dual cell structure used in some proofs of Poincaré duality — for example, see [15, §64], which calls this space the "dual $(d - k - 1)$-skeleton" of the manifold.

A related duality result is the Alexander duality theorem, which has direct implications for co-skeletons.

Theorem 7 (Alexander duality, [15, §71, particularly Theorem 71.1 and Exercise 4]). *Suppose X is a polytopal complex where $\dim \Delta = d$ and $|X|$ is homeomorphic to a sphere, and suppose Y is a proper, non-empty subcomplex of X. Then*

$$\widetilde{H}^i(|Y|) \cong \widetilde{H}_{d-i-1}(|X| \setminus |Y|) \qquad and \qquad \widetilde{H}_i(|Y|) \cong \widetilde{H}^{d-i-1}(|X| \setminus |Y|).$$

Taking $Y = \mathrm{Skel}_k X$ gives us this corollary:

Corollary 8. *If X is a d-dimensional polytopal complex where $|X|$ is homeomorphic to a sphere, then for $k = 0, \ldots, d - 1$,*

$$\widetilde{H}^i(|\mathrm{Skel}_k X|) \cong \widetilde{H}_{d-i-1}(|\mathrm{Skel}_k^c X|) \qquad and \qquad \widetilde{H}_i(|\mathrm{Skel}_k X|) \cong \widetilde{H}^{d-i-1}(|\mathrm{Skel}_k^c X|).$$

For a first example of an application for co-skeletons, let us consider neighbourly simplicial spheres. Suppose Δ is a simplicial sphere (that is, a simplicial complex where $|\Delta|$ is homeomorphic to a sphere) with n vertices. Then Δ is *t-neighbourly* if every set of t vertices is the vertex set of a face, or equivalently, if its $(t - 1)$-skeleton is isomorphic to the $(t - 1)$-skeleton of a simplex with n vertices. For example, the boundaries of d-dimensional cyclic polytopes are $\lfloor d/2 \rfloor$-neighbourly. A major reason why neighbourly spheres have been studied is their connection with the Upper Bound Conjecture (see e.g. [2]).

The following well-known fact lets us characterise skeletons of simplices by their homology.

Lemma 9. *Suppose Δ is a simplicial complex with n vertices, $n \geqslant 1$, and $\dim \Delta = k < n$. Then*

$$\mathrm{rank}\, \widetilde{H}_k(|\Delta|; \mathbb{Z}) \leqslant \binom{n-1}{k+1},$$

with equality if and only if Δ is isomorphic to the k-skeleton of a simplex with n vertices. If the coefficient ring is a field instead of $\mathbb{Z}$, the same holds with rank replaced by dimension.

(This follows without much difficulty from [6, Theorem 3.1] and [13, Equation (3.6)]. Alternatively, for a self-contained proof sketch, let v be a vertex of Δ and consider the Mayer–Vietoris long exact sequence with $A := |\mathrm{star}\, v|$, $B := |\Delta| \setminus \{v\} \simeq |\Delta - \{v\}|$. The sequence contains this segment:

$$\widetilde{H}_k(|\Delta - \{v\}|) \longrightarrow \widetilde{H}_k(|\Delta|) \longrightarrow \widetilde{H}_{k-1}(|\mathrm{link}\, v|).$$

The rank of the middle term is thus bounded by the sum of the ranks of the outer terms; now, use induction on n.)

This lemma tells us that "being a skeleton of a simplex" is a homological property, if dimension and number of vertices are fixed. Combining this with Corollary 8, we obtain the following characterisations of neighbourly spheres in terms of their skeletons and their co-skeletons:

Corollary 10. *Suppose Δ is a d-dimensional simplicial sphere with n vertices. Then for $t = 1, \ldots, d$, the following are equivalent:*

- *Δ is t-neighbourly,*

- *rank $\widetilde{H}_{t-1}(|\mathrm{Skel}_{t-1}\, \Delta|; \mathbb{Z}) = \binom{n-1}{t}$,*

- *rank $\widetilde{H}^{d-t}(|\mathrm{Skel}_{t-1}^c\, \Delta|; \mathbb{Z}) = \binom{n-1}{t}$.*

Proof. By definition, Δ is t-neighbourly if and only if $\mathrm{Skel}_{t-1}\, \Delta$ is isomorphic to the $(t-1)$-skeleton of a simplex on n vertices. Lemma 9 says that this occurs if and only if rank $\widetilde{H}_{t-1}(|\mathrm{Skel}_{t-1}\, \Delta|; \mathbb{Z}) = \binom{n-1}{t}$. And Corollary 8 says that

$$\widetilde{H}_{t-1}(|\mathrm{Skel}_{t-1}\, \Delta|; \mathbb{Z}) \cong \widetilde{H}^{d-t}(|\mathrm{Skel}_{t-1}^c\, \Delta|; \mathbb{Z}). \qquad \square$$

4 The key long exact sequence

We now reach the main tool of this paper, which we will make much use of in the upcoming sections.

Theorem 11. *If X is a d-dimensional polytopal complex and $0 \leqslant k \leqslant d$, then there is the following long exact sequence:*

$$\cdots \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\mathrm{link}_X\, \sigma|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_k^c\, X|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_{k-1}^c\, X|)$$

$$\longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_{i-1}(|\mathrm{link}_X\, \sigma|) \longrightarrow \cdots.$$

Proof. We will begin with the case $k = d$. The link of any d-dimensional face is empty, as is $\mathrm{Skel}_d^c\, X$. The $(d-1)$th co-skeleton $\mathrm{Skel}_{d-1}^c\, X$ consists only of the interiors of the d-dimensional faces, so it retracts onto $f_d(X)$ points. Therefore, the non-zero part of the claimed long exact sequence looks like this:

$$0 \longrightarrow \widetilde{H}_0(\mathrm{Skel}_{d-1}^c) \longrightarrow \bigoplus_{\sigma \in X_d} \widetilde{H}_{-1}(\mathrm{link}\, \sigma) \longrightarrow \widetilde{H}_{-1}(\mathrm{Skel}_d^c\, X) \longrightarrow 0.$$

$$\| \wr \qquad\qquad\qquad\qquad \| \wr \qquad\qquad\qquad\qquad \| \wr$$

$$R^{f_d(X)-1} \qquad\qquad\quad R^{f_d(X)} \qquad\qquad\qquad\quad R$$

(a) $A = \bigsqcup_{\sigma \in X_0} |\mathrm{star}_{\mathrm{bary}(X)} v_\sigma|$

(b) $B = |\mathrm{Skel}_0^c X|$

(c) $A \cap B = \bigsqcup_{\sigma \in X_0} |\mathrm{star}_{\mathrm{bary}(X)} v_\sigma| \setminus |\sigma|$

(d) $A \cup B = |\mathrm{Skel}_{-1}^c X| = |X|$
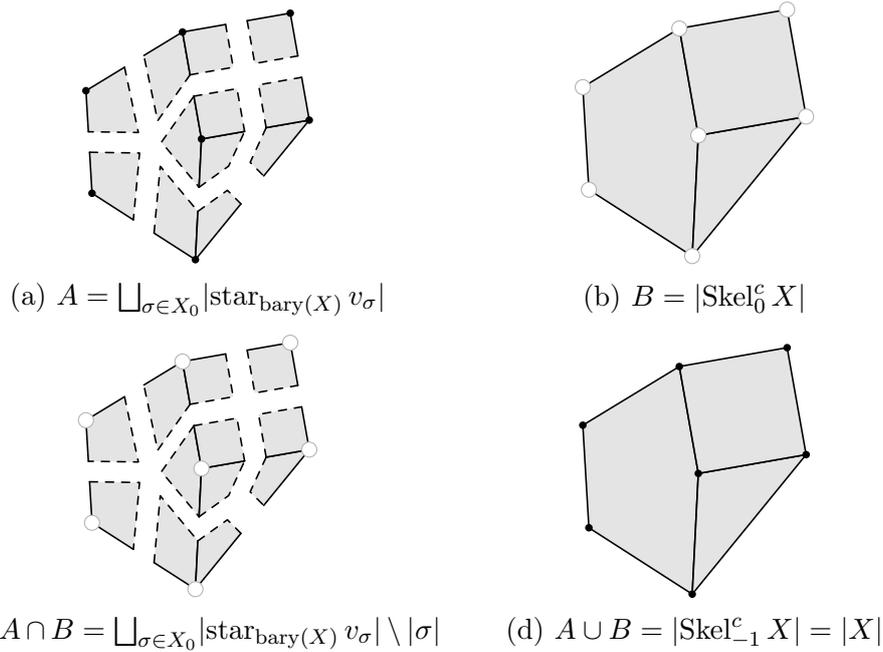
Figure 5: The sets A, B, $A \cap B$ and $A \cup B$ in the proof of Theorem 11, when $k = 0$ and X is the polytopal complex in Fig. 1

These homology groups do indeed form an exact sequence, with appropriate maps. (We won't need to know what the maps are in the rest of this paper, just that they exist.)

