Triangles and subgraph probabilities
in random regular graphs

Pu Gao

Abstract

We improve the estimates of the subgraph probabilities in a random regular graph. Using the improved results, we further improve the limiting distribution of the number of triangles in random regular graphs.

Mathematics Subject Classifications: 05C80, 05C30

1 Introduction

Research in random graph theory started from the study of subgraphs [2]. The distributional results for small subgraphs and for large subgraphs are very different in nature. The distributions of small subgraphs in $\mathcal{G}(n,p)$ are well understood [18, 19]. Under some mild conditions, the number of subgraphs isomorphic to a fixed graph H, denoted by Z_H, is asymptotically normally distributed when p exceeds some critical value p_H. Similar results hold for $\mathcal{G}(n,m)$ as well [18]. In particular, if H is a balanced graph and p is not too close to 1, then Z_H is asymptotically normally distributed in $\mathcal{G}(n,p)$, and is thus highly concentrated around its expectation $\mathbb{E}Z_H$, when $\mathbb{E}Z_H \to \infty$ as $n \to \infty$. The picture for large subgraphs is very different. Take Z_M, the number of perfect matchings (assuming n is even) for an example. The threshold of its appearance is at $p = \log n/n$. However, $\mathbb{E}Z_M \to \infty$ already when $p = C/n$ for $C > e$. This implies no concentration of Z_M around its expectation for $p = O(1/n)$, and indeed no such concentration for larger p either. It has been proved that Z_M is asymptotically log-normally distributed for $p \gg n^{-1/2}$ [13]. For smaller values of p its distribution is unknown. Similar results have been proved for the number of spanning trees and the number of Hamilton cycles [13], the number of d-factors [4], and the number of triangle factors [5]. What happens to the number of subgraphs whose size is between constant size and size of order $\Omega(n)$? For the number of matchings of size ℓ, it turns out that its limiting distribution transitions from normal to log-normal at some critical point of p [9], and this distributional transition may be a general phenomenon for other subgraphs as well.

University of Waterloo (`pu.gao@uwaterloo.ca`).

Another extensively studied random graph model is the model of random regular graphs. Let $\mathcal{G}(n,d)$ denote a graph chosen uniformly at random from the set of all d-regular graph on vertex set $[n]$. Much less is known about the distribution of Z_H in $\mathcal{G}(n,d)$ where H is of fixed size, or of size $\Omega(n)$. For $d = O(1)$, $\mathcal{G}(n,d)$ is locally tree like and thus, the only nontrivial connected subgraphs of fixed sizes are cycles (see [21] for more discussions of properties of $\mathcal{G}(n,d)$ for fixed d). In $\mathcal{G}(n,d)$ the joint distribution of the number of cycles of any fixed length tends asymptotically to the distribution of independent Poisson random variables [1, 20]. This continues to hold for slowly growing d [17]. For instance, the Poisson like distribution for the number of triangles is proved to hold for all $d = o(n^{1/5})$. Z. Gao and Wormald [11] proved normal distribution for strictly balanced graphs, which, in cases of short cycles, permits slightly larger d than in [17]. For the number of triangles, their result holds for $d = o(n^{2/7})$. For large subgraphs, Janson [12] determined the limiting distribution of the number of perfect matchings and the number of Hamilton cycles in $\mathcal{G}(n,d)$ for constant d. For growing d, we are not aware of any results when the size of H is beyond $\log n$. The recent development on the sandwich conjecture [6, 3] allows to translate properties and graph parameters from $\mathcal{G}(n,p)$ to $\mathcal{G}(n,d)$, but in general the distributional results do not translate.

There are two main obstacles in proving the limiting distribution of Z_H, even for H of fixed size. The first obstacle lies in the computation of the moments of Z_H. To prove normality, one can compute its central j-th moment for each fixed j. However, due to the appearance of both positive and negative terms in the central moments computation, the leading terms cancel each other, and the errors in the subgraph probabilities quickly become the dominating terms when d grows. (Take the triangle $H = C_3$ for an example. We only manage to compute the variance of Z_{C_3} for $d = o(n^{2/5})$, and computing higher moments requires even smaller d.) The other standard method is computing the k-th raw moment for k around $\mathbb{E}Z_H / \sigma(Z_H)$, where $\sigma(Z_H)$ is the standard deviation of Z_H. For such large k (which typically grows quickly with d), it is difficult to bound the number of k-tuples of copies of H sharing a specific number of edges. The second obstacle is to estimate the probabilities of subgraphs in $\mathcal{G}(n,d)$. Specifying a set of edges $H \subseteq \binom{[n]}{2}$, what is the probability that $H \subseteq \mathcal{G}(n,d)$ (i.e. $\mathcal{G}(n,d)$ contains all edges in H)? McKay [14, 15] gave an estimate of this probability which has a relative error $O(|H|d/n)$, and to our knowledge, this has been the best estimate to date. This immediately limits the study of Z_H for large H, or the computation of high moments of Z_H even for H of constant size, which is required when proving the limiting distribution of Z_H. Take $H = C_3$ as an example. To determine the limiting distribution of Z_{C_3}, we need to estimate the k-th moment of Z_{C_3} where $k \approx d^{3/2}$. This requires an asymptotic joint probability estimate of $\Theta(d^{3/2})$ edges. With errors in [14, 15], d is required to be $o(n^{2/5})$.

For the second obstacle, we manage to improve the error in the subgraph probabilities in [14, 15]. See Theorem 6 in Section 2 for the precise statement. This new result allows us to estimate, for instance, the asymptotic probability that $\mathcal{G}(n,d)$ contains a specific perfect matching for d up to $n^{1/2}$. As we only study small subgraphs in this paper, it turns out that a less precise form would be sufficient, which is stated in Theorem 2 as a corollary of Theorem 6.

For the first obstacle, Z. Gao and Wormald bypassed it by smartly considering a new variable $\widetilde{Z}_H$, which counts isolated copies of H, i.e. copies of H that do not share edges with other copies of H. In the range of d where the number of non-isolated copies of H is so small that it does not affect the standard deviation of $\widetilde{Z}_H$, the distribution of $\widetilde{Z}_H$ will immediately yield the distribution of Z_H. Working on $\widetilde{Z}_H$ avoids having to deal with tuples of heavily intersecting copies of H. However, it also limits the range of d for which the proof method can apply. For the case $H = C_3$, $d = o(n^{2/7})$ is required in [11], which did not even reach the natural stopping point $d = o(n^{2/5})$ (explained above). To relax the condition on d, it is possible to modify the definition of $\widetilde{Z}_H$. Instead of counting isolated triangles, we may, for instance, count triangles that intersect only a bounded number of other triangles. With the new definition and with the new subgraph probabilities in Theorem 2, it is possible to get beyond $d = o(n^{2/7})$. However it is not sufficient to reach $d = n^{1/2-o(1)}$. In this paper, we handle Z_H directly, and prove the normality of the number of triangles for all $d = O(n^{1/2})$ where $d \to \infty$.

We also computed the variance of Z_H when H is a fixed strictly balanced graph. Surprisingly, although the variance seemingly carries less information than the limiting distribution and thus computing it should be easier than determining the limiting distribution, we are not able to determine the variance of the number of triangles for d up to $n^{1/2}$. In fact, we can only compute the variance for $d = o(n^{2/5})$. Note that the limiting distribution does not imply the variance, as it is not sensitive to a lottery effect (events affecting Z_H that occur with a tiny probability), whereas the variance is, although we do not expect such a lottery effect for the variance either. Another surprising discovery is that the number of triangles seems to behave more like the sum of a set of independent variables in $\mathcal{G}(n, d)$ than in $\mathcal{G}(n, p)$, whereas the intuition is the other way around. Let X_{ijk} be the indicator variable that the three vertices i, j, k induce a triangle. Then the number of triangles is $\sum X_{ijk}$ where the sum is over all 3-subsets of $[n]$. In $\mathcal{G}(n, p)$, X_{ijk} is independent of all but those X_{uvw} such that $|\{i, j, k\} \cap \{u, v, w\}| = 2$. In $\mathcal{G}(n, d)$, due to the dependency between edges, all these indicator variables are correlated. However, surprising cancellations of leading terms appear in the calculations for the variance of the number of triangles (see Section 4.2 for details). The variance of the number of triangles turns out smaller in $\mathcal{G}(n, d)$ than in $\mathcal{G}(n, d/n)$.

One of the challenges in computing the moments of the number of triangles is to bound the number of k-tuples of triangles where $k \approx d^{3/2}$, such that these k triangles induce a large number of "holes" (see Definition 16 in Section 5.2). The bound and the arguments in this paper do not easily extend to other strictly balanced graphs in general. Extending the distributional result for the number of triangles to other strictly balanced graphs requires a smarter treatment for the holes.

2 Main results

2.1 Improved subgraph probabilities

Let $\mathbf{d} = (d_1, \ldots, d_n)$ be a degree sequence and let $\mathcal{G}(n, \mathbf{d})$ denote a uniformly random graph with degree sequence $\mathbf{d}$. Let H_1 and H_2 be two disjoint graphs on $[n]$, i.e. $E(H_1) \cap E(H_2) = \emptyset$. Let H_1^+ denote the event that $H_1 \subseteq \mathcal{G}(n, \mathbf{d})$ and let H_2^- denote the event that $H_2 \cap \mathcal{G}(n, \mathbf{d}) = \emptyset$. The conditional edge probability $\mathbb{P}(uv \in \mathcal{G}(n, \mathbf{d}) \mid H_1^+, H_2^-)$ was first given by McKay [14, 15], which immediately applies to give an asymptotic estimate of $\mathbb{P}(H \subseteq \mathcal{G}(n, \mathbf{d}))$ when H is not too large. This result was recently extended by Ohapkin and the author [8] to more general degree sequences, with the same order of the error term $O(\Delta(\mathbf{d})/n)$ as in [8] ($\Delta(\mathbf{d})$ denotes the maximum degree in $\mathbf{d}$). Given a graph H on $[n]$, let $\mathbf{d}^H = (d_1^H, \ldots, d_n^H)$ denote the degree sequence of H and let $|H|$ denote the number of edges in H. Let Δ_H denote the maximum degree of H. Given two degree sequences $\mathbf{d}$ and $\mathbf{d}'$, we say $\mathbf{d} \preceq \mathbf{d}'$ if $d_i \leqslant d_i'$ for every $1 \leqslant i \leqslant n$. Let $\Delta = \Delta(\mathbf{d})$ be the maximum component of $\mathbf{d}$ and let $M = \sum_{i=1}^{n} d_i$. The following result follows as a corollary of [8, Theorem 1].

Theorem 1. *Let $\mathbf{d}$ be such that $\Delta^2 = o(M)$. Let H_1 and H_2 be two disjoint graphs on $[n]$ where $\mathbf{d}^{H_1} \preceq \mathbf{d}$. Suppose that $\Delta_{H_2} = O(\Delta)$ and $M - 2|H_1| = \Omega(M)$. Suppose further that $uv \notin H_1 \cup H_2$. Then,*

$$\mathbb{P}(uv \in \mathcal{G}(n, \mathbf{d}) \mid H_1^+, H_2^-) = \left(1 + O\left(\frac{\Delta^2}{M}\right)\right) \frac{(d - d_u^{H_1})(d - d_v^{H_1})}{M - 2|H_1|}.$$

One of the main results of this paper is an estimate of the above conditional probability with a relative error $\Delta^5 n/M^3$ (See Theorem 6). Due to the technical statement of the theorem, we start our discussion from a special and simpler case where $\mathbf{d} = (d, \ldots, d)$ and $H_2 = \emptyset$. Given graph H and $u \in [n]$, let $\mathcal{N}_H(u)$ denote the set of vertices that are adjacent to u in H. The theorem below gives an approximation of $\mathbb{P}(uv \in \mathcal{G}(n, d) \mid H^+)$ with a less sharp error compared to Theorem 6 below. However, it is sufficient for studying Z_H in $\mathcal{G}(n, d)$ when H is not too big.

Theorem 2. *Let H be a graph on $[n]$ where $d_i^H \leqslant d$ for every i. Suppose that $d = o(n)$, $dn - |H| = \Omega(dn)$, and that $uv \notin H$. Let $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}^H$ where $\mathbf{d} = (d, \ldots, d)$. Then,*

$$\mathbb{P}(uv \in \mathcal{G}(n, d) \mid H^+) = \frac{\widetilde{d}_u \widetilde{d}_v}{dn} \left(1 - \frac{\phi_H(uv)}{dn}\right) \left(1 + O\left(\frac{|H|}{n^2} + \frac{|H|^2}{d^2 n^2} + \frac{d^2}{n^2}\right)\right), \quad (1)$$

where

$$\phi_H(uv) = -d - 2|H| - (d-1)(d_u^H + d_v^H) + d_u^H d_v^H + \sum_{x \in \mathcal{N}_H(u)} d_x^H + \sum_{y \in \mathcal{N}_H(v)} d_y^H. \quad (2)$$

Remark 3. By the definition of $\phi_H(uv)$ it is immediate that if the maximum degree of H is $O(1)$ then

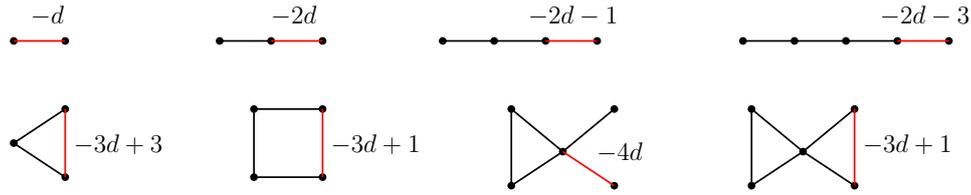$$\phi_H(uv) = -2|H| - d(1 + d_u^H + d_v^H) + O(1), \quad (3)$$

Figure 1: $\phi_H(uv)$

and this bound is sufficiently accurate in many applications. Given H, $\phi_H(uv)$ can be easily computed by examining the neighbourhood of u and v in H. In Figure 1 we give a few examples of the value of $\phi_H(uv)$. Edge uv is coloured red, and edges in H are coloured black. The value of $\phi_H(uv)$ is given next to the edge uv.

