A Comparison of Integer Partitions Based on Smallest Part

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Abstract

For positive integers n, L and s, consider the following two sets that both contain partitions of n with the difference between the largest and smallest parts bounded by L: the first set contains partitions with smallest part s, while the second set contains partitions with smallest part at least $s+1$. Let $G_{L,s}(q)$ be the generating series whose coefficient of q^n is difference between the sizes of the above two sets of partitions. This generating series was introduced by Berkovich and Uncu (2019). Previous results concentrated on the nonnegativity of $G_{L,s}(q)$ in the cases $s=1$ and $s=2$. In the present paper, we show the eventual positivity of $G_{L,s}(q)$ for general s and also find a precise nonnegativity result for the case $s=3$.

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1 Introduction

Let n be a positive integer. A *partition* $\pi = (\pi_1, \pi_2, \dots)$ *of* n is a weakly decreasing list of positive integers whose sum is n, and we write $|\pi| = n$ to indicate this. Each π_i is known as a *part* of π. A standard way to visualize π is through its *Ferrers diagram*, which is a collection of left justified rows of boxes, with the i^{th} row containing π_i boxes.

In the present article, it is more convenient to use the notation that expresses the number of parts of each size in a partition. In this notation, we write $\pi = (1^{f_1}, 2^{f_2}, \dots)$, where f_i is the *frequency* of i or the number of times a part i occurs in π. Thus, each frequency f_i is a nonnegative integer, and if $f_i = 0$, then π has no part of size i. When the frequency of a number is 0, it may or may not be omitted in the expression. In the latter notation, it is clear that $|\pi| = \sum_i i \cdot f_i$. Thus $(4,4,2,2,1)$, $(1^1, 2^2, 3^0, 4^2, 6^0)$, and $(1^1, 2^2, 4^2, 5^0)$ all represent the same partition of 13. We let $s(\pi)$ and $l(\pi)$ denote the smallest and largest parts of π, respectively, and $\mho$ denotes the set of partitions.

For indeterminates a and q, and positive integers n and k, define

$$(k)_n := k \cdot (k+1) \cdots (k+n-1)$$

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and

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The central objects of this article are the following series. For positive integers L, s and k, define

- $G_{L,s}(q)$ to be the generating series

$$G_{L,s}(q) := \sum_{\substack{\pi\in\mho,\\ s(\pi)=s,\\ l(\pi)-s(\pi)\leqslant L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geqslant s+1,\\ l(\pi)-s(\pi)\leqslant L}} q^{|\pi|}, \tag{1}$$

  and

- $H_{L,s,k}(q)$ to be the generating series

$$H_{L,s,k}(q) := \frac{q^s(1-q^k)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right). \tag{2}$$

A series $\sum_{n\geqslant 0} a_n q^n$ is said to be *eventually positive* if there exists some $l \in \mathbb{N}$ such that $a_n > 0$ for all $n \geqslant l$.

Berkovich and Uncu conjectured that $H_{L,s,k}(q)$ is a eventually positive for all positive integers triples $L \geqslant 3$, s and $k \geqslant s+1$. This conjecture was recently proved independently by Zang and Zeng [ZZ20, Theorem 1.3] and by the present authors [BR21, Section 3]. The two proofs are substantially different. While the proof of Zang and Zeng was partly combinatorial and partly analytic, the proof of the present authors was entirely combinatorial. The present authors in fact proved a stronger result. To state this, we need the following notation. For positive integers L and s, define:

- $$\gamma(L,s) := \left(\binom{L+1}{2} + Ls\right)\cdot \left((s+1)_L^{\left((s+1)_L^2-1\right)L+2} + \left(\left((s+1)_L^2-1\right)L-2\right)(s+1)_L\right), \tag{3}$$
- $$\Gamma(s) := \gamma(3s+2, s). \tag{4}$$

Note that $\binom{L+1}{2} + Ls = (s+1)+\cdots+(s+L)$. The following theorem appears in [BR21, Theorem 5].

Theorem 1. *For positive integers L, s and k, with $L \geqslant 3$ and $k \geqslant s+1$, the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geqslant \Gamma(s)$.*

We emphasize that Theorem 1 is stronger than the conjecture of Berkovich and Uncu about $H_{L,s,k}(q)$ as it explicitly gives a bound for when $H_{L,s,k}(q)$ is positive, and it further states this bound depends only on s.

Berkovich and Uncu [BU19, Theorems 5.1 and 5.2] showed that the series $G_{L,s}(q)$ and $H_{L,s,k}(q)$, for $s=1$ and $s=2$ and $L \geqslant 1$, satisfy the following simple relationship:

$$G_{L,s}(q) = \frac{H_{L,s,L}(q)}{1-q^L}. \tag{5}$$

A series $S(q) = \sum_{n \geqslant 0} a_n q^n$ is said to be *nonnegative* if $a_n \geqslant 0$ for all n. The nonnegativity of the series $S(q)$ is denoted by $S(q) \succeq 0$.

Berkovich and Uncu [BU19, Theorem 5.1] prove, using (5), that $G_{L,1}(q) \succeq 0$. Also using (5), they conjectured Theorem 2 below, which pertains to the nonnegativity of $G_{L,2}(q)$. Theorem 2 was proved by the present authors [BR21, Section 3].

Theorem 2. *For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0;$$

for $L = 4$,

$$G_{L,2}(q) + q^3 + q^9 \succeq 0;$$

and for $L \geqslant 5$,

$$G_{L,2}(q) + q^3 \succeq 0.$$

In the present article, we explore the nonnegativity properties of $G_{L,s}(q)$ for general positive integers s. We begin the study of this series with the next result.

Lemma 3. *For positive integers $L \geqslant 1$,*

$$G_{L,s}(q) = \frac{H_{L,s,L}(q)}{1-q^L}.$$

We prove Lemma 3 in Section 2, but we emphasize that the proof is essentially the one given by Berkovich and Uncu for the cases $s=1$ and $s=2$, with only minor modifications. We include this proof for completeness. Next we show that for any $L \geqslant s+1$, the series $G_{L,s}(q)$ is eventually positive, and the bound after which the coefficient of q^N is positive can be written explicitly in terms of s only. Define the quantities

$$\begin{aligned} &\delta(s) := e^{3\Gamma(s)}, \\ \text{and} \quad &\delta'(s) := 10s + (s+2)(s+3)(\delta(s)+1). \end{aligned} \tag{6}$$

Obviously $\delta'(s) > \delta(s)$ for all positive s.

Theorem 4. *If s and $L \geqslant s+1$ are positive integers, then the coefficient of q^n in $G_{L,s}(q)$ is positive whenever $n \geqslant \delta'(s)$, so $G_{L,s}(q)$ is eventually positive.*

We prove Theorem 4 in Section 2. Then we focus on the case $s=3$ and obtain an extension of Theorem 2; that is, we show that, with the exception of a few small terms, the series $G_{L,s}(q)$ is nonnegative. The next result states this precisely, and its proof is in Section 3.

Theorem 5. *For $L \geqslant 10$,*

$$G_{L,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + q^{16} \succeq 0.$$

For $5 \leqslant L \leqslant 9$, the following hold.

$$\begin{aligned} G_{9,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} \succeq 0. \\ G_{8,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} + q^{20} \succeq 0. \\ G_{7,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + 2q^{14} + q^{16} + q^{20} \succeq 0. \\ G_{6,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{13} + 2q^{14} + 2q^{16} + q^{18} + 2q^{20} + q^{22} \succeq 0. \\ G_{5,3}(q) + q^4 + q^5 + q^8 + q^{10} + 2q^{12} + q^{13} + q^{14} + 2q^{16} + q^{17} + q^{18} + 3q^{20} \\ + q^{22} + q^{24} + q^{28} \succeq 0. \end{aligned}$$

and for $L = 4$,

$$\begin{aligned} G_{4,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{11} + 2q^{12} + 2q^{14} + 3q^{16} + q^{17} \\ + 2q^{18} + q^{19} + 4q^{20} + 3q^{22} + q^{23} + 4q^{24} + q^{25} + 4q^{26} + 5q^{28} \\ + q^{29} + 3q^{30} + 6q^{32} + 3q^{34} + 4q^{36} + 2q^{38} + 4q^{40} + 2q^{44} \succeq 0. \end{aligned}$$

We point out that the bound in Theorem 4 is likely far from optimal. Take, for example, the case $s = 3$. According to Theorem 4, the coefficient of q^n in $G_{L,3}(q)$ is nonnegative whenever $n \geqslant \delta'(3)$, where $\delta'(3)$ is is extremely large. However, from Theorem 5, the coefficient of q^n in $G_{L,3}(q)$ is nonnegative whenever $n \geqslant 45$. This suggests that the bound in Theorem 4 can be improved greatly.

Lemma 3 and Theorems 4 and 5 can also be found in the PhD. thesis of the first author [Bin21].

2 Proofs of Lemma 3 and Theorem 4

We begin by proving Lemma 3.

