Generating New Partition Identities via a Generalized Continued Fraction Algorithm

Wael Baalbaki[a]   Thomas Garrity[b]

Abstract

Using the slow triangle map (a type of multi-dimensional continued fraction algorithm), we exhibit a method for generating any number of new identities for subsets of integer partitions.

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1   Introduction

Andrew and Eriksson's introduction to integer partitions [2] starts with discussing Euler's identity, "*Every number has as many integer partitions into odd parts as into distinct parts.*" As they state, this is quite surprising if you have never seen it before. There are, though, many other equally if not more surprising partition identities. For all there are two basic questions. First, how to even guess the existence of any potential partition identities. Then, once a possible potential identity is conjectured, how to prove it.

In [5], Bonanno, Del Vigna, Isola and the two authors developed a link between traditional continued fractions and the slow triangle map (a type of multi-dimensional continued fraction algorithm) with integer partitions of numbers into two or three distinct parts, with multiplicity. The triangle map was initially introduced for number theoretic reasons and has over the years exhibited interesting dynamical properties. We will see that the slow triangle map, when extended to higher dimensions, will provide a natural map T from the set of integer partitions to itself. Further, we will split the set of integer partitions into three disjoint subsets, which we will denote by $\triangle_0, \triangle_D$ and $\triangle_1$. We will see that the triangle map T will be one-to-one on $\triangle_0$ and $\triangle_1$. Thus if we have any subset S of $\triangle_0 \cup \triangle_1$, then its image $T(S)$ will have to have the same size. This will be used to generate many new partition identities, which appear in section four. In section six, the corresponding generating functions are listed.

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[a]Fu Foundation School of Engineering and Applied Sciences, Columbia University, New York, NY 10025, U.S.A. (`wb2393@columbia.edu`).

[b] Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, U.S.A. (`tgarrity@williams.edu`).

2   Partitions

For a general background on partition numbers, see Andrews [1]. Given a positive integer n, the partition number $p(n)$ is the number of ways of writing n as the sum of less than or equal t positive integers (ordering not mattering). Thus $p(4) = 5$ since we can write 4 as

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$$

while $p(7) = 15$, since we can write 7 as

7	$6 + 1$	$5 + 2$	$5 + 1 + 1$
$4 + 3$	$4 + 2 + 1$	$4 + 1 + 1 + 1$	$3 + 3 + 1$
$3 + 2 + 2$	$3 + 2 + 1 + 1$	$3 + 1 + 1 + 1 + 1$	$2 + 2 + 2 + 1$
$2 + 2 + 1 + 1 + 1$	$2 + 1 + 1 + 1 + 1 + 1$	$1 + 1 + 1 + 1 + 1 + 1 + 1.$	

We can write a given partition of n as a descending sequence of positive integers $(\lambda_1, \ldots, \lambda_m)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ with $n = \sum_{i=1}^m \lambda_i$.

It is common to concatenate those λ_i that are equal, and write the number of times a given λ_i appears as an exponent. Thus the above partitions of 7 would be written as

$$
\begin{array}{cccc}
(7) & (6,1) & (5,2) & (5,1^2) \\
(4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\
(3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\
(2^2,1^3) & (2,1^5) & (1^7).
\end{array}
$$

Then a partition of n would be some $(\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m})$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, each $k_i > 0$ and $n = \sum_{i=1}^m k_i \lambda_i$. We call the λ_i the *parts* of the partition and the k_i the *multiplicities* of the partition. We will always use λ_i to denote parts and k_i to denote multiplicities.

For our purposes, we will write partitions as follows: a partition λ of a positive integer n is of the form

$$
\begin{aligned}
\lambda &= (\overline{\lambda}) \times [\overline{k}] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]
\end{aligned}
$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, each $k_i > 0$ and $n = \sum_{i=1}^m k_i \lambda_i$ is the dot product of the vector of parts $\overline{\lambda}$ with the vector of multiplicities $\overline{k}$.