Remark 4. We check that $\mathbb{P}(uv \in \mathcal{G}(n,d))$ from Theorem 2 agrees with what we know about this probability. By Theorem 2,

$$\mathbb{P}(uv \in \mathcal{G}(n,d)) = \frac{d}{n}\left(1 + \frac{d}{dn}\right)(1 + O(d^2/n^2)) = \frac{d}{n-1}(1 + O(d^2/n^2)),$$

whereas by symmetry, we know that $\mathbb{P}(uv \in \mathcal{G}(n,d)) = d/(n-1)$. Hence the estimate from the theorem is accurate up to a relative error $O(d^2/n^2)$, as it is supposed to.

The following corollary on the upper bound of $\mathbb{P}\big(uv \in \mathcal{G}(n,d) \mid H^+\big)$ will be useful in the study of the distribution of the number of triangles. The proof of the corollary is given in Section 5.1.

Corollary 5. *Let $\mathbf{d} = (d, d, \ldots, d)$ where dn is even and $d = o(n)$. Let H be a graph on $[n]$ with degree sequence $\mathbf{d}^H \preceq \mathbf{d}$ such that $dn - 2|H| = \Omega(dn)$. Suppose $F \subseteq K_n \setminus H$ where $|F| = O(1)$. Let $\lambda_F = \mathbb{P}(F \subseteq \mathcal{G}(n,d))$ and $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}^H$. Then,*

$$\mathbb{P}\Big(F \subseteq \mathcal{G}(n,d) \mid H^+\Big) \;\leqslant\; \lambda_F\left(1 + O\left(\frac{1}{n} + \frac{|H|}{dn} + \frac{d^2}{n^2}\right)\right). \tag{4}$$

Theorem 2 is a special and less precise version of the following more general result. Given $\mathbf{d}$, recall that $M = M(\mathbf{d}) = \sum_{i=1}^{n} d_i$. Let $M_j = M_j(\mathbf{d}) = \sum_{i=1}^{n}(d_i)_j$ for any integer $j \geqslant 2$.

Theorem 6. *Let $\mathbf{d} = (d_1, \ldots, d_n)$ be such that M is even and $\Delta^2 = o(M)$, where $\Delta = \Delta(\mathbf{d})$. Let H_1 and H_2 be disjoint graphs with degree sequences $\mathbf{d}^{H_1} \preceq \mathbf{d}$ and $\Delta_{H_2} = O(\Delta)$. Let $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}^{H_1}$ and let $\widetilde{M} = M(\widetilde{\mathbf{d}})$ and $\widetilde{M_j} = M_j(\widetilde{\mathbf{d}})$ for $j \geqslant 2$. Suppose $\widetilde{M} = \Omega(M)$, and $uv \notin H_1 \cup H_2$. Then*

$$\mathbb{P}\Big(uv \in \mathcal{G}(n,\mathbf{d}) \mid H_1^+, H_2^-\Big)$$
$$= \left(1 + O\left(\frac{\Delta^5 n}{M^3}\right)\right)\frac{\widetilde{d_u}\widetilde{d_v}}{\widetilde{M}}\left(1 - \frac{\widetilde{M_2}^2}{\widetilde{M}^3} - \frac{\widetilde{M_2}}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}}\frac{\widetilde{d_x}\widetilde{d_y}}{\widetilde{M}^2}\right)\left(1 - \frac{\overline{\phi}_{H_1,H_2}(uv)}{\widetilde{M}}\right),$$

where

$$\bar{\phi}_{H_1,H_2}(uv) = -2(\widetilde{d}_u + \widetilde{d}_v) + 2 - \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x - \sum_{y \in \mathcal{N}_{H_1 \cup H_2}(v)} \widetilde{d}_y - \frac{(\widetilde{d}_u + \widetilde{d}_v - 2)\widetilde{M_2}}{\widetilde{M}} + \widetilde{d}_u \widetilde{d}_v,$$

and $\mathcal{W}$ is the set of (x, y) such that $xy \in H_1 \cup H_2$, $xu, yv \notin H_1 \cup H_2$, and $\{x, y\} \cap \{u, v\} = \emptyset$.

Remark 7. To study Z_H for large H, for instance, if H is a perfect matching, then the error $O(|H|^2/d^2n^2)$ in Theorem 2 is too large, as the cumulative error becomes $O(|H|^3/d^2n^2) = O(n/d^2)$ which is $\Omega(1)$ for $d = O(\sqrt{n})$. In such cases Theorem 6 needs to be applied.

Remark 8. A result by McKay [16, Theorem 4.6] applies to find the probability $\mathbb{P}(H \subseteq \mathcal{G}(n, d))$ with a relative error d^3/n. Theorem 6 applies to find this probability with a smaller relative error $d^2|H|/n^2$.

2.2 Proof of Theorem 2

We prove that Theorem 2 follows as a corollary of Theorem 6. Apply Theorem 6 with $M = dn$, $H_1 = H$ and $H_2 = \emptyset$, and note that $\widetilde{M} = M - 2|H|$, $\widetilde{M_2} = M_2 + O(d|H|)$, and thus,

$$\frac{\widetilde{M_2^2}}{\widetilde{M}^3} = \frac{(d-1)^2}{dn}\left(1 + O\left(\frac{|H|}{dn}\right)\right), \quad \frac{\widetilde{M_2}}{\widetilde{M}^2} = \frac{d-1}{dn}\left(1 + O\left(\frac{|H|}{dn}\right)\right),$$

and

$$\frac{\widetilde{M_2}}{\widetilde{M}} = (d-1)\left(1 - O\left(\frac{|H|}{dn}\right)\right), \quad \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y}{\widetilde{M}^2} = O\left(\frac{d^2|H|}{M^2}\right) = O\left(\frac{|H|}{n^2}\right).$$

By letting $\bar{\phi}_H = \bar{\phi}_{H,\emptyset}$ and using the approximation of $\widetilde{M_2}/\widetilde{M}$ above, it is straightforward to find that $\bar{\phi}_H(uv) = \phi_H(uv) + d - d^2 + 2|H| + O(d|H|/n)$. Hence,

$$\frac{1}{dn - 2|H|}\left(1 - \frac{\bar{\phi}_H(uv)}{dn - 2|H|}\right)$$
$$= \frac{1}{dn}\left(1 + \frac{2|H|}{dn} + O\left(\frac{|H|^2}{d^2n^2}\right)\right)\left(1 - \frac{\bar{\phi}_H(uv)}{dn} + O\left(\frac{\bar{\phi}_H(uv)}{dn}\frac{|H|}{dn}\right)\right)$$
$$= \frac{1}{dn}\left(1 - \frac{\bar{\phi}_H(uv) - 2|H|}{dn} + O\left(\frac{|H|^2}{d^2n^2} + \frac{|H|}{n^2}\right)\right)$$
$$= \frac{1}{dn}\left(1 - \frac{\phi_H(uv) + d - d^2 + O(d|H|/n)}{dn} + O\left(\frac{|H|^2}{d^2n^2} + \frac{|H|}{n^2}\right)\right)$$

as $\bar{\phi}_H(uv) = O(d^2)$. Moreover,

$$1 - \frac{\widetilde{M_2^2}}{\widetilde{M}^3} - \frac{\widetilde{M_2}}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y}{\widetilde{M}^2} = \left(1 - \frac{d-1}{n}\right)\left(1 + O\left(\frac{|H|}{n^2}\right)\right).$$

Theorem 2 follows by noting further that $d^5n/M^3 = O(d^2/n^2)$ and

$$\left(1 - \frac{\phi_H(uv) + d - d^2}{dn}\right)\left(1 - \frac{d-1}{n}\right) = \left(1 - \frac{\phi_H(uv)}{dn}\right)\left(1 + O\left(\frac{d^2}{n^2}\right)\right). \quad \blacksquare$$

2.3 Expectation and variance of strictly balanced subgraphs

Let H be a fixed graph. Let

$$t = |V(H)| \quad \text{and} \quad h = |H|.$$

Let Z_H denote the number of subgraphs of $\mathcal{G}(n, d)$ that are isomorphic to H; each such subgraph is called a copy of H. Let

$$\mu_H = \mathbb{E}Z_H, \quad \text{and} \quad \sigma_H^2 = \mathbf{Var}Z_H.$$

Define, for each $1 \leqslant j \leqslant h - 1$,

$$\rho_H(j) = \sup\left\{\rho : |V(H')| \geqslant t\left(\frac{|H|'}{|H|} + \rho\right) \text{ for all subgraphs } H' \text{ of } H \text{ where } |H'| = j\right\}.$$

See Example 10 below for values of ρ_H where H is a cycle or a clique.

Given a real number x and a nonnegative integer k, let $(x)_k = \prod_{j=0}^{k-1}(x - j)$. Let $\mathrm{aut}(H)$ denote the size of the automorphism group of H.

Theorem 9. *Let H be a fixed graph with t vertices and h edges. Then, the expectation of Z_H is*

$$\mu_H = \frac{(n)_t}{\mathrm{aut}(H)}\frac{\varphi(d, H)}{(dn)^h}\left(1 - \frac{A_H d + B_H}{dn} + O(d^2/n^2)\right),$$

where

$$\varphi(d, H) = \prod_{v \in V(H)}(d)_{d_v^H},$$

and A_H and B_H are constants depending only on H. Suppose further that H is strictly balanced and

$$\frac{d^{h-1}}{n^{h-t+1}} = o(1), \quad \frac{d^{h+2}}{n^{h-t+2}} = o(1), \quad \frac{\mu_H^{1-j/h}}{n^{t\rho_H(j)}} = o(1), \text{ for every integer } 1 \leqslant j \leqslant h - 1.$$

Then,

$$\sigma_H^2 \sim \mu_H \sim \frac{(n)_t}{\mathrm{aut}(H)}\frac{\varphi(d, H)}{(dn)^h}.$$

Example 10. *(Cycles)* Let $H = C_\ell$ where $\ell \geqslant 3$. Then, $\rho_{C_\ell}(j) = 1/\ell$ for every $1 \leqslant j \leqslant \ell - 1$. The assumptions of Theorem 9 are satisfied when $d^{\ell-1} = o(n)$ and $d^{\ell+2} = o(n^2)$. In particular, for $H = K_3$, the assumptions of Theorem 9 are satisfied when $d = o(n^{2/5})$.

(Cliques) Let $H = K_t$ where $t \geqslant 3$. Then, $\rho_{K_t}(j) = \frac{s(t-1)-2j}{t(t-1)}$ for every $\binom{s-1}{2} < j \leqslant \binom{s}{2}$. The assumptions of Theorem 9 are equivalent to

$$d \ll n^{1-2/(t+1)}, \quad d \ll n^{1-2t/(t^2-t+4)}, \quad d \ll n^{(t+s-3)/(t+s-1)},$$

which reduce to $d \ll n^{1-2t/(t^2-t+4)}$.

(Trees) Theorem 9 does not cover trees, as $d^{h-1} = o(n^{h-t+1})$ cannot be satisfied. Indeed the variance of Z_H cannot be of order μ_H when H is a tree. Consider $H = P_3$ to be a path of length three. Then $Z_H = d(d-1)^2 n - 6Z_{C_3}$, and therefore $\sigma_{P_3}^2 = 36\sigma_{C_3}^2 \sim 36\mu_{C_3}$, which is much smaller than μ_{P_3}.

2.4 Distribution of the number of triangles

Theorem 11. *Suppose $d = O(\sqrt{n})$ and $d = \omega(1)$. Let Z_{C_3} be the number of triangles in $\mathcal{G}(n, d)$. Then,*

$$\frac{Z_{C_3} - \mu_{C_3}}{\sqrt{\mu_{C_3}}} \xrightarrow{d} \mathcal{N}(0, 1), \quad as \ n \to \infty.$$

We will prove Theorem 6 in Section 3. The variance of Z_H for strictly balanced H will be estimated in Section 4. Finally, the limiting normal distribution of the number of triangles will be proved in Section 5. The proof for the case where $d = o(\sqrt{n})$ is given in Section 5.2 and the case where $d = \Theta(\sqrt{n})$ is treated in Section 5.3. The proof for Corollary 5 is presented in Section 5.1.

3 Proof of Theorem 6

The proof is obtained by expressing $\mathbb{P}(uv \in \mathcal{G}(n, \mathbf{d}) \mid H_1^+, H_2^-)$ by a function involving such conditional probabilities for other pairs $u'v'$. Applying Theorem 1 for pairs $u'v'$ yields a new estimate for $\mathbb{P}(uv \in \mathcal{G}(n, \mathbf{d}) \mid H_1^+, H_2^-)$ with improved error. By repeatedly applying this argument it is possible to further improve the relative error and to relax conditions on H_1 and H_2. However, the expression of the conditional probability would become more complicated after each iteration. We did not attempt this in the paper.

Let $\mathcal{G}$ be the class of graphs G in $\mathcal{G}(n, \mathbf{d})$ such that $H_1 \subseteq G$ and $H_2 \cap G = \emptyset$. Let $\mathcal{G}^+ \subseteq \mathcal{G}$ be the set of graphs containing edge uv and let $\mathcal{G}^- = \mathcal{G} \setminus \mathcal{G}^+$. We will estimate $|\mathcal{G}^+|/|\mathcal{G}^-|$ by defining switchings that relate graphs in $\mathcal{G}^+$ to graphs in $\mathcal{G}^-$ and counting the number of switchings that can be applied to a graph $G \in \mathcal{G}^+$, and the number of switchings that can produce a graph $G' \in \mathcal{G}^-$.

