An embedding of the skein action on set partitions into the skein action on matchings

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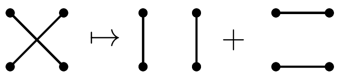
Abstract

Rhoades defined a skein action of the symmetric group on the linear span of noncrossing set partitions which generalized an action of the symmetric group on the linear span of matchings. The $\mathfrak{S}_n$-action on matchings is made possible via the Ptolemy relation, while the action on set partitions is defined in terms of a set of skein relations that generalize the Ptolemy relation. The skein action on noncrossing set partitions has seen applications to coinvariant theory and coordinate rings of partial flag varieties. In this paper, we will show how Rhoades' $\mathfrak{S}_n$-module can be embedded into the $\mathfrak{S}_n$-module generated by matchings, thereby explaining how Rhoades' generalized skein relations all arise from the Ptolemy relation.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

This paper concerns two actions of $\mathfrak{S}_n$. The first, due to Rhoades [7], is on the vector space with basis given by the set of noncrossing set partitions of $[n] := \{1, 2, \ldots, n\}$. We will refer to this action as the *skein action* on noncrossing set partitions as it is defined in terms of three *skein relations*, the simplest of which is the *Ptolemy relation* shown below.



The second is a well-known action on noncrossing matchings first studied by Rumer, Teller, and Weyl, then further developed by Temperley and Lieb, Jones, Kauffman, Kuperberg, and others [1, 2, 4, 10, 11]. If V is the defining representation of SL_2, then the SL_2 invariants of $V^{\otimes n}$ have a basis, called the SL_2 *web basis* or *Temperley-Lieb basis*, indexed by noncrossing matchings. The $\mathfrak{S}_n$ action on $V^{\otimes n}$ which permutes tensor factors thus induces a $\mathfrak{S}_n$-action on the linear span of noncrossing matchings. Combinatorially, this action can be understood via the Ptolemy relation. A permutation in $\mathfrak{S}_n$ acts on

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a matching by swapping elements, then, if crossings were introduced, resolving those crossings via the Ptolemy relation.

The skein action on noncrossing set partitions was originally defined to provide a representation theoretic proof of a cyclic sieving result on noncrossing set partitions. Noncrossing set partitions of $[n]$ are counted by the Catalan numbers, and noncrossing set partitions of $[n]$ with exactly $n-k$ blocks are counted by the Narayana numbers:

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Reiner, Stanton and White [6] showed that a q-analogue of the Narayana numbers:

$$N(n,k,q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}$$

exhibits the cyclic sieving phenomenon for the natural cyclic action on noncrossing set partitions with $n-k$ blocks. Their proof proceeded via direct calculation of $N(n,k,q)$ and sizes of fixed point sets; the skein action allowed for an alternate proof utilizing Springer's theorem on regular elements [7, 9]. The skein action has since been found within coinvariant rings and coordinate rings of certain partial flag varieties [3, 5], strengthening the claim that it is an action worth studying in its own right.

The skein action on noncrossing set partitions is defined combinatorially in an analogous way to the action on noncrossing perfect matchings. In fact, since noncrossing perfect matchings are a subset of noncrossing set partitions, it can be considered a generalization of the matching action to all noncrossing set partitions. To act by a transposition $(i, i+1)$ on a noncrossing matching, swap i and $i+1$, then if a crossing was introduced, use one of the following skein relations to resolve it, depending on the sizes of the blocks that cross:

Rhoades was able to determine the $\mathfrak{S}_n$-irreducible structure of the skein action on $mathbbC[NCP(n)]$, the span of noncrossing set partitions [7]. In particular, $\mathbb{C}[NCP(n)_0]$, the span of all singleton-free noncrossing set partitions with exactly k blocks is an $\mathfrak{S}_n$-irreducible of shape $(k,k,1^{n-2k})$, and the span of all noncrossing set partitions with exactly s singletons and exactly k non-singleton blocks is isomorphic to an induction product of $S^{(k,k,1^{n-2k-s})}$ with the sign representation of $\mathfrak{S}_s$. The structure of the noncrossing matching action is similar; the submodule spanned by noncrossing matchings with exactly k pairs is isomorphic to the induction product of $S^{(k,k)}$ and a sign representation of $\mathfrak{S}_{n-2k}$. By the Pieri rule, this induction product is a direct sum of three irreducible submodules, one of which is isomorphic to $S^{(k,k,1^{n-2k})}$, so there exists a unique embedding of $\mathbb{C}[NCP(n)_0]$, the span of all singleton-free noncrossing set partitions in $\mathbb{C}[NCP(n)]$, into $\mathbb{C}[NCM(n)]$. The first main result of this paper (appearing as Theorem 15 in section 3) explicitly describes the embedding as follows:

Theorem 1. *The linear map* $f_n : \mathbb{C}[NCP(n)_0] \to \mathbb{C}[NCM(n)]$ *defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