For the remainder of this proof, we will assume that $k < d$. We will use the Mayer–Vietoris theorem. Define the sets A and B by:

$$A := \bigcup_{\sigma \in X_k} |\mathrm{star}_{\mathrm{bary}(X)} v_\sigma|,$$
$$B := |\mathrm{Skel}_k^c X| = |X| \setminus |\mathrm{Skel}_k X|.$$

See Figs. 5 and 6 for an example. Both A and B are open in $|X|$: A is a union of open sets, and B is the complement of a closed set.

First, we claim that the union in A is in fact a *disjoint* union. If two stars, say $\mathrm{star}_{\mathrm{bary}(X)} v_\sigma$ and $\mathrm{star}_{\mathrm{bary}(X)} v_\tau$, were to intersect, they would have a face of $\mathrm{bary}(X)$ in common, and by the definition of an open star, this face would need to contain both v_σ and v_τ. But faces in $\mathrm{bary}(X)$ correspond to chains in the face poset of X; therefore, no face in $\mathrm{bary}(X)$ can contain both v_σ and v_τ, since σ and τ are both k-dimensional and are thus incomparable in the face poset. So A is a disjoint union of stars. Since these stars are open, each one thus forms a connected component of A.

Next, let us consider $A \cap B$. Since $\mathrm{star}_{\mathrm{bary}(X)} v_\sigma$ does not intersect any faces of X of dimension less than k, and the only k-dimensional face it intersects is σ, we conclude that

$$A \cap B = \bigsqcup_{\sigma \in X_k} |\mathrm{star}_{\mathrm{bary}(X)} v_\sigma| \setminus |\sigma|.$$

(a) $A = \bigsqcup_{\sigma \in X_1} |\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma|$

(b) $B = |\mathrm{Skel}_1^c X|$

(c) $A \cap B = \bigsqcup_{\sigma \in X_1} |\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma| \setminus |\sigma|$

(d) $A \cup B = |\mathrm{Skel}_0^c X|$
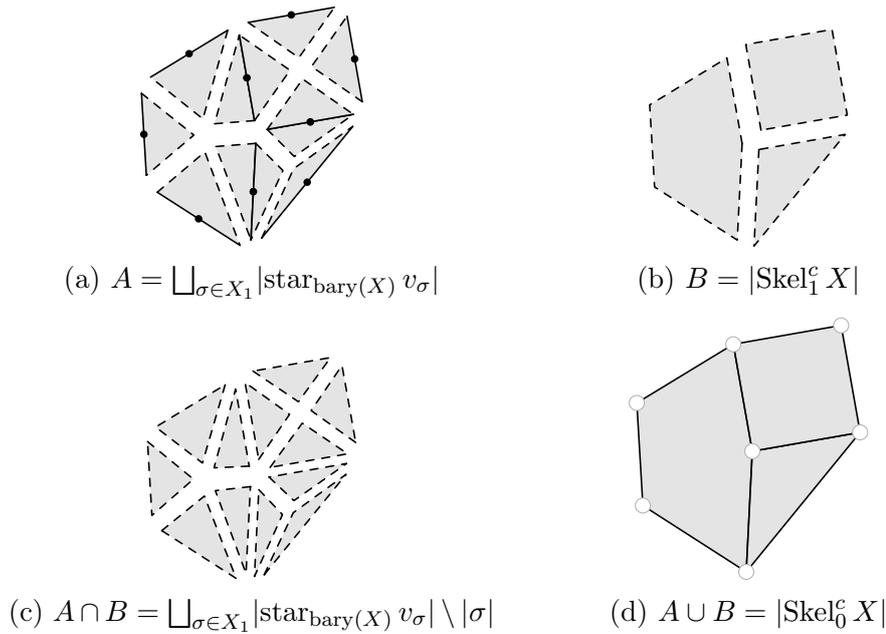
Figure 6: The sets A, B, $A \cap B$ and $A \cup B$ in the proof of Theorem 11, when $k = 1$ and X is the polytopal complex in Fig. 1

Now for $A \cup B$. Suppose τ is an arbitrary face of X. If $\dim \tau \leqslant k - 1$, then $|\tau|$ is disjoint from both A and B; if $\dim \tau = k$, then $|\tau|$ is a subset of $|\mathrm{star}_{\mathrm{bary}(X)}\, v_\tau|$ hence $|\tau| \subseteq A$, and if $\dim \tau \geqslant k + 1$ then $|\tau|$ is a subset of B. Therefore, $A \cup B$ consists of the relative interiors of all faces of dimension k and greater; that is,

$$A \cup B = |\mathrm{Skel}_{k-1}^c X|.$$

We can now apply the Mayer–Vietoris theorem, with unreduced homology. This gives us the following long exact sequence:

$$\cdots \longrightarrow H_i\left(\bigsqcup_{\sigma \in X_k} |\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma| \setminus |\sigma| \right) \longrightarrow H_i\left(\bigsqcup_{\sigma \in X_k} |\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma| \right) \oplus H_i(|\mathrm{Skel}_k^c X|)$$

$$\longrightarrow H_i(|\mathrm{Skel}_{k-1}^c X|) \longrightarrow \cdots .$$

Since unreduced homology of a disconnected space can be decomposed as a direct sum over components, we can simplify this long exact sequence:

$$\cdots \longrightarrow \bigoplus_{\sigma \in X_k} H_i\big(|\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma| \setminus |\sigma|\big) \longrightarrow \left(\bigoplus_{\sigma \in X_k} H_i(|\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma|) \right) \oplus H_i(|\mathrm{Skel}_k^c X|)$$

$$\longrightarrow H_i(|\mathrm{Skel}_{k-1}^c X|) \longrightarrow \cdots .$$

While this is a perfectly good long exact sequence, there are some steps we can do to "clean it up" and make it more convenient for later proofs. The first step is to convert

from unreduced to reduced homology, which affects this sequence in degrees 0 and -1. For each face $\sigma \in X_k$, there are two cases to consider: either σ is not a facet, or it is.

If σ is not a facet, then $|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|$ is non-empty, so $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|) \neq 0$. The map from $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|)$ to $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma|) = R$ takes generators to generators, so we can remove a summand of R from each of these two groups while preserving exactness; the result is that these unreduced homology groups become reduced.

On the other hand, suppose σ is a facet: then $|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|$ is empty, so $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|) = 0$. Since $H_0(|\text{Skel}^c_{k-1}\, X|)$ is non-zero, and the map from $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma|)$ to $H_0(|\text{Skel}^c_{k-1}\, X|)$ takes generators to generators, we can remove a summand of R from each of these two groups. If we also add a summand of R to each of $H_0(|\text{Skel}^c_{k-1}\, X|)$ and $H_{-1}(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|) = 0$, then exactness is preserved, no net change has occurred to $H_0(|\text{Skel}^c_{k-1}\, X|)$, and the two homology groups $H_{-1}(|\text{star}_{\text{bary}(X)}\, v_\sigma|$ and $H_0(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|)$ become reduced.

There is one more change to make in converting from unreduced to reduced homology: the map from $H_0(|\text{Skel}^c_k\, X|)$ to $H_0(|\text{Skel}^c_{k-1}\, X|)$ takes generators to generators (both groups are non-zero because we assume $k < d$), so we may remove a summand of R from each to replace both of these unreduced homology groups with reduced homology.

With the observation that $\widetilde{H}_i(|\text{star}_{\text{bary}(X)}\, v_\sigma|) = 0$ for all i, our long exact sequence now becomes:

$$\cdots \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_i\big(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|\big) \longrightarrow \widetilde{H}_i(|\text{Skel}^c_k\, X|) \longrightarrow \widetilde{H}_i(|\text{Skel}^c_{k-1}\, X|) \longrightarrow \cdots .$$

There is one last thing to do to this long exact sequence: simplifying $\widetilde{H}_i\big(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|\big)$. To do this, let us call on the Mayer–Vietoris theorem again. Assume that σ is not a facet, so $|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|$ is non-empty. Define two new spaces:

$$A' := |\text{star}_{\text{bary}(X)}\, v_\sigma|,$$
$$B' := |\text{star}_X\, \sigma| \setminus |\sigma| = |\text{link}_X\, \sigma|$$

Since $|\sigma| \subseteq |\text{star}_{\text{bary}(X)}\, v_\sigma| \subseteq |\text{star}_X\, \sigma|$ — for example, see Fig. 7 — the intersection $A' \cap B'$ is $|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|$, and the union $A' \cup B'$ is simply $|\text{star}_X\, \sigma|$. Therefore, the Mayer–Vietoris long exact sequence with reduced homology for A' and B' is the following:

$$\cdots \longrightarrow \widetilde{H}_{i+1}(|\text{star}_X\, \sigma|) \longrightarrow \widetilde{H}_i(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|)$$

$$\longrightarrow \widetilde{H}_i(|\text{star}_{\text{bary}(X)}\, v_\sigma|) \oplus \widetilde{H}_i(|\text{star}_X\, \sigma| \setminus |\sigma|) \longrightarrow \widetilde{H}_i(|\text{star}_X\, \sigma|) \longrightarrow \cdots .$$

But stars of non-empty faces are contractible, so this becomes:

$$\cdots \longrightarrow 0 \longrightarrow \widetilde{H}_i(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|) \longrightarrow \widetilde{H}_i(|\text{star}_X\, \sigma| \setminus |\sigma|) \longrightarrow 0 \longrightarrow \cdots .$$

Therefore, $\widetilde{H}_i(|\text{star}_{\text{bary}(X)}\, v_\sigma| \setminus |\sigma|) \cong \widetilde{H}_i(|\text{star}_X\, \sigma| \setminus |\sigma|) = \widetilde{H}_i(|\text{link}_X\, \sigma|)$ for all non-facets σ. Finally, note that if σ is a facet, this isomorphism still holds, since all spaces involved are empty.