Define the switching as follows. Given $G \in \mathcal{G}^+$, a forward switching specifies an ordered pair of vertices (x, y) such that

(a) xy in $G \setminus H_1$ and u, v, x and y are all distinct;

(b) none of ux and vy is in $G \cup H_2$.

Then the forward switching replaces edges uv and xy by ux and vy. The resulting graph G' is obviously in $\mathcal{G}^-$. The inverse operation which converts G' to G is called a backward switching. See Figure 2 for an illustration.

Let $f(G)$ be the number of forward switchings that can be applied to G. The number of ways to choose (x, y) such that $xy \in G \setminus H_1$ is $\widetilde{M}$. Among these, the number of choices where $x \in \{u, v\}$ is $\widetilde{d}_u + \widetilde{d}_v$, and there are the same number of choices where $y \in \{u, v\}$. The number of choices where $\{x, y\} = \{u, v\}$ is 2. Thus, by inclusion-exclusion, the number of choices for (x, y) satisfying (a) is $\widetilde{M} - 2(\widetilde{d}_u + \widetilde{d}_v) + 2$. To bound $f(G)$ we will subtract the number of choices for (x, y) satisfying (a) but not (b).

Let X_u be the number of choices of (x, y) satisfying (a) where ux is an edge in $G \cup H_2$, and let X_v be the number of choices of (x, y) satisfying (a) where vy is an edge in $G \cup H_2$.
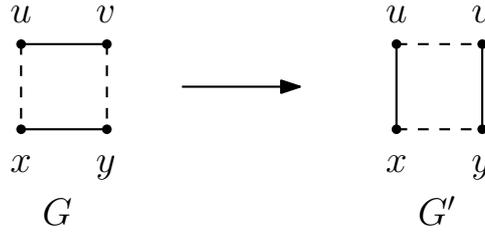
Figure 2: Forward switching

Let X_{uv} be the number of choices of (x, y) satisfying (a) where both ux and vy are edges in $G \cup H_2$ and $xy \in G \setminus H_1$. Then,

$$f(G) = \widetilde{M} - 2(\widetilde{d}_u + \widetilde{d}_v) + 2 - X_u - X_v + X_{uv}.$$

Claim 12. $\mathbb{E}X_{uv} = O(\Delta^4/M)$ and

$$\mathbb{E}X_u = \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x + \frac{(\widetilde{d}_u - 1)\widetilde{M}_2}{\widetilde{M}} + O(\Delta^4/M),$$

$$\mathbb{E}X_v = \sum_{y \in \mathcal{N}_{H_1 \cup H_2}(v)} \widetilde{d}_y + \frac{(\widetilde{d}_v - 1)\widetilde{M}_2}{\widetilde{M}} + O(\Delta^4/M).$$

By the claim,

$$\mathbb{E}\, f(G) = \widetilde{M} - 2(\widetilde{d}_u + \widetilde{d}_v) + 2 - \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x - \sum_{y \in \mathcal{N}_{H_1 \cup H_2}(v)} \widetilde{d}_y$$

$$- \frac{(\widetilde{d}_u + \widetilde{d}_v - 2)\widetilde{M}_2}{\widetilde{M}} + O\left(\frac{\Delta^4}{M}\right)$$

$$= \left(1 + O\left(\frac{\Delta^4}{M^2}\right)\right) \cdot \Sigma,$$

where

$$\Sigma = \widetilde{M} - 2(\widetilde{d}_u + \widetilde{d}_v) + 2 - \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x - \sum_{y \in \mathcal{N}_{H_1 \cup H_2}(v)} \widetilde{d}_y - \frac{(\widetilde{d}_u + \widetilde{d}_v - 2)\widetilde{M}_2}{\widetilde{M}}.$$

Next, we estimate $b(G')$, the number of backward switchings that can be applied to G'. To perform a backward switching, we choose x and y such that $xu \in G' \setminus H_1$ and $yv \in G' \setminus H_1$ and $xy \notin G' \cup H_2$. Then replace ux, yv by uv and xy. The number of ways to choose (x, y) such that $xu \in G' \setminus H_1$ and $yv \in G' \setminus H_1$ is $\widetilde{d}_u \widetilde{d}_v$. Thus, $b(G') = \widetilde{d}_u \widetilde{d}_v - Y_1 - Y_2$ where Y_1 is the number of pairs (x, y) such that $xu \in G' \setminus H_1$, $yv \in G' \setminus H_1$ and $xy \in G' \cup H_2$, and Y_2 is the number of vertices x such that $xu \in G' \setminus H_1$ and $xv \in G' \setminus H_1$. To obtain upper and lower bounds for $\mathbb{E}b(G')$ it is sufficient to estimate $\mathbb{E}Y_1$ and $\mathbb{E}Y_2$.

Claim 13.

$$\mathbb{E}Y_1 = \frac{\widetilde{d}_u\widetilde{d}_v\widetilde{M_2^2}}{\widetilde{M}^3} + \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y\widetilde{d}_u\widetilde{d}_v}{\widetilde{M}^2} + O\left(\widetilde{d}_u\widetilde{d}_v\frac{\Delta^5 n}{M^3}\right)$$

$$\mathbb{E}Y_2 = \frac{\widetilde{d}_u\widetilde{d}_v}{\widetilde{M}^2}\widetilde{M}_2 + O\left(\widetilde{d}_u\widetilde{d}_v\frac{\Delta^3}{M^2}\right).$$

By the claim and noting that $\Delta^3/M^2 = O(\Delta^5 n/M^3)$ we have

$$\mathbb{E}\,b(G') = \widetilde{d}_u\widetilde{d}_v - \frac{\widetilde{d}_u\widetilde{d}_v\widetilde{M_2^2}}{\widetilde{M}^3} - \frac{\widetilde{d}_u\widetilde{d}_v\widetilde{M}_2}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y\widetilde{d}_u\widetilde{d}_v}{\widetilde{M}^2} + O\left(\widetilde{d}_u\widetilde{d}_v\frac{\Delta^5 n}{M^3}\right)$$

$$= \widetilde{d}_u\widetilde{d}_v\left(1 - \frac{\widetilde{M_2^2}}{\widetilde{M}^3} - \frac{\widetilde{M}_2}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y}{\widetilde{M}^2}\right)\left(1 + O\left(\frac{\Delta^5 n}{M^3}\right)\right).$$

Note that $\sum_{G\in\mathcal{G}^+} f(G) = \sum_{G'\in\mathcal{G}^-} b(G')$ and thus $|\mathcal{G}^+|\cdot\mathbb{E}f(G) = |\mathcal{G}^-|\cdot\mathbb{E}b(G')$. Therefore, noting that $\Delta^4/M^2 = O(\frac{\Delta^5 n}{M^3})$ we have

$$\mathbb{P}\left(uv \in \mathcal{G}(n,d) \mid H_1^+, H_2^-\right) = \frac{|\mathcal{G}^+|}{|\mathcal{G}^+| + |\mathcal{G}^-|} = \frac{\mathbb{E}\,b}{\mathbb{E}\,f + \mathbb{E}\,b}$$

$$= \frac{\widetilde{d}_u\widetilde{d}_v\left(1 - \frac{\widetilde{M_2^2}}{\widetilde{M}^3} - \frac{\widetilde{M}_2}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y}{\widetilde{M}^2}\right)\left(1 + O\left(\frac{\Delta^5 n}{M^3}\right)\right)}{\widetilde{M} - 2(\widetilde{d}_u + \widetilde{d}_v) + 2 - \sum_{x\in\mathcal{N}_{H_1\cup H_2}(u)} \widetilde{d}_x - \sum_{y\in\mathcal{N}_{H_1\cup H_2}(v)} \widetilde{d}_y - \frac{(\widetilde{d}_u+\widetilde{d}_v-2)\widetilde{M}_2}{\widetilde{M}} + \widetilde{d}_u\widetilde{d}_v}$$

$$= \frac{\widetilde{d}_u\widetilde{d}_v}{\widetilde{M}}\left(1 - \frac{\widetilde{M_2^2}}{\widetilde{M}^3} - \frac{\widetilde{M}_2}{\widetilde{M}^2} - \sum_{(x,y)\in\mathcal{W}} \frac{\widetilde{d}_x\widetilde{d}_y}{\widetilde{M}^2}\right)\left(1 - \frac{\bar{\phi}_{H_1,H_2}(uv)}{dn - 2|H_1|}\right)\left(1 + O\left(\frac{\Delta^5 n}{M^3}\right)\right). \quad \blacksquare$$

Now we prove Claims 12 and 13.

Proof of Claim 12. First we estimate $\mathbb{E}X_u$. Let $X_{u,1}$ be the number of (x,y) such that $ux \in H_1 \cup H_2$, $xy \in G \setminus H_1$ and $y \notin \{u,v\}$, $X_{u,2}$ be the number of (x,y) such that $ux \in G \setminus H_1$ and $xy \in G \setminus H_1$ and $y \neq u$, and $X_{u,3}$ be the number of (x,y) such that $ux \in G \setminus H_1$ and $xy \in G \setminus H_1$ and $y = v$. Then $X_u = X_{u,1} + X_{u,2} - X_{u,3}$. Obviously,

$$X_{u,1} = \sum_{x\in\mathcal{N}_{H_1\cup H_2}(u)} \widetilde{d}_x - X',$$

where X' is the number of (x,y) such that $ux \in H_1 \cup H_2$, $xy \in G \setminus H_1$ and $y = v$. Then, by Theorem 1,

$$\mathbb{E}X' = \sum_{\substack{x\in\mathcal{N}_{H_1\cup H_2}(u) \\ xv\notin H_1}} \mathbb{P}(xv \mid (H_1 \cup \{uv\})^+, H_2^-) = O(\Delta_{H_1\cup H_2}\Delta^2/M) = O(\Delta^3/M).$$

Thus, $\mathbb{E} X_{u,1} = \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x + O(\Delta^3/M)$. On the other hand,

$$\mathbb{E} X_{u,2} = \sum_{x \in [n] \setminus (\{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))} \mathbb{P}(ux \mid (H_1 \cup \{uv\})^+, H_2^-) \cdot (\widetilde{d}_x - 1).$$

Again by Theorem 1,

$$\mathbb{E} X_{u,2} = \sum_{x \in [n] \setminus (\{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))} \frac{(\widetilde{d}_u - 1)\widetilde{d}_x}{\widetilde{M}} (1 + O(\Delta^2/M)) \cdot (\widetilde{d}_x - 1)$$

$$= (\widetilde{d}_u - 1) \sum_{x \in [n] \setminus (\{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))} \frac{(\widetilde{d}_x)_2}{\widetilde{M}} + O(\Delta^4/M)$$

$$= \frac{\widetilde{d}_u - 1}{\widetilde{M}} \left(\widetilde{M}_2 - \sum_{z \in \{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u)} (\widetilde{d}_z)_2 \right) + O(\Delta^4/M)$$

$$= \frac{(\widetilde{d}_u - 1)\widetilde{M}_2}{\widetilde{M}} + O\left(\Delta^4/M\right).$$

Finally, by Theorem 1

$$\mathbb{E} X_{u,3} = \sum_{x \in [n] \setminus (\{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))} \mathbb{P}(ux, vx \mid (H_1 \cup \{uv\})^+, H_2^-)$$

$$\leqslant (1 + O(\Delta^2/M)) \sum_{x \in [n]} \frac{d_u d_v (d_x)_2}{\widetilde{M}^2} = O(\Delta^3/M).$$

Combining all together, we have

$$\mathbb{E} X_u = \sum_{x \in \mathcal{N}_{H_1 \cup H_2}(u)} \widetilde{d}_x + \frac{(\widetilde{d}_u - 1)\widetilde{M}_2}{\widetilde{M}} + O\left(\frac{\Delta^4}{M}\right),$$

as $\Delta_{H_1} \leqslant \Delta$ and $\Delta_{H_2} = O(\Delta)$ by assumption. By symmetry,

$$\mathbb{E} X_v = \sum_{y \in \mathcal{N}_{H_1 \cup H_2}(v)} \widetilde{d}_y + \frac{(\widetilde{d}_v - 1)\widetilde{M}_2}{\widetilde{M}} + O\left(\frac{\Delta^4}{M}\right).$$

Finally, we bound $\mathbb{E} X_{uv}$ by $O(\Delta^4/M)$. First we expose the neighbours of u and v in G. Next, for each $x \in \mathcal{N}_{G \cup H_2}(u)$ and $y \in \mathcal{N}_{G \cup H_2}(v)$, if $x \neq y$ and $xy \notin H_1$, the probability that $xy \in \mathcal{G}(n, \mathbf{d})$, conditional on H_1^+, H_2^-, and the presence of the exposed edges incident with u and v, is at most $(1 + O(\Delta^2/M))d^2/(\widetilde{M} - 4\Delta) = O(\Delta^2/M)$. There are at most $O(\Delta^2)$ such pairs of (x, y), as $\Delta_{H_2} = O(\Delta)$. Thus,

$$\mathbb{E} X_{uv} = O(\Delta^2) \cdot O(\Delta^2/M) = O(\Delta^4/M). \quad \blacksquare$$

Proof of Claim 13. Let $Y_{1,1}$ be the number of pairs (x, y) such that $xu \in G' \setminus H_1$ and $yv \in G' \setminus H_1$ and $xy \in H_1 \cup H_2$ and $Y_{1,2}$ be the number of pairs (x, y) such that $xu \in G' \setminus H_1$ and $yv \in G' \setminus H_1$ and $xy \in G' \setminus H_1$. Then $Y_1 = Y_{1,1} + Y_{1,2}$. We first estimate $\mathbb{E} Y_{1,1}$. Recall that $\mathcal{W}$ is the set of (x, y) such that $xy \in H_1 \cup H_2$, $xu, yv \notin H_1 \cup H_2$, and $\{x, y\} \cap \{u, v\} = \emptyset$.