Proof of Lemma 3. The definition of $G_{L,s}(q)$ in (1) is given as the difference of two generating series. We begin by finding a rational expression for the first generating series. The partitions counted by this generating series have smallest part equal to s and largest part at most $L + s$. Hence we obtain for the first generating series the expression

$$\sum_{\substack{\pi \in \mho, \\ s(\pi)=s, \\ l(\pi)-s(\pi)\leqslant L}} q^{|\pi|} = \frac{q^s}{(1-q^s)(1-q^{s+1})\cdots(1-q^{L+s})} = \frac{q^s}{(q^s;q)_{L+1}}. \tag{7}$$

For the second generating series in the definition of $G_{L,s}(q)$, we fix the number of parts of the partition to be n and then sum over all n. Suppose π is a partition with n parts, where each part is at least $s+1$. Then, in the Ferrers diagram of π, the whole column over the smallest part of π is generated by the q-factor

$$\frac{q^{(s+1)n}}{1-q^n}.$$

Stripping the columns above the smallest part from the far left of the Ferrers diagram of π, we are left with a new partition that has at most $n-1$ parts and largest part bounded above by L. It is well known (see for example [Aig07, Proposition 1.1]) that these partitions are generated by the q-binomial coefficient

$$\begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q := \frac{(q;q)_{L+n-1}}{(q;q)_L (q;q)_{n-1}}.$$

Thus, for the second generating series in the definition of $G_{L,s}(q)$, we have

$$\sum_{\substack{\pi \in \mho, \\ s(\pi) \geqslant s+1, \\ l(\pi)-s(\pi) \leqslant L}} q^{|\pi|} = \sum_{n=1}^{\infty} \frac{q^{(s+1)n}}{1-q^n} \begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q .$$

Note that

$$\frac{1}{1-q^n} \begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q = \frac{1}{1-q^L} \frac{(q^L;q)_n}{(q;q)_n}.$$

Therefore,

$$\begin{aligned} \sum_{\substack{\pi \in \mho, \\ s(\pi) \geqslant s+1, \\ l(\pi)-s(\pi) \leqslant L}} q^{|\pi|} &= \frac{1}{1-q^L} \sum_{n=1}^{\infty} q^{(s+1)n} \frac{(q^L;q)_n}{(q;q)_n} \\ &= \frac{1}{1-q^L} \left(\frac{1}{(q^{s+1};q)_L} - 1 \right), \end{aligned} \tag{8}$$

where the last step follows from the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

(see [BU19, (2.1)]) with $a = q^L$ and $z = q^{s+1}$. Substituting (7) and (8) into the definition of $G_{L,s}(q)$ gives us

$$\begin{aligned} G_{L,s}(q) &= \frac{q^s}{(q^s;q)_{L+1}} - \frac{1}{1-q^L} \left(\frac{1}{(q^{s+1};q)_L} - 1 \right) \\ &= \frac{1}{1-q^L} \left(\frac{q^s(1-q^L)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1 \right) \right) \\ &= \frac{1}{1-q^L} H_{L,s,L}(q), \end{aligned}$$

as required. □

Lemma 3 expresses $G_{L,s}(q)$ in terms of $H_{L,s,L}(q)$, while Theorem 1 gives the explicit bound $\Gamma(s)$ after which coefficients in the series $H_{L,s,L}(q)$ are positive. We use these to show nonnegativity properties of $G_{L,s}(q)$. We first show in Theorem 6 that there is a bound M, which depends only on L and s, such that the coefficient of q^n in $G_{L,s}(q)$ is nonnegative whenever $n \geqslant M$.

For positive integers s and $L \geqslant s+1$, let $H_{L,s,L}(q) = \sum_{n\geqslant 0} a_{L,n} q^n$ and $G_{L,s}(q) = \sum_{n\geqslant 0} b_{L,n} q^n$. Then Lemma 3 implies

$$b_{L,n} = a_{L,n} + a_{L,n-L} + a_{L,n-2L} + \cdots = \sum_{\substack{m\leqslant n \\ m\equiv n \pmod{L}}} a_{L,m}. \tag{9}$$

We introduce some more notation:

- $\eta_1(L,s) = \sum_{n<\Gamma(s)} |a_{L,n}|$;
- $\eta_2(L,s) = \max(\eta_1(L,s), \Gamma(s))$; and
- $\eta_3(L,s) = (L+1)\eta_2(L,s)$.

Theorem 6. *Let s and $L \geqslant s+1$ be positive integers. Then the coefficient of q^n in $G_{L,s}(q)$ is nonnegative whenever $n \geqslant \eta_3(L,s)$.*

Proof. Suppose $n \geqslant \eta_3(L,s)$. We can rewrite (9) as

$$b_{L,n} = \sum_{\substack{\eta_2(L,s)\leqslant m\leqslant n \\ m\equiv n \pmod{L}}} a_{L,m} + \sum_{\substack{\Gamma(s)\leqslant m<\eta_2(L,s) \\ m\equiv n \pmod{L}}} a_{L,m} + \sum_{\substack{m<\Gamma(s) \\ m\equiv n \pmod{L}}} a_{L,m}. \tag{10}$$

Note that the second sum may be empty. Since $n \geqslant \eta_3(L,s)$, the first sum on the right hand side of (10) contains at least $\eta_2(L,s)$ terms, all of which are positive by Theorem 1. Thus

$$\sum_{\substack{\eta_2(L,s)\leqslant m\leqslant n \\ m\equiv n \pmod{L}}} a_{L,m} \geqslant \eta_2(L,s). \tag{11}$$

For the second sum in the right hand side of (10), Theorem 1 gives

$$\sum_{\substack{\Gamma(s)\leqslant m<\eta_2(L,s) \\ m\equiv n \pmod{L}}} a_{L,m} \geqslant 0. \tag{12}$$

For the third sum in the right hand side of (10), using the triangle inequality, we obtain

$$\left| \sum_{\substack{m<\Gamma(s) \\ m\equiv n \pmod{L}}} a_{L,m} \right| \leqslant \sum_{\substack{m<\Gamma(s) \\ m\equiv n \pmod{L}}} |a_{L,m}| \leqslant \sum_{m<\Gamma(s)} |a_{L,m}| = \eta_1(L,s) \leqslant \eta_2(L,s),$$

and thus

$$\sum_{\substack{m<\Gamma(s)\\ m\equiv n \pmod{L}}} a_{L,m} \geqslant -\eta_2(L,s). \tag{13}$$

The result now follows immediately from (10), (11), (12) and (13). □

The bound $\eta_3(L,s)$ in Theorem 6, which guarantees when the coefficients of $G_{L,s}(q)$ are nonnegative, depends on both L and s. To prove Theorem 4, we need to find a bound that only depends on s, and this is our next goal. We again use the connection between $G_{L,s}(q)$ and $H_{L,s,L}(q)$ in Lemma 3; while Theorem 1 guarantees the series $H_{L,s,L}(q)$ is eventually positive, we also need a lower bound on the size of the coefficients of $H_{L,s,L}(q)$. This is the content of Theorem 8 below, a strengthening of Theorem 1 in the case $k = L$.

Let

- $D_{L,s}$ denote the set of nonempty partitions with parts in the set $\{s+1,\dots,L+s\}$, and
- $I_{L,s,L}$ be the set of partitions where the smallest part is s, all parts are $\leqslant L+s$, and L does not appear as a part.

From the definition of the series $H_{L,s,L}(q)$ in (2), when $L \geqslant s+1$, elementary partition theory gives the coefficient of q^N in $H_{L,s,L}(q)$ as the difference

$$|\{\pi \in I_{L,s,L} : |\pi| = N|\}| - |\{\pi \in D_{L,s} : |\pi| = N\}|. \tag{14}$$

Thus to prove nonnegativity of the coefficient of q^N in $H_{L,s,L}(q)$, it suffices to show there exists an injection ϕ such that

$$\phi : \{\pi \in D_{L,s} : |\pi| = N\} \to \{\pi \in I_{L,s,L} : |\pi| = N\} \tag{15}$$

Equations (14) and (15) are central to proving our theorems below in this section and the next. We also need the following result.

Proposition 7. *For given positive integers a, b and n with $\gcd(a,b) = 1$, the number of solutions of $ax + by = n$ in nonnegative integer pairs (x, y) is either $\lfloor \frac{n}{ab} \rfloor$ or $\lfloor \frac{n}{ab} \rfloor + 1$.*

For an elementary proof of Proposition 7 see [NZM91, Chapter 5]. See also [Tri00].

Theorem 8. *For positive integers $L \geqslant 3$ and s, with $L \geqslant s+1$, the coefficient of q^N in $H_{L,s,L}(q)$ is greater than or equal to $\left\lfloor \frac{N-10s}{(s+2)(s+3)} \right\rfloor$ whenever $N \geqslant \Gamma(s)$.*

Proof. The proof of Theorem 1 in [BR21] uses the combinatorial interpretation of the coefficients of $H_{L,s,L}(q)$ in (14); the proof there constructs for $N \geqslant \Gamma(s)$ an injection as in (15) to show nonnegativity of (14). To show positivity of (14), elements of the codomain that are not in the range of ϕ are given. The details of these injections and elements are dealt with in different cases and theorems of [BR21] depending on the relative sizes of L and s.[1] For example, when $L \geqslant 2s+3$, there is no partition of the form $(s^{10},(s+1)^x,(s+2)^y)$ in the range of ϕ, but such partitions are in the codomain. To achieve the present result, we count the number of such partitions; this will then give the desired lower bound for the difference (14) in each case. We then compare the results of all cases.