In this notation, the partitions of 7 are

$$
\begin{array}{cccc}
(7) \times [1] & (6,1) \times [1,1] & (5,2) \times [1,1] & (5,1) \times [1,2] \\
(4,3) \times [1,1] & (4,2,1) \times [1,1,1] & (4,1) \times [1,3] & (3,1) \times [2,1] \\
(3,2) \times [1,2] & (3,2,1) \times [1,1,2] & (3,1) \times [1,4] & (2,1) \times [3,1] \\
(2,1) \times [2,3] & (2,1) \times [1,5] & (1) \times [7].
\end{array}
$$

Let $\mathcal{P}$ denote the set of all possible partitions $\lambda = (\overline{\lambda}) \times [\overline{k}]$. If λ is a partition of the

natural number n, meaning that $n = \sum_{i=1}^{m} k_i \lambda_i$, we say that λ has *size n*, and write it as

$$\lambda \vdash n \text{ and } |\lambda| = \sum_{i=1}^{m} k_i \lambda_i = n.$$

If $\lambda = (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$, we say that the *dimension* of λ is m. We denote the dimension as $\dim(\lambda)$ or as $\dim((\overline{\lambda}) \times [\overline{k}])$ The dimension of a partition will be quite important for this paper, unlike in most other work on partition numbers. It is important enough that we denote

$$\mathcal{P}_m = \{(\overline{\lambda}) \times [\overline{k}] \in \mathcal{P} : \dim((\overline{\lambda}) \times [\overline{k}]) = m\}$$

and

$$\mathcal{P}_{\geqslant m} = \{(\overline{\lambda}) \times [\overline{k}] \in \mathcal{P} : \dim((\overline{\lambda}) \times [\overline{k}]) \geqslant m\}.$$

Given a subset S of $\mathcal{P}$, we define

$$p_S(n) = \#\{\lambda \in S : \lambda \vdash n\}.$$

As an example of this rhetoric, Euler's identity is

Theorem 1. *Let*

$$\mathcal{O} = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P} : \text{for all } i, \lambda_i \text{ odd}\}$$

and

$$\mathcal{D} = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P} : \forall i, k_i = 1\}.$$

Then for all n,

$$p_{\mathcal{O}}(n) = p_{\mathcal{D}}(n).$$

2.1 The slow triangle map

This section is an exposition on the slow triangle map, a type of multidimensional continued fraction. The fast version originally appeared in [10, 4], where the concern was the number-theoretic Hermite problem. Messaoudi, Nogueira, and Schweiger [13] showed that the fast map is ergodic. Further dynamical properties were discovered by Berthé, Steiner and Thuswaldner [6] and by Fougeron and Skripchenko [9]. Bonanno, Del Vigna and Munday [7] and Bonanno and Del Vigna [8] recently used the $\mathbb{R}^3$ slow triangle map to develop a tree structure of rational pairs in the plane. In a recent preprint, Ito [11] showed that the fast map is self-dual (in section three of that paper). These papers are all primarily motivated by questions from dynamics. For general background in multidimensional continued fraction algorithms, see Karpenkov [12] and Schweiger [14].

Here we will simply give the basic definitions for the m dimensional triangle map. Let

$m \geqslant 2$. Start with the cone

$$\triangle = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_1 > x_2 > \cdots > x_m > 0\}$$

with the two subcones

$$\begin{aligned} \triangle_0 &= \{(x_1, x_2, \ldots, x_m) \in \triangle : x_1 < x_2 + x_m\} \\ \triangle_1 &= \{(x_1, x_2, \ldots, x_m) \in \triangle : x_1 > x_2 + x_m\} \end{aligned}$$

and the lower dimensional cone

$$\triangle_D = \{(x_1, x_2, \ldots, x_m) \in \triangle : x_1 = x_2 + x_m\}.$$

The subscript D is for "diagonal."

When $m = 2$, the $x_2 + x_m$ term is $2x_2$.