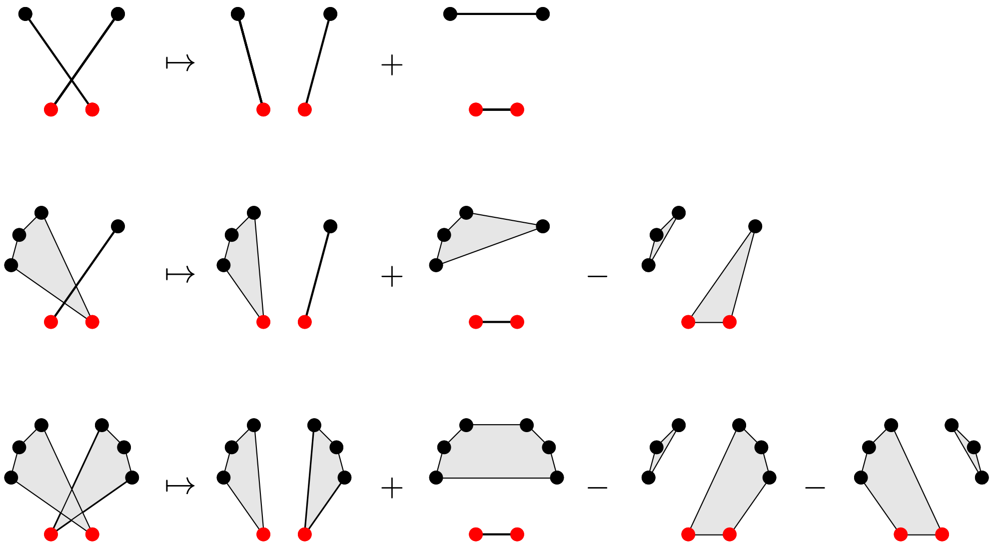

Figure 1: The three skein relations defining the action of $\mathfrak{S}_n$ on $\mathbb{C}[NCP(n)]$. The red vertices are adjacent indices $i, i+1$ and the shaded blocks have at least three elements. The symmetric 4-term relation obtained by reflecting the middle relation across the y-axis is not shown.

is an $\mathfrak{S}_n$-equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m.

For an example of this map, let $\pi = \{\{1,2,3\},\{4,5\}\}$ then

$$f_n(\pi) = \{\{1,2\},\{4,5\}\} + \{\{1,3\},\{4,5\}\} + \{\{2,3\},\{4,5\}\}$$

is a sum of 3 matchings in $\mathbb{C}[NCM(n)]$. The proof of Theorem 1 also gives an alternate proof that the skein action on noncrossing set partitions is well-defined, see Remark16. The skein action being well-defined was originally shown through a laborious verification of the braid relations [7].

The second main result of this paper (appearing as Theorem 20 in section 4) is to then describe the image of this map within $\mathbb{C}[NCM(n)]$. For this purpose, as well as the purpose of simplifying the proof of Theorem 1, it is helpful to introduce a multiplicative structure to $\mathbb{C}[NCM(n)]$, where multiplication corresponds to union when matchings are disjoint, and gives 0 otherwise. With this added structure, the image of f_n is a principal ideal:

Theorem 2. *Let H_n be the ideal of $\mathbb{C}[NCM(n)]$ generated by $f_n([n])$. Then*

$$\mathrm{im}(f_n) = H_n.$$

The SL_2 web basis has generalizations to other Lie groups. The first generalizations, to Lie groups of rank two and their quantum deformations was given by Kuperberg, with indexing sets given by certain planar diagrams embedded in a disk [4]. We propose a set of combinatorial objects which might serve as an analog of noncrossing set partitions for the SL_3 web basis, as their enumeration conjecturally matches the dimension of the Specht module $S^{(k^3,n-3k)}$.

The rest of the paper is organized as follows. Section 2 will provide necessary background information. Section 3 will prove our first main result, the embedding from $\mathbb{C}[NCP(n)_0]$ to $\mathbb{C}[NCM(n)]$. Section 4 will determine the image of this embedding within $\mathbb{C}[NCM(n)]$. Section 5 will describe the conjectural analog for the SL_3 web basis.

2 Background

2.1 Noncrossing matchings

A *matching* of $[n]$ is a collection of disjoint size-two subsets of $[n]$. A matching is *noncrossing* if it does not contain two subsets $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. Let $M(n)$ denote the set of all matchings of $[n]$, and let $NCM(n)$ denote the set of all noncrossing matchings of $[n]$. The symmetric group $\mathfrak{S}_n$ acts naturally on $M(n)$ as follows. If $\sigma \in \mathfrak{S}_n$ and $m = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$ is a matching, then

$$\sigma \circ m = \operatorname{sign}(\sigma)\{\{\sigma(a_1), \sigma(b_1)\}, \dots, \{\sigma(a_k), \sigma(b_k)\}\}. \tag{1}$$

We can extend this action to an action on $\mathbb{C}[M(n)]$, the $\mathbb{C}$-vector space with basis given by matchings of $[n]$. The action on matchings does not descend to an action on $NCM(n)$ since permuting elements in a noncrossing matching could introduce crossings. However, we can linearize and define an action on $\mathbb{C}[NCM(n)]$, the $\mathbb{C}$-vector space with basis given by noncrossing matchings of $[n]$. For any noncrossing matching m and adjacent transposition $s_i = (i, i + 1)$, define

$$s_i \cdot m = \begin{cases} s_i \circ m & s_i \circ m \text{ is noncrossing} \\ m + m' & \text{otherwise.} \end{cases} \tag{2}$$

Here $\circ$ denotes the action on all matchings and m' is the matching where the subsets of m containing i and $i + 1$, call them $\{i, a\}$ and $\{i + 1, b\}$ have been replaced with $\{i, i + 1\}$ and $\{a, b\}$ and all other subsets remain the same. In other words, $s_i \circ m$, m, and m' form a trio of matchings that differ only in a Ptolemy relation. It can be shown that this definition satisfies the braid relations and thus gives an action of the symmetric group on $\mathbb{C}[NCM(n)]$. There exists an $\mathfrak{S}_n$-equivariant linear projection $p_M : \mathbb{C}[M(n)] \to \mathbb{C}[NCM(n)]$ given for any matching m by

$$m \mapsto w^{-1} \cdot (w \circ m), \tag{3}$$

where w is any permutation for which $w \circ m$ is noncrossing. This projection can be thought of as a way to"resolve" crossings in a matching and obtain a sum of noncrossing matchings. The following proposition is not new, but we were unable to find a suitable reference and thus include a proof for completeness.