Table 1 lists the cases, the theorem from [BR21] showing positivity of (14), the partitions in the codomain but not the range of the relevant injection ϕ in (15), and a lower bound for the enumeration of these partitions. To count the number of partitions in

Table 1: The first column describes the case, the second the theorem from [BR21] that proves positivity in this case, the third the partitions in the codomain that are not in the range of ϕ in (15), and the last column contains the minimum number of partitions of the type in Column 3

	Case of L, s	Theorem of [BR21]	Partitions in codomain not in range of ϕ	min. num. of partitions
1	$L \geqslant 2s+3$	10	$(s^{10},(s+1)^x,(s+2)^y)$	$\left\lfloor \frac{N-10s}{(s+1)(s+2)} \right\rfloor$
2	$s+3 \leqslant L \leqslant 2s+2$	12	$(s^1,(s+1)^x,(s+2)^y)$	$\left\lfloor \frac{N-s}{(s+1)(s+2)} \right\rfloor$
3	$L = s+1$	12	$(s^1,(s+2)^x,(s+3)^y)$	$\left\lfloor \frac{N-s}{(s+2)(s+3)} \right\rfloor$
4	s even, $L = s+2$	12	$(s^1,(s+1)^x,(s+3)^y)$	$\left\lfloor \frac{N-s}{(s+1)(s+3)} \right\rfloor$
5	s odd, $L = s+2$, N odd	12	$(s^1,(s+1)^x,(s+3)^y)$	$\left\lfloor \frac{2(N-s)}{(s+1)(s+3)} \right\rfloor$
6	$s \geqslant 1$ odd, $L = s+2$, N even	12	$(s^2,(s+1)^x,(s+3)^y)$	$\left\lfloor \frac{2(N-2s)}{(s+1)(s+3)} \right\rfloor$
7	$s = 1$, $L = 3$, N even	12	$(1^6,2^x,4^y)$	$\left\lfloor \frac{(N-6)}{4} \right\rfloor$

Column 3 in Table 1, in Rows 1-4 a straight forward application of Proposition 7 to $n = N - ts$, where t is the number of parts of s, and a and b are set to be the remaining parts, which are coprime, gives the numbers in Column 4.

The remaining rows require a slightly more work. For Row 5, the numbers $s+1$ and $s+3$ have greatest common factor 2, so we can apply Proposition 7 to $n = \frac{N-s}{2}$, $a = \frac{s+1}{2}$ and $b = \frac{s+3}{2}$. The count in Column 4 then follows. The analysis for Rows 6 and 7 are similar.

The result is now obtained by observing that all the values in Column 4 exceed $\left\lfloor \frac{N-10s}{(s+2)(s+3)} \right\rfloor$. □

[1] As indicated by Theorem 1, the proof in [BR21] for the positivity of $H_{L,s,k}(q)$ applies for all $k \geqslant s+1$. In particular, a more general $I_{L,s,k}$ is defined than the one given in (14). Here we are only interested in the case $k = L$.

For a positive integer m, let $p(m)$ be the number of partitions of m. We need the following result of de Azevedo Pribitkin [dAP09].

Theorem 9. *Let m be a positive integer. Then $p(m) \leqslant e^{3\sqrt{m}}$.*

Proof of Theorem 4. As before, let $a_{L,n}$ and $b_{L,n}$ be the coefficient of q^n in $H_{L,s,L}(q)$ and $G_{L,s}(q)$, respectively. Further, recall the definitions of $\Gamma(s), \delta(s)$, and $\delta'(s)$ in (3) and (6).

Suppose $n \geqslant \delta'(s)$. Again from (9), we have

$$b_{L,n} = \sum_{\substack{\delta'(s)\leqslant m\leqslant n\\ m\equiv n \pmod{L}}} a_{L,m} + \sum_{\substack{\Gamma(s)\leqslant m<\delta'(s)\\ m\equiv n \pmod{L}}} a_{L,m} + \sum_{\substack{m<\Gamma(s)\\ m\equiv n \pmod{L}}} a_{L,m}. \tag{16}$$

For $m \geqslant \delta'(s)$, we have $m \geqslant \Gamma(s)$ and $\lfloor \frac{m-10s}{(s+2)(s+3)} \rfloor \geqslant \delta(s)$, so $a_{L,m} \geqslant \delta(s)$ by Theorem 8. The first sum in the right hand side of (16) contains at least 1 term (the term $m = n$) and each term in the sum is greater than or equal to $\delta(s)$. Thus

$$\sum_{\substack{\delta'(s)\leqslant m\leqslant n\\ m\equiv n \pmod{L}}} a_{L,m} \geqslant \delta(s). \tag{17}$$

For the second sum in the right hand side of (16), from Theorem 1 it follows

$$\sum_{\substack{\Gamma(s)\leqslant m<\delta'(s)\\ m\equiv n \pmod{L}}} a_{L,m} \geqslant 0. \tag{18}$$

For the third sum on the right hand side of (16), the combinatorial interpretation of $H_{L,s,L}(q)$ in (14) gives $a_{L,m} \geqslant -p(m)$ for any $m \in \mathbb{N}$. By Theorem 9, we then have

$$a_{L,m} \geqslant -p(m) \geqslant -e^{3\sqrt{m}} \geqslant -e^{3m}.$$

Therefore,

$$\sum_{\substack{m<\Gamma(s)\\ m\equiv n \pmod{L}}} a_{L,m} \geqslant - \sum_{\substack{m<\Gamma(s)\\ m\equiv n \pmod{L}}} e^{3m} \geqslant - \sum_{m<\Gamma(s)} e^{3m} > -\delta(s), \tag{19}$$

where the last inequality follows from the familiar formula for finite geometric sums and the definition of $\delta(s)$. The theorem now follows immediately from (16), (17), (18) and (19). □

3 Proof of Theorem 5

To prove Theorem 5, we use the connection between $G_{L,3}(q)$ and $H_{L,3,L}(q)$ given in Lemma 3. To use this relationship, we need to understand the coefficients of q^N in $H_{L,3,L}(q)$ for small N, so we need a result, in the case $s = 3$ and $L \geqslant s + 1$, stronger than Theorem 1.

Our strategy for proving the coefficient of q^N in $H_{L,3,L}(q)$ is nonnegative for small N is to show that the coefficient of q^N in $H_{L,3,L}(q)$ is nonnegative when N is larger than a small bound; this is done in Lemmas 15 - 19. Then we use these lemmas along with machine computation and Lemma 3 to prove Theorem 5 at the end of the section.

Recall the following fundamental result of Sylvester [Syl82].

Theorem 10 (Sylvester's theorem). *Let a and b be positive coprime integers. Then the equation $ax+by=n$ has a nonnegative integer solution (x,y) whenever $n \geqslant (a-1)(b-1)$.*

We additionally need the following lemmas.

Lemma 11. *Let $n \geqslant 4$ be a positive integer such that $n \neq 7$. Then the equation $4x + 5y + 6z = n$ has a solution in nonnegative integer triples (x,y,z).*

Lemma 12. *Let $n \geqslant 5$ be a positive integer such that $n \neq 8, 9$. Then the equation $5x + 6y + 7z = n$ has a solution in nonnegative integer triples (x,y,z).*

Lemma 13. *Let $n \geqslant 4$ be a positive integer. Then the equation $4x + 5y + 6z + 7u = n$ has a solution in nonnegative integer tuples (x,y,z,u).*

The proofs of these lemmas are all simple applications of Sylvester's theorem. For example, the proof of Lemma 11 can be obtained by applying Sylvester's theorem to $a = 4$ and $b = 5$, which establishes the conclusion for all $n \geqslant 12$, and then each smaller n can be dealt with individually. The proofs of Lemmas 12 and 13 are similar, so we omit the details.

We also need another lemma.

Lemma 14. *Suppose the equation $4x+5y+6z = n$ has a solution (α,β,γ) in nonnegative integer triples. Then the equation $4x + 5y + 6z = n + 6$ has a solution different from $(\alpha,\beta,\gamma+1)$ whenever $n \geqslant 4$ and $n \neq 5$.*

Proof. First suppose $\alpha \geqslant 1$. Then $(\alpha-1, \beta+2, \gamma)$ is a required solution. Next suppose $\alpha = 0$. If $\gamma \geqslant 1$, then $(\alpha+3, \beta, \gamma-1)$ is a required solution. If $\gamma = 0$, then $\beta \geqslant 2$ because of the restriction on n, and $(\alpha+4, \beta-2, \gamma)$ is a required solution. $\square$

The next lemma shows that the coefficient of q^N in $H_{L,3,L}(q)$ is positive for most values of L and small N. Positivity, as opposed to nonnegativity, is needed in this case to prove Theorem 5 at the end of the section.