Hence we obtain the desired long exact sequence. $\qquad\square$

(a) Three faces of X

(b) $|\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma|$ for $\sigma = \sigma_1, \sigma_2, \sigma_3$

(c) $|\mathrm{star}_X\, \sigma|$ for $\sigma = \sigma_1, \sigma_2, \sigma_3$
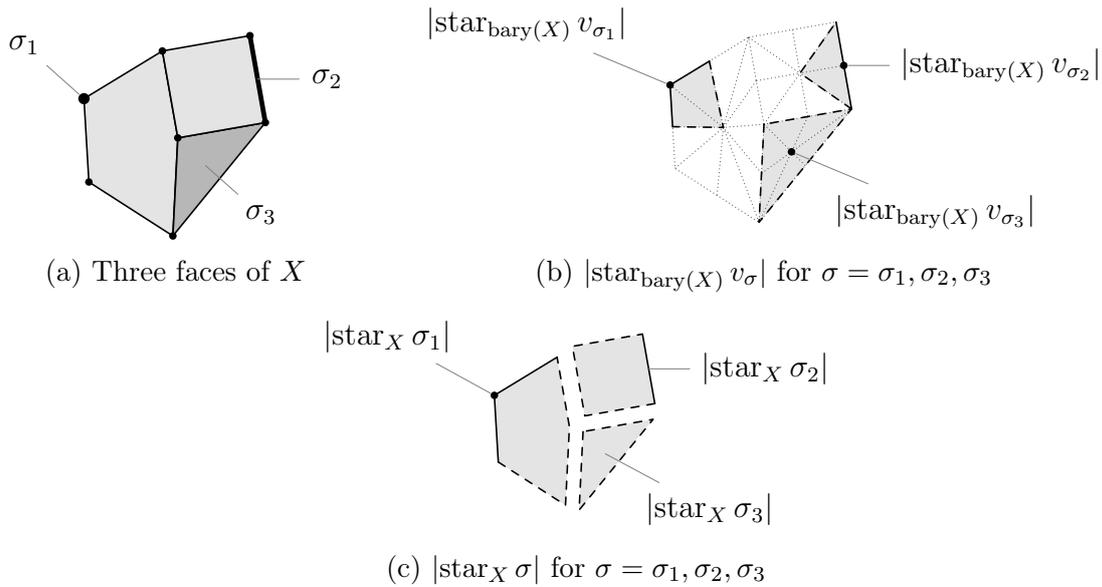
Figure 7: The spaces $|\mathrm{star}_{\mathrm{bary}(X)}\, v_\sigma|$ and $|\mathrm{star}_X\, \sigma|$ in the proof of Theorem 11, for various choices of σ

5 Complexes characterised by link conditions

Now that we have a long exact sequence relating co-skeletons and links, in this section we will consider some particular families of polytopal complexes that are characterised by links of faces having zero homology in some degrees.

5.1 Cohen–Macaulay complexes

A pure, d-dimensional polytopal complex X is *Cohen–Macaulay* if for every face σ, including $\sigma = \varnothing$, and every $i < d - \dim \sigma - 1$, the homology $\widetilde{H}_i(|\mathrm{link}_X \sigma|)$ is 0. If $\Delta := X$ is a simplicial complex, this condition is equivalent to the Stanley–Reisner ring of Δ being a Cohen–Macaulay ring [5, p. 1855].

Remark 12. Cohen–Macaulayness is also a topological condition: X is Cohen–Macaulay if and only if $\widetilde{H}_i(|X|)$ and $\widetilde{H}_i(|X|, |X| \setminus \{p\})$ are both 0 for all $i < \dim X$ and all points $p \in |X|$. This was proved by Munkres [16, Corollary 3.4] for simplicial complexes, but a similar proof applies to polytopal complexes as well, with the following modified version of [16, Lemma 3.3]:

Lemma 13. *If X is a polytopal complex, σ a face of X and p a point in the relative interior of σ, then $\widetilde{H}_i(|\mathrm{link}_X \sigma|) \cong \widetilde{H}_{i+\dim \sigma}(|\mathrm{link}_{\mathrm{bary}(X)} v_\sigma|) \cong \widetilde{H}_{i+\dim \sigma+1}(|X|, |X| \setminus \{p\})$.*

Proof. We saw in the proof of Theorem 11 that

$$\widetilde{H}_i(|\mathrm{link}_X \sigma|) \cong \widetilde{H}_i(|\mathrm{star}_{\mathrm{bary}(X)} v_\sigma| \setminus |\sigma|)$$

$$= \widetilde{H}_i\big(|\mathrm{link}_{\mathrm{bary}(X)} v_\sigma| \setminus (|\sigma| \setminus \{v_\sigma\})\big)$$

$$\cong \widetilde{H}_i(|\text{s-link}_{\mathrm{bary}(X)} v_\sigma| \setminus |\mathrm{bary}(\partial\sigma)|),$$

where "s-link" denotes the simplicial link. Since $\mathrm{bary}(\partial\sigma)$ is an induced subcomplex of the simplicial complex s-link$_{\mathrm{bary}(X)} v_\sigma$, Lemma 1 says that

$$|\text{s-link}_{\mathrm{bary}(X)} v_\sigma| \setminus |\mathrm{bary}(\partial\sigma)| \simeq |\text{s-link}_{\mathrm{bary}(X)} v_\sigma - \mathrm{bary}(\partial\sigma)|.$$

It is not hard to check that

$$\text{s-link}_{\mathrm{bary}(X)} v_\sigma = \big(\text{s-link}_{\mathrm{bary}(X)} v_\sigma - \mathrm{bary}(\partial\sigma)\big) * \mathrm{bary}(\partial\sigma),$$

where $*$ denotes simplicial join [15, §62], so [15, Theorem 62.5] implies that

$$\widetilde{H}_i(|\text{s-link}_{\mathrm{bary}(X)} v_\sigma - \mathrm{bary}(\partial\sigma)|) \cong \widetilde{H}_{i+\dim\sigma}(|\text{s-link}_{\mathrm{bary}(X)} v_\sigma|).$$

Finally, since $\mathrm{bary}(X)$ is a simplicial complex, we can apply [16, Lemma 3.3] to conclude that

$$\widetilde{H}_{i+\dim\sigma}(|\text{s-link}_{\mathrm{bary}(X)} v_\sigma|) \cong \widetilde{H}_{i+\dim\sigma+1}(|X|, |X| \setminus \{v_\sigma\}),$$

and note that although v_σ is typically defined to be the barycentre of σ, it can be chosen to be any point in the relative interior of σ. $\qquad\square$

The following theorem allows us to determine whether a complex is Cohen–Macaulay by considering its co-skeletons.

Theorem 14. *A d-dimensional polytopal complex X is Cohen–Macaulay if and only if it is pure and $\widetilde{H}_i(|\mathrm{Skel}_k^c X|) = 0$ for all $i < d - k - 1$ and all $k = -1, \ldots, d$.*

Note that this theorem combined with Corollary 5 means that the only non-zero homology of $|\mathrm{Skel}_k^c X|$ is in degree $d - k - 1$.

Proof of Theorem 14. To begin with, assume X is Cohen–Macaulay. We will prove that $\widetilde{H}_i(|\mathrm{Skel}_k^c X|) = 0$ for $i < d - k - 1$ by induction on k.

For the base case, when $k = -1$, we have

$$\widetilde{H}_i(|\mathrm{Skel}_{-1}^c X|) = \widetilde{H}_i(|X|) = \widetilde{H}_i(|\mathrm{link}_X \varnothing|),$$

which is zero for all $i < d$ since X is Cohen–Macaulay.

Now, suppose $k > -1$, and consider this part of the long exact sequence from Theorem 11:

$$\bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\mathrm{link}\,\sigma|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_k^c X|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_{k-1}^c X|).$$
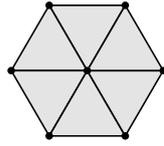
(a) A Cohen–Macaulay simplicial complex
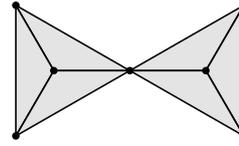
(b) A non-Cohen–Macaulay simplicial complex

Figure 8: Two simplicial complexes whose skeletons have isomorphic homology groups

Since X is Cohen–Macaulay and $i < d - k - 1$, we know that $\widetilde{H}_i(|\text{link}\,\sigma|)$ is 0 for all $\sigma \in X_k$. Also, since i is less than $d - k - 1$, it is certainly less than $d - (k-1) - 1$, so by induction, $\widetilde{H}_i(|\text{Skel}^c_{k-1}\,X|)$ is 0. Therefore, $\widetilde{H}_i(|\text{Skel}^c_k\,X|)$ must be 0 too, and we are done with this direction of the proof.