Proposition 3. *The kernel of the projection $p_M : \mathbb{C}[M(n)] \to \mathbb{C}[NCM(n)]$ is spanned by elements of the form*

$$\{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} + \{\{a_1, a_3\}, \{a_2, a_4\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} + \{\{a_1, a_4\}, \{a_2, a_3\}, \{a_5, a_6\}, \ldots, \{a_{2k-1}, a_{2k}\}\} \quad (4)$$

for any $a_1, \ldots, a_{2k} \in [n]$, i.e. sums of three matchings which differ by a Ptolemy relation.

Proof. Let β denote the set of all elements of the form given in (4). To see that the span of β is contained in the kernel of p_M, note that by the $\mathfrak{S}_n$-equivariance of p_M it suffices to check that applying p_M gives 0 in the case where $a_i = i$ for all i. In this case, we have

$$\begin{aligned} p_M(\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\} &+ \{\{1,3\},\{2,4\}\ldots,\{2k-1,2k\}\} \\ &+ \{\{1,4\},\{2,3\}\ldots,\{2k-1,2k\}\}) \\ = \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\} &+ (2,3)\cdot(-\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}) \\ &+ \{\{1,4\},\{2,3\},\ldots,\{2k-1,2k\}\} = 0 \end{aligned} \quad (5)$$

To see that the kernel is contained in the span, note that since p_M is a projection, the kernel is spanned by $m - p_M(m)$ for any matching m. Let t denote the minimum number of transpositions $s_{i_1}, \ldots, s_{i_t}$ for which $(s_{i_1} \cdots s_{i_t}) \circ m$ is noncrossing, and let $w = s_{i_1} \cdots s_{i_t}$. We will show by induction on t that $m - p_M(m) \in \text{span}(\beta)$. When $t = 0$, then $m - p_M(m) = 0$, so the claim is true. Otherwise, assume the claim holds for $t - 1$. We have $m - p_M(m) = s_{i_1} \circ (s_{i_1} \circ m) - s_{i_1} \cdot p_M(s_{i_1} \circ m)$. By our inductive hypothesis, $s_{i_1} \circ m - p_M(s_{i_1} \circ m)$ lies in the span of β, so it suffices to verify for any $b \in \beta$, that if we apply $s_{i_1} \circ (-)$ to every crossing term of b and apply either $s_{i_1} \cdot (-)$ or $s_{i_1} \circ (-)$ to every noncrossing term of b, we remain in the span of β. This is true because β is closed under the $\circ$ action, and for every noncrossing matching m_1, either

$$s_{i_1} \circ m = s_{i_1} \cdot m$$

or

$$s_{i_1} \cdot m_1 = s_{i_1} \circ m_1 - (s_{i_1} \circ m_1 + m_1 + m_1')$$

where m_1' is obtained by replacing the sets $\{i, a\}$ and $\{i+1, b\}$ with the sets $\{i, i+1\}$ and $\{a, b\}$. In the second case, $s_{i_1} \circ m_1 + m_1 + m_1'$ is in β. □

2.2 The skein action

A set partition of $[n]$ is a collection of disjoint subsets of $[n]$ whose union is $[n]$. A set partition is *noncrossing* if there do not exist distinct blocks A and B and elements $a, c \in A$, $b, d \in B$ with $a < b < c < d$. Let $\Pi(n)$ denote the set of all set partitions of n, and let $NCP(n)$ denote the set of all noncrossing set partitions of $[n]$. We can define an action of $\mathfrak{S}_n$ on $\mathbb{C}[\Pi(n)]$ analogous to the action on $\mathbb{C}[M(n)]$. Rhoades defined an action of $\mathfrak{S}_n$ on

$\mathbb{C}[NCP(n)]$ as follows [7]. For any noncrossing set partition π and adjacent transposition s_i,

$$s_i \cdot \pi = \begin{cases} -\pi & i \text{ and } i+1 \text{ are in the same block of } \pi \\ -\pi' & \text{at least one of } i \text{ and } i+1 \text{ is in a singleton block of } \pi \\ \sigma(\pi') & i \text{ and } i+1 \text{ are in different size 2 or larger blocks of } \pi \end{cases}$$

where π' is the set partition obtained by swapping which blocks i and $i+1$ are in, and σ is defined for any almost-noncrossing (i.e. the crossing can be removed by a single adjacent transposition) partition π by $\sigma(\pi) = \pi + \pi_2 - \pi_3 - \pi_4$ where, if the crossing blocks in σ are $\{i, a_1, \ldots, a_k\}$ and $\{i+1, b_1, \ldots, b_l\}$, then π_2, π_3 and π_4 are obtained from π by replacing these blocks with

- $\{i, i+1\}$ and $\{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ for π_2
- $\{i, i+1, a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$ for π_3
- $\{i, i+1, b_1, \ldots, b_l\}$ and $\{a_1, \ldots, b_k\}$ for π_3

when $k, l \geqslant 2$. If $k = 1$ then $\pi_4 = 0$ instead and if $l = 1$ then $\pi_3 = 0$ instead. The sum of partitions given by $\sigma(\pi)$ is best described with a picture, see Figure 1 in the introduction. The three possibilities (depending on whether $k, l \geqslant 2$) are the three skein relations mentioned in the introduction. A more detailed description of this action can be found in [7].

We again have an $\mathfrak{S}_n$-equivariant linear projection $p_\Pi : \mathbb{C}[\Pi(n)] \to \mathbb{C}[NCP(n)]$ given for any set partition π by

$$\pi \mapsto w^{-1} \cdot (w \circ \pi), \tag{6}$$

where w is any permutation for which $w \circ \pi$ (here $\circ$ denotes the action of $\mathfrak{S}_n$ on all set partitions) is noncrossing. We have the following proposition, analogous to Proposition 3, and with an analagous proof.

Proposition 4. *The kernel of the projection $p_\Pi : \mathbb{C}[\Pi(n)] \to \mathbb{C}[NCP(n)]$ is spanned by elements of the form*

$$w \circ (s_i \circ \pi + \sigma(\pi))$$

for any permutation w and singleton-free almost noncrossing set partition π, i.e. sums of set partitions which differ by a skein relation.