Lemma 15. *For $L \geqslant 22$ and $N \geqslant 21$, the coefficient of q^N in $H_{L,3,L}(q)$ is positive.*

Proof. Fix $L \geqslant 22$ and $N \geqslant 21$. Recall the combinatorial interpretation of the coefficients of $H_{L,3,L}(q)$ in (14). We prove nonnegativity of the coefficients of $H_{L,3,L}(q)$ by constructing an injective map ϕ as in (15) for $s = 3$. Positivity will then be shown at the end of the proof by displaying an element of the codomain of ϕ that is not in its range.

Let $\pi = \left(4^{f_4}, \ldots, L^{f_L}, \ldots, (L+3)^{f_{L+3}}\right)$ be a partition of N in $D_{L,3}$ and let f denote f_L. Recall that partitions of N in $I_{L,3,L}$, the codomain of ϕ, have parts in the set $\{3, \ldots, L+3\}$, the number 3 must occur as a part, and L does not occur as a part. Our proof of injectivity of ϕ is as follows.

- We define $\phi(\pi)$ in cases chiefly determined by f in π, with several subcases that depend on the frequencies of other parts of π. In each case, it will usually be readily apparent that ϕ is injective, but we will provide some justification for more complicated cases.

- When we analyze why ϕ is injective overall, we will gather cases by the frequency of 3 in the image of ϕ. Thus each case is labelled twice: first by its case determined by the frequency f of L in π (see below, for example, Case 2(c)(ii)(α)) and second, in parentheses, by the frequency of 3 in $\phi(\pi)$ (for example, (B2)). Once the cases are gathered by their frequency of 3 in the image of ϕ, we analyze, for each fixed i, all cases where the frequency of 3 is i in the image of ϕ, and we argue why ϕ is injective collectively in these cases. For example, on Page 15 the cases (A$*$) are all the cases where the frequency of 3 is 1 in the image of ϕ.

- We then observe that ϕ must be injective overall because distinct cases where in the image of ϕ the frequency of 3 in one case is i and the frequency of 3 in the other case is j, where $i \neq j$, cannot contain common elements, so two partitions π and π' pertaining to distinct cases i and j cannot have the same image.

We now define ϕ.

Case 1 (F1, K1) (this case, exceptionally, has many values for the frequency of 3 in a partition in the image of ϕ, so it has more than one pink label): $f \geqslant 1$. Notice $(L-18)i \geqslant 4$ for all $i \geqslant 1$, so the equation

$$(L-18)i = 4x_i + 5y_i + 6z_i + 7u_i \tag{20}$$

has a nonnegative integer solution by Lemma 13. For each $i \geqslant 1$, fix such a solution x_i, y_i, z_i and u_i. Define

$$\phi(\pi) = \left(3^{6f}, 4^{f_4+x_f}, 5^{f_5+y_f}, 6^{f_6+z_f}, 7^{f_7+u_f}, \ldots, L^0, \ldots\right).$$

The function ϕ is injective in this case. Given a partition $\phi(\pi) = \left(3^{6f}, 4^a, 5^b, 6^c, 7^d, \ldots, L^0, \ldots\right)$ in the range ϕ, we can infer π comes from this case (no cases below have the same frequency of 3). From the frequency of 3 in $\phi(\pi)$, we can infer f; then, from (20), we can infer x_f, y_f, z_f and u_f; finally, from f and x_f, y_f, z_f and u_f, we can reconstruct π.

Case 2: $f = 0$. We have the following subcases. Recall that the smallest part of π is denoted by $s(\pi)$.

Case 2(a) (B1): $s(\pi) = L+3$. Then $\pi = ((L+3)^{f_{L+3}})$. Define

$$\phi(\pi) = \left(3^2, 4^2, 5^1, (L-16), \ldots, (L+3)^{f_{L+3}-1}\right).$$

Note $L - 16 \geqslant 6$ because $L \geqslant 22$.

Case 2(b) (A1): $7 \leqslant s(\pi) < L+3$. Define

$$\phi(\pi) = \left(3^1, (s(\pi)-3)^1, (s(\pi)^{f_{s(\pi)}-1}), \ldots,\right).$$

Note $s(\pi) - 3 \neq L$, so no part of size L is created.

Case 2(c): $s(\pi) \leqslant 6$. We have the following subcases.

Case 2(c)(i) (C1): $f_4 \geqslant 1$ and $f_5 \geqslant 1$. Define

$$\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^{f_5-1}, 6^{f_6}, \ldots\right).$$

Case 2(c)(ii): $f_4 = 0$ or $f_5 = 0$. We have the following subcases.

Case 2(c)(ii)(α) (B2): $f_6 \geqslant 1$. Define

$$\phi(\pi) = (3^2, 4^{f_4}, 5^{f_5}, 6^{f_6-1}, \ldots).$$

Case 2(c)(ii)(β): $f_6 = 0$. Thus in this subcase either $f_4 = f_6 = 0$ or $f_5 = f_6 = 0$, and since $s(\pi) \leqslant 6$, precisely one of these two conditions holds. We have further subcases.

Case 2(c)(ii)(β)(I): $f_4 = f_6 = 0$. Then $\pi = \left(5^{f_5}, 7^{f_7}, \ldots\right)$.

Case 2(c)(ii)(β)(I)(A) (E1): $f_5 \geqslant 3$. Define

$$\phi(\pi) = (3^5, 5^{f_5-3}, 7^{f_7}, \ldots).$$

Case 2(c)(ii)(β)(I)(B): $f_5 = 1$. So $\pi = \left(5^1, 7^{f_7}, \ldots, \right)$. Let $m_1 \geqslant 7$ be the least number with a nonzero frequency in π, which must exist because $N \geqslant 21$.

Case 2(c)(ii)(β)(I)(B)(i) (A2): $m_1 \neq 7, 11, 12$. Then $m_1 - 3 \geqslant 5$ and $m_1 - 3 \neq 8, 9$. By Lemma 12 there exist some nonnegative integers u_{m_1-3}, v_{m_1-3} and w_{m_1-3} such that

$$m_1 - 3 = 5u_{m_1-3} + 6v_{m_1-3} + 7w_{m_1-3}. \tag{21}$$

Define

$$\phi(\pi) = \left(3^1, 5^{1+u_{m_1-3}}, 6^{v_{m_1-3}}, 7^{w_{m_1-3}}, m_1^{f_{m_1}-1}, \ldots\right).$$

Our explanation for why ϕ is injective in this case is similar to that in Case 1. Suppose that we are given a member of the range of ϕ of the form $\phi(\pi) = \left(3^1, 5^{1+A}, 6^B, 7^C, m_1^{f_{m_1}-1}, \ldots\right)$. Then, using (21), we can find $m_1 - 3$ from A, B and C, and from there m_1 can be recovered. From m_1, we can reconstruct the partition π uniquely.

There are other cases below where the reasoning that ϕ is injective is similar to Case 1, so we omit the details there.

Case 2(c)(ii)(β)(I)(B)(ii): $m_1 = 7$.

Case 2(c)(ii)(β)(I)(B)(ii)(a) (E2): $f_7 \geqslant 2$. Then define

$$\phi(\pi) = \left(3^5, 4^1, 5^0, 7^{f_7-2}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(ii)(b): $f_7 = 1$. Then $\pi = \left(5^1, 7^1, 8^{f_8}, \ldots\right)$. Let $m_2 \geqslant 8$ be the least number with a nonzero frequency in π.

Case 2(c)(ii)(β)(I)(B)(ii)(b)(i) (B3): $m_2 = 8$. Define

$$\phi(\pi) = \left(3^2, 4^1, 5^2, 7^0, 8^{f_8-1}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(ii)(b)(ii) (B4): $9 \leqslant m_2 < L+3$. Define

$$\phi(\pi) = \left(3^2, 4^1, 5^1, (m_2-3)^1, m_2^{f_{m_2}-1}, \ldots\right).$$

To be clear, our notation indicates that the frequency of 7 in $\phi(\pi)$ is 0 unless $m_2 = 10$.

Case 2(c)(ii)(β)(I)(B)(ii)(b)(iii) (B5): $m_2 = L+3$. Then $\pi = \left(5^1, 7^1, \ldots (L+3)^{f_{L+3}}\right)$. Define

$$\phi(\pi) = \left(3^2, 4^2, 5^2, 7^0, (L-9)^1, (L+3)^{f_{L+3}-1}\right).$$

Case 2(c)(ii)(β)(I)(B)(iii): $m_1 = 11$. Then $\pi = \left(5^1, 11^{f_{11}}, \ldots\right)$. We have further subcases.