The slow triangle map T is a map

$$T : \triangle_0 \cup \triangle_1 \to \triangle$$

defined as

$$\begin{aligned} T(x_1, \ldots, x_m) &= \begin{cases} T_0(x_1, \ldots, x_m) & \text{if } x_1 < x_2 + x_m \\ T_1(x_1, \ldots, x_m) & \text{if } x_1 > x_2 + x_m \end{cases} \\ &= \begin{cases} (x_2, x_3, \ldots, x_m, x_1 - x_2) & \text{if } x_1 < x_2 + x_m \\ (x_1 - x_m, x_2, \ldots, x_m) & \text{if } x_1 > x_2 + x_m \end{cases} \end{aligned}$$

Reflecting its roots in dynamical systems, the map is often written as a map $T : \triangle \to \triangle$, with the convention that any $(x_1, \ldots, x_m) \in \triangle_D$ is ignored, as $\triangle_D$ is a set of measure zero.

As discussed in [5, 10], the $\mathbb{R}^2$ triangle map acting on some (x_1, x_2) can be interpreted as a means of finding the continued fraction expansion of the ratio x_2/x_1. (This connection is not obvious if you have not seen it before.) Also, for the expert, the fast version is just the slow version when one concatenates the T_1 and then applies T_0. In the $\mathbb{R}^2$ case, the fast version is the classical Gauss map for continued fractions while the slow version is the Farey map.

We will throughout be concentrating on the slow version. Thus for the rest of the paper, the "slow triangle map" will simply be called the triangle map.

3 Linking the triangle map to partitions

3.1 The basic link

We now link the triangle map to a map from the set of integer partitions of dimension at least two to the set of partitions. Thus we will define a map

$$T : \mathcal{P}_{\geqslant 2} \to \mathcal{P}.$$

Recall that we are denoting the set of all partition as $\mathcal{P}$ and write each partition as $\lambda = (\bar{\lambda}) \times [\bar{k}] = (\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m]$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and each $k_i > 0$.

By a slight abuse of notation, we set

$$
\begin{aligned}
\triangle_0 &= \{(\bar{\lambda}) \times [\bar{k}] \in \mathcal{P}_{\geqslant 2} : \lambda_1 < \lambda_2 + \lambda_m\} \\
\triangle_1 &= \{(\bar{\lambda}) \times [\bar{k}] \in \mathcal{P}_{\geqslant 2} : \lambda_1 > \lambda_2 + \lambda_m\} \\
\triangle_D &= \{(\bar{\lambda}) \times [\bar{k}] \in \mathcal{P}_{\geqslant 2} : \lambda_1 = \lambda_2 + \lambda_m\}
\end{aligned}
$$

(As before, when $m = 2$, then $\lambda_2 + \lambda_m$ is $2\lambda_2$.) Now to define $T : \mathcal{P}_{\geqslant 2} \to \mathcal{P}$. This map, this almost internal symmetry, will allow us in the next section to find many new partition identities.

Define

$$
T((\bar{\lambda}) \times [\bar{k}]) = \begin{cases}
T_0((\bar{\lambda}) \times [\bar{k}]) & \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_0 \\
T_1((\bar{\lambda}) \times [\bar{k}]) & \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_1 \\
T_D((\bar{\lambda}) \times [\bar{k}]) & \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_D
\end{cases}
$$

$$
= \begin{cases}
(\lambda_2, \dots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \dots, k_m, k_1] \\
\quad \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_0 \\
(\lambda_1 - \lambda_m, \lambda_2, \dots, \lambda_m) \times [k_1, k_2, \dots, k_{m-1}, k_1 + k_m] \\
\quad \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_1 \\
(\lambda_2, \dots, \lambda_m) \times [k_1 + k_2, k_3, \dots, k_{m-1}, k_1 + k_m] \\
\quad \text{if } (\bar{\lambda}) \times [\bar{k}] \in \triangle_D \cap \mathcal{P}_{\geqslant 3} \\
(\lambda_2) \times [2k_1 + k_2] \\
\quad \text{if } (\bar{\lambda}) \times [\bar{k}] \text{ is } (2\lambda_2, \lambda_2) \times [k_1, k_2] \in \triangle_D
\end{cases}
$$