2.3 $\mathfrak{S}_n$-representation theory

For $n \in \mathbb{Z}_{\geqslant 0}$, a *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 + \cdots + \lambda_k = n$. A partitions of n can be represented by a *Young diagram*, which is an arrangement of square boxes into n left-justified rows, with the i^{th} row containing λ_i boxes.

Irreducible representations of the symmetric group $\mathfrak{S}_n$ are naturally indexed by partitions of n. Let S^λ denote the $\mathfrak{S}_n$-irreducible corresponding to partition λ. Given two representations V and W of $\mathfrak{S}_{m_1}$ and $\mathfrak{S}_{m_2}$ respectively, with $m_1 + m_2 = n$, the induction product $V \circ W$ is given by

$$V \circ W = \mathrm{Ind}_{\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}}^{\mathfrak{S}_n} V \otimes W$$

where $\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$ is identified with the parabolic subgroup of $\mathfrak{S}_n$ which permutes the first m_1 elements, $\{1, \ldots, m_1\}$, and last m_2 elements, $\{m_1 + 1, \ldots, n\}$, separately. When V is an irreducible representation S^μ for some partition μ of m_1 and W is a sign representation of $\mathfrak{S}_{m_2}$, the dual Pieri rule describes how to express $V \circ W$ in terms of irreducibles,

$$S^\mu \circ \mathrm{sign}_{\mathfrak{S}_{m_2}} \cong \sum_\lambda S^\lambda \tag{7}$$

where the sum is over all partitions λ whose young diagram can be obtained from that of μ by adding m_2 boxes, no two in the same row. For further background, see [8].

3 The embedding

In order to prove that our map is an embedding, it will be helpful to introduce a multiplicative structure to work with. To do so we will introduce three commutative graded $\mathbb{C}$-algebras R_n, A_n, and M_n, all with $\mathfrak{S}_n$-actions. If we forget the multiplicative structure, the underlying $\mathfrak{S}_n$-modules of R_n, A_n, and M_n will contain a copy $\mathbb{C}[\Pi(n)]$, $\mathbb{C}[M(n)]$, and $\mathbb{C}[NCM(n)]$ respectively. In the case of M_n, this copy will be all of M_n. The structure of this proof is best explained via a commutative diagram, see Figure 2. We will define a map $h_n \circ \iota_\Pi : \mathbb{C}[\Pi(n)_0] \to M_n$, and show that its kernel is equal to the kernel of p_Π. The desired embedding f_n then follows from the first isomorphism theorem.

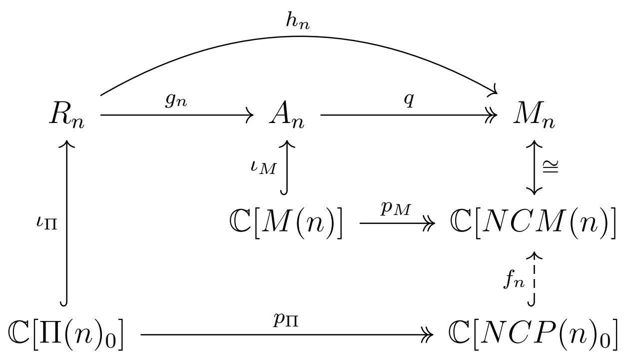

Figure 2: A commutative diagram of the maps used in the following proofs. All maps shown are $\mathfrak{S}_n$-equivariant linear maps. Maps between R_n, A_n, and M_n are also morphisms of $\mathbb{C}$-algebras. The desired embedding is shown as a dashed arrow.

We begin with the definition of R_n.

Definition 5. Let $n \in \mathbb{N}$. Define R_n to be the unital commutative $\mathbb{C}$-algebra generated by nonempty subsets of $[n]$. Define a degree-preserving action of $\mathfrak{S}_n$ on R_n by

$$\pi \cdot \{a_1, \ldots, a_k\} = \mathrm{sign}(\pi)\{\pi(a_1), \ldots, \pi(a_k)\}$$

for any permutation $\pi \in \mathfrak{S}_n$ and generator $\{a_1, \ldots, a_k\} \in R_n$.

The ring R_n can be thought of as the ring of multiset collections of subsets of $[n]$ with multiplication given by union of collections and addition purely formal. It is in this sense that it contains a copy of $\mathbb{C}[\Pi(n)]$, as set partitions of n are particular collections of subsets of $[n]$. To be precise, there exists an $\mathfrak{S}_n$-module embedding $\iota_\Pi : \mathbb{C}[\Pi(n)_0] \hookrightarrow R_n$, given by sending any singleton-free set partition π to the product of its blocks. For the proofs in this section, the main benefit of working with R_n instead of $\mathbb{C}[\Pi(n)]$ is that it allows us to work with only those two parts of a set partition which vary between terms in the skein relations, rather than carrying around excess notation for the unchanging parts.

The ring A_n is a subring of R_n designed to model matchings in much the same way which R_n models set partitions. It is defined as follows.

Definition 6. Let $n \in \mathbb{N}$ and define A_n to be the $\mathfrak{S}_n$-invariant subalgebra of R_n generated by the size two subsets of $[n]$. The subring A_n is invariant under the $\mathfrak{S}_n$-action of R_n, and thus inherits a graded $\mathfrak{S}_n$-action from R_n.

Like R_n, the ring A_n can be thought of as the ring of multiset collections of size-two subsets of $[n]$. As matchings are particular collections of size-two subsets of $[n]$, we again have an $\mathfrak{S}_n$-module embedding $\iota_M : \mathbb{C}[M(n)] \hookrightarrow A_n$, given by

$$\{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\} \longmapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any matching $\{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\})$.

Our final ring, M_n, is defined as a quotient of A_n in the following way.