Case 2(c)(ii)(β)(I)(B)(iii)(a) (I1): $f_{11} \geqslant 2$. Define

$$\phi(\pi) = \left(3^9, 5^0, 11^{f_{11}-2}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(iii)(b) (H1): $f_{11} = 1$. Then $\pi = \left(5^1, 11^1, 12^{f_{12}}, \ldots\right)$. Let $m_3 \geqslant 12$ be the least number with a nonzero frequency in π. Then define

$$\phi(\pi) = \left(3^8, (m_3-8)^1, m_3^{f_{m_3}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(iv): $m_1 = 12$. Then $\pi = \left(5^1, 12^{f_{12}}, \ldots\right)$. We have further subcases.

Case 2(c)(ii)(β)(I)(B)(iv)(a) (G1): $f_{12} \geqslant 2$. Define

$$\phi(\pi) = \left(3^7, 4^2, 5^0, 12^{f_{12}-2}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(iv)(b): $f_{12} = 1$. Then $\pi = \left(5^1, 12^1, 13^{f_{13}}, \ldots\right)$. Let $m_4 \geqslant 13$ be the least number with a nonzero frequency in π, so $\pi = \left(5^1, 12^1, m_4^{f_{m_4}}, \ldots\right)$.

Case 2(c)(ii)(β)(I)(B)(iv)(b)(i) (D1): $m_4 = 13$. Define

$$\phi(\pi) = \left(3^4, 6^3, 13^{f_{13}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(B)(iv)(b)(ii) (I2): $m_4 \geqslant 14$. Then $m_4 - 10 \geqslant 4$, and by Lemma 13 there exist nonnegative integers $X_{m_4-10}, Y_{m_4-10}, Z_{m_4-10}$ and U_{m_4-10} such that

$$m_4 - 10 = 4X_{m_4-10} + 5Y_{m_4-10} + 6Z_{m_4-10} + 7U_{m_4-10}.$$

For each $m_4 \geqslant 14$, fix a solution to the above equation and define

$$\phi(\pi) = \left(3^9, 4^{X_{m_4-10}}, 5^{Y_{m_4-10}}, 6^{Z_{m_4-10}}, 7^{U_{m_4-10}}, m_4^{f_{m_4}-1}, \ldots,\right).$$

Case 2(c)(ii)(β)(I)(C): $f_5 = 2$. Thus $\pi = \left(5^2, 7^{f_7}, \ldots,\right)$. Let $m_5 \geqslant 7$ be the least number with a nonzero frequency in π.

Case 2(c)(ii)(β)(I)(C)(i) (A3): $m_5 \neq 10$. Then $m_5 - 3 \geqslant 4$ and $m_5 - 3 \neq 7$. By Lemma 11 there are nonnegative integers x_{m_5-3}, y_{m_5-3} and z_{m_5-3} of the equation

$$m_5 - 3 = 4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3}.$$

For each $m_5 \geqslant 7$ such that $m_5 \neq 10$, fix a solution to the above equation and define

$$\phi(\pi) = \left(3^1, 4^{1+x_{m_5-3}}, 5^{y_{m_5-3}}, 6^{1+z_{m_5-3}}, m_5^{f_{m_5}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(C)(ii): $m_5 = 10$. Then $\pi = (5^2, 10^{f_{10}}, \ldots)$.

Case 2(c)(ii)(β)(I)(C)(ii)(a) (J1): $f_{10} \geqslant 2$. Then define

$$\phi(\pi) = \left(3^{10}, 10^{f_{10}-2}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(C)(ii)(b): $f_{10} = 1$. Then $\pi = (5^2, 10^1, 11^{f_{11}}, \ldots)$. Let $m_6 \geqslant 11$ be the least number with a nonzero frequency in π.

Case 2(c)(ii)(β)(I)(C)(ii)(b)(i) (G2): m_6 is odd. Then define

$$\phi(\pi) = \left(3^7, \left(\frac{m_6 - 1}{2}\right)^2, m_6^{f_{m_6}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(I)(C)(ii)(b)(ii) (G3): m_6 is even. Then define

$$\phi(\pi) = \left(3^7, \left(\frac{m_6}{2} - 1\right)^1, \left(\frac{m_6}{2}\right)^1, m_6^{f_{m_6}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(II): $f_5 = f_6 = 0$. Since $s(\pi) \leqslant 6$, we have $f_4 \geqslant 1$. Thus $\pi = \left(4^{f_4}, 7^{f_7}, \ldots\right)$.

Case 2(c)(ii)(β)(II)(A) (F2): $f_4 \geqslant 3$. Define

$$\phi(\pi) = \left(3^4, 4^{f_4-3}, 7^{f_7}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(B): $f_4 = 1$. Thus $\pi = \left(4^1, 7^{f_7}, \ldots\right)$. Let $m_7 \geqslant 7$ be the least number with a nonzero frequency in π. Thus $\pi = \left(4^1, m_7^{f_{m_7}}, \ldots\right)$.

Case 2(c)(ii)(β)(II)(B)(i) (A4): $m_7 \neq 10, 14$. Then $m_7 - 3 \geqslant 4$ and $m_7 - 3 \neq 7, 11$. By Lemma 11 there is a triple $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3})$ such that

$$m_7 - 3 = 4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3}.$$

Crucially, to avoid injectivity problems with the Case 2(c)(ii)(β)(I)(C)(i), if $m_7 = m_5 + 6$, using Lemma 14, we choose a solution with $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$. Note here that $m_7 \neq 14$, so $m_5 - 3 \neq 5$, which is required to use Lemma 14.

Define

$$\phi(\pi) = \left(3^1, 4^{1+x_{m_7-3}}, 5^{y_{m_7-3}}, 6^{z_{m_7-3}}, \ldots, m_7^{f_{m_7}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(B)(ii): $m_7 = 10$. Then $\pi = \left(4^1, 10^{f_{10}}, \ldots\right)$.

Case 2(c)(ii)(β)(II)(B)(ii)(a) (D3): $f_{10} \geqslant 2$. Define

$$\phi(\pi) = \left(3^4, 6^2, 10^{f_{10}-2}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(B)(ii)(b) (G4): $f_{10} = 1$. Thus $\pi = \left(4^1, 10^1, 11^{f_{11}}, \ldots\right)$. Let $m_8 \geqslant 11$ be the least number with a nonzero frequency in π. Then define

$$\phi(\pi) = \left(3^7, (m_8 - 7)^1, m_8^{f_{m_8}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(B)(iii) (F2): $m_7 = 14$. So $\pi = \left(4^1, 14^{f_{14}}, \ldots\right)$. Define

$$\phi(\pi) = \left(3^6, 4^0, 14^{f_{14}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(C): $f_4 = 2$. Thus $\pi = \left(4^2, 7^{f_7}, \ldots\right)$. Let $m_9 \geqslant 7$ be the least number with a nonzero frequency in π.

Case 2(c)(ii)(β)(II)(C)(i) (A5): m_9 is odd. Define

$$\phi(\pi) = \left(3^1, \left(\frac{m_9 + 5}{2}\right)^2, m_9^{f_{m_9}-1}, \ldots\right).$$

Case 2(c)(ii)(β)(II)(C)(ii) (A6): m_9 is even. Define

$$\phi(\pi) = \left(3^1, \left(\frac{m_9}{2} + 2\right)^1, \left(\frac{m_9}{2} + 3\right)^1, m_9^{f_{m_9}-1}, \ldots\right).$$

To prove the injectivity of the map ϕ, we organize the cases based on the various frequencies of 3 in $\phi(\pi)$.

First we organize the cases where the frequency of 3 in $\phi(\pi)$ is 1.

A1. Case 2(b): $\phi(\pi) = \left(3^1, (s(\pi) - 3)^1, (s(\pi)^{f_{s(\pi)}-1}), \ldots, \right)$, where $s(\pi) \geqslant 7$.

A2. Case 2(c)(ii)(β)(I)(B)(i): $\phi(\pi) = \left(3^1, 5^{1+A}, 6^B, 7^C, m_1^{f_{m_1}-1}, \ldots\right)$, where $m_1 \geqslant 8$, $m_1 \neq 11, 12$, and A, B and C are some nonnegative integers such that at least one of these is positive.

A3. Case 2(c)(ii)(β)(I)(C)(i): $\phi(\pi) = \left(3^1, 4^{1+x_{m_5-3}}, 5^{y_{m_5-3}}, 6^{1+z_{m_5-3}}, m_5^{f_{m_5}-1}, \ldots\right)$, where $m_5 \geqslant 7, m_5 \neq 10$, and $4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3} = m_5 - 3$.

A4. Case 2(c)(ii)(β)(II)(B)(i): $\phi(\pi) = \left(3^1, 4^{1+x_{m_7-3}}, 5^{y_{m_7-3}}, 6^{z_{m_7-3}}, \ldots, m_7^{f_{m_7}-1}, \ldots\right)$, where $m_7 \geqslant 7, m_7 \neq 10, 14$, and $4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3} = m_7 - 3$. Moreover, if $m_7 = m_5 + 6$, then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$.

A5. Case 2(c)(ii)(β)(II)(C)(i): $\phi(\pi) = \left(3^1, \left(\frac{m_9+5}{2}\right)^2, m_9^{f_{m_9}-1}, \ldots\right)$, where $m_9 \geqslant 7$ is odd.

