On Some Extremal and Probabilistic Questions for Tree Posets

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Abstract

Given two posets P, Q we say that Q is P-free if Q does not contain a copy of P. The size of the largest P-free family in $2^{[n]}$, denoted by $La(n,P)$, has been extensively studied since the 1980s. We consider several related problems. For posets P whose Hasse diagrams are trees and have radius at most 2, we prove that there are $2^{(1+o(1))La(n,P)}$ P-free families in $2^{[n]}$, thereby confirming a conjecture of Gerbner, Nagy, Patkós and Vizer [Electronic Journal of Combinatorics, 2021] in this case. For such P we also resolve the random version of the P-free problem, thus generalising the random version of Sperner's theorem due to Balogh, Mycroft and Treglown [Journal of Combinatorial Theory Series A, 2014], and Collares Neto and Morris [Random Structures and Algorithms, 2016]. Additionally, we make a general conjecture that, roughly speaking, asserts that subfamilies of $2^{[n]}$ of size sufficiently above $La(n,P)$ robustly contain P, for any poset P whose Hasse diagram is a tree.

Mathematics Subject Classifications: 05D05, 06A06

1 Introduction

In this paper we will consider several related extremal and probabilistic questions for posets. All posets we consider will be viewed as finite collections of finite subsets of $\mathbb{N}$ equipped with the containment relation. Indeed, we will usually be working inside the power set $2^{[n]}$ for some $n \in \mathbb{N}$.

Let P, Q be posets. A *poset homomorphism* from P to Q is a function $\phi: P \to Q$ such that for every $A, B \in P$, if $A \subseteq B$ then $\phi(A) \subseteq \phi(B)$. We say that P is a *subposet* of Q if there is an injective poset homomorphism from P to Q; otherwise, Q is said to be *P-free*. The size of the largest P-free family in $2^{[n]}$ is denoted by $La(n,P)$. Further we say P is an *induced subposet* of Q if there is an injective poset homomorphism ϕ from P to Q such that for every $A, B \in P$, $\phi(A) \subseteq \phi(B)$ if and only if $A \subseteq B$; otherwise, Q is said to be *induced P-free*.

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The systematic study of $La(n, P)$ was initiated by Katona and Tarján in 1983 [14], though Sperner's classical theorem [20] was the first result in the area. The latter asserts that the largest C_2-free family[1] in $2^{[n]}$ has size $\binom{n}{\lfloor n/2 \rfloor}$. Whilst the asymptotic value of $La(n, P)$ has been determined for a range of posets P, the general problem remains open; see [10] or Chapter 7 of [8] for a survey on the topic and [3, 12] for a conjecture on the asymptotic value of $La(n, P)$ for all posets P.

The *Hasse diagram* of a poset P is the directed graph with vertex set P and for $A, B \in P$, there is a directed edge from A to B in the Hasse diagram if $A \subset B$ and there does not exist a $C \in P$ such that $A \subset C \subset B$. The *undirected Hasse diagram* is the (undirected) graph obtained from the Hasse diagram by removing all orientations on the edges. A *tree poset* is a poset whose undirected Hasse diagram is a tree. A tree poset P is *upward (resp. downward) monotone* if there exists an element $B \in P$ with $B \subseteq A$ (resp. $A \subseteq B$) for any $A \in P$. A tree poset is called *monotone* if it is either upward or downward monotone.

The following result of Bukh [3] determines $La(n, P)$ asymptotically for all tree posets P. Given a poset P, we write $h(P)$ for the *height of* P, i.e., the length t of the longest chain C_t in P.

Theorem 1 (Bukh [3])**.** *If P is a tree poset then*

$$La(n, P) = (h(P) - 1)\binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)).$$

A phenomenon often exhibited in combinatorial problems is the property that once one is sufficiently above a 'threshold' for guaranteeing the existence of a structure, one can actually 'robustly' find this structure. This behaviour is often articulated in terms of *supersaturation*. For example, in recent years there has been significant attention on an old conjecture of Kleitman on the minimum number of chains C_t of length t in a subfamily of $2^{[n]}$ of a given size. Indeed, after earlier progress [2, 5, 6], this conjecture was proven by Samotij [19].

The first question we consider takes a different viewpoint on robustness. For this we need the following definition.

Definition 2. Let P be a tree poset and $x \in P$, and let $d \geqslant 2$ be a positive integer. The *d-blow-up $P(x, d)$ rooted at x* is the tree poset whose Hasse diagram is defined as follows: for each $u \in P$, if u is at distance ρ from x in the undirected Hasse diagram of P, then u is replaced with d^ρ elements $u^1, u^2, \dots, u^{d^\rho}$; furthermore, if uv is an edge of the Hasse diagram of P with v being at distance $\rho - 1$ to x in P, then the u^is are partitioned into $d^{\rho-1}$ pairwise disjoint sets $U^1, U^2, \dots, U^{d^{\rho-1}}$ each of size d, and for every $j \in [d^{\rho-1}]$, v^j is joined to all members of U^j. The orientation of all such edges is the same as that of uv.

The following conjecture states that once sufficiently above the threshold in Theorem 1, one can robustly guarantee a copy of a given tree poset P.

[1]We write C_t to denote the chain on t elements. Thus, a C_2-free family is simply an antichain.

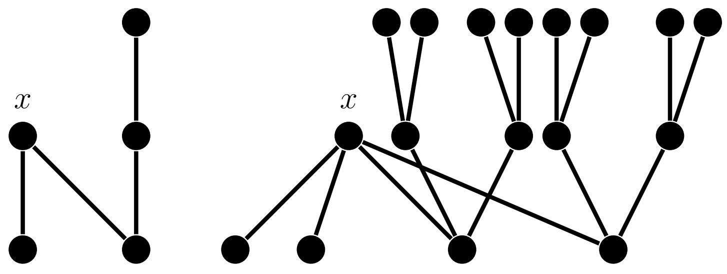

Figure 1: A poset rooted at x and its 2-blow-up.

Conjecture 3. Let P be any tree poset of height h. Given any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds for any sufficiently large $n \in \mathbb{N}$ and any $x \in P$. Consider the δn-blow-up $P(x, \delta n)$ of P rooted at x. If $\mathcal{F} \subseteq 2^{[n]}$ has size at least $(h-1+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$, then $\mathcal{F}$ contains a copy of $P(x, \delta n)$.

Remark 4. First observe that, by considering the complement family $\overline{\mathcal{F}} := \{[n] \setminus F : F \in \mathcal{F}\}$, it is clear that if the conjecture holds for some poset P, then it also holds for the dual poset P^d obtained from P by reversing all relations.

Note that at the cost of decreasing δ by a constant factor, if Conjecture 3 holds for a 'simple' tree poset P, then it holds for some more complicated tree posets P'. More precisely, suppose u, v are leaves of the Hasse diagram of P', w is an element of P' lying on the paths from x to both u and v, and for the unique paths $w = v_0, v_1 v_2, \ldots, v_k = v$ from w to v and $w = u_0, u_1, u_2, \ldots, u_\ell = u$ from w to u with $k \leqslant \ell$ we have $v_{i-1} \subseteq v_i$ if and only if $u_{i-1} \subseteq u_i$ for all $i \in [k]$. Finally, suppose that v_i has degree 2 in the Hasse diagram of P for all $i \in [k-1]$. Then if we obtain P from P' by removing $v_1, v_2, \ldots, v_k$, then $P'(x, \delta n/2) \subset P(x, \delta n)$ for sufficiently large n; so it is enough to verify Conjecture 3 for P.

Similarly, suppose P_1, P_2 are connected components of $P' \setminus \{x\}$ such that $y_1 \in P_1, y_2 \in P_2$ are the only elements of $P_1 \cup P_2$ adjacent to x in the Hasse diagram of P', and also $x \subseteq y_1$ if and only if $x \subseteq y_2$. Suppose further P_2 contains a copy of P_1 with y_2 playing the role of y_1. If we obtain P from P' by removing P_1, then again $P'(x, \delta n/2) \subset P(x, \delta n)$ for sufficiently large n; so it is enough to verify Conjecture 3 for P.

Perhaps the most interesting case of Conjecture 3 is when P is a monotone tree. For example, it implies that if one is sufficiently above the threshold for guaranteeing a binary monotone tree of height h (or even a chain of length h), then actually one can find a δn-ary monotone tree of height h.

As well as being interesting in its own right, Conjecture 3 has implications to a couple of other well-studied questions. The first of these concerns counting P-free families in $2^{[n]}$. Kleitman [15] proved that the number of C_2-free families in $2^{[n]}$ is $2^{(1+o(1))\binom{n}{\lfloor n/2 \rfloor}}$; this result was further refined by Korshunov [16]. More generally, a trivial lower bound on the number of P-free families in $2^{[n]}$ is $2^{La(n,P)}$. The following conjecture from [7] states that this lower bound is close to being tight.

Conjecture 5 (Gerbner, Nagy, Patkós and Vizer [7])**.** The number of P-free families in $2^{[n]}$ is

$$2^{(1+o(1))La(n,P)}.$$

Our first result asserts that if Conjecture 3 is true for a given tree poset P, and for a specific choice of some $x \in P$, then Conjecture 5 is true for this P.

Theorem 6. *Suppose P is a tree poset of height h for which Conjecture 3 holds for some choice of $x \in P$. Then the number of P-free families in $2^{[n]}$ is*

$$2^{(h-1+o(1))\binom{n}{\lfloor n/2 \rfloor}}.$$

In Section 3.1 we prove Theorem 6 via a variation on the standard *graph container algorithm.* The following result gives a class of tree posets P for which Conjecture 3 holds for some choices of $x \in P$.

Theorem 7. *Conjecture 3 holds for all tree posets P and any $x \in P$ for which all elements of P are within distance at most 2 of x in the (undirected) Hasse diagram of P.*

In Section 4 we discuss why it seems challenging to extend Theorem 7 to arbitrary tree posets. The next corollary follows immediately from Theorems 6 and 7. Here we say a tree poset P has *radius at most t* if there is some $x \in P$ so that every vertex in the (undirected) Hasse diagram is at distance at most t from x.

Corollary 8. *Given any tree poset P of height $h \leqslant 5$ and radius at most 2, the number of P-free families in $2^{[n]}$ is*

$$2^{(h-1+o(1))\binom{n}{\lfloor n/2 \rfloor}}.$$

Notice that every tree poset P of radius at most 2 has height at most 5; we only include the $h \leqslant 5$ condition in Corollary 8 (and in the statement of other results below) to make this restriction on the height explicit. Note that Corollary 8 resolves Conjecture 5 for various natural posets P including all monotone trees of height at most 3 and the poset N (see Figure 2).

We also prove the following special version of Theorem 7 that ensures a larger blow-up of P.

Lemma 9. *Let P be any tree poset of height $h \leqslant 5$ and radius at most 2. Let $x \in P$ such that every element of P is of distance at most two in the (undirected) Hasse diagram of P. Given any $\varepsilon > 0$, the following holds for any sufficiently large $n \in \mathbb{N}$. Consider the $n^{1.9}$-blow-up $P(x, n^{1.9})$ of P rooted at x. If $\mathcal{F} \subseteq 2^{[n]}$ is such that $|\mathcal{F}| \geqslant 4(h-1+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$, then $\mathcal{F}$ contains a copy of $P(x, n^{1.9})$.*

This lemma is a crucial tool in the proof of Theorem 11. Note that we have not tried to optimise the lower bound on $|\mathcal{F}|$ in Lemma 9 as the result suffices for our purposes.