Now,
$$\mathbb{E} Y_{1,1} = \sum_{(x,y) \in \mathcal{W}} \mathbb{P}(ux, vy \in \mathcal{G}(n, d) \mid H_1^+, (H_2 \cup \{uv\})^-).$$

By Theorem 1,

$$\mathbb{E} Y_{1,1} = (1 + O(\Delta^2/M)) \sum_{(x,y) \in \mathcal{W}} \frac{\widetilde{d}_u \widetilde{d}_x}{\widetilde{M}} \frac{\widetilde{d}_y \widetilde{d}_v}{\widetilde{M}} = \sum_{(x,y) \in \mathcal{W}} \frac{\widetilde{d}_u \widetilde{d}_x}{\widetilde{M}} \frac{\widetilde{d}_y \widetilde{d}_v}{\widetilde{M}} + O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^5 n}{M^3}\right), \quad (5)$$

since $|\mathcal{W}| = O(\Delta n)$. Similarly,

$$\mathbb{E} Y_{1,2} = \sum \frac{\widetilde{d}_u (\widetilde{d}_x)_2 (\widetilde{d}_y)_2 \widetilde{d}_v}{\widetilde{M}^3} \left(1 + O\left(\frac{\Delta^2}{M}\right)\right) = \sum \frac{\widetilde{d}_u (\widetilde{d}_x)_2 (\widetilde{d}_y)_2 \widetilde{d}_v}{\widetilde{M}^3} + O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^5 n}{M^3}\right),$$

where the summation is over all pairs (x, y) such that $x \in [n] \setminus (\{u, v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))$ and $y \in [n] \setminus (\{u, v, x\} \cup \mathcal{N}_{H_1 \cup H_2}(v) \cup \mathcal{N}_{H_1 \cup H_2}(x))$. Thus,

$$\mathbb{E} Y_{1,2} = O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^5 n}{M^3}\right) + \frac{\widetilde{d}_u \widetilde{d}_v}{\widetilde{M}^3}\left(\widetilde{M}_2^2 - a_1 - a_2\right),$$

where

$$a_1 = \widetilde{M}_2 \left(\sum_{x \in \{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u)} (\widetilde{d}_x)_2 + \sum_{j \in \{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(v)} (\widetilde{d}_y)_2\right)$$
$$- \sum_{x \in \{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u)} (\widetilde{d}_x)_2 \sum_{j \in \{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(v)} (\widetilde{d}_y)_2 = O(\Delta^5 n),$$

and $a_2 = \sum (\widetilde{d}_x)_2 (\widetilde{d}_y)_2$, where the sum is over all pairs (x, y) such that $y \in \{x\} \cup \mathcal{N}_{H_1 \cup H_2}(x)$ in addition to $x \in [n] \setminus (\{u, v\} \cup \mathcal{N}_{H_1 \cup H_2}(u))$ and $y \in [n] \setminus (\{u, v\} \cup \mathcal{N}_{H_1 \cup H_2}(v))$. Trivially, $a_2 = O(\Delta^4 |H_1 \cup H_2|) = O(\Delta^5 n)$. Thus,

$$\mathbb{E} Y_{1,2} = \frac{\widetilde{d}_u \widetilde{d}_v \widetilde{M}_2^2}{\widetilde{M}^3} + O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^5 n}{M^3}\right).$$

It follows now that

$$\mathbb{E} Y_1 = \frac{\widetilde{d}_u \widetilde{d}_v \widetilde{M}_2^2}{\widetilde{M}^3} + \sum_{(x,y) \in \mathcal{W}} \frac{\widetilde{d}_u \widetilde{d}_v \widetilde{d}_x \widetilde{d}_y}{\widetilde{M}^2} + O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^5 n}{M^3}\right).$$

With a similar but simpler argument, we have

$$
\begin{aligned}
\mathbb{E}Y_2 &= (1 + O(\Delta^2/M)) \sum_{x \in [n] \setminus (\{u,v\} \cup \mathcal{N}_{H_1 \cup H_2}(u) \cup \mathcal{N}_{H_1 \cup H_2}(v))} \frac{\widetilde{d}_u \widetilde{d}_v (\widetilde{d}_x)_2}{\widetilde{M}^2} \\
&= (1 + O(\Delta^2/M)) \frac{\widetilde{d}_u \widetilde{d}_v}{\widetilde{M}^2} \left(\widetilde{M}_2 - O(\Delta^3) \right) = \frac{\widetilde{d}_u \widetilde{d}_v \widetilde{M}_2}{\widetilde{M}^2} + O\left(\widetilde{d}_u \widetilde{d}_v \frac{\Delta^3}{M^2} \right). \quad \blacksquare
\end{aligned}
$$

4 Variance of the number of strictly balanced subgraphs

Let H be a fixed graph with t vertices and h edges. Recall that Z_H denotes the number of subgraphs of $\mathcal{G}(n,d)$ that are isomorphic to H, and $\mu_H = \mathbb{E}Z_H$, $\sigma_H^2 = \mathbf{Var}Z_H$. The goal of this section is to estimate σ_H^2 for strictly balanced H. We start by estimating μ_H.

4.1 Expectation

Given a graph H, let

$$
\varphi(d, H) = \prod_{v \in V(H)} (d)_{d_v^H}.
$$

Let J be a copy of H in K_n, and let

$$
\lambda_H = \mathbb{P}(J \subseteq \mathcal{G}(n,d)).
$$

By Theorem 2,

$$
\lambda_H = \frac{\varphi(d, H)}{(dn)^h} \left(1 - \frac{A_H d + B_H}{dn} + O(d^2/n^2) \right), \tag{6}
$$

where A_H and B_H are constants that depend only on H. The values of A_H and B_H are given for H being a cycle in Example 14 below. By linearity of expectation,

$$
\mu_H = \frac{(n)_t}{\mathrm{aut}(H)} \lambda_H, \tag{7}
$$

where $\mathrm{aut}(H)$ denotes the size of the automorphism group of H. Hence,

$$
\mu_H = \Theta(n^t (d/n)^h) = \Theta(d^h n^{t-h}).
$$

Example 14. Let $H = C_\ell$, a cycle of length $\ell \geqslant 3$. By Theorem 2,

$$
\lambda_{C_\ell} = \left(\frac{d-1}{n} \right)^\ell \left(1 - \sum_{i=1}^\ell \frac{\phi_i}{dn} + O(d^2/n^2) \right),
$$

where

$$
\begin{aligned}
\phi_1 &= -d, \\
\phi_2 &= -2d, \\
\phi_i &= -2d - 2(i-3) - 1 \text{ for all } 3 \leqslant i \leqslant \ell - 1, \\
\phi_\ell &= -3d + 3 - 2(\ell - 3).
\end{aligned}
$$

In particular,

$$\lambda_{C_3} = \left(\frac{d-1}{n}\right)^3 \left(1 + \frac{6d-3}{dn} + O(d^2/n^2)\right).$$ (8)

4.2 Variance of Z_{C_3}

In this section, we calculate $\sigma_{C_3}^2$ in detail. In the next section, we will briefly sketch how to extend the argument for C_3 to other strictly balanced graphs. Let $N = \binom{n}{3}$, and let $H_1, \ldots, H_N$ be an enumeration of all copies of triangles in K_n. Let $X_i = 1_{\{H_i \subseteq \mathcal{G}(n,d)\}}$. Then $Z_{C_3} = \sum_{i=1}^N X_i$. Moreover, $\mathbb{E}X_i = \lambda$ for every i where $\lambda = \lambda_{C_3}$ has been estimated in (8). Hence,

$$\sigma_{C_3}^2 = \mathbb{E}(Z_{C_3} - \mu_{C_3})^2 = \mathbb{E}\left(\sum_{i=1}^N (X_i - \lambda)\right)^2 = \sum_{(i,j)\in[N]^2} (\mathbb{E}X_iX_j - \lambda^2).$$ (9)

We split the above sum into four parts according to how H_i and H_j intersect.

(a) H_i and H_j are vertex disjoint. For such pairs (i,j), by Theorem 2 and (2),

$$\mathbb{E}X_iX_j = \left(\frac{d-1}{n}\right)^6 \left(1 + \frac{12d+12}{dn} + O(d^2/n^2)\right).$$

Thus,

$$\mathbb{E}X_iX_j - \lambda^2 = \left(\frac{d-1}{n}\right)^6 \frac{18}{dn} + O(d^8/n^8).$$

The number of vertex disjoint pairs (H_i, H_j) is $\binom{n}{3}\binom{n-3}{3} = (n)_6/36$. Hence, the contribution to (9) from part (a) is

$$\frac{(n)_6}{36}\left(\left(\frac{d-1}{n}\right)^6 \frac{18}{dn} + O(d^8/n^8)\right) = \left(\frac{d-1}{n}\right)^6 \frac{(n)_6}{2dn} + O(d^8/n^2).$$ (10)

(b) H_i and H_j share exactly one vertex. For such pairs,

$$\begin{aligned}
\mathbb{E}X_iX_j &= \frac{(d(d-1))^4 (d)_4}{(dn)^6}\left(1 + \frac{16d-4}{dn} + O\left(d^2/n^2\right)\right) \\
&= \frac{(d-1)^5(d-2)(d-3)}{dn^6}\left(1 + \frac{16d-4}{dn} + O\left(d^2/n^2\right)\right).
\end{aligned}$$

Thus,

$$\mathbb{E}X_iX_j - \lambda^2 = -\frac{2(2d-3)(d-1)^5}{dn^6} + O(d^6/n^7 + d^8/n^8).$$

The number of pairs (H_i, H_j) sharing exactly one vertex is $\binom{n}{3}3\binom{n-3}{2} = (n)_5/4$. Thus, the contribution to (9) from part (b) is

$$\frac{(n)_5}{4}\left(-\frac{2(2d-3)(d-1)^5}{dn^6} + O\left(\frac{d^6}{n^7} + \frac{d^8}{n^8}\right)\right)$$

$$= -\frac{(n)_5(2d-3)(d-1)^5}{2dn^6} + O\left(\frac{d^6}{n^2} + \frac{d^8}{n^3}\right). \qquad (11)$$

(c) H_i and H_j share exactly two vertices. For such pairs,

$$\mathbb{E}X_iX_j = \frac{(d)_2^2(d)_3^2}{(dn)^5}\left(1 + \frac{13d-11}{dn}\right) = \frac{(d-1)^4(d-2)^2}{dn^5}\left(1 + \frac{13d-11}{dn}\right).$$

Thus,

$$\mathbb{E}X_iX_j - \lambda^2 = \frac{(d-1)^4(d-2)^2}{dn^5} + O(d^6/n^6).$$

The number of such pairs (H_i, H_j) is $\binom{n}{3}3(n-3) = (n)_4/2$. Thus, the contribution to (9) from part (c) is

$$\frac{(n)_4}{2}\left(\frac{(d-1)^4(d-2)^2}{dn^5} + O(d^6/n^6)\right) = \frac{(n)_4(d-1)^4(d-2)^2}{2dn^5} + O(d^6/n^2). \qquad (12)$$

(d) $H_i = H_j$. For such pairs,

$$\mathbb{E}X_iX_j = \mathbb{E}X_i = \lambda.$$

Thus,

$$\mathbb{E}X_iX_j - \lambda^2 = \lambda - \lambda^2 = \left(\frac{d-1}{n}\right)^3\left(1 + \frac{6d-3}{dn} + O(d^2/n^2)\right) + O(d^6/n^6).$$

There are $(n)_3/6$ pairs (H_i, H_j) where $H_i = H_j$. Thus, the contribution to (9) from part (d) is

$$\frac{(n)_3}{6}\left(\left(\frac{d-1}{n}\right)^3\left(1 + \frac{6d-3}{dn}\right) + O(d^5/n^5)\right)$$

$$= \frac{(n)_3}{6}\left(\frac{d-1}{n}\right)^3\left(1 + \frac{6d-3}{dn}\right) + O\left(\frac{d^5}{n^2}\right). \qquad (13)$$

By (10)–(12), the total contribution to (9) from parts (a)–(c) is

$$-\frac{(d-1)^4(d-2)(n-1)_3}{2dn^4} + O(d^8/n^2).$$

Combining with (13) and by (9) we have

$$\sigma_{C_3}^2 = \frac{(n)_3}{6}\left(\frac{d-1}{n}\right)^3\left(1+\frac{6d-3}{dn}\right) - \frac{(d-1)^4(d-2)(n-1)_3}{2dn^4} + O(d^8/n^2)$$

$$= (1+o(1))\frac{(d-1)^3}{6} + O(d^8/n^2).$$

Since $d = o(n^{2/5})$, we have

$$d^8/n^2 = o(d^3).$$

It follows now that

$$\sigma_{C_3}^2 \sim \mu_{C_3} \sim \frac{(d-1)^3}{6}.$$

4.3 Variance of Z_H: proof of Theorem 9

Recall that

$$\rho_H(j) = \sup\left\{\rho : |V(H')| \geqslant t\left(\frac{|H|'}{|H|} + \rho\right) \; \forall H' \subseteq H \; s.t. \; H' \neq H, |H'| = j\right\}$$

and $\eta_H(j)$ is positive for every $1 \leqslant j \leqslant h-1$ as H is strictly balanced.