Definition 7. Define I_n to be the ideal of A_n generated by elements of the following forms

- $\{a, b\} \cdot \{a, b\}$
- $\{a, b\} \cdot \{a, c\}$
- $\{a, b\} \cdot \{c, d\} + \{a, c\} \cdot \{b, d\} + \{a, d\} \cdot \{b, c\}$

for any distinct $a, b, c, d \in [n]$. Then I_n is a $\mathfrak{S}_n$-invariant ideal of A_n, so define M_n to be the $\mathfrak{S}_n$-module $M_n := A_n/I_n$. Let $q : A_n \to M_n$ be the quotient map.

The first two types of elements listed in the definition of I_n serve the purpose of removing collections of size-two subsets which are not actually matchings. The third is the Ptolemy relation used to define the action of $\mathfrak{S}_n$ on $\mathbb{C}[NCM(n)]$, so quotienting by this ideal gives an $\mathfrak{S}_n$-module isomorphic to $\mathbb{C}[NCM(n)]$, as per the following argument.

Proposition 8. *There is an $\mathfrak{S}_n$-module isomorphism from $\mathbb{C}[NCM(n)]$ to M_n, given by*

$$\{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\} \longmapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any noncrossing matching $\{\{a_1, b_1\}, \ldots, \{a_k, b_k\}\}$.

Proof. Let $q : A_n \to M_n$ be the quotient map. Consider the map $q \circ \iota_M : \mathbb{C}[M(n)] \to M_n$. The kernel of $q \circ \iota_M$ is the preimage $\iota_M^{-1}(I_n)$. The image of ι_M is the span of all monomials consisting of nonintersecting generators, so $I_n \cap \iota_M$ is the linear span of elements of the form

$$(\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\})m$$

where $a, b, c, d \in [n]$ are distinct and m is a monomial not containing a, b, c, d. The kernel of $q \circ \iota_M$ is therefore spanned by the preimage of these elements. This is equivalent to the description of $\ker(p_M)$ given in Proposition 3, so the kernel of $q \circ \iota_M$ is equal to the kernel of p_M. The image of $q \circ \iota_M$ is all of M_n. To see this, note that products of generators of A_n form a vector space basis for A_n, and every such basis element is either in the image of ι_M or in I_n. We therefore have

$$\mathbb{C}[NCM(n)] \cong \mathbb{C}[M(n)]/\ker(p_M) = \mathbb{C}[M(n)]/\ker(q \circ \iota_M) \cong \operatorname{im}(q \circ \iota_M) = M_n \tag{8}$$

where the isomorphism on the left is induced by the map p_M and the isomorphism on the right is induced by the map $q \circ \iota_M$. Composing these isomorphisms gives the stated map. □

The following definition is the key idea behind our main theorem.

Definition 9. Let $n \in \mathbb{N}$. Define the $\mathbb{C}$-algebra map $g_n : R_n \to A_n$ by

$$g_n(A) = \sum_{\{a,b\} \subseteq A} \{a,b\}$$

for generators $A \in R_n$. Singleton sets are sent to 0 by g_n. Define $h_n := q \circ g_n$ where q is the quotient map $A_n \to M_n$.

We give the definition in terms of R_n, A_n, and M_n for simplicity and ease of proofs later, but the map we really care about is $h_n \circ \iota_\Pi : \mathbb{C}[\Pi(n)] \to M_n$. Under this map, a set partition π is sent to the product of its blocks, then each block is sent to the sum of all size-two subsets it contains. After distributing, we get a sum of all ways to pick a size two subset from each block. Composing with the isomorphism between M_n and $\mathbb{C}[NCM(n)]$ we get the sum of all matchings such that each block of π contains exactly one pair of the matching, as in Theorem 15.

We will now show that $h_n \circ \iota_\Pi$ factors through the projection map p_Π to produce an injective map. To do so, we will show that the kernels of these two maps agree. To show that the kernel of $h_n \circ \iota_\Pi$ contains the kernel of p_Π, we introduce an element of R_n modelling the five term skein relation depicted in Figure 1.

Definition 10. Let $i, j \geqslant 2$ and let $p_1, p_2, \ldots, p_i$ and $q_1, q_2, \ldots, q_j$ be distinct in $[n]$. Define $\kappa_n \in R_n$ by

$$\begin{aligned}\kappa_n := \; & \{p_1, \ldots, p_i\} \cdot \{q_1, \ldots, q_j\} - \{p_1, \ldots, p_{i-1}\} \cdot \{q_1, \ldots, q_j, p_i\} \\ & - \{p_1, \ldots, p_i, q_j\} \cdot \{q_1, \ldots, q_{j-1}\} + \{p_1, \ldots, p_{i-1}, q_j\} \cdot \{q_1, \ldots, q_{j-1}, p_i\} \\ & + \{p_1, \cdots, p_{i-1}, q_1, \cdots, q_{j-1}\} \cdot \{p_i, q_j\}\end{aligned} \tag{9}$$

Note that κ_n is implicitly depending on a choice of $p_1, \dots, p_i$, and $q_1, \dots, q_j$, we omit these from the notation to avoid clutter.

When $i, j > 2$, the element κ_n corresponds to the five-term skein relation depicted in Figure 1. If i equals 2, then $\{p_1, \cdots p_{i-1}\} = \{p_1\}$ is a one element set and therefore sent to 0 by h_n, removing the term containing $\{p_1\}$ corresponds to the four-term skein relation depicted in Figure 1. Similarly, if j equals 2 or i and j both equal two, removing the terms in κ_n which are individually sent to 0 corresponds to the four or three-term skein relation depicted in Figure 1.

We have the following calculation.