A6. Case 2(c)(ii)(β)(II)(C)(ii): $\phi(\pi) = \left(3^1, \left(\frac{m_9}{2} + 2\right)^1, \left(\frac{m_9}{2} + 3\right)^1, m_9^{f_{m_9}-1}, \ldots\right)$, where $m_9 \geqslant 7$ is even.

Each case above is individually injective (we can find π from $\phi(\pi)$). We explain why the map ϕ is injective overall so far, and we do this by confirming that no two distinct cases contain common partitions. In Case A1, the second smallest and the third smallest parts differ by at least 3, and the frequency of the second smallest part is 1. This distinguishes it from all the other cases. In Case A2, the number 4 is not present as a part, which distinguishes it from Cases A3 and A4, and it contains 5 as a part, which distinguishes it from Cases A5 and A6. In Cases A3 and A4, the number 4 is present as a part, which distinguishes it from Cases A5 and A6. Cases A5 and A6 are distinguished by the frequency of the second smallest part.

What remains is to show that Cases A3 and A4 can be distinguished, and we show this by demonstrating the cases have no common element in the image of ϕ. Suppose, to the contrary, that Cases A3 and A4 have a common element. Then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) = (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$. From the relations $4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3} = m_5 - 3$ and $4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3} = m_7 - 3$, we obtain $m_7 = m_5 + 6$. But if $m_7 = m_5 + 6$, then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$, giving the required contradiction.

We follow the same reasoning below. We collect cases according to the frequency of 3 in partitions in the range of ϕ. We then explain why all the distinct cases with fixed frequency of 3 have no partitions in common in their range. We leave the verification that each of the different cases for fixed frequency of 3 in the range of ϕ are individually injective to the reader.

Next we organize the cases where the frequency of 3 in $\phi(\pi)$ is 2.

B1. Case 2(a) : $\phi(\pi) = \left(3^2, 4^2, 5^1, (L-16), \ldots, (L+3)^{f_{L+3}-1}\right)$.

B2. Case 2(c)(ii)(α): $\phi(\pi) = (3^2, 4^{f_4}, 5^{f_5}, \ldots)$, where $f_4 = 0$ or $f_5 = 0$.

B3. Case 2(c)(ii)(β)(I)(B)(ii)(b)(i): $\phi(\pi) = \left(3^2, 4^1, 5^2, 8^{f_8-1}, \ldots\right)$, where $f_8 \geqslant 1$.

B4. Case 2(c)(ii)(β)(I)(B)(ii)(b)(ii): $\phi(\pi) = \left(3^2, 4^1, 5^1, (m_2-3)^1, m_2^{f_{m_2}-1}, \ldots\right)$, where $9 \geqslant m_2 < L+3$.

B5. Case 2(c)(ii)(β)(I)(B)(ii)(b)(iii): $\phi(\pi) = \left(3^2, 4^2, 5^2, 7^0, (L-9)^1, (L+3)^{f_{L+3}-1}\right)$.

These cases are distinguished by their frequencies of 4 and 5.

There is only one case where the frequency of 3 in $\phi(\pi)$ is 3:

C1. Case 2(c)(i): $\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^{f_5-1}, 6^{f_6}, \ldots\right)$.

It is therefore distinguishable from other cases.

Next we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 4.

D1. Case 2(c)(ii)(β)(I)(B)(iv)(b)(i): $\phi(\pi) = \left(3^4, 6^3, 13^{f_{13}-1}, \ldots\right)$, where $f_{13} \geqslant 1$.

D2. Case 2(c)(ii)(β)(II)(A): $\phi(\pi) = \left(3^4, 4^{f_4-3}, 7^{f_7}, \ldots\right)$, where $f_4 \geqslant 3$.

D3. Case 2(c)(ii)(β)(II)(B)(ii)(a): $\phi(\pi) = \left(3^4, 6^2, 10^{f_{10}-2}, \ldots\right)$, where $f_{10} \geqslant 2$.

Thus, when the frequency of 3 in $\phi(\pi)$ is 4, these cases are distinguishable by the frequency of 6 in the image. Next we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 5.

E1. Case 2(c)(ii)(β)(I)(A): $\phi(\pi) = \big(3^5, 5^{f_5-3}, 7^{f_7}, \ldots\big)$, where $f_5 \geqslant 3$.

E2. Case 2(c)(ii)(β)(I)(B)(ii)(a): $\phi(\pi) = \big(3^5, 4^1, 7^{f_7-2}, \ldots\big)$, where $f_7 \geqslant 2$.

Thus, when the frequency of 3 in $\phi(\pi)$ is 5, these cases are distinguishable by the frequency of 4 in the image. Next we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 6.

F1. Case 1 with $f = 1$: $\phi(\pi) = \big(3^6, 4^\alpha, 5^\beta, 6^\gamma, 7^\delta, \ldots\big)$, where α, β, γ and δ are nonnegative integers with at least one positive.

F2. Case 2(c)(ii)(β)(II)(B)(iii): $\phi(\pi) = \big(3^6, 14^{f_{14}-1}, \ldots\big)$, where $f_{14} \geqslant 1$.

Thus, when the frequency of 3 in $\phi(\pi)$ is 6, these cases are distinguishable. Next we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 7.

G1. Case 2(c)(ii)(β)(I)(B)(iv)(a): $\phi(\pi) = \big(3^7, 4^2, 12^{f_{12}-2}, \ldots\big)$, where $f_{12} \geqslant 2$.

G2. Case 2(c)(ii)(β)(I)(C)(ii)(b)(i): $\phi(\pi) = \Big(3^7, \big(\frac{m_6-1}{2}\big)^2, m_6^{f_{m_6}-1}, \ldots\Big)$, where $m_6 \geqslant 11$ is odd.

G3. Case 2(c)(ii)(β)(I)(C)(ii)(b)(ii): $\phi(\pi) = \Big(3^7, \big(\frac{m_6}{2} - 1\big)^1, \big(\frac{m_6}{2}\big)^1, m_6^{f_{m_6}-1}, \ldots\Big)$, where $m_6 \geqslant 11$ is even.

G4. Case 2(c)(ii)(β)(II)(B)(ii)(b): $\phi(\pi) = (3^7, (m_8 - 7)^1, m_8^{f_{m_8}-1}, \ldots)$, where $m_8 \geqslant 11$.

Thus, when the frequency of 3 in $\phi(\pi)$ is 7, the next parts after 3 and their frequencies distinguish the various cases.

There is only one case in which the frequency of 3 in $\phi(\pi)$ is 8:

H1. Case 2(c)(ii)(β)(I)(B)(iii)(b): $\phi(\pi) = \Big(3^8, (m_3 - 8)^1, m_3^{f_{m_3}-1}, \ldots\Big)$, where $m_3 \geqslant 12$ and $f_{m_3} \geqslant 1$.

Next we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 9.

I1. Case 2(c)(ii)(β)(I)(B)(iii)(a): $\phi(\pi) = \big(3^9, 11^{f_{11}-2}, \ldots\big)$, where $f_{11} \geqslant 2$

I2. Case 2(c)(ii)(β)(I)(B)(iv)(b)(ii): $\phi(\pi) = \Big(3^9, 4^\alpha, 5^\beta, 6^\gamma, 7^\delta, m_4^{f_{m_4}-1}, \ldots, \Big)$, where α, β, γ and δ are nonnegative integers such that at least one of these is positive and $m_4 \geqslant 14$.

Thus, when the frequency of 3 in $\phi(\pi)$ is 9, these cases are distinguishable.

There is only case in which the frequency of 3 in $\phi(\pi)$ is 10:

J1. Case 2(c)(ii)(β)(I)(C)(ii)(a): $\phi(\pi) = \left(3^{10}, 10^{f_{10}-2}, \ldots\right)$, where $f_{10} \geqslant 2$.

Finally, there is only one case in which the frequency of 3 in $\phi(\pi)$ is $6f$ for some $f \geqslant 2$.

K1. Case 1 with $f \geqslant 2$. $\phi(\pi) = \left(3^6, 4^\alpha, 5^\beta, 6^\gamma, 7^\delta, \ldots\right)$, where α, β, γ and δ are some nonnegative integers such that at least one of these is positive.

Hence all the cases are distinguishable, and the map ϕ is injective. This shows nonnegativity of the coefficient of q^N in $H_{L,3,L}(q)$ when $L \geqslant 22$ and $N \geqslant 21$. To show these coefficients are positive, we find an element of the codomain of ϕ that is not in its range. In all cases such an element will have frequency of 3 equalling 4, and these can be compared to (D1) - (D3) to ensure they are not in the range of ϕ. If $N = 21$ and $N = 22$, the partitions $\pi_{21} = (3^4, 4^1, 5^1)$ and $\pi_{22} = (3^4, 5^2)$ are partitions not in the range but in the codomain of ϕ.