Now, for the reverse direction, assume that $\widetilde{H}_i(|\text{Skel}^c_k\,X|) = 0$ for all $i < d - k - 1$ and all $k = -1, \ldots, d$. Consider this part of the long exact sequence for $k \geqslant 0$:

$$\widetilde{H}_{i+1}(|\text{Skel}^c_{k-1}\,X|) \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\text{link}\,\sigma|) \longrightarrow \widetilde{H}_i(|\text{Skel}^c_k\,X|)$$

By our assumption, $\widetilde{H}_i(|\text{Skel}^c_k\,X|) = 0$. Also, since $i < d - k - 1$, we have $i + 1 < d - (k-1) - 1$, so our assumption tells us that $\widetilde{H}_{i+1}(|\text{Skel}^c_{k-1}\,X|) = 0$ too. Therefore, $\bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\text{link}\,\sigma|)$ is 0, so $\widetilde{H}_i(|\text{link}\,\sigma|)$ must be 0 for all $i < d - k - 1 = d - \dim\sigma - 1$.

This is true for all faces σ of dimension $k \geqslant 0$, which leaves the case $\sigma = \varnothing$. In this case,

$$\widetilde{H}_i(|\text{link}\,\sigma|) = \widetilde{H}_i(|X|) = \widetilde{H}_i(|\text{Skel}^c_{-1}\,X|) = 0.$$

Therefore, X is Cohen–Macaulay. $\qquad\square$

Remark 15. This theorem has potential computational applications. Naïvely, to check whether some complex X is Cohen–Macaulay, one must compute the homology groups of the link of every face of X — if $\dim X = d$, there are at least 2^{d+1} faces. But with this result, one only needs to compute homologies of the kth co-skeletons of X for the $d + 2$ values of k between -1 and d, or equivalently (by Lemma 4) the simplicial complexes $\text{bary}(X) - \text{bary}(\text{Skel}_k\,X)$. The tradeoff is that these simplicial complexes are much larger and more complicated than the links of faces: for instance, the number of vertices of $\text{bary}(X) - \text{bary}(\text{Skel}_k\,X)$ is at least on the order of 2^d.

Remark 16. We saw in Corollary 10 that neighbourly spheres are characterised by the homology groups of either their skeletons or their co-skeletons. However, for Cohen–Macaulay complexes, the homology groups of skeletons are not sufficient for a characterisation: for example, Fig. 8 shows two complexes, one Cohen–Macaulay and one not, whose k-skeletons have isomorphic ith homology groups for all k and i.

Theorem 14 says that if X is Cohen–Macauly, almost all of the long exact sequence from Theorem 11 is 0. The remaining non-zero terms are captured in the following corollary:

Corollary 17. *If X is a Cohen–Macaulay polytopal complex and $k = 0, \ldots, d$, we have the following short exact sequence:*

$$0 \longrightarrow \widetilde{H}_{d-k}(|\mathrm{Skel}^c_{k-1} X|) \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_{d-k-1}(|\mathrm{link}\,\sigma|) \longrightarrow \widetilde{H}_{d-k-1}(|\mathrm{Skel}^c_k X|) \longrightarrow 0.$$

Proof. Take the following segment of the long exact sequence from Theorem 11:

$$\widetilde{H}_{d-k}(|\mathrm{Skel}^c_k X|) \longrightarrow \widetilde{H}_{d-k}(|\mathrm{Skel}^c_{k-1} X|) \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_{d-k-1}(|\mathrm{link}\,\sigma|)$$

$$\longrightarrow \widetilde{H}_{d-k-1}(|\mathrm{Skel}^c_k X|) \longrightarrow \widetilde{H}_{d-k-1}(|\mathrm{Skel}^c_{k-1} X|).$$

The first term, $\widetilde{H}_{d-k}(|\mathrm{Skel}^c_k X|)$, is 0 by Corollary 5. The last term, $\widetilde{H}_{d-k-1}(|\mathrm{Skel}^c_{k-1} X|)$, is 0 by Theorem 14. $\square$

We can combine these short exact sequences into a long exact sequence, using the following homological algebra fact:

Lemma 18. *Suppose we have two exact sequences:*

$$\cdots \longrightarrow A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} C \longrightarrow 0 \qquad and \qquad 0 \longrightarrow C \xrightarrow{b_0} B_{-1} \xrightarrow{b_{-1}} B_{-2} \longrightarrow \cdots.$$

Then the horizontal sequence in the following diagram is exact:

Proof. Since a_1 is surjective, $\mathrm{im}(b_0 \circ a_1) = \mathrm{im}(b_0) = \ker(b_{-1})$. Since b_0 is injective, $\ker(b_0 \circ a_1) = \ker a_1 = \mathrm{im}\, a_2$. $\square$

Corollary 19. *If X is Cohen–Macaulay, we have the following long exact sequence:*

$$0 \longrightarrow \widetilde{H}_d(|X|) \longrightarrow \bigoplus_{\sigma \in X_0} \widetilde{H}_{d-1}(|\mathrm{link}\,\sigma|) \longrightarrow \bigoplus_{\sigma \in X_1} \widetilde{H}_{d-2}(|\mathrm{link}\,\sigma|) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\sigma \in X_d} \widetilde{H}_{-1}(|\mathrm{link}\,\sigma|) \longrightarrow R \longrightarrow 0.$$

Proof. Use Lemma 18 to stitch together the short exact sequences from Corollary 17. The start of the sequence looks like this:

$$
\begin{array}{ccccccc}
0 & & & & 0 & & \\
& \searrow & & \nearrow & & & \\
& & \widetilde{H}_{d-1}(|\mathrm{Skel}_0^c\, X|) & & & & \\
& & & \searrow & & & \\
0 \to \widetilde{H}_d(|\mathrm{Skel}_{-1}^c\, X|) \to \bigoplus_{\sigma \in X_0} \widetilde{H}_{d-1}(|\mathrm{link}\,\sigma|) \dashrightarrow \bigoplus_{\sigma \in X_1} \widetilde{H}_{d-2}(|\mathrm{link}\,\sigma|) \dashrightarrow \bigoplus_{\sigma \in X_2} \widetilde{H}_{d-3}(|\mathrm{link}\,\sigma|) \dashrightarrow \cdots & & & & & & \\
& & & & \searrow & \nearrow & \\
& & & & \widetilde{H}_{d-2}(|\mathrm{Skel}_1^c\, X|) & & \\
& & & & \nearrow & \searrow & \\
& & & & 0 & & 0
\end{array}
$$

And the end like this:

$$
\begin{array}{ccccccc}
0 & & & & 0 & & \\
& \searrow & & \nearrow & & & \\
& & \widetilde{H}_1(|\mathrm{Skel}_{d-2}^c\, X|) & & & & \\
& & & \nearrow & & & \\
\cdots \dashrightarrow \bigoplus_{\sigma \in X_{d-2}} \widetilde{H}_1(|\mathrm{link}\,\sigma|) \dashrightarrow \bigoplus_{\sigma \in X_{d-1}} \widetilde{H}_0(|\mathrm{link}\,\sigma|) \dashrightarrow \bigoplus_{\sigma \in X_d} \widetilde{H}_{-1}(|\mathrm{link}\,\sigma|) \to \widetilde{H}_{-1}(|\mathrm{Skel}_d^c\, X|) \to 0 & & & & & & \\
& & & \searrow & & \nearrow & \\
& & & & \widetilde{H}_0(|\mathrm{Skel}_{d-1}^c\, X|) & & \\
& & & \nearrow & & \searrow & \\
& & & 0 & & & 0
\end{array}
$$

$\square$

Note that this exact sequence is reminiscent of the "partition complex", a chain complex defined in [1, Definition 25].

Remark 20. If the definition of a Cohen–Macaulay complex is altered to allow $\widetilde{H}_i(|\mathrm{link}_X\,\sigma|)$ to be non-zero when $\sigma = \varnothing$, we get the definition of a *Buchsbaum* complex. Buchsbaum complexes can also be characterised by homological properties of their co-skeletons, although the statement is not as simple as for Cohen–Macaulay complexes in Theorem 14: a complex X is Buchsbaum if and only if the map $\widetilde{H}_i(|\mathrm{Skel}_k^c\, X|) \to \widetilde{H}_i(|X|)$ induced by the inclusion of topological spaces is an isomorphism for all $i < d - k - 1$ and a surjection for $i = d - k - 1$, for all $k = -1, \ldots, d$. This follows fairly easily from the long exact sequence in Theorem 11, although we leave the details to the reader.

5.2 Leray complexes

A simplicial complex Δ is *r-Leray* if every induced subcomplex Λ of Δ has $\widetilde{H}_i(|\Lambda|) = 0$ for $i \geqslant r$. For example, if Δ is the nerve of a family of convex open subsets of $\mathbb{R}^r$, then it

follows from the nerve theorem (Theorem 3) that Δ is r-Leray [14].