Proposition 11. *The element $\kappa_n \in R_n$ lies in the kernel of h_n.*

Proof. Applying h_n to κ_n gives

$$\begin{aligned} h_n(\kappa_n) = & \sum_{\substack{\{a,b\} \subseteq \{p_1, \dots, p_i\} \\ \{c,d\} \subseteq \{q_1, \dots, q_j\}}} \{a, b\} \cdot \{c, d\} \\ & - \sum_{\substack{\{a,b\} \subseteq \{p_1, \dots, p_{i-1}\} \\ \{c,d\} \subseteq \{q_1, \dots, q_j, p_i\}}} \{a, b\} \cdot \{c, d\} \\ & - \sum_{\substack{\{a,b\} \subseteq \{p_1, \dots, p_i, q_j\} \\ \{c,d\} \subseteq \{q_1, \dots, q_{j-1}\}}} \{a, b\} \cdot \{c, d\} \\ & + \sum_{\substack{\{a,b\} \subseteq \{p_1, \dots, p_{i-1}, q_j\} \\ \{c,d\} \subseteq \{q_1, \dots, q_{j-1}, p_i\}}} \{a, b\} \cdot \{c, d\} \\ & + \sum_{\substack{\{a,b\} \subseteq \{p_1, \dots, p_{i-1}, q_1, \cdots, q_{j-1}\} \\ \{c,d\} \subseteq \{p_i, q_j\}}} \{a, b\} \cdot \{c, d\} \end{aligned} \tag{10}$$

Note that the pairs of sets defining the first and second summations in the above expression differ only in the location of p_i, and similarly for the third and fourth. Since these summations come with opposite signs, the $\{a, b\}, \{c, d\}$ terms in the above expression will cancel unless one of a, b, c, d is equal to p_i. Similarly, comparing the first and third sums and the second and fourth sums we find cancellation unless at least one of a, b, c, d is equal to q_j. If the remaining two elements of a, b, c, d are both p's or both q's, then $\{a, b\} \cdot \{c, d\}$ also cancels. Therefore we have

$$h_n(\kappa_n) = \sum_{\substack{a \in \{p_1, \dots, p_{i-1}\} \\ b \in \{q_1, \dots, q_{j-1}\}}} \{a, p_i\} \cdot \{b, q_j\} + \{a, q_j\} \cdot \{b, p_i\} + \{a, b\} \cdot \{p_i, q_j\} \tag{11}$$

which is manifestly a sum of the definining relations of M_n. □

To show that the kernel of $h_n \circ \iota_\Pi$ is no larger than the kernel of p_π, we will show that the images of singleton-free noncrossing set partitions under $h_n \circ \iota_\Pi$ are linearly independent. To do so, we introduce a term order on M_n.

Definition 12. Define a total order on the generators of M_n as follows by

- If $a < b$, $c < d$, and $a < c$, then $\{a, b\} < \{c, d\}$
- If $a < b$, $c < d$, $a = c$ and $b > d$, then $\{a, b\} < \{c, d\}$

Let $\leqslant$ denote lexicographic order on monomials of M_n with respect to this order on the generators. Note that $b > d$ in the second condition is not a typo, earlier generators have small smallest element and large largest element, e.g. $\{1, n\}$ is the first in this total order.

With this monomial order we have the following.

Proposition 13. *The set* $\{h_n \circ \iota_\Pi(\pi) \mid \pi \in NCP(n)_0\}$ *is linearly independent.*

Proof. By Proposition 8, M_n has a basis consisting of monomials corresponding to noncrossing matchings. We claim that the leading term of $h_n \circ \iota_\Pi(\pi)$ when expanded in this basis is unique. By the definition of the term order, the leading term of $h_n \circ \iota_\Pi(\pi)$ is the noncrossing matching obtained by matching the smallest element of each block of π to the largest element of the same block. We can recover π be placing every unmatched element j in a block with the matched pair $\{i, k\}$ for which $i < j < k$ and $k - i$ is minimal, and the result follows. □

Corollary 14. *The kernel of* $h_n \circ \iota_\Pi$ *is spanned by the set of all elements of the form* $w \circ (s_i \circ \pi + \sigma(\pi))$ *(the skein relations) for any permutation* w*, adjacent transposition* s_i*, and singleton-free almost noncrossing set partition* π*.*

Proof. By Proposition 11, all such elements lie in the kernel. By Proposition 13 and a dimension count it is no larger. □

We can now prove our main result.

Theorem 15. *The linear map* $f_n : \mathbb{C}[NCP(n)_0] \to \mathbb{C}[NCM(n)]$ *defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is a $\mathfrak{S}_n$*-equivariant embedding of vector spaces. Here* $M_\pi(n)$ *is defined to be the set of all matchings* m *in* $M(n)$ *for which each block of* π *contains exactly one pair in* m*.*

Proof. By Corollary 14 and Proposition 4, the kernel of $h \circ \iota_\Pi$ is equal to the kernel of p_Π. So we have

$$\mathbb{C}[NCP(n)_0] \cong \mathbb{C}[\Pi_0(n)]/\ker(p_\Pi) \cong \operatorname{im}(h \circ \iota_\Pi) \subset M_n \cong \mathbb{C}[NCM(n)] \tag{12}$$

where the isomorphism on the left is induced by p_Π and the isomorphism on the right is induced by $h \circ \iota_\Pi$. Chasing these isomorphisms and inclusions results in the map f_n. □

Remark 16. Theorem 15 gives an alternate proof that the skein action is well defined. Instead of defining the skein action via the skein relations and checking that it satisfies the braid relations, we can instead define it as the pullback of the action on M_n through f_n. Corollary 14 shows that this pullback can then be interpreted via the skein relations.

4 The image

We have an embedding $f_n : \mathbb{C}[NCP(n)_0] \hookrightarrow \mathbb{C}[NCM(n)]$, so it is a natural question to ask for a description of the image of f_n within $\mathbb{C}[NCM(n)]$. Via the commutative diagram in Figure 2,we have an isomorphism of images

$$\mathrm{im}(h_n) \cong \mathrm{im}(f_n). \tag{13}$$

So it is equivalent to describe the image of h_n, and the multiplicative structure of M_n will make describing the image of h_n easier. This section will show that the image of h_n has a simple description as a principal ideal, the proof of which will require the following lemmas.