Our other application of Conjecture 3 is to the random version of the P-free problem. Let $\mathcal{P}(n,p)$ be obtained from $2^{[n]}$ by selecting each element of $2^{[n]}$ independently at random

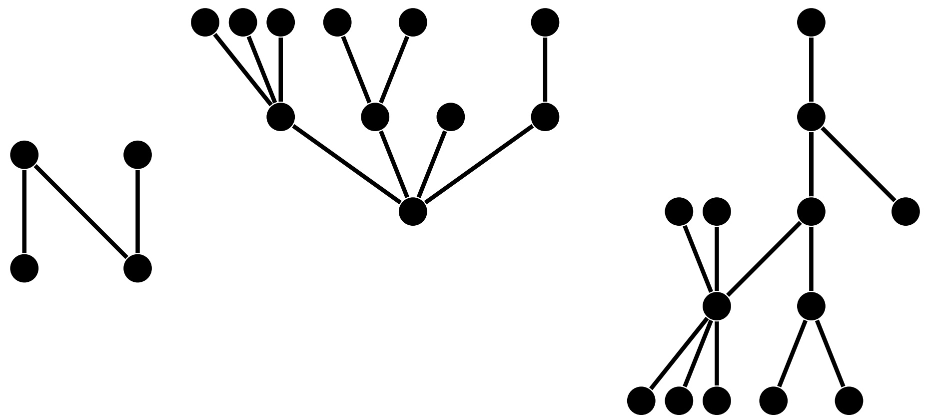

Figure 2: Some posets of radius 2: the N poset, a monotone tree, and a more complicated one.

with probability p. The model $\mathcal{P}(n,p)$ was first investigated by Rényi [18] who determined the probability threshold for the property that $\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. There has also been interest in the problem of determining when the largest P-free family of $\mathcal{P}(n,p)$ has size $p\cdot(1+o(1))La(n,P)$ with high probability (w.h.p.).[2] For example, the next result resolves this problem when $P = C_2$.

Theorem 10 (Balogh, Mycroft and Treglown [1] and Collares Neto and Morris [4])**.** *For any $\varepsilon > 0$ there exists a constant C such that if $p > C/n$ then w.h.p. the largest C_2-free family (i.e., antichain) in $\mathcal{P}(n,p)$ has size $(1 \pm \varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$.*

Osthus [17] had earlier proven Theorem 10 in the case when $pn/\log n \to \infty$. Moreover, Osthus showed that, for a fixed $c > 0$, if $p = c/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at least $(1+o(1))(1+e^{-c/2})p\binom{n}{\lfloor n/2 \rfloor}$. So the bound on p in Theorem 10 is best-possible up to the constant C.

The results in [1, 4] actually yield an analogous result for C_t-free families (for any fixed $t \in \mathbb{N}$). Hogenson [13] also adapted the proof of Theorem 10 from [1] to obtain an analogous result for the so-called t-fork. In [7, Conjecture 7] a general conjecture was made on the threshold for the P-free problem in $\mathcal{P}(n,p)$ and the corresponding problem for induced P was also considered. Our next result resolves the random version of the P-free problem for all tree posets of radius at most 2.

Theorem 11. *Let P be any tree poset of height $h \leqslant 5$ and radius at most 2. Given any $\varepsilon > 0$, there exists $C = C(\varepsilon, P) > 0$ such that the following holds. If $p > C/n$ then w.h.p. the largest P-free family in $\mathcal{P}(n,p)$ has size $(h - 1 \pm \varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$.*

Notice that Theorem 11 implies the random version of Sperner's theorem (Theorem 10). Furthermore, the lower bound on p in Theorem 11 is essentially tight. Indeed,

[2]Here, by *with high probability* we mean with probability tending to 1 as n tends to infinity.

for such P, [7, Corollary 6] implies that if $p = o(1/n)$ then w.h.p. the largest P-free family in $\mathcal{P}(n,p)$ has size at least $(h - o(1))p\binom{n}{\lfloor n/2 \rfloor}$. The proof of Theorem 11 applies Theorem 7, Lemma 9 and the graph container method. As noted by one of the referees of the paper, it may be possible to prove Theorem 11 via the *hypergraph* container method. In fact, the analogous result for C_t-free families was proven using the hypergraph container method in [4]. Our proofs of Theorems 6 and 11 show that variants of the graph container algorithm have the potential to attack problems that cannot be naturally stated in the language of independent sets in graphs. This philosophy is perhaps the most important message for the reader to take from our work.

The paper is organised as follows. In Section 2 we introduce some useful results. The proofs of Theorems 6, 7 and 11 and Lemma 9 are presented in Section 3. In Section 4 we provide some further discussion and make a few simple observations.

Notation. Throughout this paper we omit floors and ceilings whenever this does not affect the argument. Given a set S and $t \in \mathbb{N}$ we write $\binom{S}{\leqslant t}$ to denote the set of all subsets of S of size at most t, and let $\binom{|S|}{\leqslant t} := \left|\binom{S}{\leqslant t}\right|$.

Given any poset P, when considering P as a Hasse diagram we will often refer to the elements of P as vertices. Consider a tree poset P with root $x \in P$. Suppose $u \in P$ is of distance ρ from x in the undirected Hasse diagram of P, and let $v \in P$ be a neighbour of u of distance $\rho + 1$ from x in the undirected Hasse diagram. Then we call u the *parent of* v and v a *child of* u. Note that every vertex other than x has precisely one parent, though every vertex can have multiple children.

2 Preliminaries

In this section, we introduce a few tools that will be used in the proofs of some of our main results.

Definition 12. The *Lubell-mass* $\lambda_n(\mathcal{F})$ of a family $\mathcal{F} \subseteq 2^{[n]}$ is $\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}$. This is the expected number of sets in $\mathcal{F}$ that are contained in a maximum chain $\mathcal{C}$ in $2^{[n]}$ chosen uniformly at random among all such chains.

Let $\mathbf{C}_n$ denote the set of all maximal chains in $2^{[n]}$. Given a family $\mathcal{F} \subseteq 2^{[n]}$, the *min-partition* of $\mathbf{C}_n$ is $\bigcup_{F\in\mathcal{F}} \mathbf{C}_F \cup \mathbf{C}^*_\emptyset$, where $\mathbf{C}_F$ contains those $\mathcal{C} \in \mathbf{C}_n$ where F is the smallest set in $\mathcal{F} \cap \mathcal{C}$, while $\mathbf{C}^*_\emptyset$ is the set of those $\mathcal{C} \in \mathbf{C}_n$ for which $\mathcal{C} \cap \mathcal{F} = \emptyset$. For a set $F \in \mathcal{F}$, we write $\mathcal{U}_\mathcal{F}(F) := \{G \setminus F : G \in \mathcal{F}, F \subseteq G\}$ and we drop the subscript $\mathcal{F}$ if it is clear from context.

As $\lambda_n(\mathcal{F})$ is the expected number of sets of $\mathcal{F}$ in a randomly selected $\mathcal{C} \in \mathbf{C}_n$, and $\lambda_{n-|F|}(\mathcal{U}(F))$ is the expected number of sets of $\mathcal{F}$ in a randomly selected chain $\mathcal{C} \in \mathbf{C}_F$, we obtain the following proposition.

Proposition 13. *Given $K > 0$, if $\lambda_{n-|F|}(\mathcal{U}(F)) \leqslant K$ for all $F \in \mathcal{F}$, then $\lambda_n(\mathcal{F}) \leqslant K$.*

Proposition 14 (Griggs, Li and Lu, Lemma 3.2 in [11])**.** *Given $K > 0$, if $\lambda_n(\mathcal{F}) \leqslant K$, then $|\mathcal{F}| \leqslant K\binom{n}{\lfloor n/2 \rfloor}$.*

The proof of the next lemma is essentially identical to that of Theorem 3.1 in [9]. Let $\nabla^j_{\mathcal{F}}(F) := |\{G \in \mathcal{F} : F \subseteq G, |G| = |F| + j\}|$.

Lemma 15. *Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be sufficiently large. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family that contains only sets of size in $[n/2 - n^{2/3}, n/2 + n^{2/3}]$. If for every $F \in \mathcal{F}$ we have (i) $\nabla^1_{\mathcal{F}}(F) \leqslant \varepsilon n$ and (ii) $\nabla^j_{\mathcal{F}}(F) \leqslant \varepsilon n^2$ for all $j \geqslant 2$, then $|\mathcal{F}| \leqslant (1 + 15\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. We consider the min-partition of $\mathbf{C}_n$ and bound $\lambda_{n-|F|}(\mathcal{U}(F))$. As all sets of $\mathcal{F}$ have size in $[n/2 - n^{2/3}, n/2 + n^{2/3}]$, we obtain that for all $F \in \mathcal{F}$,

$$\lambda_{n-|F|}(\mathcal{U}(F)) \leqslant 1 + \frac{\nabla^1_{\mathcal{F}}(F)}{n - |F|} + \frac{\nabla^2_{\mathcal{F}}(F)}{\binom{n-|F|}{2}} + \sum_{j=3}^{2n^{2/3}} \frac{\nabla^j_{\mathcal{F}}(F)}{\binom{n-|F|}{j}} \leqslant 1 + 14\varepsilon + O\left(\frac{\varepsilon}{n}\right),$$

where here we used properties (i) and (ii). Propositions 13 and 14 complete the proof. □

Lemma 16. *Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be sufficiently large. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family that contains only sets of size in $[n/2 - n^{2/3}, n/2 + n^{2/3}]$. If for every $F \in \mathcal{F}$ we have $\nabla^j_{\mathcal{F}}(F) \leqslant \varepsilon n^4$ for all $j \geqslant 4$, then $|\mathcal{F}| \leqslant (4 + 400\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. We again consider the min-partition of $\mathbf{C}_n$ and bound $\lambda_{n-|F|}(\mathcal{U}(F))$. For all $F \in \mathcal{F}$,

$$\lambda_{n-|F|}(\mathcal{U}(F)) \leqslant 4 + \frac{\nabla^4_{\mathcal{F}}(F)}{\binom{n-|F|}{4}} + \sum_{j=5}^{2n^{2/3}} \frac{\nabla^j_{\mathcal{F}}(F)}{\binom{n-|F|}{j}} \leqslant 4 + 399\varepsilon + O\left(\frac{\varepsilon}{n}\right).$$

Propositions 13 and 14 complete the proof. □

3 The proofs of our main results

3.1 Proof of Theorem 6

To prove Theorem 6 we apply the following graph container result.

Lemma 17. *Suppose P is a tree poset of height h for which Conjecture 3 holds for some choice of $x \in P$. Given any $\varepsilon > 0$, let $\delta > 0$ be as in Conjecture 3 and suppose $n \in \mathbb{N}$ is sufficiently large. Then there exists a function $f : \binom{2^{[n]}}{\leqslant |P| 2^n/\delta n} \to \binom{2^{[n]}}{\leqslant (h-1+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}}$ such that for any P-free family $\mathcal{F}$ in $2^{[n]}$ there is a subfamily $\mathcal{H} \subseteq \mathcal{F}$ so that $\mathcal{H} \in \binom{2^{[n]}}{\leqslant |P| 2^n/\delta n}$ and $\mathcal{F} \subseteq \mathcal{H} \cup f(\mathcal{H})$.*

Given any $\mathcal{H} \in \binom{2^{[n]}}{\leqslant |P| 2^n/\delta n}$ we refer to the family $\mathcal{H} \cup f(\mathcal{H})$ produced by Lemma 17 as a *container*. We call $\mathcal{H}$ the *fingerprint* of the container $\mathcal{H} \cup f(\mathcal{H})$. With Lemma 17 at hand, Theorem 6 now follows easily.