For example,

$$
\begin{aligned}
T((6, 5, 4, 2) \times [k_1, k_2, k_3, k_4] &= (5, 4, 2, 6 - 5) \times [k_1 + k_2, k_3, k_4, k_1] \\
&= (5, 4, 2, 1) \times [k_1 + k_2, k_3, k_4, k_1] \\
T((9, 5, 4, 2) \times [k_1, k_2, k_3, k_4] &= (9 - 2, 5, 4, 2) \times [k_1, k_2, k_3, k_1 + k_4] \\
&= (7, 5, 4, 2) \times [k_1, k_2, k_3, k_1 + k_4] \\
T((6, 5, 4, 1) \times [k_1, k_2, k_3, k_4] &= (5, 4, 1) \times [k_1 + k_2, k_3, k_1 + k_4] \\
T(6, 3) \times [k_1, k_2] &= (3) \times [2k_1 + k_2].
\end{aligned}
$$

A straightforward calculation gives us the key to this paper:

Theorem 2. *For any partition $(\overline{\lambda}) \times [\overline{k}]$ of dimension at least two we have*

$$(\overline{\lambda}) \times [\overline{k}] \vdash n \text{ if and only if } T((\overline{\lambda}) \times [\overline{k}]) \vdash n.$$

We can see

Proposition 3. *For all $\lambda \in \triangle_0$ and for all $\lambda \in \triangle_1$, we have*

$$dim(\lambda) = dim(T(\lambda)).$$

For all $\lambda \in \triangle_D$, the dimension under the map T is reduced by one:

$$dim(\lambda) = dim(T(\lambda)) + 1.$$

The three subsets $\triangle_0, \triangle_1$ and $\triangle_D$ are all defined in terms of the parts λ_i of the partition. There are also natural subsets of $\mathcal{P}_{\geqslant 2}$ defined in terms of the multiplicities k_i, such as

$$
\begin{aligned}
M_0 &= \{(\overline{\lambda}) \times [\overline{k}] \in \mathcal{P} : k_1 > k_m\} \\
M_1 &= \{(\overline{\lambda}) \times [\overline{k}] \in \mathcal{P} : k_1 < k_m\}
\end{aligned}
$$

These sets correspond to $\triangle_0$ and $\triangle_1$, as seen in:

Theorem 4. *The map*
$$T_0 : \triangle_0 \to M_0$$

and the map
$$T_1 : \triangle_1 \to M_1$$

are each one-to-one and onto maps.

Proof. This can be shown simply by explicitly finding the explicit inverses to T_0 and to T_1:

$$
\begin{aligned}
T^{-1}((\overline{\lambda}) \times [\overline{k}]) &= \begin{cases} T_0^{-1}((\overline{\lambda}) \times [\overline{k}]) & \text{if } (\overline{\lambda}) \times [\overline{k}] \in M_0 \\ T_1^{-1}((\overline{\lambda}) \times [\overline{k}]) & \text{if } (\overline{\lambda}) \times [\overline{k}] \in M_1 \end{cases} \\[2ex]
&= \begin{cases} (\lambda_1 + \lambda_m, \lambda_1, \ldots, \lambda_{m-1}) \times [k_m, k_1 - k_m, k_2 \ldots, k_{m-1}] \\ \qquad \text{if } (\overline{\lambda}) \times [\overline{k}] \in M_0 \\[1ex] (\lambda_1 + \lambda_m, \lambda_2, \ldots, \lambda_m) \times [k_1, k_2, \ldots, k_{m-1}, k_m - k_1] \\ \qquad \text{if } (\overline{\lambda}) \times [\overline{k}] \in \triangle_1 \end{cases}
\end{aligned}
$$

We are using throughout that the parts are strictly decreasing and that the multiplicities are all positive. $\square$

The map T acts on both the parts λ_i and on the multiplicities k_i. The way both T_0 and T_1 act on the parts comes from the triangle map as described in Section 2.1. The

action of each on the multiplicities is of course determined by the desire for the size of the partition to not be changed. But once written down, it is apparent that these actions correspond to what is called in dynamical systems "the natural extension of a map." This is described in [5]. For background on natural extensions, see Arnoux and Nogueira [3].