Equivalently, Δ is r-Leray if the homology $\widetilde{H}_i(|\mathrm{link}_\Delta \sigma|)$ is 0 for every face $\sigma \in \Delta$ (including the empty face) and every $i \geqslant r$ [14, Proposition 3.1]. We will use this condition to generalise r-Leray-ness to polytopal complexes: we will say a complex X is r-Leray if $\widetilde{H}_i(|\mathrm{link}_X \sigma|) = 0$ for all faces σ (including $\sigma = \varnothing$) and all $i \geqslant r$. Note that this condition is not equivalent to the condition on induced subcomplexes in the non-simplicial case: for example, if X is a square, every non-empty link is contractible so X is 0-Leray, but the subcomplex induced by a pair of diagonally opposite vertices is not connected.

The following theorem provides a characterisation of r-Leray complexes in terms of their co-skeletons.

Theorem 21. *A polytopal complex X is r-Leray if and only if $\widetilde{H}_i(|\mathrm{Skel}_k^c X|) = 0$ for all $i \geqslant r$ and all $k = -1, \ldots, d$.*

Proof. For the forward direction, assume X is r-Leray and let $i \geqslant r$. When $k = -1$,

$$\widetilde{H}_i(|\mathrm{Skel}_{-1}^c X|) = \widetilde{H}_i(|X|) = \widetilde{H}_i(|\mathrm{link}_X \varnothing|),$$

which is 0, since X is r-Leray.

For $k \geqslant 0$, consider the following subsequence of the long exact sequence from Theorem 11:

$$\bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\mathrm{link}\, \sigma|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_k^c X|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_{k-1}^c X|).$$

Since X is r-Leray, the first term in this subsequence is 0. Therefore, $\widetilde{H}_i(|\mathrm{Skel}_k^c X|)$ injects into $\widetilde{H}_i(|\mathrm{Skel}_{k-1}^c X|)$. Since this is true for all $k = 0, \ldots, d$, we get a series of injections:

$$\widetilde{H}_i(|\mathrm{Skel}_k^c X|) \hookrightarrow \widetilde{H}_i(|\mathrm{Skel}_{k-1}^c X|) \hookrightarrow \widetilde{H}_i(|\mathrm{Skel}_{k-2}^c X|) \hookrightarrow \cdots$$

$$\cdots \hookrightarrow \widetilde{H}_i(|\mathrm{Skel}_0^c X|) \hookrightarrow \widetilde{H}_i(|\mathrm{Skel}_{-1}^c X|).$$

But we saw above that $\widetilde{H}_i(|\mathrm{Skel}_{-1}^c X|) = 0$. Therefore, $\widetilde{H}_i(|\mathrm{Skel}_k^c X|)$ injects into 0, so it must itself be 0, for all $k = 0, \ldots, d$.

For the reverse direction, assume that $\widetilde{H}_i(|\mathrm{Skel}_k^c X|) = 0$ for all $k = -1, \ldots, d$ and all $i \geqslant r$. When $k \geqslant 0$, consider this part of the long exact sequence:

$$\widetilde{H}_{i+1}(|\mathrm{Skel}_{k-1}^c X|) \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_i(|\mathrm{link}\, \sigma|) \longrightarrow \widetilde{H}_i(|\mathrm{Skel}_k^c X|).$$

By our assumptions, the two outer terms here are both 0, hence the middle term is 0 as well, so $\widetilde{H}_i(|\mathrm{link}\, \sigma|) = 0$ for all k-faces σ. This works for all $k = 0, \ldots, d$, which only leaves $k = -1$: the only (-1)-dimensional face is $\varnothing$, and

$$\widetilde{H}_i(|\mathrm{link}_X \varnothing|) = \widetilde{H}_i(|X|) = \widetilde{H}_i(|\mathrm{Skel}_{-1}^c X|)$$

which we assumed to be 0.

Therefore, $\widetilde{H}_i(|\mathrm{link}_X \sigma|) = 0$ for all faces σ and all $i \geqslant r$, so X is r-Leray. $\qquad \square$

5.3 Stacked balls

A *homology sphere* is a polytopal complex X of dimension d where for every face σ including $\sigma = \varnothing$,

$$\widetilde{H}_i(|\text{link}\,\sigma|) = \begin{cases} 0 & \text{if } i \neq d - \dim\sigma - 1 \\ R & \text{if } i = d - \dim\sigma - 1. \end{cases}$$

A *homology manifold* is defined similarly but only considering $\sigma \neq \varnothing$. A *homology ball* of dimension d is a complex X where:

- $\widetilde{H}_i(|X|) = 0$ for all i,

- for every non-empty face σ,

$$\widetilde{H}_i(|\text{link}\,\sigma|) = \begin{cases} 0 & \text{if } i \neq d - \dim\sigma - 1 \\ R \text{ or } 0 & \text{if } i = d - \dim\sigma - 1, \end{cases}$$

  and

- the set of faces σ with $\widetilde{H}_{d-\dim\sigma-1}(|\text{link}\,\sigma|) = 0$ forms a subcomplex of X that is a $(d-1)$-dimensional homology sphere.

If $\widetilde{H}_{d-\dim\sigma-1}(|\text{link}\,\sigma|)$ is 0, σ is called a *boundary face*, and if this homology is R, σ is an *interior face*.

Suppose X is a homology ball of dimension d. If every face of X of dimension less than or equal to $d - s - 1$ is a boundary face, then X is said to be *s-stacked*. For example, if X is 1-stacked, then the interior faces must all have dimension d or $d-1$; this condition is sometimes simply called "stacked" for simplicial complexes (e.g. [12]), or "capped" for cubical complexes (e.g. [7]). Simplicial *s*-stacked balls are well studied due to their connection to the Lower Bound Conjecture — see e.g. [17].

Just like Cohen–Macaulay and *r*-Leray complexes, we can characterise *s*-stacked balls using their co-skeletons.

Theorem 22. *Suppose X is a homology ball with dimension d. Then X is s-stacked if and only if $\widetilde{H}_{d-k-1}(|\text{Skel}_k^c X|) = 0$ for all $k \leqslant d - s - 1$ (or equivalently $\widetilde{H}_j(|\text{Skel}_{d-j-1}^c X|) = 0$ for all $j \geqslant s$, where $j = d - k - 1$).*

Proof. By definition, a k-face σ is a boundary face if and only if $\widetilde{H}_{d-k-1}(|\text{link}\,\sigma|)$ is 0. Therefore, X is *s*-stacked if and only if

$$\bigoplus_{\sigma \in X_k} \widetilde{H}_{d-k-1}(|\text{link}\,\sigma|) = 0$$

for all $k \leqslant d - s - 1$.

(a) A 1-stacked sphere, the boundary of the ball made from three stacked 3-simplices

(b) A sphere that is not 1-stacked, the boundary of an octahedron

Figure 9: Two spheres with homeomorphic co-skeletons

We could apply this fact directly to the long exact sequence from Theorem 11; however, since X is a homology ball, it is Cohen–Macaulay, so we can take a shortcut by using the short exact sequence from Corollary 17:

$$0 \longrightarrow \widetilde{H}_{d-k}(|\mathrm{Skel}^c_{k-1} X|) \longrightarrow \bigoplus_{\sigma \in X_k} \widetilde{H}_{d-k-1}(|\mathrm{link}\,\sigma|) \longrightarrow \widetilde{H}_{d-k-1}(|\mathrm{Skel}^c_k X|) \longrightarrow 0.$$

The inner term of this short exact sequence is 0 for $k \leqslant d - s - 1$ if and only if both outer terms are 0 for the same range of k, which is equivalent to the claimed condition. $\square$

Remark 23. A closely related notion is an *s-stacked sphere*, which is a complex that is the boundary of some s-stacked ball. Unfortunately, s-stacked spheres cannot be distinguished by topological features of their co-skeletons, at least for simplicial complexes. For example, the two spheres shown in Fig. 9 have homeomorphic kth co-skeletons for all k, but only one of the spheres is 1-stacked as a simplicial complex.

6 CAT(0) cubical complexes

For a further application of co-skeletons, let us consider some families of cubical complexes.

One of the key features of a cubical complex is its *hyperplanes*. If we associate an r-dimensional cube with the space $[0,1]^r$ in $\mathbb{R}^r$, then the *ith hyperplane* of that cube is the subspace where $x_i = \frac{1}{2}$ — see Fig. 10(b). A hyperplane in a cubical complex $\square$ is a maximally connected cubical complex obtained by glueing together hyperplanes of its component cubes where they meet along faces — see Fig. 11. (We should note that in general, issues can arise when a hyperplane "intersects itself", in which case the hyperplane is not a true polytopal complex by our definitions. However, the focus of this section will be on CAT(0) cubical complexes, in which hyperplanes behave nicely — see e.g. Lemma 29 — so we will not dwell on this issue.)