Proof of Theorem 6. Suppose P is a tree poset of height h for which Conjecture 3 holds for some choice of $x \in P$. By Theorem 1, there exists a P-free family $\mathcal{F}$ in $2^{[n]}$ of size $(h-1+o(1))\binom{n}{\lfloor n/2 \rfloor}$. By considering all subfamilies of $\mathcal{F}$ we see that there are at least $2^{(h-1+o(1))\binom{n}{\lfloor n/2 \rfloor}}$ P-free families in $2^{[n]}$. To prove the theorem, note that it therefore suffices to prove the following: given any $\varepsilon > 0$, if $n \in \mathbb{N}$ is sufficiently large then there are at most $2^{(h-1+3\varepsilon)\binom{n}{\lfloor n/2 \rfloor}}$ P-free families in $2^{[n]}$.

Given any $\varepsilon > 0$ let $\delta > 0$ be as in Conjecture 3 and suppose $n \in \mathbb{N}$ is sufficiently large. Let $\mathcal{C}$ be the collection of all the containers $\mathcal{H} \cup f(\mathcal{H})$ (for all $\mathcal{H} \in \binom{2^{[n]}}{\leqslant |P|2^n/\delta n}$) obtained when applying Lemma 17. Thus,

$$|\mathcal{C}| = \binom{2^n}{\leqslant |P|2^n/\delta n} \leqslant 2^{\frac{2|P|2^n}{\delta n} \log n} \leqslant 2^{\varepsilon \binom{n}{\lfloor n/2 \rfloor}},$$

where the last inequality follows as n is sufficiently large. Moreover, by Lemma 17, $|C| \leqslant (h-1+2\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$ for every $C \in \mathcal{C}$. Since every P-free family in $2^{[n]}$ is a subset of some container $C \in \mathcal{C}$, we immediately obtain that there are at most

$$|\mathcal{C}| \cdot 2^{(h-1+2\varepsilon)\binom{n}{\lfloor n/2 \rfloor}} \leqslant 2^{(h-1+3\varepsilon)\binom{n}{\lfloor n/2 \rfloor}}$$

P-free families in $2^{[n]}$, as desired. □

It remains to prove Lemma 17. The proof uses a modified version of the graph container algorithm. In particular, there are some specific subtleties we need to take care of.

In the proof of Lemma 17 we will be constructing our containers through a process of analysing copies of t-blow-ups $P(x,t)$ in $2^{[n]}$. For this, it will be helpful to consider the following process of 'constructing' a copy of P in $P(x,t)$ with root x. Suppose the children of x in P are $u_1, \dots, u_s$. Then for each of these children u_i there is a set U_i of size t in $P(x,t)$ where each $x_i \in U_i$ can 'play the role' of u_i in a copy of P rooted at x. That is, there is an edge between x_i and x in the Hasse diagram of $P(x,t)$ and it is oriented in the 'same' direction as the edge between u_i and x in the Hasse diagram of P.

Fix some choice of $x_i \in U_i$ for each $i \in [s]$; so x_i will play the role of u_i in the copy of P we are constructing. Fix any $i \in [s]$, and let $u_{1,i}, \dots, u_{s_i,i}$ denote the children of u_i in P. Again, for each of these children $u_{j,i}$ there is a set $U_{j,i}$ of size t in $P(x,t)$ where each $x_{j,i} \in U_{j,i}$ can play the role of $u_{j,i}$ in a copy of P rooted at x that contains $x_1, \dots, x_s$. In particular, every vertex $x_{j,i} \in U_{j,i}$ is adjacent to x_i in the Hasse diagram of $P(x,t)$ and the edge between $x_{j,i}$ and x_i is oriented in the 'same' direction as the edge between $u_{j,i}$ and u_i in the Hasse diagram of P. For each $i \in [s]$ and $j \in [s_i]$, fix some $x_{j,i} \in U_{j,i}$ to play the role of $u_{j,i}$ in the copy of P we are constructing. We repeat this process to construct a copy of P rooted at x in $P(x,t)$: at every step we consider the children u of every vertex selected in the previous step; we have t vertices in $P(x,t)$ that are able to play the role of u in the copy of P we are constructing, so we can continue this process to obtain our desired copy of P.

Suppose now $\mathcal{F}$ is a P-free family in $2^{[n]}$ and consider a copy of $P(x,t)$ in $2^{[n]}$ for some $t \in \mathbb{N}$ and $x \in P$. Run the process described in the previous two paragraphs, however, now at every step insist that any vertex in $P(x,t)$ selected must be in $\mathcal{F}$. Since $\mathcal{F}$ is a P-free family this process cannot finish and produce a copy of P. That is, either the root of our copy of $P(x,t)$ does not belong to $\mathcal{F}$, or at some step of the process we must encounter a set U of t vertices that corresponds to some $u \in P$ and where $U \cap \mathcal{F} = \emptyset$. As we will now see, this simple observation is crucial to the success of running our version of the graph container algorithm.

Proof of Lemma 17. Suppose P is a tree poset of height h for which Conjecture 3 holds for some choice of $x \in P$. Set $p := |P|$. Given any $\varepsilon > 0$, let $\delta > 0$ be as in Conjecture 3 and suppose $n \in \mathbb{N}$ is sufficiently large.

Fix a total order $\mathcal{O}_{2^{[n]}}$ of the elements of $2^{[n]}$. Let $\mathcal{O}_P$ be a total order of the vertices of P such that the first vertex is x; the next vertices are the children of x; the next vertices are those vertices of distance two from x in the undirected Hasse diagram of P, and so forth. We write $y_1, \dots, y_p$ for the vertices of P ordered as in $\mathcal{O}_P$; so $y_1 = x$. Further, let $\mathcal{P}_{\text{blow}}$ denote the set of all copies of $P(x,t)$ in $2^{[n]}$ for all $t \in \mathbb{N}$. Let $\mathcal{O}_{\text{blow}}$ be a total order of the elements of $\mathcal{P}_{\text{blow}}$.

We now run our modified version of the graph container algorithm. The input of the algorithm is a P-free family $\mathcal{F} \subseteq 2^{[n]}$. The algorithm will output a fingerprint $\mathcal{H}$ and a container $\mathcal{H} \cup f(\mathcal{H})$ where $\mathcal{H} \subseteq \mathcal{F} \subseteq \mathcal{H} \cup f(\mathcal{H})$.

Initially we set $\mathcal{G}^0 := 2^{[n]}$ and $\mathcal{H}^0 := \emptyset$. We will add vertices from $2^{[n]}$ to $\mathcal{H}^0$ and remove vertices from $\mathcal{G}^0$ through the following iterative process, beginning at Step 1.

At Step i, let $P_1(x,t) \in \mathcal{P}_{\text{blow}}$ be a copy of $P(x,t)$ in $\mathcal{G}^{i-1} \subseteq 2^{[n]}$ where we choose t to be as large as possible. If there is more than one copy of $P(x,t)$ in $\mathcal{G}^{i-1}$ for this choice of t we choose $P_1(x,t)$ to be the copy of $P(x,t)$ appearing earliest in the total order $\mathcal{O}_{\text{blow}}$. Let x_1 be the vertex of $P_1(x,t)$ that plays the role of x.

- Suppose $x_1 \notin \mathcal{F}$. Then define $\mathcal{G}^i := \mathcal{G}^{i-1} \setminus \{x_1\}$ and $\mathcal{H}^i := \mathcal{H}^{i-1}$. Proceed to Step $i+1$.

- Suppose $x_1 \in \mathcal{F}$ and $t \geqslant \delta n$. We will update $\mathcal{G}^{i-1}$ and $\mathcal{H}^{i-1}$ in several *subphases* to obtain $\mathcal{G}^i$ and $\mathcal{H}^i$ respectively.

  - *Subphase* 1. Delete x_1 from $\mathcal{G}^{i-1}$ and add it to $\mathcal{H}^{i-1}$. Proceed to Subphase 2.
  - *Subphase* 2. Recall that we write $y_1, \dots, y_p$ for the vertices of P ordered as in $\mathcal{O}_P$. So $y_1 = x$ and y_2 is a child of x in P. In $P_1(x,t)$ there is a set of t vertices $U_2 \subseteq \mathcal{G}^{i-1}$ so that each $x_2 \in U_2$ is adjacent to x_1 in the Hasse diagram of $P_1(x,t)$ and further the edge between x_1 and x_2 is oriented in the same direction as the edge between $y_1 (= x)$ and y_2 in the Hasse diagram for P.

    Look through the vertices in $U_2 \subseteq 2^{[n]}$ one by one, following the total order $\mathcal{O}_{2^{[n]}}$. If we run through the entire set U_2 without finding an element from $\mathcal{F}$ then we delete all t vertices in U_2 from $\mathcal{G}^{i-1}$ and stop Step i.

Otherwise, let $x_2 \in U_2$ denote the very first element from $\mathcal{F}$ that we discover in U_2. Add x_2 to $\mathcal{H}^{i-1}$ and remove from $\mathcal{G}^{i-1}$ both x_2 and all elements of U_2 that occur before x_2 in the total order $\mathcal{O}_{2^{[n]}}$. So here we have deleted between 1 and t elements of $\mathcal{G}^{i-1}$, and of these only x_2 lies in $\mathcal{F}$. Proceed to Subphase 3.

- *Subphase j (for $3 \leqslant j \leqslant p$).* In the previous subphases we have defined $x_1, \dots, x_{j-1} \in P_1(x,t)$ corresponding to $y_1, \dots, y_{j-1} \in P$.

  By definition of $\mathcal{O}_P$, the unique parent y_k of y_j in P must be one of $y_1, \dots, y_{j-1}$. Furthermore, by definition of $P(x,t)$, there is a set of t vertices $U_j \subseteq \mathcal{G}^{i-1}$ in $P_1(x,t)$ corresponding to y_j so that every $x_j \in U_j$ is adjacent to x_k in the Hasse diagram of $P_1(x,t)$, and moreover, the edge between x_k and x_j is oriented in the same direction as the edge between y_k and y_j in the Hasse diagram for P.

  Look through the vertices in $U_j \subseteq 2^{[n]}$ one by one, following the total order $\mathcal{O}_{2^{[n]}}$. If we run through the entire set U_j without finding an element from $\mathcal{F}$ then we delete all t vertices in U_j from $\mathcal{G}^{i-1}$ and stop Step i.

  Otherwise, let $x_j \in U_j$ denote the very first element from $\mathcal{F}$ that we discover in U_j. Add x_j to $\mathcal{H}^{i-1}$ and remove from $\mathcal{G}^{i-1}$ both x_j and all elements of U_j that occur before x_j in the total order $\mathcal{O}_{2^{[n]}}$. So here we have deleted at most t elements of $\mathcal{G}^{i-1}$, of which only x_j lies in $\mathcal{F}$. Proceed to Subphase $j+1$.

At the end of these subphases relabel $\mathcal{G}^{i-1}$ as $\mathcal{G}^i$ and $\mathcal{H}^{i-1}$ as $\mathcal{H}^i$; then proceed to Step $i+1$. Since $\mathcal{F}$ is P-free, for some $j \in [p]$, Subphase j must consider a set U_j where $U_j \cap \mathcal{F} = \emptyset$. Thus, in Step i we have deleted at least t vertices from $\mathcal{G}^{i-1}$ to obtain $\mathcal{G}^i$ and have added at most $p-1$ vertices to $\mathcal{H}^{i-1}$ (at most one in each of the first $p-1$ subphases) to obtain $\mathcal{H}^i$.