3.2 The curious case of the diagonal map T_D

The above map T_D is a new map. It is not natural if we are concerned only with the dynamics of the original triangle map. In fact, in the study of the triangle map as a multidimensional continued fraction algorithm, points on the diagonal (where $x_1 = x_2 + x_m$) are usually ignored, as these points form sets of measure zero. (These "boundary" types points are also similarly ignored in most other multidimensional continued fraction algorithms.) Here though we defined T_D on the partitions $\lambda = (\overline{\lambda}) \times [\overline{k}]$ where $\lambda_1 = \lambda_2 + \lambda_m$ for $m \geqslant 3$ and also for the quite special case of when $\lambda = (\overline{\lambda}) \times [\overline{k}] = (2\lambda_2, \lambda_2) \times [k_1, k_2]$. We want to see why our definition of T_D is the "right one."

Start with some $(\overline{\lambda}) \times [\overline{k}]$ where $\lambda_1 = \lambda_2 + \lambda_m$, where $m \geqslant 3$. In some sense, we can think of this point as on both the boundary of $\triangle_0$ (where $\lambda_1 < \lambda_2 + \lambda_m$) and on the boundary of $\triangle_1$ (where $\lambda_1 > \lambda_2 + \lambda_m$). Let us start with thinking of $(\overline{\lambda}) \times [\overline{k}]$ as actually being on $\triangle_0$ and act on it by T_0:

$$\begin{aligned}
T_0((\overline{\lambda}) \times [\overline{k}]) &= (\lambda_2, \ldots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \ldots, k_m, k_1] \\
&= (\lambda_2, \ldots, \lambda_m, \lambda_m) \times [k_1 + k_2, k_3, \ldots, k_m, k_1].
\end{aligned}$$

As the last two parts are equal, in the context of integer partitions it is natural to concatenate the last two parts, getting

$$(\lambda_2, \ldots, \lambda_m) \times [k_1 + k_2, k_3, \ldots, k_m + k_1],$$

which is precisely how we defined the map T_D.

But what if we initially think of $(\overline{\lambda}) \times [\overline{k}]$ as on $\triangle_1$ and now act on it by T_1:

$$\begin{aligned}
T_1((\overline{\lambda}) \times [\overline{k}]) &= (\lambda_1 - \lambda_m, \lambda_2, \ldots, \lambda_m) \times [k_1, k_2, \ldots, k_{m-1}, k_1 + k_m] \\
&= (\lambda_2, \lambda_2, \ldots, \lambda_m) \times [k_1, k_2, \ldots, k_{m-1}, k_1 + k_m].
\end{aligned}$$

Now the first two parts are equal. When we concatenate these first two terms, we get

$$(\lambda_2, \ldots, \lambda_m) \times [k_1 + k_2, k_3, \ldots, k_m + k_1],$$

which is once again $T_D((\overline{\lambda}) \times [\overline{k}]$.

And again, while this concatenation is extremely natural for integer partitions, it is unnatural if one is concerned with the underlying dynamics of the map, which is why the map T_D has never been written down before appearing in this paper and in [5].