Since each hyperplane of a cubical complex is itself a cubical complex, it also has its own hyperplanes. Therefore, we will define an *iterated hyperplane* of $\square$ to be a hyperplane of a hyperplane of ... of $\square$. We will say the original hyperplanes are the "1st iterated hyperplanes", the hyperplanes of the hyperplanes are the "2nd iterated hyperplanes", and so on. See Figs. 10(a) to 10(d). As these figures show, an iterated hyperplane can

(a) A 3-cube

(b) The hyperplanes (aka the 1st iterated hyperplanes)

(c) The 2nd iterated hyperplanes

(d) The 3rd iterated hyperplane
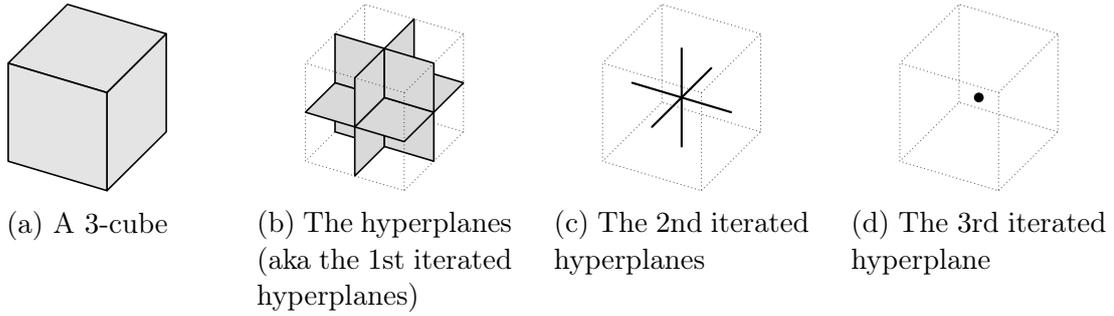
Figure 10: Hyperplanes and iterated hyperplanes of a 3-cube

(a) □

(b) The six hyperplanes of □
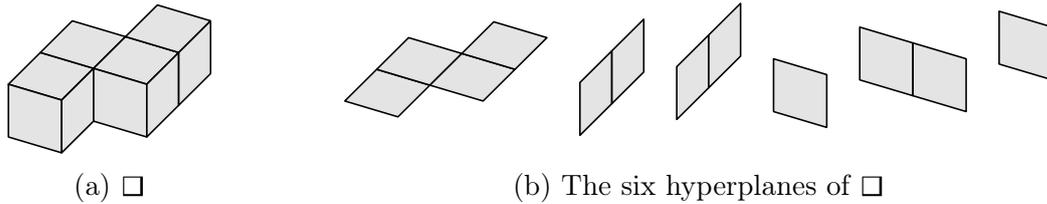
Figure 11: A cubical complex and its hyperplanes

sometimes be obtained in more than one way, when considered as a subspace of □ — for example, each line segment in Fig. 10(c) is a hyperplane of two different squares in Fig. 10(b).

The following proposition reveals the connection between co-skeletons and iterated hyperplanes.

Proposition 24. *Suppose* □ *is a cubical complex. Then* $|\mathrm{bary}(\square) - \mathrm{bary}(\mathrm{Skel}_k \square)|$ *is equal to the union of the* $(k+1)$*th iterated hyperplanes of* □*, considered as subspaces of* □*.*

For example, compare Fig. 10(b) and Fig. 12. Note that in combination with Lemma 4, this proposition tells us that $|\mathrm{Skel}_k^c \square|$ is homotopy equivalent to the union of the $(k+1)$th iterated hyperplanes.

Proof of Proposition 24. We will show that the intersections of $|\mathrm{bary}(\square) - \mathrm{bary}(\mathrm{Skel}_k \square)|$ and of the union of the $(k+1)$th iterated hyperplanes with each cube of □ are the same.

First, consider the union of the $(k+1)$th hyperplanes. In each cube, identified with $[0,1]^r$, a single $(k+1)$th iterated hyperplane is the result of setting $(k+1)$ of the coordinates to be $\frac{1}{2}$; therefore, the union of all $(k+1)$th iterated hyperplanes is the space

$$Z := \left\{ x \in |[0,1]^r| : \text{at least } k+1 \text{ coordinates of } x \text{ are } \tfrac{1}{2} \right\}.$$

Now, consider $|\mathrm{bary}(\square) - \mathrm{bary}(\mathrm{Skel}_k \square)|$. The intersection of this space with some cube of □, identified with $|[0,1]^r|$, is

$$B := |\mathrm{bary}([0,1]^r) - \mathrm{bary}(\mathrm{Skel}_k[0,1]^r)|.$$
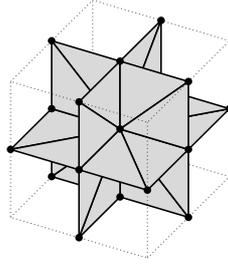
Figure 12: $\mathrm{bary}([0,1]^3) - \mathrm{bary}(\mathrm{Skel}_0[0,1]^3)$

For each face σ of $[0,1]^r$, we have a vertex v_σ of $\mathrm{bary}([0,1]^r)$ at the barycenter of σ; the coordinates of v_σ will all be 0, 1 or $\frac{1}{2}$, and the number of coordinates equal to $\frac{1}{2}$ is $\dim \sigma$. The remaining vertices of $\mathrm{bary}([0,1]^r)$ after deleting $\mathrm{bary}(\mathrm{Skel}_k[0,1]^r)$, therefore, are precisely those which have at least $k+1$ coordinates equal to $\frac{1}{2}$.

For the rest of this proof, we will show that $B = Z$. First, we claim that $B \subseteq Z$. For each face $\{v_{\sigma_1}, \ldots, v_{\sigma_m}\}$ of B, the set $\{\sigma_1, \ldots, \sigma_m\}$ is a chain, so it has a minimal element — without loss of generality, say this is σ_1. By construction, the dimension of σ_1 is at least $k+1$, so there is some index set I of at least $k+1$ elements so that $(v_{\sigma_1})_i = \frac{1}{2}$ for all $i \in I$. Since $\{\sigma_1, \ldots, \sigma_m\}$ is a chain, every v_{σ_j} for $j = 1, \ldots, m$ must also satisfy $(v_{\sigma_j})_i = \frac{1}{2}$ for all $i \in I$. Therefore, the entire set $\{v_{\sigma_1}, \ldots, v_{\sigma_m}\}$ lies in the affine subspace $\{x \in \mathbb{R}^r : x_i = \frac{1}{2} \text{ for all } i \in I\}$, so the convex hull of this set is contained in Z. Thus $B \subseteq Z$.

Conversely, let z be a point in Z. By definition, there is an index set I of size $k+1$ so that $z_i = \frac{1}{2}$ for all $i \in I$. The set of all points of $[0,1]^r$ satisfying $x_i = \frac{1}{2}$ for $i \in I$ is an iterated hyperplane of $[0,1]^r$, and its barycentric subdivision is the subcomplex of $\mathrm{bary}([0,1]^r) - \mathrm{bary}(\mathrm{Skel}_k[0,1]^r)$ generated by vertices satisfying $(v_\sigma)_i = \frac{1}{2}$ for $i \in I$. Therefore, z is in B, so $Z \subseteq B$.

In summary, the space $|\mathrm{bary}(\square) - \mathrm{bary}(\mathrm{Skel}_k \square)|$ and the union of the $(k+1)$th iterated hyperplanes agree within each cube of $\square$, so they must be the same subspace. $\qquad\square$

There is a class of cubical complexes where hyperplanes and co-skeletons work particularly well. In general, the link of any vertex in a cubical complex is isomorphic as a poset to the non-empty faces of a simplicial complex. Suppose $\square$ is a cubical complex in which every cube is *isometric* to a standard cube $[0,1]^r$ — such a complex is said to be *CAT(0)* if:

- it is simply connected, and

- the link of every vertex is isomorphic to a *flag* simplicial complex.

A simplicial complex is flag if the vertex set of every clique in its 1-skeleton is the vertex set of a face.

CAT(0) cubical complexes were first studied from the perspective of metric spaces: equivalently, a cubical complex is CAT(0) if and only if it is simply connected and has non-positive curvature at every point (see [8, Theorem 5.4 & Theorem 5.18]; see also [9]).

(a) $\square$, with hyperplanes labelled   (b) Cross($\square$)
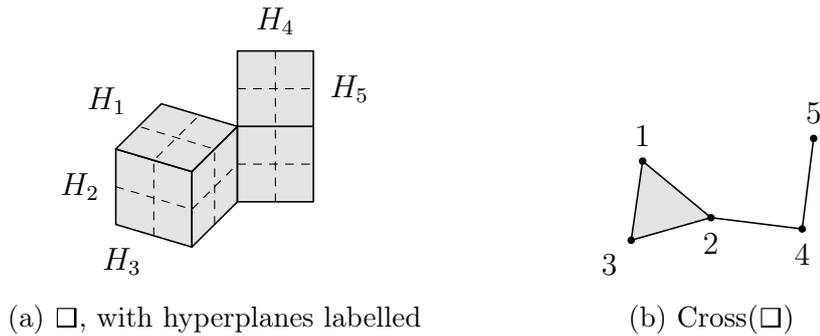
Figure 13: A CAT(0) cubical complex and its crossing complex

However, finite CAT(0) cubical complexes are also interesting from a purely combinatorial and topological perspective [3, 10, 20, 18, 19].

One particular construction associated to a CAT(0) cubical complex is its *crossing complex*, introduced in [19]: given a CAT(0) cubical complex $\square$, its crossing complex, here denoted Cross($\square$), is a simplicial complex whose vertices are in bijection with the hyperplanes of $\square$, and a set of vertices forms a face if the intersection of the corresponding hyperplanes is non-empty. (In other words, the crossing complex is the nerve of the hyperplanes, as defined in Theorem 3.)