- Suppose $x_1 \in \mathcal{F}$ and $t < \delta n$. Then define $\mathcal{H} := \mathcal{H}^{i-1} \cup \{x_1\}$ and $f(\mathcal{H}) := \mathcal{G}^{i-1} \setminus \{x_1\}$ and terminate the algorithm.

Note that in every step of the algorithm we only add elements to $\mathcal{H}^{i-1}$ that lie in $\mathcal{F}$. So certainly $\mathcal{H} \subseteq \mathcal{F}$. Similarly, by construction, $\mathcal{F} \subseteq \mathcal{H} \cup f(\mathcal{H})$. In every step of the algorithm, except the final step, if we add (at most $p-1$) elements to $\mathcal{H}^{i-1}$ we delete at least δn elements from $\mathcal{G}^{i-1} \subseteq 2^{[n]}$. Therefore,

$$|\mathcal{H}| \leqslant \frac{(p-1)2^n}{\delta n} + 1 \leqslant \frac{p \cdot 2^n}{\delta n}.$$

Furthermore, by construction of the algorithm, the family $f(\mathcal{H}) \subseteq 2^{[n]}$ does not contain a copy of the δn blow-up $P(x, \delta n)$. Thus, by the assumption of the lemma, we have that

$$|f(\mathcal{H})| \leqslant (h-1+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}.$$

Therefore all that remains to check is that the function f is well-defined. That is, suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are P-free families in $2^{[n]}$ such that, on input $\mathcal{F}_i$, our algorithm outputs the container $\mathcal{H}_i \cup f(\mathcal{H}_i)$ for each $i \in [2]$; if $\mathcal{H}_1 = \mathcal{H}_2$ then we require that $f(\mathcal{H}_1) = f(\mathcal{H}_2)$. This follows though immediately from the definition of the algorithm. Indeed, suppose one is only presented with the fingerprint that is outputted by the algorithm. Then one can completely identify every action taken during the algorithm. Thus, two applications of the algorithm yielding the same fingerprint must also yield the same container. $\square$

3.2 Proofs of Theorem 7 and Lemma 9

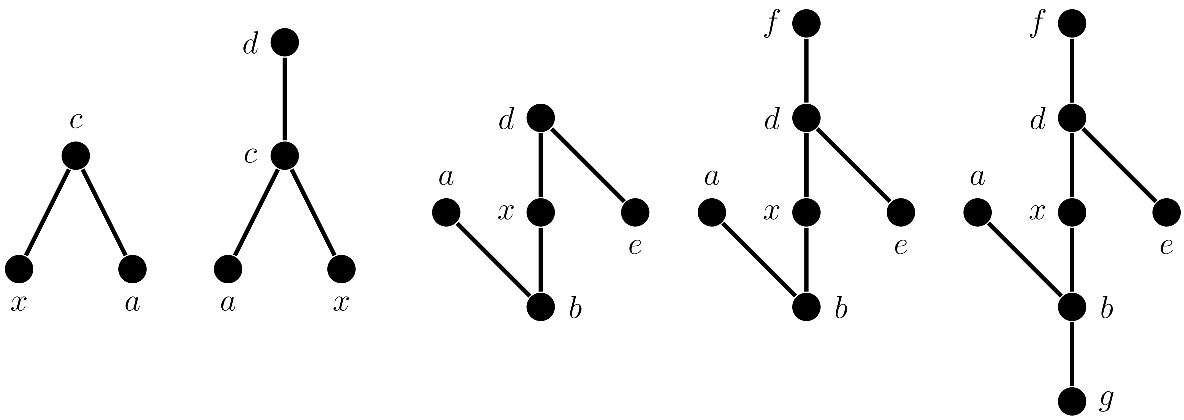

Figure 3: The Hasse diagrams of the $\bigwedge$, Y^d, S, S^+, and S^{++} posets.

For the proofs, we need to define five specific posets. See their Hasse diagrams in Figure 3. Let

- $\bigwedge$ be the three element poset on $\{a, x, c\}$ with $a, x < c$, and a, x unrelated;
- Y be the four element poset on $\{a, x, c, d\}$ with $a, x > c > d$ being all its cover relations;
- $\mathcal{S}$ be the five element poset on $\{a, b, x, d, e\}$ with $a > b < x < d > e$ being all its cover relations;
- $\mathcal{S}^+$ be the six element poset on $\{a, b, x, d, e, f\}$ with $a > b < x < d < f$ and $d > e$ being all its cover relations;
- $\mathcal{S}^{++}$ be the seven element poset on $\{a, b, x, d, e, f, g\}$ with $a > b$, $g < b < x < d < f$, and $d > e$ being all its cover relations.

Proof of Theorem 7. Let $\varepsilon > 0$; note that it suffices to prove the theorem under the assumption that ε is sufficiently small and $n \in \mathbb{N}$ is sufficiently large. Observe that if $x \in P$ is within distance at most 2 of all other elements of P, then a connected component of $P \setminus \{x\}$ is either an isolated vertex or contains a vertex u that is contained in all edges of the Hasse diagram of that component, and where u is adjacent to x in the Hasse diagram of P. This and Remark 4 imply that it is enough to prove the statement for the posets $\bigwedge, Y^d, \mathcal{S}, \mathcal{S}^+, \mathcal{S}^{++}$ and x as given in their definition and in Figure 3. For example, if P is of height 2 and $\gamma_1 > 0$, then there exists a $\gamma_2 > 0$ such that there is a choice of $y \in P$ so that $P(y, \gamma_2 n) \subseteq \bigwedge(x, \gamma_1 n)$ or $P(y, \gamma_2 n) \subseteq \bigwedge^d(x, \gamma_1 n)$ for all sufficiently large $n \in \mathbb{N}$.

Consider first the poset $\bigwedge$; its height is 2. It suffices to consider $\mathcal{F} \subseteq 2^{[n]}$ of size exactly $(1+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$. Let $\delta := \frac{\varepsilon^2}{(1+\varepsilon)120}$. As n is large enough, we can and will assume (here and in the cases of the four other posets as well) that all sets in $\mathcal{F}$ have size between $n/2 - n^{2/3}$ and $n/2 + n^{2/3}$. Indeed, there are $o(\binom{n}{\lfloor n/2 \rfloor})$ sets in $2^{[n]}$ that do not have size between $n/2 - n^{2/3}$ and $n/2 + n^{2/3}$, so we can ignore any such sets in $\mathcal{F}$. We say that $F \in \mathcal{F}$ is of

- *Type 1* if there exist at least $\varepsilon n/30$ sets $G \in \mathcal{F}$ with $|G| = |F| - 1$ and $G \subset F$;
- *Type 2* if, for some $j \geqslant 2$, there exist at least $\varepsilon n^2/30$ sets $G \in \mathcal{F}$ with $|G| = |F| - j$ and $G \subset F$;
- *Type 3* otherwise.

Let $\mathcal{F}^*$ denote the subfamily of $\mathcal{F}$ of sets of type 1 or 2. Applying Lemma 15 'upside down', we obtain that the subfamily $\mathcal{F} \backslash \mathcal{F}^*$ of sets of type 3 has size at most $(1+\varepsilon/2)\binom{n}{\lfloor n/2 \rfloor}$. Therefore, $|\mathcal{F}^*| \geqslant \varepsilon\binom{n}{\lfloor n/2 \rfloor}/2$. Consider the bipartite graph B with parts $\mathcal{F}$ and $\mathcal{F}^*$, where there is an edge between $G \in \mathcal{F}$ and $F \in \mathcal{F}^*$ if and only if $G \subset F$. By definition of the types, the number of edges in B is at least $\varepsilon n|\mathcal{F}^*|/30 \geqslant \varepsilon^2 n\binom{n}{\lfloor n/2 \rfloor}/60$; so there exists a set $F_x \in \mathcal{F}$ with degree at least $2\delta n$ in B. F_x will play the role of x in the copy of $P(x, \delta n)$ that we will construct. At least δn of F_x's neighbours in B are of the same type (i.e., all of type 1 or all of type 2); they will play the role of the many copies of c in $P(x, \delta n)$. Let $\mathcal{F}_c \subseteq \mathcal{F}^*$ denote a family of such sets so that $|\mathcal{F}_c| = \delta n$.

Suppose first that all sets in $\mathcal{F}_c$ are of type 1. Each $F \in \mathcal{F}_c$ has at least $\varepsilon n/30$ subsets in $\mathcal{F}$ of size $|F| - 1$; so there are at least $\varepsilon n/30 - |\mathcal{F}_c| - 1 \geqslant \varepsilon n/60$ such subsets of F that do not belong to $\mathcal{F}_c \cup \{F_x\}$. If $F, F' \in \mathcal{F}_c$ have different sizes, then the subsets of F and F' that we consider are clearly distinct. If F, F' have the same size, then they share at most one common subset of size $|F| - 1$. Therefore, for all $F \in \mathcal{F}_c$ we can pick $\varepsilon n/60 - |\mathcal{F}_c| \geqslant \delta n$ distinct subsets of F to obtain a copy of $\bigwedge(x, \delta n)$ in $\mathcal{F}$.

Suppose next that $\mathcal{F}_c$ consists only of sets of type 2. Then each of them contains at least $\varepsilon n^2/30 - |\mathcal{F}_c| - 1 \geqslant \delta n|\mathcal{F}_c|$ subsets that do not lie in $\mathcal{F}_c \cup \{F_x\}$. So we can pick greedily δn distinct subsets for each $F \in \mathcal{F}_c$ to obtain a copy of $\bigwedge(x, \delta n)$.

Consider next the poset Y^d. In this case it suffices to consider $\mathcal{F} \subseteq 2^{[n]}$ of size exactly $(2+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$. Let δ_1 be the output of the theorem on input $\varepsilon/2$, $\bigwedge$ and x; thus, $\delta_1 = \varepsilon^2/(480(1+\varepsilon/2))$. For Y^d we show that we can take $\delta := \delta_1/2$. Now we say that $F \in \mathcal{F}$ is of

- *Type 1* if there exist at least $\varepsilon n/30$ sets $G \in \mathcal{F}$ with $|G| = |F| + 1$ and $G \supset F$;
- *Type 2* if, for some $j \geqslant 2$, there exist at least $\varepsilon n^2/30$ sets $G \in \mathcal{F}$ with $|G| = |F| + j$ and $G \supset F$;

- *Type 3* otherwise.

Applying Lemma 15, we obtain the subfamily of $\mathcal{F}$ of sets of type 3 has size at most $(1+\varepsilon/2)\binom{n}{\lfloor n/2 \rfloor}$. Therefore, the subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ of sets of type 1 or 2 has size at least $(1+\varepsilon/2)\binom{n}{\lfloor n/2 \rfloor}$. By the previous case, we obtain a copy of $\bigwedge(x, \delta_1 n)$ in $\mathcal{F}^*$. Let F_x be the set corresponding to x, $\mathcal{F}_c$ be the family of sets corresponding to the copies of c, and $\mathcal{F}_a$ be the family of sets corresponding to copies of a. Suppose that there exist $F_c \in \mathcal{F}_c, F_a \in \mathcal{F}_a$ with $F_a \supset F_c$. Then we can exchange the roles of F_a and F_c and we still have a copy of $\bigwedge(x, \delta n)$. As after such a swap, the number of pairs $F_a \supset F_c$ decreases, after a finite number of changes, we can assume that no $F_a \in \mathcal{F}_a$ contains any $F_c \in \mathcal{F}_c$.