For the remaining cases, we need the following result: for any positive integer $n \geqslant 6$, the equation

$$6x_6 + 7x_7 + \cdots + 11x_{11} = n$$

has a solution where $x_6, \ldots, x_{11}$ are nonnegative integers (see [BR21, Lemma 8] for a proof of a more general result). If $N \geqslant 23$, then $N - 17 \geqslant 6$, so we can fix a solution with nonnegative integers to the equation

$$6x_6 + 7x_7 + \cdots + 11x_{11} = N - 17.$$

Then the partition $\pi_N = (3^4, 5^1, 6^{x_6}, \ldots, 11^{x_{11}}, \ldots)$ is not in the range but in the codomain of ϕ. □

We are left with the cases $4 \leqslant L \leqslant 21$. We deal with $7 \leqslant L \leqslant 21$ in Lemma 16 below. The proof of this lemma is similar in spirit to the proof of [BR21, Theorem 19].

Lemma 16. *Let $7 \leqslant L \leqslant 21$ and $N_L = L^2 + 8L + 7$. Then the coefficient of q^N in $H_{L,3,L}(q)$ is nonnegative whenever $N \geqslant N_L$.*

Proof. Again it suffices to show that for fixed $7 \leqslant L \leqslant 21$ and $N \geqslant N_L$, there is an injective map ϕ as in (15). Let $\pi = \left(4^{f_4}, \ldots, L^{f_L}, \ldots, (L+3)^{f_{L+3}}\right)$ be a partition of N in $D_{L,3}$ and let $f = f_L$. We define $\phi(\pi)$ in cases depending on the value of f.

Case 1: f is positive and $f \equiv 0 \pmod 3$. Then define

$$\phi(\pi) = \left(3^{\frac{Lf}{3}}, 4^{f_4}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 2: $f \equiv 1 \pmod 3$. Then define

$$\phi(\pi) = \left(3^{L\left(\frac{f-1}{3}\right)+1}, 4^{f_4}, \ldots, (L-3)^{f_{L-3}+1}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 3: $f \equiv 2 \pmod 3$. Then define

$$\phi(\pi) = \left(3^{L\left(\frac{f-2}{3}\right)+2}, 4^{f_4}, \ldots, (L-3)^{f_{L-3}+2}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 4: $f = 0$. Since $N \geqslant N_L$ is large enough, either $f_{L+2} \geqslant 6$ or there exists an $i \neq L+2$ such that $4 \leqslant i \leqslant L+3$ and $f_i \geqslant 3$. Note that the condition on N is in fact tight for this to happen. We have further subcases.

Case 4(i): $f_{L+2} \geqslant 6$. Then define

$$\phi(\pi) = \left(3^{2L+4}, 4^{f_4}, \ldots L^0, (L+2)^{f_{L+2}-6}, (L+3)^{f_{L+3}}\right).$$

Case 4(ii): $f_{L+2} \leqslant 5$, and there exists an $i \neq L+2$ such that $4 \leqslant i \leqslant L+3$ and $f_i \geqslant 3$. Let i_0 be the least such number. Note $i_0 \neq L$ since $f = 0$. We have further subcases depending on whether $i_0 = L+1$ or not.

Case 4(ii)(a): $i_0 \neq L+1$. Then define

$$\phi(\pi) = \left(3^{i_0}, 4^{f_4}, \ldots, i_0^{f_{i_0}-3}, \ldots L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 4(ii)(b): $i_0 = L+1$. Then define

$$\phi(\pi) = \left(3^3, 4^{f_4}, \ldots, (L-2)^{f_{L-2}+3}, L^0, (L+1)^{f_{L+1}-3}, \ldots, (L+3)^{f_{L+3}}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 modulo L in the image distinguishes the cases, with the exception of Cases 4(i) and 4(ii)(a) (when $i_0 = 4$): in these two cases the frequency of 3 is 4 modulo L, but in Case 4(ii)(a) the frequency is precisely 4, whereas in Case 4(i) the frequency is at least $L+4$. Thus all cases are distinguishable. Hence the map ϕ is injective. $\square$

We handle the cases $4 \leqslant L \leqslant 6$ in the next three lemmas.

Lemma 17. *The coefficient of q^N in $H_{6,3,6}(q)$ is nonnegative whenever $N \geqslant 67$.*

Proof. Again it suffices to show that for $L = 6$ and fixed $N \geqslant 67$, there is an injective map ϕ as in (15). Recall that partitions of N in $I_{6,3,6}$, the codomain of ϕ, have smallest part 3, no part equal to 6, and largest part at most 9. Let $\pi = \left(4^{f_4}, \ldots, 6^{f_6}, \ldots, 9^{f_9}\right)$ be a partitions of N in $D_{6,3}$ and let $f = f_6$. We define $\phi(\pi)$ in cases depending on the value of f.

Case 1: $f > 0$. Define

$$\phi(\pi) = \left(3^{2f}, 4^{f_4}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9}\right).$$

Case 2: $f = 0$. Then $\pi = (4^{f_4}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9})$. Since $N \geqslant 67$, there exists $4 \leqslant i \leqslant 9$ such that $i \neq 6$ and $f_i \geqslant 3$. Let i_0 be the least such number. Note that the condition on N is tight for this to happen.

Case 2(i): i_0 is odd. So i_0 is $5, 7$ or 9. Define

$$\phi(\pi) = \left(3^{i_0}, \ldots, 6^0, \ldots i_0^{f_{i_0}-3} \ldots\right).$$

Case 2(ii): $i_0 = 4$. Define

$$\phi(\pi) = \left(3^1, 4^{f_4-3}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9+1}\right).$$

Case 2(iii): $i_0 = 8$. Define

$$\phi(\pi) = \left(3^3, 4^{f_4}, 5^{f_5+3}, 6^0, 7^{f_7}, 8^{f_8-3}, 9^{f_9}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. Hence the map ϕ is injective. □

Lemma 18. *The coefficient of q^N in $H_{5,3,5}(q)$ is nonnegative whenever $N \geqslant 164$.*

Proof. Again it suffices to show that for $L = 5$ and fixed $N \geqslant 164$, there is an injective map ϕ as in (15). Recall that partitions of N in $I_{5,3,5}(q)$, the codomain of ϕ, have smallest part 3, no part equal to 5, and largest part at most 8. Let $\pi = \left(4^{f_4}, 5^{f_5}, \ldots, 8^{f_8}\right)$ be a partition of N in $D_{5,3}$ and let f denote f_5. We define $\phi(\pi)$ in cases depending on the value of f.

Case 1: f is a positive number with $f \equiv 0 \pmod 3$. Define

$$\phi(\pi) = \left(3^{\frac{5f}{3}}, 4^{f_4}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 2: $f > 1$ and $f \equiv 1 \pmod 3$. Define

$$\phi(\pi) = \left(3^{5\left(\frac{f-4}{3}\right)+4}, 4^{f_4+2}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 3: $f \equiv 2 \pmod 3$. Define

$$\phi(\pi) = \left(3^{5\left(\frac{f-2}{3}\right)+1}, 4^{f_4}, 5^0, 6^{f_6}, 7^{f_7+1}, 8^{f_8}\right).$$

We are left with the cases $f = 0$ and $f = 1$.

Case 4: Suppose $f = 0$. Then $\pi = \left(4^{f_4}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right)$. Since $N \geqslant 164$ is large enough, at least one of the following conditions is true: (i) $f_4 \geqslant 6$; (ii) $f_6 \geqslant 1$; (iii) $f_7 \geqslant 3$; or (iv) $f_8 \geqslant 12$. We deal with each case below. Note that the condition on N is not tight. The bound of 164 will be required in Case 5.

Case 4(i): $f_4 \geqslant 6$. Define

$$\phi(\pi) = \left(3^8, 4^{f_4-6}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 4(ii): $f_4 \leqslant 5$ and $f_6 \geqslant 1$. Define

$$\phi(\pi) = \left(3^2, 4^{f_4}, 5^0, 6^{f_6-1}, 7^{f_7}, 8^{f_8}\right).$$

Case 4(iii): $f_4 \leqslant 5$, $f_6 = 0$ and $f_7 \geqslant 3$. Define

$$\phi(\pi) = \left(3^7, 4^{f_4}, 5^0, 7^{f_7-3}, 8^{f_8}\right).$$

Case 4(iv): $f_4 \leqslant 5$, $f_6 = 0$, $f_7 \leqslant 2$ and $f_8 \geqslant 12$. Define

$$\phi(\pi) = \left(3^{32}, 4^{f_4}, 5^0, 7^{f_7}, 8^{f_8-12}\right).$$

Case 5: $f = 1$. Then $\pi = \left(4^{f_4}, 5^1, 6^{f_6}, 7^{f_7}, 8^{f_8}\right)$. Since $N \geqslant 164$ is large enough, at least one of the following conditions is true: (i) $f_4 \geqslant 1$; (ii) $f_6 \geqslant 11$; (iii) $f_7 \geqslant 7$; or (iv) $f_8 \geqslant 8$. We deal with each case below. Note that the condition on N is tight here. Case 5(i): $f_4 \geqslant 1$. Define

$$\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 5(ii): $f_4 = 0$ and $f_6 \geqslant 11$. Define

$$\phi(\pi) = \left(3^{13}, 4^8, 5^0, 6^{f_6-11}, 7^{f_7}, 8^{f_8}\right).$$

Case 5(iii): $f_4 = 0$, $f_6 \leqslant 10$ and $f_7 \geqslant 7$. Define

$$\phi(\pi) = \left(3^{18}, 4^0, 5^0, 6^{f_6}, 7^{f_7-7}, 8^{f_8}\right).$$

Case 5(iv): $f_4 = 0$, $f_6 \leqslant 10$, $f_7 \leqslant 6$ and $f_8 \geqslant 8$. Define

$$\phi(\pi) = \left(3^{23}, 4^0, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8-8}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. In Cases 1, 2 and 3, the frequency of 3 is $0, 4$ and 1 modulo 5, respectively. In Cases 4 and 5, it is always 2 or 3 modulo 5 and different for each subcase. Hence the map ϕ is injective. ☐