We record here some basic facts about CAT(0) cubical complexes and their crossing complexes.

Lemma 25 ([8, Corollary 1.5], [9, Section 4]). *Any CAT(0) cubical complex is contractible (not just simply connected).*

Lemma 26 ([10, Lemma 2.15]). *Cross($\square$) is always a flag simplicial complex. In particular, it is the clique complex of the "crossing graph" defined by Hagen [10, Definition 2.16].*

Lemma 27 ([19, Proposition 6.4, Corollary 6.5]). *There is a bijection between the facets of $\square$ and the facets of Cross($\square$). If a facet of $\square$ has dimension k, the corresponding facet of Cross($\square$) has dimension $k - 1$. Consequently, $\dim \text{Cross}(\square) = \dim \square - 1$, and $\square$ is pure if and only if Cross $\square$ is pure.*

Lemma 28 ([20, Theorem 4.11]). *Every hyperplane (and thus every iterated hyperplane) in a CAT(0) cubical complex is also CAT(0).*

Ardila et al. [3, p. 9] described a way to embed a CAT(0) cubical complex with m hyperplanes into the cube $[0,1]^m$ as a subcomplex. This embedding is essentially canonical, up to the choice of a "root vertex" and a choice of labelling for the hyperplanes as $H_1, \ldots, H_m$.

Lemma 29 ([19, Proposition 5.7]). *In this embedding of $\square$ into $[0,1]^m$, the hyperplane H_i of $\square$ is the intersection of $\square$ with the ith hyperplane of $[0,1]^m$.*

The resulting embedding of a hyperplane H_i into the ith hyperplane of $[0,1]^m$ agrees with Ardila et al.'s canonical embedding of H_i as a CAT(0) cubical complex into a cube; thus the iterated hyperplanes of $\square$ are themselves each an intersection of $\square$ with some set of hyperplanes of $[0,1]^m$. Therefore, if S is a subset of $\{1,\ldots,m\}$ of size $(k+1)$, we define

$$H_S := \square \cap \bigcap_{i \in S} (i\text{th hyperplane of } [0,1]^m)$$

$$= \bigcap_{i \in S} \left(\square \cap (i\text{th hyperplane of } [0,1]^m) \right) = \bigcap_{i \in S} H_i.$$

Either this is a $(k+1)$th iterated hyperplane of $\square$, or it is empty, and all $(k+1)$th iterated hyperplanes of $\square$ are of this form. By definition of $\mathrm{Cross}(\square)$, H_S is an iterated hyperplane if and only if S is a face of $\mathrm{Cross}(\square)$; therefore, the $(k+1)$th iterated hyperplanes of $\square$ correspond to k-faces of $\mathrm{Cross}(\square)$, for $k = 0,\ldots,d-1$.

We can now begin to prove some new results about CAT(0) cubical complexes. The following proposition generalises [19, Theorem 6.3], which was the case of $k = 0$.

Proposition 30. *If $\square$ is a d-dimensional CAT(0) cubical complex and $\Delta := \mathrm{Cross}(\square)$ is its crossing complex, then $|\mathrm{Skel}_k^c \square| \simeq |\mathrm{Skel}_{k-1}^c \Delta|$ for all $k = 0,\ldots,d$.*

Proof. If $k = d$, then both $|\mathrm{Skel}_d^c \square|$ and $|\mathrm{Skel}_{d-1}^c \Delta|$ are empty, since $\dim \Delta = d - 1$ by Lemma 27. Therefore, for the rest of this proof, assume that $k < d$.

We will apply the nerve theorem (Theorem 3) to each of $|\mathrm{Skel}_k^c \square|$ and $|\mathrm{Skel}_{k-1}^c \Delta|$, to show that they are both homotopy equivalent to the same nerve.

First, consider $|\mathrm{Skel}_k^c \square|$, and define the family $\mathcal{H} := \{H_\sigma : \sigma \in \Delta_k\}$, where $H_\sigma := \bigcap_{i \in \sigma} H_i$ as above. By Proposition 24 and the above discussion, $|\mathrm{Skel}_k^c \square|$ is homotopy equivalent to the union of $\mathcal{H}$. Each H_σ is a closed set in $\square$, and by Proposition 24 their union is a triangulable space. Any intersection of spaces in $\mathcal{H}$ is itself an iterated hyperplane if it is non-empty, which is CAT(0) by Lemma 28 and thus contractible by Lemma 25. Therefore, we may apply the nerve theorem to $\mathcal{H}$, and conclude that $|\mathrm{Skel}_k^c \square| \simeq |N(\mathcal{H})|$.

But what is $N(\mathcal{H})$? The vertices are in bijection with k-faces σ of Δ. A set of vertices, say $\{\sigma_1,\ldots,\sigma_m\}$, forms a face of $N(\mathcal{H})$ whenever the intersection $H_{\sigma_1} \cap \cdots \cap H_{\sigma_m}$ is non-empty. This intersection is the set

$$H_{\sigma_1} \cap \cdots \cap H_{\sigma_m} = \bigcap_{i \in \sigma_1} H_i \cap \cdots \cap \bigcap_{i \in \sigma_m} H_i$$

$$= \bigcap_{i \in \sigma_1 \cup \cdots \cup \sigma_m} H_i$$

$$= H_{\sigma_1 \cup \cdots \cup \sigma_m}.$$

So $\{\sigma_1,\ldots,\sigma_m\}$ is a face of $N(\mathcal{H})$ if and only if $\sigma_1 \cup \cdots \cup \sigma_m$ is a face of Δ.

Now, let us turn our attention to $\mathrm{Skel}_{k-1}^c \Delta$, the set of faces of Δ of dimension at least k. For each k-face σ of Δ, let S_σ denote $|\mathrm{star}_\Delta \sigma|$, and define $\mathcal{S} := \{S_\sigma : \sigma \in \Delta_k\}$. Every

face of Δ of dimension at least k is contained in $\mathrm{star}_\Delta \sigma$ for some k-face σ, and no face of dimension less than k is, so the union of $\mathcal{S}$ is $|\mathrm{Skel}^c_{k-1} \Delta|$. Each S_σ is open in Δ, and any non-empty intersection of stars is itself a star and thus contractible. Therefore, we may apply the nerve theorem to $\mathcal{S}$ as well: $|\mathrm{Skel}^c_{k-1} \Delta| \simeq N(\mathcal{S})$.

Let us examine $N(\mathcal{S})$. The vertices are in bijection with k-faces σ of Δ, and a set $\{\sigma_1, \ldots, \sigma_m\}$ forms a face of $N(\mathcal{S})$ whenever $\mathrm{star}_\Delta \sigma_1 \cap \cdots \cap \mathrm{star}_\Delta \sigma_m$ is non-empty. This happens if and only if there exists a face of Δ that contains all of $\sigma_1, \ldots, \sigma_m$, which happens whenever $\sigma_1 \cup \cdots \cup \sigma_m$ is a face of Δ.

But this means that $N(\mathcal{S}) = N(\mathcal{H})$! Therefore,

$$|\mathrm{Skel}^c_k \square| \simeq |N(\mathcal{H})| = |N(\mathcal{S})| \simeq |\mathrm{Skel}^c_{k-1} \Delta|. \qquad \square$$

Next, we will examine some applications of this result, but first we need one more construction.

There is a standard bijection from the non-empty faces σ of the cube $[0,1]^r$ to vectors χ^σ in $\{0, 1, *\}^r$, where the ith coordinate χ^σ_i of χ^σ is 0 or 1 if all points x in σ have $x_i = 0$ or $x_i = 1$ respectively, and χ^σ_i is $*$ otherwise. If σ and τ are two faces, then $\sigma \subseteq \tau$ if and only if $\chi^\sigma_i \preceq \chi^\tau_i$ for all i, where $0 \prec *$ and $1 \prec *$ are the only non-equality relations on 0, 1 and $*$.

Suppose Δ is an abstract simplicial complex with vertex set $\{1, \ldots, n\}$. We define the *cubical cone* over Δ, denoted $\mathrm{Cone}(\Delta)$, to be the subcomplex of $[0,1]^n$ consisting of all faces σ such that the set $\{i : \chi^\sigma_i \in \{1, *\}\}$ is a face of Δ (including the empty face). For example, see Fig. 14. The reason for the name "cubical cone" is the following lemma, analogous to properties of a simplicial cone.

Lemma 31. *The cubical cone over Δ is the unique subcomplex of a cube (up to isomorphism) with the property that there is a vertex v_0 (the "cone point", which is the point $(0, \ldots, 0)$ in the standard labelling) such that:*

- *every facet contains v_0, and*

- *the link of v_0 is (isomorphic as a poset to the non-empty faces of) Δ.*

Proof. First, let us check that the cubical cone does satisfy these properties. If σ is a face of $\mathrm{Cone}(\Delta)$, we can replace every 1 in χ^σ with $*$ to obtain the vector of a face of $\mathrm{Cone}(\Delta)$ that contains σ; therefore, the vectors of facets of $\mathrm{Cone}(\Delta)$ consist only of 0 and $*$, so every facet contains $v_0 = (0, \ldots, 0)$. Conversely, the faces that properly contain v_0 are precisely the faces $\sigma \neq v_0$ such that χ^σ consists of 0 and $*$, and these faces are in bijection with non-empty faces of Δ, so the link of v_0 is indeed Δ.