Now we are ready to use our copy of $\bigwedge(x, \delta_1 n)$ to obtain a copy of $Y^d(x, \delta_1 n/2)$ by adding $\delta_1 n/2$ distinct supersets of each $F_c \in \mathcal{F}_c$. We distinguish two cases according to the type of sets in $\mathcal{F}_c$; at the price of working with $\bigwedge(x, \delta_1 n/2)$ rather than $\bigwedge(x, \delta_1 n)$, we can assume that all sets in $\mathcal{F}_c$ are of the same type.

Suppose first that all sets in $\mathcal{F}_c$ are of type 1. Then, by our assumption that no $F_a \in \mathcal{F}_a$ contains any $F_c \in \mathcal{F}_c$, any $F_c \in \mathcal{F}_c$ has at least $(\varepsilon/30 - \delta_1/2)n$ supersets of size $|F_c|+1$ in $\mathcal{F}$ that are not used in our copy of $\bigwedge(x, \delta_1 n/2)$. If F_c and F'_c have different sizes, then these supersets are distinct because they are also of different size. If F_c, F'_c are of the same size, then they can share at most one common superset of size $|F_c|+1$; thus, as δ_1 is much smaller than ε, we can pick $\delta_1 n/2$ distinct supersets of each $F_c \in \mathcal{F}_c$ to obtain $Y^d(x, \delta_1 n/2)$. Suppose next that all $F_c \in \mathcal{F}_c$ are of type 2. Then as $\varepsilon n^2/30 > \delta_1 n|\mathcal{F}_c|$, we can pick the supersets of each F_c greedily to obtain a copy of $Y^d(x, \delta_1 n/2)$.

Finally, we can consider the posets S, S^+, and S^{++}. As the proofs are almost identical, we only show the statement for S^+. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of size exactly $(3+\varepsilon)\binom{n}{\lfloor n/2 \rfloor}$. Let δ_1 be the value of δ that we showed to exist for $\varepsilon/3$, $\bigwedge$ and x, and let δ_2 be the value of δ that we showed to exist for $\varepsilon/3$, Y^d and x. Set $\delta^* := \min\{\delta_1, \delta_2\}$. Let $\mathcal{F}_1$ be the subfamily of $\mathcal{F}$ consisting of those sets $F \in \mathcal{F}$ for which there exists no copy of $\bigvee(x, \delta^* n) \subset \mathcal{F}$ in which F plays the role of x. (The poset $\bigvee$ is the dual of $\bigwedge$.) Let $\mathcal{F}_2$ be the subfamily of $\mathcal{F}$ consisting of those sets $F \in \mathcal{F}$ for which there exists no copy of $Y^d(x, \delta^* n) \subset \mathcal{F}$ in which F plays the role of x. By definition of δ_1, δ_2 and δ^*, we have $|\mathcal{F}_1| \leqslant (1+\varepsilon/3)\binom{n}{\lfloor n/2 \rfloor}$ and $|\mathcal{F}_2| \leqslant (2+\varepsilon/3)\binom{n}{\lfloor n/2 \rfloor}$; so there exists a set $F_x \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. By definition of $\mathcal{F}_1$ and $\mathcal{F}_2$, F_x plays the role of x both in a copy $\mathcal{G}_1$ of $\bigvee(x, \delta^* n)$ and in a copy $\mathcal{G}_2$ of $Y^d(x, \delta^* n)$. In $\mathcal{G}_1 \cup \mathcal{G}_2$, the containment relations are as required as in a copy of $S^+(x, \delta^* n)$ with F_x playing the role of x; the only problem that can occur is that $\mathcal{G}_1$ and $\mathcal{G}_2$ might overlap. We overcome this problem by decreasing the value of δ^* and filtering out sets from $\mathcal{G}_1 \cup \mathcal{G}_2$.

For $y \in \{d, e, f\}$ let $\mathcal{G}_y \subset \mathcal{G}_2$ denote the family of sets corresponding to y in $\mathcal{G}_2$ and similarly for $y \in \{a, b\}$ let $\mathcal{G}_y \subset \mathcal{G}_1$ denote the family of sets corresponding to y in $\mathcal{G}_1$. Therefore, $\mathcal{G}_d, \mathcal{G}_e, \mathcal{G}_f$ are pairwise disjoint and so are $\mathcal{G}_a, \mathcal{G}_b$. Also, $\mathcal{G}_b \cap (\mathcal{G}_d \cup \mathcal{G}_f) = \emptyset$ as all sets in $\mathcal{G}_b$ are proper subsets of F_x, while all sets in $\mathcal{G}_d \cup \mathcal{G}_f$ are proper supersets of F_x. For a set $G_b \in \mathcal{G}_b$, we write $\mathcal{E}(G_b) \subset \mathcal{G}_a$ for the family of sets that correspond to the component of $\mathcal{G}_1 \setminus \{F_x\}$ containing G_b, and for a set $G_d \in \mathcal{G}_d$ we write $\mathcal{E}(G_d) \subset \mathcal{G}_e \cup \mathcal{G}_f$ for the family of sets that correspond to the component of $\mathcal{G}_2 \setminus \{F_x\}$ containing G_d.

- Consider a subfamily $\mathcal{G}_b' \subset \mathcal{G}_b$ with $|\mathcal{G}_b'| = \frac{1}{2}|\mathcal{G}_b|$. Remove from $\mathcal{G}_1$ (and thus $\mathcal{G}_b$ and $\mathcal{G}_a$ respectively) all elements of $\mathcal{G}_b'$ as well as all elements from $\{\mathcal{E}(G_b') : G_b' \in \mathcal{G}_b'\}$. As we removed $\delta^* n/2$ sets from $\mathcal{G}_b$, for every set $G_d \in \mathcal{G}_d$ we have $|\mathcal{E}(G_d) \cap \mathcal{G}_b| \leqslant \delta^* n/2$. We may therefore throw away exactly half of $\mathcal{E}(G_d) \cap \mathcal{G}_e$ and exactly half of $\mathcal{E}(G_d) \cap \mathcal{G}_f$ for every $G_d \in \mathcal{G}_d$, so that now $\mathcal{E}(G_d)$ is disjoint from $\mathcal{G}_b$. We now must have that any overlaps between $\mathcal{G}_1$ and $\mathcal{G}_2$ occur between $\mathcal{G}_a$ and $\mathcal{G}_d \cup \mathcal{G}_e \cup \mathcal{G}_f$. Note that currently we have that $|\mathcal{G}_b| = \delta^* n/2$; $|\mathcal{E}(G_b)| = \delta^* n$ for all $G_b \in \mathcal{G}_b$; $|\mathcal{G}_d| = \delta^* n$; $|\mathcal{E}(G_d) \cap \mathcal{G}_e| = \delta^* n/2$ and $|\mathcal{E}(G_d) \cap \mathcal{G}_f| = \delta^* n/2$ for all $G_d \in \mathcal{G}_d$.

- Consider a subfamily $\mathcal{G}_d' \subset \mathcal{G}_d$ with $|\mathcal{G}_d'| = \frac{99}{100}|\mathcal{G}_d|$. Remove from $\mathcal{G}_2$ (and thus $\mathcal{G}_d$ and $\mathcal{G}_e \cup \mathcal{G}_f$ respectively) all elements of $\mathcal{G}_d'$ as well as all elements from $\{\mathcal{E}(G_d') : G_d' \in \mathcal{G}_d'\}$. For every set $G_b \in \mathcal{G}_b$ we therefore now have that $|\mathcal{E}(G_b) \cap \mathcal{G}_d| \leqslant \delta^* n/100$. We can thus throw away exactly half of $\mathcal{E}(G_b)$ for every $G_b \in \mathcal{G}_b$, so that now $\mathcal{E}(G_b)$ is disjoint from $\mathcal{G}_d$. We now must have that any overlaps between $\mathcal{G}_1$ and $\mathcal{G}_2$ occur between $\mathcal{G}_a$ and $\mathcal{G}_e \cup \mathcal{G}_f$. Note that currently we have that $|\mathcal{G}_b| = \delta^* n/2$; $|\mathcal{E}(G_b)| = \delta^* n/2$ for all $G_b \in \mathcal{G}_b$; $|\mathcal{G}_d| = \delta^* n/100$; $|\mathcal{E}(G_d) \cap \mathcal{G}_e| = \delta^* n/2$ and $|\mathcal{E}(G_d) \cap \mathcal{G}_f| = \delta^* n/2$ for all $G_d \in \mathcal{G}_d$.

- We say that a $G_b \in \mathcal{G}_b$ is *destroyed* if $|\mathcal{E}(G_b) \cap (\mathcal{G}_e \cup \mathcal{G}_f)| \geqslant 99|\mathcal{E}(G_b)|/100 = 99\delta^* n/200$. Notice that at most $99|\mathcal{G}_b|/100$ $G_b \in \mathcal{G}_b$ are destroyed. Indeed, suppose not. Then recalling that the sets $\mathcal{E}(G_b)$ for $G_b \in \mathcal{G}_b$ are disjoint, we obtain that

$$|\mathcal{G}_e \cup \mathcal{G}_f| \geqslant \sum_{G_b \in \mathcal{G}_b} |\mathcal{E}(G_b) \cap (\mathcal{G}_e \cup \mathcal{G}_f)| \geqslant \frac{99|\mathcal{G}_b|}{100} \times \frac{99\delta^* n}{200} = \left(\frac{99\delta^* n}{200}\right)^2.$$

  This is a contradiction though as

$$|\mathcal{G}_e \cup \mathcal{G}_f| = |\mathcal{G}_d|\delta^* n = \frac{(\delta^* n)^2}{100} < \left(\frac{99\delta^* n}{200}\right)^2.$$

  Remove all destroyed G_b from $\mathcal{G}_b$ and also remove from $\mathcal{G}_a$ everything from $\{\mathcal{E}(G_b) : G_b \text{ is destroyed}\}$. So now $|\mathcal{G}_b| \geqslant \delta^* n/200$. For each $G_b \in \mathcal{G}_b$ delete from $\mathcal{E}(G_b)$ all elements from $\mathcal{E}(G_b) \cap (\mathcal{G}_e \cup \mathcal{G}_f)$. Since each $G_b \in \mathcal{G}_b$ is not destroyed, we still have that $|\mathcal{E}(G_b)| = \delta^* n/200$. Recall also that $|\mathcal{G}_d| = \delta^* n/100$, and $|\mathcal{E}(G_d) \cap \mathcal{G}_e| = \delta^* n/2$ and $|\mathcal{E}(G_d) \cap \mathcal{G}_f| = \delta^* n/2$ for all $G_d \in \mathcal{G}_d$.

  We have ensured that $\mathcal{G}_a$ and $\mathcal{G}_e \cup \mathcal{G}_f$ are disjoint, and thus $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint. Therefore, $\mathcal{G}_1 \cup \mathcal{G}_2$ contains a copy of $\mathcal{S}^+(x, \delta n)$ where $\delta := \delta^*/200$, as desired.

□

Proof of Lemma 9. The proof closely follows that of Theorem 7, but instead of using Lemma 15 we apply Lemma 16. Again, as in the case of Theorem 7, by Remark 4, it is enough to only prove the result for $P = \bigwedge, Y^d, S, S^+, S^{++}$, but with the slightly stronger conclusion that we seek a copy of $P(x, n^{1.91})$. As x is at distance at most 2 from any element of P, every element $y \in P$ has at most $n^{3.82} \ll n^4$ sets corresponding to y in a copy of $P(x, n^{1.91})$. This combined with the fact that we now consider families $\mathcal{F}$ that are four times as large as those in Theorem 7 means the proof is actually cleaner than that of Theorem 7. As such, we only explicitly show the case of $\bigwedge$.