Recall λ_H and μ_H from (6) and (7). Let $N = \binom{(n)_t}{\text{aut}(H)}$, and let $H_1, \ldots, H_N$ be an enumeration of all copies of H in K_n. Let $X_i = 1_{\{H_i \subseteq \mathcal{G}(n,d)\}}$. Then, $Z_H = \sum_{i=1}^N X_i$. Moreover, $\mathbb{E}X_i = \lambda$ for every i where $\lambda = \lambda_H$. Hence,

$$\sigma_H^2 = \mathbb{E}(Z_H - \mu_H)^2 = \mathbb{E}\left(\sum_{i=1}^N (X_i - \lambda)\right)^2 = \sum_{(i,j)\in[N]^2}(\mathbb{E}X_iX_j - \lambda^2). \qquad (14)$$

Similarly to the case $H = C_3$, we split the above sum into four parts according to the intersection of H_i and H_j. We will give only upper bounds for the contributions from the first three parts, and estimate asymptotically the contribution from the last part. Under the hypotheses of the theorem, we verify that the contribution from the last part dominates. The calculations are similar to the case where $H = C_3$ and thus we only briefly sketch the proof.

(a) H_i and H_j are vertex disjoint. There are $O(n^{2t})$ such pairs. For each such pair,

$$\mathbb{E}X_iX_j - \lambda^2 = O(1/dn + d^2/n^2)\left(\frac{d}{n}\right)^{2h}.$$

Hence the total contribution to (14) from this part is $\mu_H \cdot O(d^{h-1}n^{t-h-1} + d^{h+2}n^{t-h-2})$.

(b) H_i and H_j are edge disjoint but share at least one vertex. There are $O(n^{2t-1})$ such pairs. For each such pair, by Theorem 2 and (3),

$$\mathbb{E}X_iX_j = \lambda \cdot \frac{\prod_{v\in H}(d - O(1))_{d_v^H}}{(dn)^h}\left(1 - \frac{O(d)}{dn} + O\left(\frac{d^2}{n^2}\right)\right)$$

$$= \lambda^2\left(1 + O\left(d^{-1} + \frac{d^2}{n^2}\right)\right).$$

Thus,

$$\mathbb{E}X_i X_j - \lambda^2 = \lambda^2 \cdot O(1/d + d^2/n^2) = \lambda \cdot O(d^{h-1}/n^h + d^{h+2}/n^{h+2}),$$

and the total contribution to (14) from this part is $\mu_H \cdot O(d^{h-1}n^{t-h-1} + d^{h+2}n^{t-h-3})$.

(c) H_i and H_j share $x \geqslant 2$ vertices and $1 \leqslant y \leqslant h - 1$ edges. There are $O(n^{2t-x})$ such pairs. For each such pair, we know $x \geqslant t(y/h + \rho_H(y))$ by definition of ρ_H. Thus,

$$\mathbb{E}X_i X_j - \lambda^2 = O(\lambda^2 + \mathbb{E}X_i X_j) = O(\lambda^2 + (d/n)^{2h-y}).$$

The total contribution to (14) from part (c) is

$$\sum_{x,y} \left(O(\lambda^2 n^{2t-x}) + O(n^{2t-x}(d/n)^{2h-y}) \right) = \mu_H \sum_{x,y} O(n^{t-x}(d/n)^{h-y})$$
$$= \mu_H \sum_{x,y} O(n^{-t\rho_H(y)} \mu_H^{1-y/h}).$$

(d) $H_i = H_j$. The total contribution to (14) from this part is

$$\frac{(n)_t}{\mathrm{aut}(H)}(\lambda - \lambda^2) \sim \mu_H \sim \frac{n^t}{\mathrm{aut}(H)} \frac{\varphi(d, H)}{(dn)^h}.$$

By the hypotheses of the theorem, the total contributions to (14) from parts (a)–(c) are $o(\mu_H)$, and the assertion of the theorem follows. ∎

5 Distribution of the number of triangles

We use the following result [10] by Z. Gao and Wormald to determine the limiting distribution of Z_{C_3}.

Theorem 15. *Let $s_n > \mu_n^{-1}$ and $\sigma_n = \sqrt{\mu_n + \mu_n^2 s_n}$, where $0 < \mu_n \to \infty$. Suppose that $\mu_n = o(\sigma_n^3)$, and a sequence (X_n) of nonnegative random variables satisfies*

$$\mathbb{E}(X_n)_k \sim \mu_n^k \exp\left(\frac{k^2 s_n}{2} \right)$$

uniformly for all integers k in the range $c\mu_n/\sigma_n \leqslant k \leqslant c'\mu_n/\sigma_n$ for some constants $c' > c > 0$. Then,

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{as } n \to \infty.$$

5.1 Proof of Corollary 5

Let $e_1, \ldots, e_{|F|}$ be an enumeration of the edges in F. Let $F_0 = \emptyset$, $H_0 = H$, $F_j = F_{j-1} \cup \{e_j\}$ and $H_j = H \cup F_j$ for every $1 \leqslant j \leqslant |F|$. Fix $1 \leqslant j \leqslant |F|$. let u and v be the ends of e_j and let $\widetilde{\mathbf{d}} = \mathbf{d} - \mathbf{d}^{H_{j-1}}$. By (1) and (3),

$$
\mathbb{P}\Big(e_j \in \mathcal{G}(n,d) \mid H_{j-1}^+\Big)
$$
$$
= \frac{\widetilde{d}_u \widetilde{d}_v}{dn} \left(1 - \frac{\phi_{H_{j-1}}(uv)}{dn}\right) \left(1 + O\left(\frac{|H_{j-1}|}{n^2} + \frac{|H_{j-1}|^2}{d^2 n^2} + \frac{d^2}{n^2}\right)\right)
$$
$$
= \left(\frac{\widetilde{d}_u \widetilde{d}_v}{dn} + O\left(\frac{d|H_{j-1}|}{n^3} + \frac{|H_{j-1}|^2}{dn^3} + \frac{d^3}{n^3}\right)\right)
$$
$$
\times \left(1 + \frac{d(1 + d_u^{H_{j-1}} + d_v^{H_{j-1}}) + O(|H_{j-1}|)}{dn}\right)
$$
$$
= \left(\frac{(d - d_u^{H_{j-1}})(d - d_v^{H_{j-1}})}{dn}\right) \left(1 + \frac{d(1 + d_u^{H_{j-1}} + d_v^{H_{j-1}})}{dn}\right) + O\left(\frac{d^3}{n^3} + \frac{|H_{j-1}|}{n^2}\right),
$$

where the last step above holds since the errors $d|H_{j-1}/n^3|$ and $|H_{j-1}|^2/dn^3$ are absorbed by $|H_{j-1}|/n^2$ as $|H_{j-1}| = O(dn)$. It is easy to verify using elementary calculus that

$$
\left(\frac{(d - d_u^{H_{j-1}})(d - d_v^{H_{j-1}})}{dn}\right) \left(1 + \frac{d(1 + d_u^{H_{j-1}} + d_v^{H_{j-1}})}{dn}\right)
$$

is a decreasing function of $d_u^{H_{j-1}}$ and $d_v^{H_{j-1}}$, and thus the above product is at most

$$
\frac{(d - d_u^{F_{j-1}})(d - d_v^{F_{j-1}})}{dn} \left(1 + \frac{d(1 + d_u^{F_{j-1}} + d_v^{F_{j-1}})}{dn}\right)
$$
$$
= \frac{(d - d_u^{F_{j-1}})(d - d_v^{F_{j-1}})}{dn} \left(1 + O(n^{-1})\right).
$$

It follows that

$$
\mathbb{P}\Big(e_j \in \mathcal{G}(n,d) \mid H_{j-1}^+\Big) \;\leqslant\; \frac{(d - d_u^{F_{j-1}})(d - d_v^{F_{j-1}})}{dn} \left(1 + O(n^{-1})\right) + O\left(\frac{d^3}{n^3} + \frac{|H_{j-1}|}{n^2}\right)
$$
$$
= \frac{(d - d_u^{F_{j-1}})(d - d_v^{F_{j-1}})}{dn} \left(1 + O\left(\frac{1}{n} + \frac{d^2}{n^2} + \frac{|H_{j-1}|}{dn}\right)\right).
$$

The assertion of the corollary follows by applying the above bound to

$$
\mathbb{P}(F \subseteq \mathcal{G}(n,d) \mid H^+) = \prod_{j=1}^{|F|} \mathbb{P}\Big(e_j \in \mathcal{G}(n,d) \mid H_{j-1}^+\Big)
$$

and then comparing the resulting expression with (6). ∎

5.2 Distribution of Z_{C_3} when $d = o(\sqrt{n})$

Let $Z = Z_{C_3}$, $\lambda = \lambda_{C_3}$ and $\mu = \mu_{C_3}$. Then, $\mu = \Theta(d^3)$. Assume $d = o(\sqrt{n})$ and $d \to \infty$. We aim to show that

$$\mathbb{E}(Z)_k \sim \mu^k,$$

for all $k = O(d^{3/2})$. Then the distribution of Z follows by Theorem 15 with $\mu_n = \mu$, $\sigma_n = \sqrt{\mu}$ and $s_n = 0$.

Let $N = \binom{n}{3}$ and let $H_1, \ldots, H_N$ be an enumeration of all copies of triangles in K_n. Let $\mathcal{F}$ be the set of ordered k-tuples $(j_1, \ldots, j_k) \in [N]^k$ that are pairwise distinct. Then, for any positive integer k,

$$\mathbb{E}(Z)_k = \sum_{(j_1, \ldots, j_k) \in \mathcal{F}} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d)\right). \tag{15}$$

Definition 16. Given a k-tuple $\mathbf{j}$, we say a triple of vertices $\{x, y, z\}$ is a *hole* induced by $j_1, \ldots, j_k$, if there exist j_{i_1}, j_{i_2} and j_{i_3}, such that $H_{j_{i_1}}$ contains xy, $H_{j_{i_2}}$ contains yz and $H_{j_{i_3}}$ contains xz, and none of $H_{j_{i_1}}$, $H_{j_{i_2}}$ and $H_{j_{i_3}}$ is xyz.

We partition the summation into the following two parts.

(a) $\mathcal{F}_1 \subseteq \mathcal{F}$: the set of $(j_1, \ldots, j_k)$ such that $H_{j_1}, \ldots, H_{j_k}$ are pairwise edge disjoint.

(b) $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$.

We will prove the following.

Lemma 17.
$$\sum_{(j_1, \ldots, j_k) \in \mathcal{F}_1} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d)\right) \sim \mu^k.$$

Lemma 18.
$$\sum_{(j_1, \ldots, j_k) \in \mathcal{F}_2} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d)\right) = o(\mu^k).$$

Now Theorem 11 follows by the above two lemmas and Theorem 15. It only remains to prove Lemmas 17 and 18.

Proof of Lemma 17. We say H_{j_i} hits v if v is incident with an edge in H_{j_i}. Given a k-tuple $H_{j_1}, \ldots, H_{j_k}$, let $h(v)$ be the number of H_j which hit v. Let h_j be the number of vertices v with $h(v) = j$. Then, given any $(j_1, \ldots, j_k) \in \mathcal{F}_1$, the number of vertices in $H^\oplus = \cup_{i=1}^k H_i$ is equal to $3k - \sum_{j \geqslant 2}(j - 1)h_j$ and the number of edges in $H^\oplus$ is $3k$.

Let $\mathcal{B}$ be the set of $(j_1, \ldots, j_k) \in [N]^k$ satisfying the following property.

$$h_j > 0 \text{ for some } j \geqslant 4; \text{ or } h_j \geqslant d^2/\sqrt{n} \text{ for some } j \in \{2, 3\}.$$

We further partition $\mathcal{F}_1$ into two parts: $\mathcal{F}_1 \setminus \mathcal{B}$ and $\mathcal{F}_1 \cap \mathcal{B}$.

Claim 19.
$$|\mathcal{F}_1 \setminus \mathcal{B}| \sim \left(\frac{(n)_3}{6}\right)^k, \text{ and } |\mathcal{F}_1 \cap \mathcal{B}| = o\left(\left(\frac{(n)_3}{6}\right)^k\right).$$

Fix an arbitrary $(j_1, \ldots, j_k) \in \mathcal{F}_1 \setminus \mathcal{B}$. Let $e_1, \ldots, e_{3k}$ be an enumeration of the edges in $H^{\oplus} = \cup_{i=1}^k H_{j_i}$, where $e_{3\ell-2}, e_{3\ell-1}, e_{3\ell}$ is the set of edges in H_{j_ℓ}. Let $F_i = \cup_{j<i} e_j$. By (1) and (3), $\phi_{F_i}(e_i) = O(i+d)$. Thus,

$$\sum_{i=1}^k \frac{\phi_{F_i}(e_i)}{dn} = \frac{O(k^2 + dk)}{dn}.$$

Since $\mathbf{j} \notin \mathcal{B}$, we have $d_u^{H^{\oplus}} = O(1)$ for every u. Hence, by Theorem 2 (with $\widetilde{d}_u, \widetilde{d}_v = d - O(1)$ and $|H| = O(k)$)

$$\mathbb{E}\left(\prod_{i=1}^k X_{H_{j_i}}\right) = \lambda^k \left(1 + O\left(\frac{h_2 + h_3}{d}\right)\right)\left(1 + O\left(\frac{k^2 + dk}{dn} + \frac{k^2 + d^2 k}{n^2} + \frac{k^3}{d^2 n^2}\right)\right) \sim \lambda^k, \tag{16}$$

where the error $O((h_2 + h_3)/d)$ accounts for the case that some H_{j_ℓ} lands on a vertex which has been occupied by some H_{j_i} where $i < \ell$ (note that for each such H_{j_ℓ}, an $O(1/d)$ error is present as $\widetilde{d}_u, \widetilde{d}_v = d - O(1)$ when Theorem 2 is applied), and the last step of asymptotic relation above holds as $k = O(d^{3/2})$ and $d = o(\sqrt{n})$. By Claim 19,

$$\sum_{(j_1,\ldots,j_k)\in\mathcal{F}_1\setminus\mathcal{B}} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) \sim \left(\frac{(n)_3}{6}\right)^k \lambda^k = \mu^k. \tag{17}$$

On the other hand, by Corollary 5, for any $(j_1, \ldots, j_k) \in \mathcal{F}_1 \cap \mathcal{B}$, we have

$$\mathbb{E}\left(\prod_{i=1}^k X_{H_{j_i}}\right) \leqslant \lambda^k \left(1 + O\left(\frac{k}{n} + \frac{k^2}{dn} + \frac{d^2 k}{n^2}\right)\right) \sim \lambda^k, \tag{18}$$

and again by Claim 19 we have

$$\sum_{(j_1,\ldots,j_k)\in\mathcal{F}_1\cap\mathcal{B}} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = o\left(\left(\frac{(n)_3}{6}\right)^k\right)\lambda^k = o(\mu^k). \tag{19}$$

Now Lemma 17 follows by (17) and (19).