As an example, consider

$$T_0(11, 8, 6, 3) \times [k_1, k_2, k_3, k_4] = (8, 6, 3, 3) \times [k_1 + k_2, k_3, k_4, k_1]$$

$$\to \quad (8,6,3) \times [k_1 + k_2, k_3, k_1 + k_4]$$

and

$$T_1(11,8,6,3) \times [k_1, k_2, k_3, k_4] = (8,8,6,3) \times [k_1, k_2, k_3, k_1 + k_4]$$
$$\to \quad (8,6,3) \times [k_1 + k_2, k_3, k_1 + k_4]$$

where in both cases $\to$ means to concatenate.

Unlike T_0 and T_1, this map T_D is not one-to-one, as seen with

$$T_D((11,8,6,3) \times [2,3,4,5] = (8,6,3) \times [5,3,9]$$
$$= T_D((11,8,6,3) \times [1,4,4,6]$$

But as seen with this example, we do have the following:

Proposition 5. *Let* $(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$ *and* $(\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m]$ *both be on the diagonal* $\triangle_D$. *If*

$$T_D((\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]) = T_D((\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m]),$$

then for all i,

$$\lambda_i = \mu_i$$

and

$$\lambda_1 + \lambda_2 = \mu_1 + \mu_2, \ \lambda_1 + \lambda_m = \mu_1 + \mu_m.$$

Thus T_D is one-to-one when thought of as a map on the parts and not as a map including the multiplicities.

Finally consider the case when our partition is of the form $\lambda = (\overline{\lambda}) \times [\overline{k}] = (2\lambda_2, \lambda_2) \times [k_1, k_2]$. Thinking of $(2\lambda_2, \lambda_2) \times [k_1, k_2]$ as being in $\triangle_0$ and acting on it by T_0 gives us

$$T_0((2\lambda_2, \lambda_2) \times [k_1, k_2]) = (\lambda_2, \lambda_2) \times [k_1 + k_2, k_1]$$
$$\to \quad (\lambda_2) \times [2k_1 + k_2],$$

where the $\to$ means to to concatenate. Note this is how we defined T_D on partitions with two parts.

And as we would suspect, now thinking of $(2\lambda_2, \lambda_2) \times [k_1, k_2]$ as being in $\triangle_1$ and acting on it by T_1 gives us

$$T_1((2\lambda_2, \lambda_2) \times [k_1, k_2]) = (\lambda_2, \lambda_2) \times [k_1, k_1 + k_2]$$
$$\to \quad (\lambda_2) \times [2k_1 + k_2],$$

where the $\to$ still means to to concatenate, which gives us the same value of T_D.

Again, T_D is natural if we want to use the triangle map to understand partitions of integers, though it is not natural if we are only interested in the underlying dynamics of the map.

4 New Partition Identities

We are now ready to start producing many new partition identities.

4.1 Applying Theorem 4

First, we have

Theorem 6. *Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ as into partitions with $k_1 > k_m$. Similarly, every number has as many integer partitions into partitions with $\lambda_1 > \lambda_2 + \lambda_m$ as into partitions with $k_1 < k_m$.*

This is just a rewriting of Theorem 4.
We can refine this theorem to be:

Theorem 7. *Let d be a positive integer. Every number has as many integer partitions into partitions with $\lambda_1 + d = \lambda_2 + \lambda_m$ as into partitions with $k_1 > k_m$ and $\lambda_{m-1} = \lambda_m + d$. Similarly, every number has as many integer partitions into partitions with $\lambda_1 = \lambda_2 + \lambda_m + d$ as into partitions with $k_1 < k_m$ and $\lambda_1 = \lambda_2 + d$.*

Proof. Set

$$\triangle_0(d) = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \triangle_0 : \lambda_1 + d = \lambda_2 + \lambda_m\}.$$

Note that in this case, $\lambda_m = (\lambda_1 - \lambda_2) + d$. We need to show that

$$T_0(\triangle_0(d)) = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : k_1 > k_m, \lambda_{m-1} = \lambda_m + d\}.$$

We know that

$$T_0((\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] = (\lambda_2, \lambda_3, \ldots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \ldots, k_m . k_1].$$

This means that the first multiplicity of $T_0((\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$ is strictly greater than the last multiplicity and that the last part is exactly d less that the next to last part, which is precisely what we wanted to show.