Let $0 < \varepsilon < 1/4$ be fixed, n be sufficiently large and $\mathcal{F} \subseteq 2^{[n]}$ be of size $(4+4\varepsilon)\binom{n}{\lfloor n/2 \rfloor} = 4(h(\bigwedge) - 1 + \varepsilon)\binom{n}{\lfloor n/2 \rfloor}$. Let $\mathcal{F}^-$ be the family of sets $F \in \mathcal{F}$ that do not contain $\varepsilon n^4/500$ other sets of $\mathcal{F}$ of size $|F| - j$ for any $j \geqslant 4$. By Lemma 16, $|\mathcal{F}^-| \leqslant (4 + \frac{4\varepsilon}{5})\binom{n}{\lfloor n/2 \rfloor}$ and thus $\mathcal{F}^+ := \mathcal{F} \setminus \mathcal{F}^-$ has size at least $3\varepsilon\binom{n}{\lfloor n/2 \rfloor}$. Consider the bipartite graph B with parts $\mathcal{F}^+$ and $\mathcal{F}$ such that $F^+ \in \mathcal{F}^+$ and $F \in \mathcal{F}$ are joined by an edge if and only if $F \subset F^+$. By definition of $\mathcal{F}^+$, the number of edges in B is at least $\varepsilon n^4|\mathcal{F}^+|/500 \geqslant \varepsilon^2 n^4\binom{n}{\lfloor n/2 \rfloor}/200$; therefore there exists $F_x \in \mathcal{F}$ with degree at least $\varepsilon^2 n^4/1000$ in B. F_x will play the role of x in the copy of $\bigwedge(x, n^{1.91})$ that we will construct, and $n^{1.91}$ of its neighbours F_c in B will play the role of x's neighbours in $\bigwedge(x, n^{1.91})$. For each of these $n^{1.91}$ neighbours F_c of F_x, we greedily pick $n^{1.91}$ distinct subsets of F_c to obtain a copy of $\bigwedge(x, n^{1.91})$ in $\mathcal{F}$. □

3.3 Proof of Theorem 11

To prove Theorem 11 we need to apply the following container result.

Lemma 18. *Let P be any tree poset of height $h \leqslant 5$ and radius at most 2. Let $0 < \varepsilon < 1/5$ and let $\delta > 0$ be as in Conjecture 3.*[3] *Suppose $n \in \mathbb{N}$ is sufficiently large and write $m := \binom{n}{\lfloor n/2 \rfloor}$. Then there exist functions $f : \binom{2^{[n]}}{\leqslant |P| n^{-1.9} 2^n} \to \binom{2^{[n]}}{\leqslant 4(h-1+\varepsilon)m}$ and $g : \binom{2^{[n]}}{\leqslant |P|(4h-3)m/\delta n} \to \binom{2^{[n]}}{\leqslant (h-1+\varepsilon)m}$ such that for any P-free family $\mathcal{F}$ in $2^{[n]}$ there are disjoint subfamilies $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{F}$ so that:*

(i) $\mathcal{H}_1 \in \binom{2^{[n]}}{\leqslant |P| n^{-1.9} 2^n}$ and $\mathcal{H}_1 \cup \mathcal{H}_2 \in \binom{2^{[n]}}{\leqslant |P|(4h-3)m/\delta n}$;

(ii) $\mathcal{H}_1 \cup \mathcal{H}_2$ and $g(\mathcal{H}_1 \cup \mathcal{H}_2)$ are disjoint;

(iii) $\mathcal{H}_2 \subseteq f(\mathcal{H}_1)$;

(iv) $\mathcal{F} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2 \cup g(\mathcal{H}_1 \cup \mathcal{H}_2)$.

Similarly to before we refer to the family $\mathcal{H}_1 \cup \mathcal{H}_2 \cup g(\mathcal{H}_1 \cup \mathcal{H}_2)$ produced by Lemma 18 as a *container*; we call $\mathcal{H}_1$ and $\mathcal{H}_2$ the *fingerprints*.

The proof of Lemma 18 closely follows that of Lemma 17 except that we analyse the container algorithm over two separate stages. The idea of a multi-stage analysis of the graph container algorithm was first introduced in [1]. In particular, as in the proof of Theorem 10 presented in [1], using a two-stage container algorithm will allow us to employ more careful calculations in the proof of Theorem 11, thereby allowing us to deal with probabilities p 'close' to $1/n$.

[3]Note that in Theorem 7 we proved that Conjecture 3 holds for such P and some $x \in P$; so we can indeed select δ as in Conjecture 3.

Proof of Lemma 18. Let P be as in the statement of the lemma and set $p := |P|$. Let $x \in P$ be of distance at most two from every other element in the Hasse diagram of P. Given any $0 < \varepsilon < 1/5$, let $\delta > 0$ be as in Conjecture 3. Suppose $n \in \mathbb{N}$ is sufficiently large and let $m := \binom{n}{\lfloor n/2 \rfloor}$.

Fix a total order $\mathcal{O}_{2^{[n]}}$ of the elements of $2^{[n]}$. Let $\mathcal{O}_P$ be a total order of the vertices of P such that the first vertex is x; the next vertices are the children of x; the next vertices are those vertices of distance two from x in the undirected Hasse diagram of P, and so forth. We write $y_1, \dots, y_p$ for the vertices of P ordered as in $\mathcal{O}_P$; so $y_1 = x$. Further, let $\mathcal{P}_{\text{blow}}$ denote the set of all copies of $P(x,t)$ in $2^{[n]}$ for all $t \in \mathbb{N}$. Let $\mathcal{O}_{\text{blow}}$ be a total order of the elements of $\mathcal{P}_{\text{blow}}$.

We now run our modified version of the graph container algorithm. The input of the algorithm is a P-free family $\mathcal{F} \subseteq 2^{[n]}$. The algorithm will output fingerprints $\mathcal{H}_1$, $\mathcal{H}_2$ and a container $\mathcal{H}_1 \cup \mathcal{H}_2 \cup g(\mathcal{H}_1 \cup \mathcal{H}_2)$ where $\mathcal{H}_1 \cup \mathcal{H}_2 \subseteq \mathcal{F} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2 \cup g(\mathcal{H}_1 \cup \mathcal{H}_2)$. We proceed in two stages.

Stage 1: Initially we set $\mathcal{G}_1^0 := 2^{[n]}$ and $\mathcal{H}_1^0 := \emptyset$. We will add vertices from $2^{[n]}$ to $\mathcal{H}_1^0$ and remove vertices from $\mathcal{G}_1^0$ through the following iterative process, beginning at Step 1.

At Step i, let $P_1(x,t) \in \mathcal{P}_{\text{blow}}$ be a copy of $P(x,t)$ in $\mathcal{G}_1^{i-1} \subseteq 2^{[n]}$ where we choose t to be as large as possible. If there is more than one copy of $P(x,t)$ in $\mathcal{G}_1^{i-1}$ for this choice of t we choose $P_1(x,t)$ to be the copy of $P(x,t)$ appearing earliest in the total order $\mathcal{O}_{\text{blow}}$. Let x_1 be the vertex of $P_1(x,t)$ that plays the role of x.

- Suppose $x_1 \notin \mathcal{F}$. Then define $\mathcal{G}_1^i := \mathcal{G}_1^{i-1} \setminus \{x_1\}$ and $\mathcal{H}_1^i := \mathcal{H}_1^{i-1}$. Proceed to Step $i+1$.

- Suppose $x_1 \in \mathcal{F}$ and $t \geqslant n^{1.9}$. We will update $\mathcal{G}_1^{i-1}$ and $\mathcal{H}_1^{i-1}$ in several *subphases* to obtain $\mathcal{G}_1^i$ and $\mathcal{H}_1^i$ respectively.

  - *Subphase* 1. Delete x_1 from $\mathcal{G}_1^{i-1}$ and add it to $\mathcal{H}_1^{i-1}$. Proceed to Subphase 2.
  - *Subphase* 2. Recall that we write $y_1, \dots, y_p$ for the vertices of P ordered as in $\mathcal{O}_P$. So $y_1 = x$ and y_2 is a child of x in P. In $P_1(x,t)$ there is a set of t vertices $U_2 \subseteq \mathcal{G}_1^{i-1}$ so that each $x_2 \in U_2$ is adjacent to x_1 in the Hasse diagram of $P_1(x,t)$ and further the edge between x_1 and x_2 is oriented in the same direction as the edge between $y_1(=x)$ and y_2 in the Hasse diagram for P.

    Look through the vertices in $U_2 \subseteq 2^{[n]}$ one by one, following the total order $\mathcal{O}_{2^{[n]}}$. If we run through the entire set U_2 without finding an element from $\mathcal{F}$ then we delete all t vertices in U_2 from $\mathcal{G}_1^{i-1}$ and stop Step i.

    Otherwise, let $x_2 \in U_2$ denote the very first element from $\mathcal{F}$ that we discover in U_2. Add x_2 to $\mathcal{H}_1^{i-1}$ and remove from $\mathcal{G}_1^{i-1}$ both x_2 and all elements of U_2 that occur before x_2 in the total order $\mathcal{O}_{2^{[n]}}$. So here we have deleted between 1 and t elements of $\mathcal{G}_1^{i-1}$, and of these only x_2 lies in $\mathcal{F}$. Proceed to Subphase 3.
  - *Subphase j (for $3 \leqslant j \leqslant p$).* In the previous subphases we have defined $x_1, \dots, x_{j-1} \in P_1(x,t)$ corresponding to $y_1, \dots, y_{j-1} \in P$.

By definition of $\mathcal{O}_P$, the unique parent y_k of y_j in P must be one of $y_1, \ldots, y_{j-1}$. Furthermore, by definition of $P(x,t)$, there is a set of t vertices $U_j \subseteq \mathcal{G}_1^{i-1}$ in $P_1(x,t)$ corresponding to y_j so that every $x_j \in U_j$ is adjacent to x_k in the Hasse diagram of $P_1(x,t)$, and moreover, the edge between x_k and x_j is oriented in the same direction as the edge between y_k and y_j in the Hasse diagram for P.

Look through the vertices in $U_j \subseteq 2^{[n]}$ one by one, following the total order $\mathcal{O}_{2^{[n]}}$. If we run through the entire set U_j without finding an element from $\mathcal{F}$ then we delete all t vertices in U_j from $\mathcal{G}_1^{i-1}$ and stop Step i.

Otherwise, let $x_j \in U_j$ denote the very first element from $\mathcal{F}$ that we discover in U_j. Add x_j to $\mathcal{H}_1^{i-1}$ and remove from $\mathcal{G}_1^{i-1}$ both x_j and all elements of U_j that occur before x_j in the total order $\mathcal{O}_{2^{[n]}}$. So here we have deleted at most t elements of $\mathcal{G}_1^{i-1}$, of which only x_j lies in $\mathcal{F}$. Proceed to Subphase $j+1$.

At the end of these subphases relabel $\mathcal{G}_1^{i-1}$ as $\mathcal{G}_1^i$ and $\mathcal{H}_1^{i-1}$ as $\mathcal{H}_1^i$; then proceed to Step $i+1$. Since $\mathcal{F}$ is P-free, for some $j \in [p]$, Subphase j must consider a set U_j where $U_j \cap \mathcal{F} = \emptyset$. Thus, in Step i we have deleted at least t vertices from $\mathcal{G}_1^{i-1}$ to obtain $\mathcal{G}_1^i$ and have added at most $p-1$ vertices to $\mathcal{H}_1^{i-1}$ (at most one in each of the first $p-1$ subphases) to obtain $\mathcal{H}_1^i$.