Lemma 17. *Let $A \subseteq [n]$. Then $h_n(A)^2 = 0$.*

Proof. Applying the definition of h_n gives

$$h_n(A)^2 = \sum_{\substack{a,b\in[n]\\ a\neq b}} \sum_{\substack{c,d\in[n]\\ c\neq d}} \{a,b\} \cdot \{c,d\} \tag{14}$$

Using the defining relation of M_n that

$$\{a,b\} \cdot \{a,c\} = 0$$

we have

$$\sum_{\substack{a,b\in[n]\\ a\neq b}} \sum_{\substack{c,d\in[n]\\ c\neq d}} \{a,b\} \cdot \{c,d\} = \frac{1}{3} \sum_{\substack{a,b,c,d\in[n]\\ a,b,c,d \,\text{distinct}}} \{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\}. \tag{15}$$

The right hand side of the above equation equals 0 because

$$\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\} = 0$$

for any distinct $a,b,c,d \in [n]$. □

Lemma 18. *Let A, B be disjoint subsets of $[n]$. Then*

$$h_n(A) \cdot \left(\sum_{\substack{a\in A\\ b\in B}} \{a,b\} \right) = 0.$$

Proof. Applying h_n gives

$$h_n(A) \cdot \left(\sum_{\substack{a\in A\\ b\in B}} \{a,b\} \right) = \frac{1}{3} \sum_{\substack{a_1,a_2,a_3\in A\\ b\in B}} \{a_1,a_2\} \cdot \{a_3,b\} + \{a_1,a_3\} \cdot \{a_2,b\} + \{a_2,a_3\} \cdot \{a_1,b\} = 0$$

□

Lemma 19. *Let $B_1, \ldots, B_k$ be the blocks of a singleton free set partition of $[n]$. Then*

$$h_n\left(\prod_{i=1}^{k} B_i\right) = h_n\left([n] \cdot \prod_{i=1}^{k-1} B_i\right)$$

Proof. We have the following calculation:

$$\begin{aligned}
h_n\left([n] \cdot \prod_{i=1}^{k-1} B_i\right) &= \left(\sum_{\substack{a,b\in[n]\\ a<b}} \{a,b\}\right) \cdot h_n\left(\prod_{i=1}^{k-1} B_i\right) \\
&= \left(\left(\sum_{i=1}^{k} h_n(B_i)\right) + \left(\sum_{1\leqslant i<j\leqslant k} \sum_{\substack{a\in B_i\\ b\in B_j}} \{a,b\}\right)\right) \cdot h_n\left(\cdot \prod_{i=1}^{k-1} B_i\right) \\
&= h_n(B_k) \cdot \left(\cdot \prod_{i=1}^{k-1} h_n(B_i)\right).
\end{aligned}$$

The last line follows by the preceeding two lemmas. Lemma 18 shows that every term in the outer sum of

$$\sum_{1\leqslant i<j\leqslant k} \sum_{\substack{a\in B_i\\ b\in B_j}} \{a,b\}$$

is annihilated by some term in the product

$$\prod_{i=1}^{k-1} h_n(B_k).$$

Similarly, Lemma 17 shows that every term except the $i = k$ term in the sum

$$\sum_{i=1}^{k} h_n(B_i)$$

is annihilated by some term in the product

$$\prod_{i=1}^{k-1} h_n(B_k).$$

□

We can now describe the image of h_n.

Theorem 20. *Let H_n be the ideal of M_n generated by $h_n([n])$. Then*

$$\mathrm{im}(h_n) = H_n.$$

Proof. It is immediate from Lemma 19 that the image of h_n is contained in H_n, so it suffices to show that H_n is no larger. We will do so by showing the dimension of H_n is no larger than the dimension of the image of h_n, i.e.

$$\dim(H_n) \leqslant \dim(\mathrm{im}(h_n)) = \dim(\mathrm{im}(f_n)) = \#NCP(n)_0 \tag{16}$$

We begin by finding a spanning set for H_n: note that for any fixed $a \in [n]$,

$$h_n([n]) \cdot \left(\sum_{\substack{b \in [n] \\ b \neq a}} \{a, b\} \right) = \frac{1}{3} \sum_{\substack{b \in [n] \\ b \neq a}} \sum_{\substack{c \in [n] \\ c \neq a}} \sum_{\substack{d \in [n] \\ d \neq a}} (\{a, b\} \cdot \{c, d\} + \{a, c\} \cdot \{b, d\} + \{a, d\} \cdot \{b, c\}) = 0$$

so

$$h_n([n]) \cdot \{1, a\} = -h_n([n]) \cdot \left(\sum_{\substack{b \in [n] \\ b \neq a, 1}} \{a, b\} \right).$$

Let $M_n^{(2)}$ denote the subspace of M_n spanned by noncrossing matchings of $\{2, \ldots, n\}$. By the above computation, H_n is spanned by elements of the form

$$h_n([n]) \cdot m$$

for $m \in M_n^{(2)}$.

The dimension of H_n is thus the rank of the map $M_n^{(2)} \to M_n$ given by multiplication by $h_n([n])$. To give an upper bound for the rank, we give a lower bound on the nullity.