A similar argument works for the second part of the theorem, but now setting

$$\triangle_1(d) = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \triangle_0 : \lambda_1 = \lambda_2 + \lambda_m + d\}$$

and then applying the one-to-one map T_1. $\square$

4.2 Images of Cylinders

We want to find interesting subsets with respect to the triangle map T. It is standard in dynamics to look at the cylinder sets. Thus the motivation for this section is coming from dynamical systems.

Here is the idea. The map T_0 is one-to-one on $\triangle_0$ and T_1 is one-to-one on $\triangle_1$. Given any subset S of either $\triangle_0$ or $\triangle_1$, we know that every number n will have as many partitions coming from S as from the appropriate $T(S)$. Thus if $S_0 \subset \triangle_0$ and if $S_1 \subset \triangle_1$, then for all positive integers n we have

$$p_{S_0}(n) = p_{T_0(S_0)}(n), \ p_{S_1}(n) = p_{T_1(S_1)}(n).$$

We would like to be able to find some easily describable and natural subsets of $\triangle_0$ and $\triangle_1$. From dynamical systems, the obvious choices would be *cylinder* sets.

The cylinder sets of length two are defined as

$$
\begin{aligned}
\triangle_{00} &= \{(\overline{\lambda}) \times [\overline{k}] \in \triangle_0 : T_0((\overline{\lambda}) \times [\overline{k}]) \in \triangle_0\} \\
\triangle_{01} &= \{(\overline{\lambda}) \times [\overline{k}] \in \triangle_0 : T_0((\overline{\lambda}) \times [\overline{k}]) \in \triangle_1\} \\
\triangle_{10} &= \{(\overline{\lambda}) \times [\overline{k}] \in \triangle_1 : T_1((\overline{\lambda}) \times [\overline{k}]) \in \triangle_0\} \\
\triangle_{11} &= \{(\overline{\lambda}) \times [\overline{k}] \in \triangle_1 : T_1((\overline{\lambda}) \times [\overline{k}]) \in \triangle_1\}
\end{aligned}
$$

By recursion, given a d-tuple $(i_1, i_2, \ldots, i_d)$ of zeros and ones, we define

$$\triangle_{(i_1, i_2, \ldots, i_d)} = \{(\overline{\lambda}) \times [\overline{k}] \in \triangle_{i_1} : T_{i_1}((\overline{\lambda}) \times [\overline{k}]) \in \triangle_{(i_2, i_3, \ldots, i_d)}\}.$$

We are defining these cylinder sets in terms of the map T, but each can be defined more intrinsically, as follows:

Proposition 8.

$$
\begin{aligned}
\triangle_{00} &= \{\overline{\lambda}) \times [\overline{k}] \in \mathcal{P}_{\geqslant 3} : \lambda_1 < \lambda_2 + \lambda_m, 2\lambda_2 < \lambda_1 + \lambda_3\} \\
&\quad \cup \{(\lambda_1, \lambda_2) \times [k_1, k_2] : \lambda_1 < 2\lambda_2, 3\lambda_2 < 2\lambda_1\} \\
\triangle_{01} &= \{\overline{\lambda}) \times [\overline{k}] : \lambda_1 < \lambda_2 + \lambda_m, 2\lambda_2 > \lambda_1 + \lambda_3\} \\
&\quad \cup \{(\lambda_1, \lambda_2) \times [k_1, k_2] : \lambda_1 < 2\lambda_2, 3\lambda_2 > 2\lambda_1\} \\
\triangle_{10} &= \{\overline{\lambda}) \times [\overline{k}] : \lambda_2 + \lambda_m < \lambda_1 < \lambda_2 + 2\lambda_m\} \\
&\quad \cup \{(\lambda_1, \lambda_2) \times [k_1, k_2] : \lambda_1 > 2\lambda_2, \lambda_1 < 3\lambda_2\} \\
\triangle_{11} &= \{\overline{\lambda}) \times [\overline{k}] : \lambda_1 > \lambda_2 + 2\lambda_m\} \\
&\quad \cup \{(\lambda_1, \lambda_2) \times [k_1, k_2] : \lambda_1 > 3\lambda_2\}
\end{aligned}
$$