- Suppose $x_1 \in \mathcal{F}$ and $t < n^{1.9}$. Then define $\mathcal{H}_1 := \mathcal{H}_1^{i-1} \cup \{x_1\}$ and $f(\mathcal{H}_1) := \mathcal{G}_1^{i-1} \setminus \{x_1\}$ and proceed to Step 1 of Stage 2.

Note that in every step of Stage 1 we only add elements to $\mathcal{H}_1^{i-1}$ that lie in $\mathcal{F}$. So certainly $\mathcal{H}_1 \subseteq \mathcal{F}$. Similarly, by construction, $\mathcal{F} \subseteq \mathcal{H}_1 \cup f(\mathcal{H}_1)$, and $\mathcal{H}_1$ and $f(\mathcal{H}_1)$ are disjoint. In every step of Stage 1 except the final step, if we add (at most $p-1$) elements to $\mathcal{H}_1^{i-1}$ we delete at least $n^{1.9}$ elements from $\mathcal{G}_1^{i-1} \subseteq 2^{[n]}$. Therefore,

$$|\mathcal{H}_1| \leqslant \frac{(p-1)2^n}{n^{1.9}} + 1 \leqslant \frac{p \cdot 2^n}{n^{1.9}}. \tag{1}$$

Furthermore, by construction of Stage 1, the family $f(\mathcal{H}_1) \subseteq 2^{[n]}$ does not contain a copy of the $n^{1.9}$-blow-up $P(x, n^{1.9})$. Thus, by Lemma 9, we have that

$$|f(\mathcal{H}_1)| \leqslant 4(h-1+\varepsilon)m.$$

Stage 2. At the start of Stage 2 we set $\mathcal{G}_2^0 := f(\mathcal{H}_1)$ and $\mathcal{H}_2^0 := \emptyset$. We will add vertices from $f(\mathcal{H}_1)$ to $\mathcal{H}_2^0$ and remove vertices from $\mathcal{G}_2^0$ through the following iterative process, beginning at Step 1.

At Step i of Stage 2, let $P_2(x,t) \in \mathcal{P}_{\text{blow}}$ be a copy of $P(x,t)$ in $\mathcal{G}_2^{i-1} \subseteq 2^{[n]}$ where we choose t to be as large as possible. If there is more than one copy of $P(x,t)$ in $\mathcal{G}_2^{i-1}$ for this choice of t we choose $P_2(x,t)$ to be the copy of $P(x,t)$ appearing earliest in the total order $\mathcal{O}_{\text{blow}}$. Let x_1 be the vertex of $P_2(x,t)$ that plays the role of x.

- Suppose $x_1 \notin \mathcal{F}$. Then define $\mathcal{G}_2^i := \mathcal{G}_2^{i-1} \setminus \{x_1\}$ and $\mathcal{H}_2^i := \mathcal{H}_2^{i-1}$. Proceed to Step $i+1$.

- Suppose $x_1 \in \mathcal{F}$ and $t \geqslant \delta n$. We then update $\mathcal{G}_2^{i-1}$ and $\mathcal{H}_2^{i-1}$ in at most p subphases to obtain $\mathcal{G}_2^i$ and $\mathcal{H}_2^i$ respectively. These subphases are identical to those subphases described in Stage 1, just with $\mathcal{G}_2^{i-1}$, $\mathcal{H}_2^{i-1}$ and $P_2(x,t)$ now playing the roles of $\mathcal{G}_1^{i-1}$, $\mathcal{H}_1^{i-1}$ and $P_1(x,t)$ respectively.

  At the end of the subphases relabel $\mathcal{G}_2^{i-1}$ as $\mathcal{G}_2^i$ and $\mathcal{H}_2^{i-1}$ as $\mathcal{H}_2^i$; then proceed to Step $i+1$ of Stage 2. Since $\mathcal{F}$ is P-free, for some $j \in [p]$, Subphase j must consider a set U_j where $U_j \cap \mathcal{F} = \emptyset$. Thus, in Step i of Stage 2, we have deleted at least t vertices from $\mathcal{G}_2^{i-1}$ to obtain $\mathcal{G}_2^i$ and have added at most $p-1$ vertices to $\mathcal{H}_2^{i-1}$ (at most one in each of the first $p-1$ subphases) to obtain $\mathcal{H}_2^i$.

- Suppose $x_1 \in \mathcal{F}$ and $t < \delta n$. Then define $\mathcal{H}_2 := \mathcal{H}_2^{i-1} \cup \{x_1\}$ and $g(\mathcal{H}_1 \cup \mathcal{H}_2) := \mathcal{G}_2^{i-1} \setminus \{x_1\}$ and terminate the algorithm.

Note that in every step of Stage 2 we only add elements to $\mathcal{H}_2^{i-1}$ that lie in $\mathcal{F} \cap f(\mathcal{H}_1)$. So certainly $\mathcal{H}_2 \subseteq \mathcal{F}$. Further, as $\mathcal{H}_1$ and $f(\mathcal{H}_1)$ are disjoint, this implies $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint. Similarly, by construction, conditions (ii)–(iv) of the lemma hold.

In every step of Stage 2 except the final step, if we add (at most $p-1$) elements to $\mathcal{H}_2^{i-1}$ we delete at least δn elements from $\mathcal{G}_2^{i-1} \subseteq 2^{[n]}$. Moreover, recall that $\mathcal{G}_2^0 := f(\mathcal{H}_1)$ and $|f(\mathcal{H}_1)| \leqslant 4(h-1+\varepsilon)m$. Therefore,

$$|\mathcal{H}_2| \leqslant \frac{(p-1)4(h-1+\varepsilon)m}{\delta n} + 1 \leqslant \frac{p \cdot 4(h-1+\varepsilon)m}{\delta n}.$$

Combining this with (1) we see that condition (i) of the lemma holds.

Note that $g(\mathcal{H}_1 \cup \mathcal{H}_2) \subseteq 2^{[n]}$ is defined so that it does not contain a copy of $P(x, \delta n)$. Thus, by Theorem 7,

$$|g(\mathcal{H}_1 \cup \mathcal{H}_2)| \leqslant (h-1+\varepsilon)m.$$

Therefore all that remains to check is that the functions f and g are well-defined. That is, if for two P-free families $\mathcal{F}$ and $\mathcal{F}'$ the algorithm outputs the same $\mathcal{H}_1$, then $f(\mathcal{H}_1)$ is defined the same, and furthermore, if $\mathcal{H}_1 \cup \mathcal{H}_2$ is the same for both runs of the algorithm then so is $g(\mathcal{H}_1 \cup \mathcal{H}_2)$. This follows though immediately from the definition of the algorithm. Indeed, suppose one is only presented with the fingerprint $\mathcal{H}_1$ that is outputted by Step 1 of the algorithm. Then one can completely identify every action taken during Step 1 of the algorithm. Similarly, if one is further given the second fingerprint $\mathcal{H}_2$, then one can completely identify every action taken during Step 2 of the algorithm. □

With Lemma 18 at hand, we can now prove Theorem 11.

Proof of Theorem 11. Let P be any tree poset of height $h \leqslant 5$ and radius at most 2. Fix $\varepsilon > 0$; it suffices to prove the theorem under the assumption that $\varepsilon < 1/5$. Let $\varepsilon_1 := \varepsilon/4$ and let $\delta > 0$ be as in Conjecture 3 on input ε_1. Define $C := 10^{10}|P|(4h-3)h^4\varepsilon^{-5}\delta^{-1}$. Let $p > C/n$ and $m := \binom{n}{\lfloor n/2 \rfloor}$.

Note that the middle $h-1$ layers of $2^{[n]}$ form a P-free subfamily of $2^{[n]}$ of size $(h-1-o(1))m$. So w.h.p. $\mathcal{P}(n,p)$ contains a P-free family of size at least $(h-1-\varepsilon)pm$.

It remains to show that, w.h.p., $\mathcal{P}(n,p)$ contains no P-free family of size greater than $(h-1+\varepsilon)pm$. We now follow the proof of Theorem 10 given in [1] closely.

Apply Lemma 18 with ε_1 playing the role of ε. Suppose for a contradiction that $\mathcal{P}(n,p)$ does contain some P-free family $\mathcal{F}$ with $|\mathcal{F}| > (h-1+\varepsilon)pm$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the fingerprints for $\mathcal{F}$ given by Lemma 18. Then since $\mathcal{H}_1 \cup \mathcal{H}_2 \subseteq \mathcal{F}$, we must have that $\mathcal{H}_1 \cup \mathcal{H}_2 \subseteq \mathcal{P}(n,p)$. Further, at least $|\mathcal{F}| - |\mathcal{H}_1 \cup \mathcal{H}_2| \geqslant (h-1+\varepsilon)pm - |P|(4h-3)m/\delta n \geqslant (h-1+\varepsilon/2)pm$ elements of $g(\mathcal{H}_1 \cup \mathcal{H}_2)$ must be in $\mathcal{P}(n,p)$.

Note that the number of possibilities for $\mathcal{H}_1$ is $\binom{2^n}{\leqslant |P|n^{-1.9}2^n}$, and for each possibility the probability that $\mathcal{H}_1 \subseteq \mathcal{P}(n,p)$ is $p^{|\mathcal{H}_1|}$. For any fixed $\mathcal{H}_1$ we have $|f(\mathcal{H}_1)| \leqslant 4(h-1+\varepsilon_1)m \leqslant (4h-3)m$ and $\mathcal{H}_2 \subseteq f(\mathcal{H}_1)$, so the number of possibilities for $\mathcal{H}_2$ is at most $\binom{(4h-3)m}{\leqslant |P|(4h-3)m/\delta n}$, and for each possibility the probability that $\mathcal{H}_2 \subseteq \mathcal{P}(n,p)$ is $p^{|\mathcal{H}_2|}$.

Furthermore, for any fixed $\mathcal{H}_1$ and $\mathcal{H}_2$ we have $g(\mathcal{H}_1 \cup \mathcal{H}_2) \leqslant (h-1+\varepsilon_1)m = (h-1+\varepsilon/4)m$, so the expected number of elements of $g(\mathcal{H}_1 \cup \mathcal{H}_2)$ selected for $\mathcal{P}(n,p)$ is at most $(h-1+\varepsilon/4)pm$. By the Chernoff bound for the binomial distribution, the probability that at least $(h-1+\varepsilon/2)pm$ elements of $g(\mathcal{H}_1 \cup \mathcal{H}_2)$ are selected for $\mathcal{P}(n,p)$ is therefore at most $e^{-\varepsilon^2 pm/(100h^2)}$.