Finally, we prove Claim 19. First, we prove

$$|\mathcal{F}_1 \cap \mathcal{B}| = o\left(\left(\frac{(n)_3}{6}\right)^k\right). \tag{20}$$

For each $j \geqslant 1$ let $\mathcal{C}_j(i)$ be the set of k-tuples with $h_j = i$. Then, $[N]^k = \cup_{i \geqslant 0} \mathcal{C}_j(i)$ for each j. Define a switching from $\mathcal{C}_j(i)$ to $\mathcal{C}_j(i-1)$ as follows. Choose v and $i_1 < i_2 < \cdots < i_j$

such that $h(v) = j$ and $H_{i_1}, \ldots, H_{i_j}$ all use v. There are exactly i ways to choose them. Then, we choose $j-1$ distinct vertices $v_1 < \cdots < v_{j-1}$, such that $h(v_\ell) = 0$ for all $1 \leqslant \ell \leqslant j-1$. Modify H_{i_ℓ} for each $2 \leqslant \ell \leqslant j$ so that v is replaced by $v_{\ell-1}$. The resulting k-tuple is in $\mathcal{C}_j(i-1)$. The number of ways to perform a switching is at least $i(n-3k)_{j-1}$. On the other hand, the number of inverse switchings is at most $(k)_j$. Hence, as $k = O(d^{3/2})$,

$$\frac{|\mathcal{C}_j(i)|}{|\mathcal{C}_j(i-1)|} \leqslant \frac{(k)_j}{i(n-3k)_{j-1}} \leqslant \frac{k^j}{i(n-3k)^{j-1}} = O\left(\frac{d^{3j/3}}{in^{j-1}}\right).$$

As $d = o(n^{1/2})$ it follows that

$$|\cup_{j \geqslant 4} \cup_{i \geqslant 1} \mathcal{C}_j(i)| \leqslant \sum_{j \geqslant 4} \sum_{i \geqslant 1} |\mathcal{C}_j(i)| = O(d^6/in^3)|[N]^k| = o(|[N]^k|).$$

For $j = 2$,

$$\frac{|\mathcal{C}_2(i)|}{|\mathcal{C}_2(i-1)|} \leqslant \frac{(k)_2}{i(n-3k)} = O\left(\frac{d^3}{in}\right).$$

It follows that the number of k-tuples where $h_2 \geqslant d^2/\sqrt{n}$ is $o(|[N]^k|)$ as $d = o(\sqrt{n})$. Similarly, the number of k-tuples where $h_3 \geqslant d^2/\sqrt{n}$ is $o(|[N]^k|)$. This confirms (20).

Next, we prove $|\mathcal{F}_1| \sim |[N]^k|$ which together with (20) confirms Claim 19. The total number of k-tuples in $[N]^k$ is $\left(\frac{(n)_3}{6}\right)^k$. It is thus sufficient to show that

$$\left|[N]^k \setminus \mathcal{F}_1\right| = o\left(\left(\frac{(n)_3}{6}\right)^k\right). \tag{21}$$

Let $\mathcal{F}_1' \subseteq [N]^k$ be the set of $(j_1, \ldots, j_k)$ where there exists $j_u < j_v$ such that H_{j_u} and H_{j_v} share at least an edge. Then, $\mathcal{F}_1' = [N]^k \setminus \mathcal{F}_1$. Obviously,

$$|\mathcal{F}_1'| \leqslant k^2 n^4 \left(\frac{(n)_3}{6}\right)^{k-2} = \left(\frac{(n)_3}{6}\right)^k \cdot O\left(\frac{k^2}{n^2}\right) = o\left(\left(\frac{(n)_3}{6}\right)^k\right),$$

as $k^2 = O(d^3) = o(n^2)$. This completes the proof for Claim 19. ∎

Proof of Lemma 18. Given $\mathbf{j} = (j_1, \ldots, j_k) \in \mathcal{F}_2$, let $\mathcal{I}_j(\mathbf{j})$ be the set of i where H_{j_i} intersects exactly j edges with $\cup_{\ell < i} H_{j_\ell}$, for $j \in \{1, 2, 3\}$. By the definition of $\mathcal{F}_2$, $|\mathcal{I}_1(\mathbf{j})| + |\mathcal{I}_2(\mathbf{j})| + |\mathcal{I}_3(\mathbf{j})| \geqslant 1$ for any $\mathbf{j} \in \mathcal{F}_2$. We will consider a set of parameters for $\mathbf{j}$ and count elements in $\mathcal{F}_2$ according to these parameters. The first three parameters are

$$i = |\mathcal{I}_1(\mathbf{j})|, \quad j = |\mathcal{I}_2(\mathbf{j})|, \quad \ell = |\mathcal{I}_3(\mathbf{j})|.$$

Let u_i, v_i, w_i be the three vertices occupied by the triangle H_{j_i}. We say that z_i, $z \in \{u, v, w\}$, creates a collision, if z_i has been occupied by some triangle H_{j_c}, $c < i$. H_{j_i} can create at most 3 collisions. Suppose $i \in \mathcal{I}_1(\mathbf{j})$ and $u_i v_i$ is an edge that has been occupied

by some triangle H_{j_c}, $c < i$, then immediately u_i and v_i each creates a collision. These collisions are called *inherent*. Suppose $i \in \mathcal{I}_2(\mathbf{j})$ then H_{j_i} creates 3 collisions all of which are called *inherent*. All other collisions are called noninherent.

Let

$$t \quad \text{denote the number of noninherent collisions.}$$

Finally, let

$$\mathcal{F}_2(i, j, \ell, t) \text{ be the set of } \mathbf{j} \text{ satisfying the set of parameters;}$$
$$\mathcal{F}_2(i, j, \ell) = \cup_{t < \log n \cdot k^2/n} \mathcal{F}_2(i, j, \ell, t).$$

Now

$$\mathcal{F}_2 = \left(\cup_{t \geqslant \log n \cdot k^2/n} \cup_{(i,j,\ell): i+j+\ell \geqslant 1} \mathcal{F}_2(i, j, \ell, t) \right) \cup \left(\cup_{(i,j,\ell): i+j+\ell \geqslant 1} \mathcal{F}_2(i, j, \ell) \right)$$

Fix $\mathbf{j} \in \mathcal{F}_2$. We show that

$$\mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d) \right) \leqslant \lambda^{k-i-j-\ell} \left(\frac{d}{n} \right)^{2i+j} \left(1 + O\left(\frac{k}{n} + \frac{k^2}{dn} + \frac{d^2 k}{n^2} \right) \right). \tag{22}$$

Since

$$\mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d) \right) = \prod_{i=1}^k \mathbb{P}\left(H_{j_i} \subseteq \mathcal{G}(n, d) \mid (\cup_{\ell < i} H_{j_\ell})^+ \right).$$

By Corollary 5, besides the relative error $O(1/n + k/dn + d^2/n^2)$, if $i \notin (\mathcal{I}_1(\mathbf{j}) \cup \mathcal{I}_2(\mathbf{j}) \cup \mathcal{I}_3(\mathbf{j}))$ then the above conditional probability is at most λ; if $i \in \mathcal{I}_1(\mathbf{j})$ then the conditional probability is at most $(d/n)^2$; if $i \in \mathcal{I}_2(\mathbf{j})$ then the conditional probability is at most d/n; and if $i \in \mathcal{I}_3(\mathbf{j})$ then the conditional probability is 1. Multiplying them together yields the main term in (22). There are $O(k)$ terms in the product, and multiply the relative errors in each term gives the error as in (22). As $k = O(d^{3/2})$ and $d = o(\sqrt{n})$ the error in (22) is $o(1)$.

Next, we prove that for some absolute constant $C > 0$:

$$|\mathcal{F}_2(i, j, \ell, t)| \leqslant \frac{k^{i+j+\ell}}{i!j!\ell!} (3kn)^i (6kd)^j X^\ell \binom{3k}{t} \left(\frac{(n)_3}{6} \right)^{k-i-j-\ell} \left(\frac{Ck}{n} \right)^t, \tag{23}$$

$$|\mathcal{F}_2(i, j, \ell)| \leqslant \frac{k^{i+j+\ell}}{i!j!\ell!} (3kn)^i (6kd)^j Y^\ell \left(\frac{(n)_3}{6} \right)^{k-i-j-\ell}, \tag{24}$$

where

$$X = C(i^{3/2} + j^{3/2} + t^{3/2}), \quad Y = C(i^{3/2} + j^{3/2} + (\log n \cdot k^2/n)^{3/2}).$$

There are at most $k^{i+j+\ell}/i!j!\ell!$ ways to fix $\mathcal{I}_1$ for $\mathcal{I}_1(\mathbf{j})$, $\mathcal{I}_2$ for $\mathcal{I}_2(\mathbf{j})$, and $\mathcal{I}_3$ for $\mathcal{I}_3(\mathbf{j})$. For every $h \in \mathcal{I}_1$, there are at most $3k$ ways to choose the edge in H_{j_h} which intersect with $\cup_{c < h} H_{j_c}$, and then at most n ways to choose the other vertex in H_{j_h}. For every $h \in \mathcal{I}_2$, there are at most $3k$ ways to choose an edge x in $\cup_{c < h} H_{j_c}$, and then at most $2d$ ways to choose another edge incident with x in $\cup_{c < h} H_{j_c}$. The upper bound $2d$ follows as we may assume that every vertex is incident with at most d edges of $\cup_{i=1}^k H_{j_i}$ — otherwise, $\mathbb{P}(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n, d)) = 0$. We claim the following bound on the number of holes.

Claim 20. *The number of holes in $\mathcal{F}_2(i, j, \ell, t)$ is at most X.*

Proof of Claim 20. First we prove that any graph with x edges has at most $3x^{3/2}$ triangles. Let $t_1, \ldots t_n$ be the degree sequence of the graph, and we call a vertex big if its degree is at least $\sqrt{x}$ and small otherwise. Then the number of big vertices is at most $2x/\sqrt{x} = 2\sqrt{x}$. A triangle in the graph either uses only big vertices, or occupies a small vertex. Hence, the number of triangles in the graph is at most

$$\binom{2\sqrt{x}}{3} + \sum_{i:t_i \leqslant \sqrt{x}} \binom{t_i}{2} \leqslant \frac{4}{3} x^{3/2} + \frac{2x}{\sqrt{x}} \frac{(\sqrt{x})_2}{2} \leqslant 3x^{3/2},$$

where the maximum of the sum above appears when all non-zero degrees of small vertices are equal to $\sqrt{x}$.

Now, we bound the holes given parameters i, j and t. Let $H = \cup_{h=1}^{k} H_{j_h}$ and we bound the largest possible number of holes that H may contain. If $h \in \mathcal{I}_1 \cup \mathcal{I}_2$ then colour all three edges in H_{j_h} red. Then the number of red edges is at most $3(i + j)$. Let T_1 be the number of holes where all their three edges are red, and let T_2 be the number of other holes. Immediately, $T_1 \leqslant 3(i + 2j)^{3/2} \leqslant C(i^{3/2} + j^{3/2})$ and $T_2 \leqslant Ct^{3/2}$. The claim follows. ∎

First consider $\mathcal{F}_2(i, j, \ell, t)$ for $t \geqslant \log n \cdot k^2/n$. By the above claim, there are at most X^{ℓ} ways to fix all j_h for $h \in \mathcal{I}_3$. Finally, there are at most $((n)_3/6)^{k-i-j-\ell}$ ways to fix $(j_i)_{i \notin (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3)}$. There are at most $\binom{3k}{t}$ ways to choose the set of noninherent collisions, and each noninherent collision contributes an additional Ck/n factor because there are at most $3k$ instead of n choices to choose the vertex where a noninherent collision occurs. Multiplying all these upper bounds together yields (23).

For (24), the argument is similar except that we do not take union bound over all possible vertex collisions. Instead, we use the fact that $X \leqslant Y = C(i^3 + j^3 + (\log n \cdot k^2/n)^{3/2})$ when $t < \log n \cdot k^2/n$. Thus the number of ways to choose j_h where $h \in \mathcal{I}_3$ is at most Y^{ℓ} for all $\mathbf{j} \in \mathcal{F}_2(i, j, \ell)$, and (24) follows.