Let $\tilde{\pi}$ be a set partition of $\{2, \ldots, n\}$. Consider the element $\tilde{f}_n(\tilde{\pi})$ of $M_n^{(2)}$ given by

$$\tilde{f}_n(\tilde{\pi}) := \prod_{B \in \tilde{\pi}} h_n(B)$$

for any singleton free noncrossing set partition $\tilde{\pi}$ of $\{2, \ldots, n\}$. The notation is meant to highlight that this is an analogous definition to the definition of f. We will show that $\tilde{f}_n(\tilde{\pi})$ is in the kernel of the multiplication by $h_n([n])$ map. Indeed, let B_1 be the block of $\tilde{\pi}$ containing 2, and let π be the set partition of $[n]$ obtained by adding 1 to block B_1. We have

$$\begin{aligned} h_n([n]) \cdot \tilde{f}_n(\tilde{\pi}) &= h_n([n]) \cdot \prod_{B \in \tilde{\pi}} h_n(B) \\ &= h_n(B_1) \cdot h_n \left([n] \cdot \prod_{\substack{B \in \pi \\ B \neq B_1 \cup \{1\}}} B \right) \end{aligned}$$

$$= h_n(B_1) \cdot h_n\left(\prod_{B\in\pi} B\right)$$

$$= h_n(B_1)h_n(B_1 \cup \{1\})h_n\left(\prod_{\substack{B\in\pi \\ B\neq B_1\cup\{1\}}} B\right)$$

$$= 0$$

The third equality follows from Lemma 19 and the final equality follows from the fact that

$$h_n(B_1)h_n(B_1 \cup \{1\}) = h_n(B_1)^2 + h_n(B_1)\left(\sum_{b\in B_1}\{1, b\}\right) = 0$$

which follows from Lemma 18 and Lemma 17. The collection of $\tilde{f}_n(\tilde{\pi})$ for singleton-free noncrossing set partitions π of $\{2, \ldots, n\}$ is linearly independent. To see this, note that any linear relation among the $\tilde{f}_n(\tilde{\pi})$ would also be a linear relation among $f_{n-1}(\pi)$ where π is the set partition of $[n-1]$ obtained by decrementing the indices in $\tilde{\pi}$. But f_{n-1} is an embedding and singleton-free noncrossing set partitions are linearly independent in $\mathbb{C}[NCP(n-1)_0]$. Thus, the dimension of the kernel of multiplication by $h_n([n])$ is at least the number of singleton-free noncrossing set partitions of $\{2, \ldots, n\}$.

The dimension of H_n is therefore bounded by

$$\dim(H_n) \leqslant \#\{\text{noncrossing matchings of } \{2, \ldots, n\}\} - \#\{\text{singleton-free noncrossing set partitions of } \{2, \ldots, n\}\}. \quad (17)$$

Noncrossing matchings of $\{2, \ldots, n\}$ are in bijection with noncrossing set partitions of $[n]$ in which only the block containing 1 may be a singleton (though it may be larger). Given a noncrossing set partition, take the matching that matches the largest and smallest element of each block not containing 1. Singleton-free noncrossing set partitions of $\{2, \ldots, n\}$ are in bijection with set partitions of $[n]$ in which $\{1\}$ is the unique singleton block. We therefore have

$$\#\{\text{singleton-free noncrossing set partitions of } [n]\} = \#\{\text{noncrossing matchings of } \{2, \ldots, n\}\} - \#\{\text{singleton-free noncrossing set partitions of } \{2, \ldots, n\}\} \quad (18)$$

and

$$\dim(H_n) \leqslant \#\{\text{singleton-free noncrossing set partitions of } [n]\}$$

as desired. □

5 Future directions

One of the goals motivating this paper is to find new combinatorially nice bases for $\mathfrak{S}_n$-irreducibles which arise from existing bases in an analogous way to the skein action. More specifically, suppose we have a basis for S^λ which is indexed by certain structures on the set $[k]$, where $k = |\lambda|$ (e.g. noncrossing perfect matchings, in the case of this paper). We can create a basis for the induction product of S^λ with a sign representation of $\mathfrak{S}_{n-k}$ indexed by all ways to put a certain structure on a k-element subset of $[n]$. The Pieri rule tells us which $\mathfrak{S}_n$ irreducibles this decomposes into. In particular, there will be one copy of $(\lambda, 1^{n-k})$. How do we isolate that irreducible?

It is perhaps optimistic to think that there will be a method that works in any sort of generality, but analogs may be found in some cases. For example, an analog might exist for the $SL(3)$-web basis for $S^{(k,k,k)}$ introduced by Kuperberg [4]. The web basis consists of planar bipartite graphs embedded in a disk with n boundary vertices all of degree 1, interior vertices are degree 3, all boundary vertices are in the same part of the bipartition, and no cycles of length less than 6 exist. One potential candidate for a basis for $S^{(k,k,k,1^{n-3k})}$ is as follows.

Conjecture 21. Let A be the set of all planar bipartite graphs embedded in a disk for which the following conditions hold

- There are n vertices on the boundary of the disk, and there exists a bipartition in which all of these vertices are in the same part.
- Every interior vertex in the same part of the bipartition as the boundary vertices is degree 3. These are called negative interior vertices.
- Every interior vertex not in the same part of the bipartition as the boundary vertices is degree at least 3. These are called positive interior vertices.
- The number of positive interior vertices minus the number of negative interior vertices is exactly k.
- No cycles of length less than 6 exist.

Then $|A|$ is equal to the dimension of $S^{(k,k,k,1^{n-3k})}$.

The set A can be thought of as consisting of webs for which the condition of interior vertices being degree 3 has been partially relaxed. The conjecture can be shown to hold for $k = 2$ and any n, as well as $n = 10, k = 3$ via direct enumeration. If the above conjecture is true, it suggests the following question.

Question 22. Does there exist a combinatorially nice action of $\mathfrak{S}_n$ on $\mathbb{C}[A]$ which creates a $\mathfrak{S}_n$ module isomorphic to $S^{(k,k,k,1^{n-3k})}$? If so, what does the unique embedding into $S^{(k,k,k)}$ induced with a sign representation of $\mathfrak{S}_{n-3k}$ look like?

A positive answer to this question might help elucidate how to apply similar methods more generally.

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