Taking a union bound, we conclude that the probability that $\mathcal{P}(n,p)$ contains a P-free family of size greater than $(h-1+\varepsilon)pm$ is at most

$$
\begin{aligned}
\Pi &:= \sum_{0 \leqslant a \leqslant |P|n^{-1.9}2^n} \; \sum_{0 \leqslant b \leqslant |P|(4h-3)m/\delta n} \binom{2^n}{a} \cdot p^a \cdot \binom{(4h-3)m}{b} \cdot p^b \cdot e^{-\varepsilon^2 pm/(100h^2)} \\
&\leqslant \left(|P|n^{-1.9}2^n + 1\right)\left(|P|(4h-3)m/\delta n + 1\right) \binom{2^n}{|P|n^{-1.9}2^n} \cdot p^{|P|n^{-1.9}2^n} \binom{(4h-3)m}{|P|(4h-3)m/\delta n} \\
&\quad \cdot p^{|P|(4h-3)m/\delta n} \cdot e^{-\varepsilon^2 pm/(100h^2)}.
\end{aligned}
$$

Note that for large n, with room to spare we have

$$
(|P|n^{-1.9}2^n + 1)(|P|(4h-3)m/\delta n + 1) \leqslant e^{\varepsilon^2 pm/(400h^2)}
$$

and

$$
\binom{2^n}{|P|n^{-1.9}2^n} \cdot p^{|P|n^{-1.9}2^n} \leqslant e^{\varepsilon^2 pm/(400h^2)}.
$$

Further, as $C = 10^{10}|P|(4h-3)h^4\varepsilon^{-5}\delta^{-1}$, for sufficiently large n we have that

$$
\binom{(4h-3)m}{|P|(4h-3)m/\delta n} \cdot p^{|P|(4h-3)m/\delta n} \leqslant e^{\varepsilon^2 pm/(400h^2)}.
$$

Therefore, the upper bound Π on the probability is $o(1)$, as required. □

4 Further discussion and observations

For tree posets P of radius at most 2, we have asymptotically determined the number of P-free families in $2^{[n]}$ (Corollary 8) and have also resolved the random version of the P-free problem (Theorem 11). In general though, these questions for other posets P remain wide open. However, as we will now see, if one has solved either of these problems for some poset P, then one can often deduce further results from this.

Let $P_{t,h}$ denote the (upward) monotone t-ary tree poset of height h; the *t-fork* $\vee_t$ is $P_{t,2}$. Let $\mathcal{D}_k$ denote the *k-diamond*; that is, the poset with a unique maximal element above all other elements, a unique minimal element below all other elements, and k (incomparable) elements in between. Note that $\Diamond := \mathcal{D}_2$ is the *diamond.*

Observation 19.

1. *Given any $t, h \in \mathbb{N}$, and $\varepsilon > 0$, there exists $C > 0$ such that the following holds. If $p > C/n$ then w.h.p. the largest $P_{t,h}$-free family in $\mathcal{P}(n,p)$ has size $(h-1\pm\varepsilon)p\binom{n}{\lfloor n/2\rfloor}$.*

2. *The number of $P_{t,h}$-free families in $2^{[n]}$ is $2^{(h-1+o(1))\binom{n}{\lfloor n/2\rfloor}}$.*

Proof. The lower bound of (1) follows by noting that, w.h.p., $\mathcal{P}(n,p)$ contains at least $(h-1-\varepsilon)p\binom{n}{\lfloor n/2\rfloor}$ elements from the $h-1$ middle layers of $2^{[n]}$. The lower bound of (2) can be seen by considering all subfamilies of the $h-1$ middle layers of $2^{[n]}$.

The proof of the upper bound of (1) proceeds by induction on h, with the base case $h = 2$ proved by Hogenson [13]. Suppose (1) holds for any choice of the height of the tree less than h where $h \geqslant 3$. Fix any $t \in \mathbb{N}$ and $\varepsilon > 0$. By the induction hypothesis, there exists $C_1 > 0$ such that if $p > C_1/n$ then, w.h.p, the largest $P_{2t^h,2}$-free family in $\mathcal{P}(n,p)$ has size at most $(1 + \varepsilon/2)p\binom{n}{\lfloor n/2\rfloor}$. Further, there exists $C_2 > 0$ such that if $p > C_2/n$ then, w.h.p, the largest $P_{t,h-1}$-free family in $\mathcal{P}(n,p)$ has size at most $(h-2+\varepsilon/2)p\binom{n}{\lfloor n/2\rfloor}$.

Let $C := \max\{C_1, C_2\}$ and consider $p > C/n$. Consider any $P_{t,h}$-free family $\mathcal{F} \subseteq \mathcal{P}(n,p)$. Define $\mathcal{F}' := \{F \in \mathcal{F} : |\mathcal{U}(F) \setminus \{F\}| < 2t^h\}$. As $\mathcal{F}'$ is $P_{2t^h,2}$-free then, w.h.p., $|\mathcal{F}'| \leqslant (1 + \varepsilon/2)p\binom{n}{\lfloor n/2\rfloor}$.

Furthermore, observe that $\mathcal{F} \setminus \mathcal{F}'$ is $P_{t,h-1}$-free, as otherwise a copy of $P_{t,h-1}$ in $\mathcal{F} \setminus \mathcal{F}'$ could be extended greedily to a copy of $P_{t,h}$ in $\mathcal{F}$. Indeed, any set F corresponding to a leaf of $P_{t,h-1}$ is contained in at least $2t^h$ many other sets of $\mathcal{F}$, at least t^h of which do not belong to the copy of $P_{t,h-1}$. As there are t^{h-2} leaves, we can pick pairwise disjoint families of t supersets for every leaf.

Thus, w.h.p., $|\mathcal{F} \setminus \mathcal{F}'| \leqslant (h-2+\varepsilon/2)p\binom{n}{\lfloor n/2\rfloor}$, yielding $|\mathcal{F}| \leqslant (h-1+\varepsilon)p\binom{n}{\lfloor n/2\rfloor}$ w.h.p., as desired.

The upper bound of (2) follows similarly by induction on h with the base case $h = 2$ proved in [13]. Any $P_{t,h}$-free family $\mathcal{F} \subseteq 2^{[n]}$ can be partitioned into $\mathcal{F}'$ and $\mathcal{F} \setminus \mathcal{F}'$ as above; so the number of $P_{t,h}$-free families in $2^{[n]}$ is at most the product of the number of $P_{t,h-1}$-free families and the number of $P_{2t^h,2}$-free families. The statement then follows by induction. □

Since any upward monotone tree poset P of height h is contained in $P_{t,h}$ for t large enough, Observation 19 holds for any such P. Moreover, it holds for a somewhat wider class of tree posets. We say that P is a *two-way monotone tree poset* if there exists $x \in P$ such that $\mathcal{U}(x) \cup \mathcal{D}(x) = P$ where $\mathcal{U}(x)$ is the set of elements of P that are supersets of x and $\mathcal{D}(x)$ is the set of elements of P that are subsets of x. It can be shown that Observation 19 holds for two-way monotone tree posets of height h as well.

In this paper we have considered the problem of determining for what probabilities p the largest P-free family in $\mathcal{P}(n,p)$ has size $p \cdot (1+o(1))La(n,P)$ with high probability. It is of course natural to consider other values of p (i.e., when the largest P-free family in $\mathcal{P}(n,p)$ has size more than $p \cdot (1+o(1))La(n,P)$); see [1, 17]. The following is such a result for the diamond.

Observation 20. *Given $\varepsilon > 0$ there exist $C_1, C_2 > 0$ such that the following holds. If $C_1/n < p < C_2/n^{2/3}$ then w.h.p. the largest $\Diamond$-free family in $\mathcal{P}(n,p)$ has size $(3\pm\varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. The upper bound follows from the fact that a $\Diamond$-free family $\mathcal{F} \subseteq \mathcal{P}(n,p)$ cannot contain a 4-chain which is a special copy of $\Diamond$. Therefore $\mathcal{F}$ can be partitioned into three antichains $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$; thus the upper bound on $|\mathcal{F}|$ follows from Theorem 10 applied to all $\mathcal{F}_i$s.

To see the lower bound, observe that the expected number E_1 of sets in the middle three layers of $\mathcal{P}(n,p)$ is $(3 \pm \varepsilon/4)p\binom{n}{\lfloor n/2 \rfloor}$. Also, the expected number E_2 of copies of $\Diamond$ in the middle three layers of $\mathcal{P}(n,p)$ is $(1\pm\varepsilon/4)p^4\frac{n^2}{8}\binom{n}{\lfloor n/2 \rfloor}$. Using the Markov and Chebyshev inequalities, it follows that if $C_2 > 0$ is small enough, then w.h.p. $E_1 \geqslant (3-\varepsilon/2)p\binom{n}{\lfloor n/2 \rfloor}$ and $E_2 \leqslant (1+\varepsilon/2)p^4\frac{n^2}{8}\binom{n}{\lfloor n/2 \rfloor} \leqslant \frac{\varepsilon}{2}p\binom{n}{\lfloor n/2 \rfloor}$. So removing one element from each copy of a $\Diamond$ leaves us with a $\Diamond$-free family of size at least $(3-\varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$ in $\mathcal{P}(n,p)$. □

As Theorem 6 articulates, Conjecture 3 has implications to the counting problem for tree posets P. The resolution of Conjecture 3 would also likely allow one to generalise Theorem 11 to all tree posets P of height h, not just those of radius 2. For this one would also need an analogue of Lemma 9.

One might wonder why our proof of Theorem 7 does not work for posets of radius larger than 2, and why Bukh's original argument [3] cannot be applied to prove Theorem 7 for arbitrary tree posets and arbitrary element x. Let us consider first the latter question. Bukh's proof uses some preliminary structural results on tree posets. Namely, he shows that any tree poset of height h is a subposet of a height h *saturated* tree poset, i.e., one in which all maximal chains contain h elements. Then he shows that any saturated tree poset T of height h can be obtained from C_h by 'adding intervals' $I_1, I_2, \ldots, I_m$, i.e., branches of the Hasse diagram. This enables him to obtain an inductive proof by showing that $T \setminus I_m$ can be embedded not only into any large enough family $\mathcal{F}$, but to nice, extendable sets of $\mathcal{F}$, where nice means that a *constant* number of sets cannot ruin the extending property. There are two points of this proof that we do not see how to save to generalise Theorem 7. As a height h tree poset can have arbitrary radius, it is not enough to extend the embedding of the blow-up by one interval, but by a polynomial (of arbitrary high degree!) number of blow-ups of intervals, and now the extension property should not be ruined by a polynomial number of sets (rather than a constant number). Even more importantly, we do not know how to start the induction. Bukh's result for chains (even with the extension property) is immediate from any supersaturation result for h-chains, but we are unable to prove Theorem 7 for C_4 and x being its minimal element.

This brings us to the other question: why our proof does not work for the blow-up of trees with larger radius. Getting the result for $\bigvee$ and $\bigwedge$ is basically an exercise if one is familiar with the notion of Lubell-mass. Then for the poset Y, we try to apply some kind of an induction: we find lots of blow-ups of $\bigvee$ and then a blow-up of $\bigvee$ in the root of the blow-ups of the first round. All we need to take care of are the overlaps of these blow-ups. This is done via introducing type 1, 2, and 3 sets and observing that if we work with type 1 sets, then the pairwise overlaps are of size at most 1 and so at the price of shrinking δ, we get our pairwise disjoint blow-ups, while if we work with type 2 sets, then we need only a linear number of sets, while the blow-ups are quadratic. The latter could work for arbitrary long chains, but type 1 sets are problematic: type 1 copies of blow-ups of Y can intersect in a linear number of sets, and we were unable to handle how to get many pairwise disjoint copies of them.

Because of the above reasoning, the full resolution of each of Conjecture 3, Conjecture 5 and the random version of the P-free problem does currently seem out of reach. It would therefore be interesting to resolve these problems for other natural classes of posets P. For example, it would be interesting to extend our results to cover all tree posets P of height 2, particularly those posets whose (undirected) Hasse diagram is a path. In [11], $La(n, \mathcal{D}_k)$ was asymptotically determined for infinitely many choices of k. It would be interesting to resolve Conjecture 5 and the random version of the P-free problem for such $\mathcal{D}_k$. It is also natural to seek induced versions of our results, including Corollary 8 and Theorem 11.

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