By (22) and (23) we have

$$\sum_{(j_1, \ldots, j_k) \in \mathcal{F}_2(i, j, \ell, t)} \mathbb{P}\left(\cup_{i=1}^{k} H_{j_i} \subseteq \mathcal{G}(n, d)\right) \leqslant (1 + o(1))|\mathcal{F}_2(i, j, \ell, t)| \cdot \lambda^{k-i-j-\ell} \left(\frac{d}{n}\right)^{2i+j}$$

$$\leqslant (1 + o(1)) \frac{k^{i+j+\ell}}{i!j!\ell!} \left(\frac{(n)_3}{6}\right)^{k-i-j-\ell} (3kn)^i (6kd)^j X^{\ell} \left(\frac{Ck^2}{tn}\right)^t \cdot \lambda^{k-i-j-\ell} \left(\frac{d}{n}\right)^{2i+j}$$

$$= O(\mu^k) \cdot \left(\frac{Ck^2 d^2}{i\mu n}\right)^i \left(\frac{Ck^2 d^2}{j\mu n}\right)^j \left(\frac{CkX}{\ell\mu}\right)^{\ell} \left(\frac{Ck^2}{tn}\right)^t$$

$$= O(\mu^k) \cdot \left(\frac{Ck^2 d^2}{i\mu n}\right)^i \left(\frac{Ck^2 d^2}{j\mu n}\right)^j \exp\left(\frac{CkX}{\mu}\right) \left(\frac{Ck^2}{tn}\right)^t$$

by defining $(x/0)^0 = 1$, and noting that $(x/\ell)^\ell \leqslant e^x$. It follows then that

$$
\sum_{(j_1,\ldots,j_k) \in \mathcal{F}_2(i,j,\ell,t)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right)
$$

$$
= O(\mu^k) \cdot \left(\frac{Ck^2 d^2}{i\mu n}\right)^i \left(\frac{Ck^2 d^2}{j\mu n}\right)^j \exp\left(\frac{Ck(i^{3/2} + j^{3/2} + b^3 + t^{3/2})}{\mu}\right) \left(\frac{Ck^2}{tn}\right)^t
$$

$$
= O(\mu^k) \cdot \exp\left(i\left(\ln(Ck^2 d^2/i\mu n) + \frac{Ck\sqrt{i}}{\mu}\right) + j\left(\ln(Ck^2 d^2/j\mu n) + \frac{Ck\sqrt{j}}{\mu}\right)\right)
$$

$$
\times \left(\frac{Ck^2}{tn}\right)^t.
$$

(25)

Since $\mu = \Theta(d^3)$, $k = \Theta(d^{3/2})$, $d = o(n^{1/2})$, and $t \geqslant \log n \cdot k^2/n$, it follows immediately that

$$
\sum_{t \geqslant \log n \cdot k^2/n} \sum_{i,j,\ell} \sum_{(j_1,\ldots,j_k) \in \mathcal{F}_2(i,j,\ell,t)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = o(\mu^k).
$$

(26)

Now combining (22) and (24) we have

$$
\sum_{(j_1,\ldots,j_k) \in \mathcal{F}_2(i,j,\ell)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) \leqslant (1 + o(1))|\mathcal{F}_2(i,j,\ell)| \cdot \lambda^{k-i-j-\ell} \left(\frac{d}{n}\right)^{2i+j}
$$

$$
\leqslant \frac{k^{i+j+\ell}}{i!j!\ell!} \left(\frac{(n)_3}{6}\right)^{k-i-j-\ell} (3kn)^i (6kd)^j (C(i^{3/2} + j^{3/2} + (\log n \cdot k^2/n)^{3/2}))^\ell
$$

$$
\times \lambda^{k-i-j-\ell} \left(\frac{d}{n}\right)^{2i+j} (1 + o(1))
$$

$$
= O(\mu^k) \cdot \left(\frac{Ck^2 d^2}{i\mu n}\right)^i \left(\frac{Ck^2 d^2}{j\mu n}\right)^j \left(\frac{Ck(i^{3/2} + j^{3/2})}{\ell\mu} + C\left(\frac{d\log n}{n}\right)^{3/2}\right)^\ell
$$

(27)

Since $\mu = \Theta(d^3)$, $k = \Theta(d^{3/2})$, $d = o(n^{1/2})$, summing the above over all (i,j,ℓ) where $i + j + \ell \geqslant 1$ yields $o(\mu^k)$. ∎

5.3 Distribution of Z_{C_3} for $d = \Theta(\sqrt{n})$

Again let $N = \binom{n}{3}$ and let $H_1, \ldots, H_N$ be an enumeration of all copies of triangles in K_n. Let $\mathcal{F}$ be the set of ordered k-tuples $(j_1, \ldots, j_k) \in [N]^k$ that are pairwise distinct. We want to estimate the following sum for $k = O(d^{3/2})$:

$$
\mathbb{E}(Z)_k = \sum_{(j_1,\ldots,j_k) \in \mathcal{F}} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right).
$$

(28)

We partition the summation into parts $\mathcal{F}_1$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$, where $\mathcal{F}_1 \subseteq \mathcal{F}$ is the set of **j** satisfying one the following:

(a) $|\mathcal{I}_3(\mathbf{j})| \geqslant 1$, or $h(v) > \log^2 n$ for some v.

(b) there exists H_{j_i} which intersects at least two edges with $\cup_{\ell<i}H_{j_\ell}$.

(c) there are more than $\log n$ i's such that H_{j_i} intersects exactly one edge with $\cup_{\ell<i}H_{j_\ell}$.

(d) there exists an edge contained in at least three triangles.

(e) there is some H_{j_i} such that at least two of its edges are each contained in some other triangles.

We will prove the following.

Lemma 21.
$$\sum_{\mathbf{j}\in\mathcal{F}_1}\mathbb{P}\left(\cup_{i=1}^k H_{j_i}\subseteq\mathcal{G}(n,d)\right)=o(\mu^k).$$

Lemma 22.
$$\sum_{\mathbf{j}\in\mathcal{F}_2}\mathbb{P}\left(\cup_{i=1}^k H_{j_i}\subseteq\mathcal{G}(n,d)\right)\sim\mu^k.$$

Proof of Lemma 22. The structure of the union of the k triangles $H_{j_1},\dots,H_{j_k}$ are simple, for $\mathbf{j}\in\mathcal{F}_2$. Except for at most $\log n$ pairs of triangles, each pair sharing exactly one edge, all the other triangles are edge disjoint. Moreover, every vertex is contained in at most $\log^2 n$ triangles. However, as we will show, there are $\Omega(d)$ vertices v with $h(v)\geqslant 2$. This means that the joint probability of a set of k edge disjoint triangles will not be asymptotic to λ^k any more, which was the case in the proof of Lemma 17.

We will partition $\mathcal{F}_2$ according to two parameters. An edge is called a *double* edge, if it is contained in two triangles. For $\mathbf{j}\in\mathcal{F}_2$, the number of double edges is exactly $|\mathcal{I}_1(\mathbf{j})|$. Recall that h_i denote the number of vertices v with $h(v)=i$. Modify $\mathcal{B}$ to state the set of $\mathbf{j}$ where

$$|h_2-9k^2/2n|>d^{2/3},\text{ or }h_3+h_4>d\cdot d^{7/2}\log n/n^2,\text{ or }h_j>0\text{ for some }j\geqslant 5.$$

Define

$$\mathcal{F}_2(i)=\{\mathbf{j}\in\mathcal{F}_2:\ |\mathcal{I}_1(\mathbf{j})|=i\},\quad\mathcal{F}_2^0(i)=\mathcal{F}_2(i)\setminus\mathcal{B},\quad\mathcal{F}_2^1(i)=\mathcal{F}_2(i)\cap\mathcal{B}$$
$$\mathcal{F}_2^0=\cup_{i=0}^{\log n}\mathcal{F}_2^0(i),\quad\mathcal{F}_2^1=\cup_{i=0}^{\log n}\mathcal{F}_2^1(i)=\mathcal{F}_2\setminus\mathcal{F}_2^0.$$

Then for every $i\leqslant\log n$,

$$
\begin{aligned}
|\mathcal{F}_2(i)| &= \binom{k}{2i}\frac{(2i)!}{2^i i!}\left(\frac{(n)_3}{6}-O(kn)\right)^{k-2i}\left(\frac{(n-O(i))^4}{2}\right)^i\\
&= \frac{k^{2i}}{4^i i!}\left(\frac{(n)_3}{6}\right)^{k-2i}n^{4i}\exp\left(O\left(\frac{k^2}{n^2}+\frac{i^2}{k}\right)\right)\\
&\sim \frac{k^{2i}n^{4i}}{4^i i!}\left(\frac{(n)_3}{6}\right)^{k-2i}.
\end{aligned}
$$

Claim 23.

$$|\mathcal{F}_2^0(i)| \sim |\mathcal{F}_2(i)| \text{ for all } i \leqslant \log n, \text{ and } \sum_{\mathbf{j} \in \mathcal{F}_2^1} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = o(\mu^k).$$

Next, fix $\mathbf{j} \in \mathcal{F}_2^0(i)$. We show that

$$\mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) \sim \lambda^{k-2i}\left(\frac{d}{n}\right)^{5i}\exp\left(-\frac{9k^2}{dn}\right). \tag{29}$$

The proof is almost the same as for (16). The accumulative error, except for that from $\phi_F(uv)/dn$, is $O\left((d^2k + k^2)/n^2 + k^3/d^2n^2\right) = o(1)$. The only two differences are listed below:

(a) The term $-2|F|$ in $\phi_F(uv)$ in (3) is no longer negligible. Its accumulative contribution after taking the product of the conditional probabilities is

$$\prod_{\ell=1}^{3k-O(i)}\left(1 + \frac{2\ell}{dn}\right) = \exp\left(\frac{9k^2}{dn} + O\left(\frac{ik}{dn} + \frac{k^3}{d^2n^2}\right)\right) \sim \exp\left(\frac{9k^2}{dn}\right).$$

(b) For each v where $h(v) = 2$, the joint probability of the two triangles hitting v cannot be approximated by λ^2 with negligible error because $h_2 = \Theta(d)$. Their accumulative contribution after taking the product of the conditional probabilities becomes

$$\left(\frac{(d-2)(d-3)}{d(d-1)}\right)^{h_2} = \exp\left(\left(-\frac{4}{d} + O(1/d^2)\right)\left(\frac{9k^2}{2n} + O(d^{2/3})\right)\right)$$
$$\sim \exp\left(-\frac{18k^2}{dn}\right).$$

Multiplying the additional factors in (a) and (b) yields (29). Finally,

$$\sum_{\mathbf{j} \in \mathcal{F}_2^0} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = \sum_{i=0}^{\log n} \sum_{\mathbf{j} \in \mathcal{F}_2^0(i)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right)$$

$$\sim \sum_{i=0}^{\log n}\left(\frac{(n)_3}{6}\right)^{k-2i}\frac{k^{2i}n^{4i}}{4^i i!}\lambda^{k-2i}\left(\frac{d}{n}\right)^{5i}\exp\left(-\frac{9k^2}{dn}\right)$$

$$\sim \mu^k \exp\left(-\frac{9k^2}{dn}\right)\sum_{i=0}^{\log n}\frac{6^{2i}}{n^{6i}}\frac{k^{2i}n^{4i}}{4^i i!}\left(\frac{n}{d}\right)^i$$

$$\sim \mu^k \exp\left(-\frac{9k^2}{dn}\right)\exp\left(\frac{9k^2}{dn}\right)$$

$$= \mu^k, \tag{30}$$

as $k = O(d^{3/2})$, $d = \Theta(\sqrt{n})$ and $\mu = \Theta(d^3)$.

It only remains to prove Claim 23.

Proof of Claim 23. The proof is basically the same as the proof for Claim 19, except that in the switching we avoid switch away the double edges — there are at most $\log n$ of them, and we estimate the number of switchings more carefully for h_2. The argument for bounds on h_j, $j \geqslant 3$ is much simpler and we skip the details. Now consider h_2, and assume that $h_3 + h_4 = o(d)$ and $h_j = 0$ for all $j \geqslant 5$. Let $\mathcal{C}(i,j)$ be the set of $\mathbf{j} \in \mathcal{F}_2$ with i double edges with $h_2 = j$. Let $\mathcal{C}(i) = \cup_{j \geqslant 0} \mathcal{C}(i,j)$. Define a switching from $\mathcal{C}(i,j)$ to $\mathcal{C}(i,j-1)$ as follows. Choose v and $i_1 < i_2$ such that $h(v) = 2$, v is not incident with a double edge, and H_{i_1} and H_{i_2} both hit v. There are exactly $j - O(i)$ ways to choose them. Then, we choose a vertex u such that $h(u) = 0$. Modify H_{i_2} so that v is replaced by u. The resulting k-tuple is in $\mathcal{C}(i,j-1)$. The number of ways to perform a switching is $(j - O(i))(n - 3k + O(\sum_j h_j))$. On the other hand, the number of inverse switchings is $9\binom{k-2(j-O(i))-\sum_{j \geqslant 3} jh_j}{2}$. Hence,

$$
\begin{aligned}
\frac{|\mathcal{C}(i,j)|}{|\mathcal{C}(i,j-1)|} &= \frac{9\binom{k-2(j-O(i))-\sum_{j \geqslant 3} jh_j}{2}}{j(n - 3k + O(\sum_j h_j))} = \frac{9(k-2j)^2}{2j(n-3k)}\left(1 + O\left(\frac{1}{\sqrt{dn}} + \frac{d}{n} + \frac{i}{j}\right)\right) \\
&= \frac{9(k-2j)^2}{2j(n-3k)}\left(1 + O\left(\frac{\log n}{\sqrt{n}}\right)\right).
\end{aligned}
\tag{31}
$$