Lemma 19. *The coefficient of q^N in $H_{4,3,4}(q)$ is nonnegative whenever $N \geqslant 1042$.*

Proof. Again it suffices to show that for $L = 4$ and fixed $N \geqslant 1042$, there is an injective map ϕ as in (15). Recall that partitions of N in $I_{4,3,4}(q)$, the codomain of ϕ, have smallest part 3, no part equal to 4, and largest part at most 7. Let $\pi = (4^{f_4}, 5^{f_5}, 6^{f_6}, 7^{f_7})$ be a partitions of N in $D_{4,3}$ and let $f = f_4$. We define $\phi(\pi)$ in cases depending on f.

For $n \geqslant 10$, Lemma 12 guarantees that there exists nonnegative integer solutions (x_n, y_n, z_n) of the equation

$$n = 5x_n + 6y_n + 7z_n. \tag{22}$$

For each $n \geqslant 10$, fix a nonnegative integer solution (x_n, y_n, z_n) to the equation.

Case 1: $10 \leqslant f < 100$. Define

$$\phi(\pi) = (3^f, 4^0, 5^{f_5+x_f}, 6^{f_6+y_f}, 7^{f_7+z_f}).$$

It is easy to see that ϕ is injective in this case: given an element $\phi(\pi)$ whose frequency is between 10 and 100, the frequency of 3 gives f, and the values of x_f, y_f and z_f can be found from (22). The partition π can then be reconstructed. Similar arguments can be used to show that ϕ is injective in the other cases below.

Case 2: $f \geqslant 100$. Define

$$\phi(\pi) = (3^{f+30}, 4^0, 5^{f_5+x_{f-90}}, 6^{f_6+y_{f-90}}, 7^{f_7+z_{f-90}}).$$

Case 3: $0 \leqslant f \leqslant 9$. Since $N \geqslant 1042$ is large enough, at least one of the following conditions is true: (i) $f_5 \geqslant 62$; (ii) $f_6 \geqslant 57$; or (iii) $f_7 \geqslant 53$. We deal with each case below. Note that the condition on N is in fact tight here.

Case 3(i): $f_5 \geqslant 62$. Define

$$\phi(\pi) = \left(3^{f+100}, 4^0, 5^{f_5-62+x_{f+10}}, 6^{f_6+y_{f+10}}, 7^{f_7+z_{f+10}}\right).$$

Case 3(ii): $f_5 \leqslant 61$ and $f_6 \geqslant 57$. Define

$$\phi(\pi) = \left(3^{f+110}, 4^0, 5^{f_5+x_{f+12}}, 6^{f_6-57+y_{f+12}}, 7^{f_7+z_{f+12}}\right).$$

Case 3(iii): $f_5 \leqslant 61$, $f_6 \leqslant 56$ and $f_7 \geqslant 53$. Define

$$\phi(\pi) = \left(3^{f+120}, 4^0, 5^{f_5+x_{f+11}}, 6^{f_6+y_{f+11}}, 7^{f_7-53+z_{f+11}}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. Hence the map ϕ is injective. ☐

Lemmas 15, 16, 17, 18 and 19 show that for N larger than a small number, the coefficient of q^N in $H_{L,s,L}(q)$ is nonnegative. With these results, the use of computer searches, and Lemma 3 (with $s = 3$), we can now prove Theorem 5.

Proof of Theorem 5. Let $H_{L,3,L}(q) = \sum_{N \geqslant 0} a_{L,N} q^N$ and $G_{L,3}(q) = \sum_{N \geqslant 0} b_{L,N} q^N$. By Lemma 3,

$$b_{L,N} = a_{L,N} + a_{L,N-L} + a_{L,N-2L} + \cdots . \tag{23}$$

First we focus on the case $L \geqslant 22$. By Lemma 15, the coefficients satisfy $a_{L,N} \geqslant 1$ whenever $N \geqslant 21$. For $N \leqslant 20$, we observe that $a_{L,N}$ is independent of L. To see why, the combinatorial interpretation of $a_{L,N}$ in (14) gives that, when $L \geqslant N + 1$, the number $a_{L,N}$ is the difference between the number of partitions of N with smallest part 3 and the number of partitions of N with smallest part at least 4; that is, the condition on the largest part of the partitions becomes superfluous. Thus, when $1 \leqslant N \leqslant 20$, since $L \geqslant N + 1$, we have $a_{L,N} = a_{N+1,N}$, and these 20 values can all be found by a computer search. The search finds that $a_{N+1,N}$ is negative only when N is one of $4, 5, 8, 10, 12, 14$ or 16, and in each case $a_{N+1,N}$ is exactly -1. Thus, for any $N \geqslant 1$, the right hand side of (23) contains at most one term equal to -1, while the rest of the terms are positive. It follows from (23) that $b_{L,N}$ is negative only when N is one of $4, 5, 8, 10, 12, 14$ or 16, and in each case $b_{L,N}$ is exactly -1. This gives Theorem 5 for $L \geqslant 22$.

The remaining cases are easier. For $7 \leqslant L \leqslant 21$, Lemma 16 renders the unknown values of $a_{L,N}$ to the cases $N \leqslant N_L$, a finite set of values that can be searched using a computer. These computations along with (23) give Theorem 5. Similarly, for $4 \leqslant L \leqslant 6$, the Lemmas 17, 18 and 19 leave only a finite number of unknown values for $a_{L,N}$ for small N, which can all be found using a computer. These values and an application of (23) complete the proof of Theorem 5. ☐

Remark. The programming for $7 \leqslant L \leqslant 21$ and $N \leqslant N_L$ turned out to be a difficult task in Magma. For example, it is hard to calculate the number of partitions of 250 with all parts in the set {4,5, ..., 17} using Magma. The command $Partitions(250, min\_part = 4, max\_part = 17).cardinality()$ in Sage also does not work (it takes too long and ultimately stops working). We overcame this problem through another related command and some mathematics. In Sage, we noticed that the command $Partitions(n, max\_part = 17).cardinality()$ is very fast even for large n (even until $n = 1000000$, it is fast!) Thus, we calculate the number of partitions with all parts in the set $\{4, 5, \dots, 17\}$ in terms of the number of partitions $p_{17}(n)$ of n with maximum part at most 17. We do this by viewing partitions with all parts in the set $\{4, 5, \dots, 17\}$ as partitions with maximum part 17 and no part 1, 2 and 3. Let A, B and C denote the set of partitions of n with maximum part 17 and also having 1, 2 and 3 as a part, respectively. Then we need to find the cardinality of the set $A^{\complement} \cap B^{\complement} \cap C^{\complement}$. Using the principle of inclusion and exclusion, we get that the number of partitions of n with all parts in the set $\{4, 5, \dots, 17\}$ is given by $p_{17}(n) - p_{17}(n-1) - p_{17}(n-2) + p_{17}(n-4) + p_{17}(n-5) - p_{17}(n-6)$, and thus can be easily computed.

4 The series $G_{L,s}(q)$ for $s \geqslant 4$ and other generalizations

For fixed L and s, let $p_{L,s}(q)$ be the smallest degree polynomial in q with smallest coefficients such that $G_{L,s}(q) + p_{L,s}(q) \succeq 0$. This paper finds $p_{L,s}(q)$ for $s = 3$ and all L, while the cases $s = 2$ (found in Theorem 2) and $s = 1$ (found in [BU19]) were found earlier. One goal is to find $p_{L,s}(q)$ for general s and L. Our numerical computations for $s = 4$ and $s = 5$ do not suggest a nice form for $p_{L,s}(q)$. A simpler problem, though also interesting, is to determine the degree of $p_{L,s}(q)$ for general L and s. Furthermore, we suspect that for fixed $s \geqslant 4$ that $p_{L,s}(q)$ stabilizes; that is, there is an L_0 such that $p_{L,s}(q) = p_{L_0,s}(q)$ for all $L \geqslant L_0$, as in the case for $s = 3$ (Theorem 5 gives $L_0 = 10$ when $s = 3$). Finding L_0 as a function of s is also another open problem.

Moreover, series analogous to $G_{L,s}(q)$ for partitions with further restrictions on parts, such as for partitions with only odd parts or for self conjugate partitions, may prove interesting.

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