Now, suppose $\square$ is an arbitrary subcomplex of a cube satisfying these properties. By the symmetry of the cube, we may assume that v_0 is the vertex $(0, \ldots, 0)$. As before, the link of v_0 is then the set of faces of $\square$ whose vectors consist only of 0 and $*$; since we assume this set is isomorphic to Δ, we may again use the symmetry of the cube to permute the coordinates and assume that this set is equal to $\mathrm{link}_{\mathrm{Cone}(\Delta)} v_0$. Since $\square$ contains these faces of the cube, it must also contain every sub-face of these faces, so $\square$ contains every face of

(a) Δ

(b) The cubical cone over Δ (embedded into $\mathbb{R}^3$), with vertices labelled
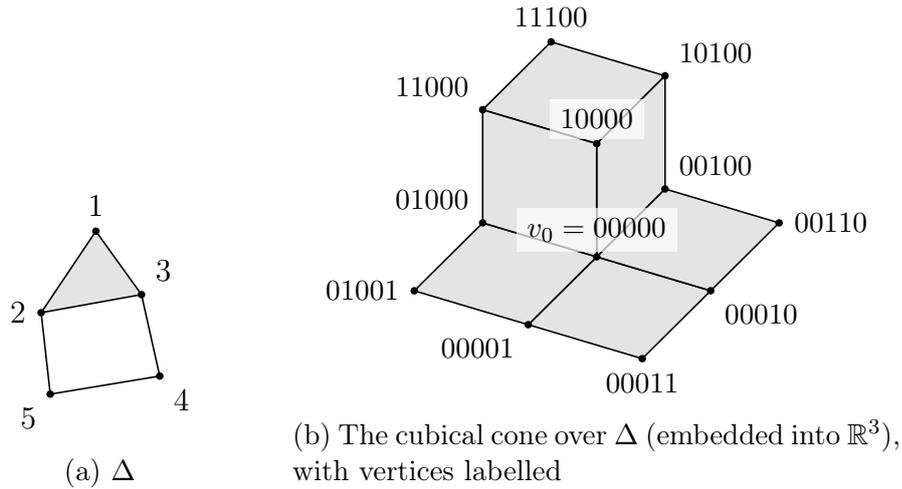
Figure 14: A cubical cone

Cone(Δ). Moreover, it cannot contain any other faces: any such face must be contained in some facet of $\square$, but that facet cannot be one that we have already accounted for in $\text{link}_{\square} v_0$, so the facet would not contain v_0. Therefore, $\square \cong \text{Cone}(\Delta)$. $\qquad \square$

While every flag simplicial complex is the crossing complex of some CAT(0) cubical complex (see [10, Proposition 2.19] and [19, Lemma 3.3], where the construction is the cubical cone), in general, there can be many CAT(0) cubical complexes with the same crossing complex. However, we can now prove that if Cross($\square$) is a connected manifold, the only possibility is $\square = \text{Cone}(\Delta)$.

Corollary 32. *Let $\square$ be a d-dimensional CAT(0) cubical complex and Δ its crossing complex. If Δ is a connected homology manifold, then $\square$ is the cubical cone over Δ.*

Proof. Examine this section of the long exact sequence from Theorem 11, taking $k = 0$ and using coefficients in $\mathbb{Z}/2$:

$$\widetilde{H}_d(|\text{Skel}^c_{-1}\, \square|) \longrightarrow \bigoplus_{v \in \square_0} \widetilde{H}_{d-1}(|\text{link}\, v|) \longrightarrow \widetilde{H}_{d-1}(|\text{Skel}^c_0\, \square|) \longrightarrow \widetilde{H}_{d-1}(|\text{Skel}^c_{-1}\, \square|).$$

Note that $|\text{Skel}^c_{-1}\, \square|$ is simply $|\square|$, which is contractible since $\square$ is CAT(0), so the first and last terms of this sequence are 0. By Proposition 30, $\widetilde{H}_{d-1}(|\text{Skel}^c_0\, \square|)$ is isomorphic to $\widetilde{H}_{d-1}(|\text{Skel}^c_{-1}\, \Delta|) = \widetilde{H}_{d-1}(|\Delta|)$, and since Δ is a connected homology manifold of dimension $d-1$, this homology group is $\mathbb{Z}/2$. Therefore,

$$\bigoplus_{v \in \square_0} \widetilde{H}_{d-1}(|\text{link}\, v|) \cong \mathbb{Z}/2.$$

Thus there must be one vertex v_0 for which $\widetilde{H}_{d-1}(|\text{link}\, v_0|) = \mathbb{Z}/2$, and all other vertices have $\widetilde{H}_{d-1}(|\text{link}\, v|) = 0$.

Every vertex link in a CAT(0) cubical complex is an induced subcomplex of the crossing complex [19, Lemma 4.9]. However, the only induced subcomplex of a connected $(d-1)$-manifold with non-zero homology in degree $d-1$ is the entire manifold. Therefore, $\operatorname{link} v_0 = \Delta$.

Lemma 27 says that there is a bijection between the facets of the cubical complex $\square$ and the facets of its crossing complex Δ. Since $\operatorname{link} v_0 = \Delta$, we also have a bijection between the facets of Δ and the facets of $\square$ that contain v_0. Therefore, every facet of $\square$ must contain v_0. Hence $\square$ is a cubical cone. $\qquad\square$

We conclude this paper by combining Proposition 30 with Theorems 14, 22 and 21, to compare the combinatorial properties discussed in Section 5 for $\square$ and $\operatorname{Cross}(\square)$.

Theorem 33. *Suppose $\square$ is a CAT(0) cubical complex and Δ is its crossing complex. Then:*

(a) *$\square$ is Cohen–Macaulay if and only if Δ is Cohen–Macaulay.*

(b) *$\square$ is r-Leray if and only if Δ is r-Leray.*

(c) *If $\square$ and Δ are both homology balls, then $\square$ is s-stacked if and only if Δ is s-stacked.*

Note that the assumption that both $\square$ and Δ are balls in (c) is essential: it is possible for one to be a ball and the other not. For example, if $\square$ consists of m squares glued in a $1 \times m$ rectangle, it is a 1-stacked ball, but its crossing complex is a star graph with m pendants, which is not a ball for $m > 2$.

Proof of Theorem 33. Let $d = \dim \square$.

(a) By Theorem 14, $\square$ is Cohen–Macaulay if and only if it is pure and $\widetilde{H}_i(|\operatorname{Skel}_k^c \square|) = 0$ for all $k = -1, \ldots, d$ and all $i < d - k - 1$. Any CAT(0) cubical complex is always contractible, so $\widetilde{H}_i(|\operatorname{Skel}_{-1}^c \square|)$ is guaranteed to be 0, hence we only need to consider $k = 0, \ldots, d$. Now, Proposition 30 says that $\widetilde{H}_i(|\operatorname{Skel}_k^c \square|) = \widetilde{H}_i(|\operatorname{Skel}_{k-1}^c \Delta|)$ for k in this range, and Lemma 27 says that $\square$ is pure if and only if Δ is pure; therefore, $\square$ is Cohen–Macaulay if and only if Δ is pure and $\widetilde{H}_i(|\operatorname{Skel}_{k-1}^c \Delta|) = 0$ for all $k = 0, \ldots, d$ and all $i < d - k - 1 = (d-1) - (k-1) - 1$. But since $\dim \Delta = d - 1$, this is exactly the condition given by Theorem 14 for Δ to be Cohen–Macaulay.

(b) By Theorem 21, $\square$ is r-Leray if and only if $\widetilde{H}_i(|\operatorname{Skel}_k^c \square|) = 0$ for all $i \geqslant r$ and all $k = -1, \ldots, d$. Because $\square$ is CAT(0), $|\operatorname{Skel}_{-1}^c \square|$ is guaranteed to be contractible, so we only need to check $k = 0, \ldots, d$. By Proposition 30, this condition is equivalent to $\widetilde{H}_i(|\operatorname{Skel}_j^c \Delta|)$ being 0 for all $i \geqslant r$ and all $j = -1, \ldots, d-1$, where $j = k - 1$. And this precisely means that Δ is r-Leray.

(c) By Theorem 22, $\square$ is s-stacked if and only if $\widetilde{H}_{d-k-1}(|\operatorname{Skel}_k^c \square|) = 0$ for all $k \leqslant d - s - 1$. Proposition 30 tells us this is equivalent to $\widetilde{H}_{d-k-1}(|\operatorname{Skel}_{k-1}^c \Delta|)$ being 0 for $k \leqslant d-s-1$. Equivalently, letting $j = k-1$, we have $\widetilde{H}_{(d-1)-j-1}(|\operatorname{Skel}_j^c \Delta|) = 0$ for $j \leqslant (d-1) - s - 1$. Since $\dim \Delta = d-1$, this is equivalent to Δ being s-stacked. $\quad\square$

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