Let j^* be the root of $9(k-2j)^2 = 2(n-3k)$. Then,

$$
j^* = \frac{5k}{12} + \frac{n}{36} - \frac{\sqrt{n^2 + 30kn - 99k^2}}{36} = \frac{9k^2}{2n} + O\left(\sqrt{d^3/n}\right).
$$

Now using (31) and following standard calculations we have

$$
\sum_{|j-j^*|>d^{2/3}} |\mathcal{C}(i,j)| = o(|\mathcal{C}(i)|)
$$

and the first assertion of the claim follows. The second assertions follows by the same calculations as for (29) and (30), except that we may use the bound in Corollary 5 which is of simpler form. For each $\mathbf{j} \in \mathcal{F}_2 \cap \mathcal{B}$,

$$
\mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) \leqslant \lambda^{k-2i}\left(\frac{d}{n}\right)^{5i}\left(1 + O\left(\frac{k}{n} + \frac{k^2}{dn} + \frac{d^2k}{n^2}\right)\right) = O\left(\lambda^{k-2i}\left(\frac{d}{n}\right)^{5i}\right).
$$

Since $|\mathcal{F}_2^1(i)| = o(|\mathcal{F}_2(i)|$ for all $i \leqslant \log n$,

$$
\begin{aligned}
\sum_{\mathbf{j} \in \mathcal{F}_2^1} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) &= \sum_{i=0}^{\log n} o(|\mathcal{F}_2(i)|) \cdot O\left(\lambda^{k-2i}\left(\frac{d}{n}\right)^{5i}\right) \\
&= o(1)\sum_{i=0}^{\log n} \frac{k^{2i}n^{4i}}{4^i i!}\left(\frac{(n)_3}{6}\right)^{k-2i}\lambda^{k-2i}\left(\frac{d}{n}\right)^{5i},
\end{aligned}
$$

which is $o(\mu^k)$ by following the same calculations as in (30). ∎

Proof of Lemma 21. Let $\mathcal{I}_j(\mathbf{j})$ be as defined before. Let $\mathcal{F}_1^1$ be the set of $\mathbf{j} \in \mathcal{F}_1$ where $\mathbf{j}$ induces a hole. With almost the same proof as in Lemma 18, we consider t: the number of noninherent collisions. For $t \geqslant \log n \cdot k^2/n$, we get (25), and thus (26). Note that the change of the range of d from $o(\sqrt{n})$ to $O(\sqrt{n})$ does not affect the analysis, as $t \geqslant \log n \cdot k^2/n$ and thus the term $(Ck^2/tn)^t$ dominates the rate the whole product vanishes. For $t < \log n \cdot k^2/n$, we get (27). Summing over all (i,j,ℓ) where $\ell \geqslant 1$ yields $o(\mu^k)$.

Next, let $\mathcal{F}_1^2 \subseteq \mathcal{F}_1 \setminus \mathcal{F}_1^1$ be the set of $\mathbf{j}$ where $i+j > \log n$ where $i = |\mathcal{I}_1(\mathbf{j})|$ and $j = |\mathcal{I}_2(\mathbf{j})|$. As $\mathbf{j} \notin \mathcal{F}_1^1$ we know $\mathcal{I}_3(\mathbf{j})$ must be empty. Let

$$\mathcal{F}_1^2(i,j) = \{\mathbf{j}: \ \mathcal{I}_1(\mathbf{j}) = i, \ \mathcal{I}_2(\mathbf{j}) = j\}.$$

We use the bound (24) (with $\ell = 0$) for $|\mathcal{F}_1^2(i,j)|$:

$$|\mathcal{F}_1^2(i,j)| \leqslant \frac{k^{i+j}}{i!j!} \left(\frac{(n)_3}{6}\right)^{k-i-j} (3kn)^i (6kd)^j.$$

Thus,

$$\sum_{i,j:i+j>\log n} \sum_{\mathbf{j}\in\mathcal{F}_1^2(i,j)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) \ \leqslant \ O(1) \cdot \mu^k \sum_{i,j:i+j\geqslant \log n} \frac{(3k^2d^2/\mu n)^i}{i!} \frac{(6k^2d^2/\mu n)^j}{j}$$

$$= \ o(\mu^k),$$

as $k = O(d^{3/2})$ and thus $3k^2d^2/\mu n = O(1)$.

Next, let $\mathcal{F}_1^3 \subseteq \mathcal{F}_1 \setminus (\mathcal{F}_1^1 \cup \mathcal{F}_1^2)$ be the set of $\mathbf{j}$ where $h(v) > \log^2 n$. The summation over $\mathcal{F}_1^3$ can be treated similarly as in $\mathcal{F}_1^1$. Fix $v \in [n]$. Since $\mathbf{j}$ induces no holes, $\mathcal{I}_3(\mathbf{j}) = 0$. Let $\mathcal{L}(\mathbf{j})$ be the set of i such that H_{j_i} hits v. Let

$$\mathcal{F}_1^3(i,j,r) \ = \ \{\mathbf{j} \in \mathcal{F}_1^3: \ \mathcal{I}_1(\mathbf{j}) = i, \ \mathcal{I}_2(\mathbf{j}) = j, \ \mathcal{L}(\mathbf{j}) = r\}.$$

Since $\mathbf{j} \notin \mathcal{F}_1^2$, $i \leqslant \log n$ and $j \leqslant \log n$. Let $r > \log^2 n$. There are at most $k^{i+j+r}/i!j!r!$ ways to fix $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{L}$ for $\mathcal{I}_1(\mathbf{j})$, $\mathcal{I}_2(\mathbf{j})$ and $\mathcal{L}(\mathbf{j})$ respectively. Given $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{L}$, let $\mathcal{L}' = \mathcal{L} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2)$ and we must have $|\mathcal{L}'| \geqslant r - 2\log n = r(1-o(1))$ as $r > \log^2 n$. Thus,

$$|\mathcal{F}(i,j,r)| \leqslant \frac{k^{i+j+r}}{i!j!r!} \left(\frac{(n)_3}{6}\right)^{k-i-j} (3kn)^i (6kd)^j n^{2r},$$

as for each $r \in \mathcal{L}'$, there are at most n^2 ways to choose the two vertices other than v. Hence,

$$\sum_{\substack{i,j,r: \\ r>\log^2 n}} \sum_{\mathbf{j}\in\mathcal{F}_1^3(i,j,\ell)} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right)$$

$$= O(1) \cdot \sum_{i,j,r:r>\log^2 n} \frac{k^{i+j+r}}{i!j!r!} \left(\frac{(n)_3}{6}\right)^{k-i-j-r(1-o(1))} (3kn)^i (6kd)^j n^{2r} \lambda^{k-i-j} \left(\frac{d}{n}\right)^{2i+j}$$

$$= O(\mu^k) \sum_{r>\log^2 n} \frac{(k/n^{1-o(1)})^r}{r!} \sum_{i\geqslant 0} \frac{(3k^2d^2/\mu n)^i}{i!} \sum_{j\geqslant 0} \frac{(6k^2d^2/\mu n)^j}{j!} = o(\mu^k),$$

as $k^2 d^2/\mu n = O(1)$ and $k/n^{1-o(1)} = o(1)$.

Next, let $\mathcal{F}_1^4 \subset \mathcal{F}_1 \setminus (\mathcal{F}_1^1 \cup \mathcal{F}_1^2 \cup \mathcal{F}_1^3)$ be the set of $\mathbf{j}$ where $\mathcal{I}_2(\mathbf{j})$ is nonempty. Let

$$\mathcal{F}_1^4(i,j) = \{\mathbf{j} \subseteq \mathcal{F}_1^4 : \ \mathcal{I}_1(\mathbf{j}) = i, \ \mathcal{I}_2(\mathbf{j}) = j\}.$$

As now $\mathbf{j} \notin \mathcal{F}_1^3$, $h(v) \leqslant \log^2 n$ for every n and thus we have the following better upper bound than (23) for $|\mathcal{F}_1^4(i,j)|$:

$$|\mathcal{F}_1^4(i,j)| \leqslant \left(\frac{(n)_3}{6}\right)^{k-i-j} (3kn)^i (12k \log^2 n)^j.$$

This is because every vertex in $\cup_{i=1}^k H_{j_i}$ is incident with at most $2\log^2 n$ instead of d edges. Using this improved upper bound, we have

$$\sum_{\mathbf{j} \in \mathcal{F}_1^4} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = O(\mu^k) \sum_{i \leqslant \log n} \frac{(3k^2 d^2/\mu n)^i}{i!} \sum_{j \geqslant 1} \frac{(6k^2 d \log^2 n/\mu n)^j}{j!} = o(\mu^k).$$

Finally, let $\mathcal{F}_1^5 = \mathcal{F}_1 \setminus (\cup_{i=1}^4 \mathcal{F}_1^i)$, the set of $\mathbf{j}$ where either an edge is contained in more than 2 triangles, or there is a triangle with two edges each contained in some other triangle. Summing over $\mathbf{j}$ in the first case we have

$$\sum_{2 \leqslant i \leqslant \log n} \frac{k^i}{i!} \left(\frac{(n)_3}{6}\right)^{k-i} i \cdot in \cdot (3kn)^{i-1} \lambda^{k-i} \left(\frac{d}{n}\right)^{2i} = O(\mu^k \cdot \log^2 n/k) = o(\mu^k).$$

Now we consider the latter case. If H_{j_u} has two edges each contained in some other triangles — call them H_{j_ℓ} and H_{j_r}. Obviously $\ell \neq r$ since otherwise H_{j_u} would be the same as H_{j_ℓ}, contradicting with them being distinct. There are two sub-cases: (a) $\ell < i$ and $r > i$; (b) $r > \ell > i$. In either case, there are at most i^3 ways to choose u, ℓ and r. Then, similar to the above, summing over $\mathbf{j}$ in the second case yields

$$\sum_{2 \leqslant i \leqslant \log n} \frac{k^i}{i!} \left(\frac{(n)_3}{6}\right)^{k-i} i^3 \cdot O(n^2) \cdot (3kn)^{i-2} \lambda^{k-i} \left(\frac{d}{n}\right)^{2i} = O(\mu^k \cdot \log^3 n/k) = o(\mu^k).$$

Combining the two cases we have $\sum_{\mathbf{j} \in \mathcal{F}_1^5} \mathbb{P}\left(\cup_{i=1}^k H_{j_i} \subseteq \mathcal{G}(n,d)\right) = o(\mu^k)$, completing the proof of the lemma. ∎

Acknowledgements

References

[1] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics*, 1(4):311–316, 1980.

[2] Paul Erdős and Alfréd Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci*, 5(1):17–60, 1960.

[3] Pu Gao, Mikhail Isaev and Brendan McKay. Kim-vu's sandwich conjecture is true for $d \geqslant \log^4 n$, *arXiv preprint arXiv:2011.09449*, 2020.

[4] Pu Gao. Distribution of the number of spanning regular subgraphs in random graphs. *Random Structures & Algorithms*, 43(3):338–353, 2013.

[5] Pu Gao. Distributions of sparse spanning subgraphs in random graphs. *SIAM Journal on Discrete Mathematics*, 27(1):386–401, 2013.

[6] Pu Gao, Mikhail Isaev, and Brendan D McKay. Sandwiching random regular graphs between binomial random graphs. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 690–701. SIAM, 2020.

[7] Pu Gao, Mikhail Isaev, and Brendan D McKay. Sandwiching random regular graphs between binomial random graphs. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 690–701. SIAM, 2020.

[8] Pu Gao and Yuval Ohapkin. Subgraph probability of random graphs with specified degrees and applications to chromatic number and connectivity. *arXiv preprint arXiv:2007.02216*, 2020.

[9] Pu Gao and Cristiane M Sato. A transition of limiting distributions of large matchings in random graphs. *Journal of Combinatorial Theory, Series B*, 116:57–86, 2016.

[10] Zhicheng Gao and Nicholas C Wormald. Asymptotic normality determined by high moments, and submap counts of random maps. *Probability theory and related fields*, 130(3):368–376, 2004.

[11] Zhicheng Gao and Nicholas C Wormald. Distribution of subgraphs of random regular graphs. *Random Structures & Algorithms*, 32(1):38–48, 2008.

[12] Svante Janson. Random regular graphs: asymptotic distributions and contiguity. *Combinatorics, Probability and Computing*, 4(4): 369 – 405, 1995.

[13] Svante Janson. The numbers of spanning trees, hamilton cycles and perfect matchings in a random graph. *Combinatorics, Probability and Computing*, 3(1):97–126, 1994.

[14] Brendan D McKay. Subgraphs of random graphs with specified degrees. *Congressus Numerantium*, 33: 213–223.1981.

[15] Brendan D McKay. Subgraphs of random graphs with specified degrees. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 2489–2501. World Scientific, 2010.

[16] Brendan D McKay. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combin* 19 (1985): 15–25.

[17] Brendan D McKay, Nicholas C Wormald, and Beata Wysocka. Short cycles in random regular graphs. *the electronic journal of combinatorics*, pages R66–R66, 2004.

[18] Andrzej Ruciński. Subgraphs of random graphs: a general approach. In *North-Holland Mathematics Studies*, volume 118, pages 221–229. Elsevier, 1985.

[19] Andrzej Ruciński. When are small subgraphs of a random graph normally distributed? *Probability Theory and Related Fields*, 78(1):1–10, 1988.

[20] Nicholas C Wormald. The asymptotic distribution of short cycles in random regular graphs. *Journal of Combinatorial Theory, Series B*, 31(2):168–182, 1981.

[21] Nicholas C Wormald. Models of random regular graphs. *London Mathematical Society Lecture Note Series*: 239–298, 1999.