Proof. We describe each cylinder set in turn, starting with looking at $\triangle_{00}$. Let $\lambda \in \triangle_{00}$ with $\dim(\lambda) \geqslant 3$. This means that

$$\lambda \in \triangle_0 \text{ and } T_0(\lambda) \in \triangle_0.$$

To be in $\triangle_0$ means that $\lambda_1 < \lambda_2 + \lambda_m$. As

$$T_0(\lambda) = (\lambda_2, \lambda_3, \ldots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \ldots, k_m, k_1],$$

to have $T(\lambda) \in \triangle_0$ means that we need

$$\lambda_2 < \lambda_3 + (\lambda_1 - \lambda_2)$$

which is the same as $2\lambda_2 < \lambda_1 + \lambda_3$

Now let $\lambda \in \triangle_{00}$ with $\dim(\lambda) = 2$. We know that for $\lambda \in \triangle_2$ in the dimension two case that $\lambda_1 < 2\lambda_2$. We have

$$T_0(\lambda) = (\lambda_2, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_1],$$

in which case to have to have $T(\lambda) \in \triangle_0$ means that

$$\lambda_2 < 2(\lambda_1 - \lambda_2)$$

which is the same as $3\lambda_2 < 2\lambda_1$.

Now turn to $\triangle_{01}$. Let $\lambda \in \triangle_{01} \cap \mathcal{P}_{\geqslant 3}$. This means that $\lambda \in \triangle_0$ and $T_0(\lambda) \in \triangle_1$. Being in $\triangle_0$ still means that $\lambda_1 < \lambda_2 + \lambda_m$. And we still have

$$T_0(\lambda) = (\lambda_2, \lambda_3, \ldots, \lambda_m, \lambda_1 - \lambda_2) \times [k_1 + k_2, k_3, \ldots, k_m, k_1].$$

But now we want $T_0(\lambda) \in \triangle_1$, which means that

$$\lambda_2 > \lambda_3 + (\lambda_1 - \lambda_2)$$

which is the same as $2\lambda_2 > \lambda_1 + \lambda_3$.

The case for $\lambda \in \triangle_{01} \cap \mathcal{P}_2$ is similar.

We now look at the third cylinder set $\triangle_{10}$. Let $\lambda \in \triangle_{10} \cap \mathcal{P}_{\geqslant 3}$. This means that $\lambda \in \triangle_1$ and $T_1(\lambda) \in \triangle_0$. Being in $\triangle_1$ means that $\lambda_1 > \lambda_2 + \lambda_m$. Then

$$T_1(\lambda) = (\lambda_1 - \lambda_m, \lambda_2, , \ldots, \lambda_m) \times [k_1, k_2, \ldots, k_{m-1}, k_1 + k_m].$$

To have $T_1(\lambda) \in \triangle_0$ means that

$$\lambda_1 - \lambda_m < \lambda_2 + \lambda_m$$

which is the same as $\lambda_1 < \lambda_2 + 2\lambda_m$, as claimed.

The case for $\lambda \in \triangle_{10} \cap \mathcal{P}_2$ is similar.

Turn to the fourth and last cylinder set $\triangle_{11}$ of the proposition. Let $\lambda \in \triangle_{11} \cap \mathcal{P}_{\geqslant 3}$. This means that $\lambda \in \triangle_1$ and $T_1(\lambda) \in \triangle_1$. Being in $\triangle_1$ means that $\lambda_1 > \lambda_2 + \lambda_m$, which is certainly true if $\lambda_1 > \lambda_2 + 2\lambda_m$. As we still have $T_1(\lambda) = (\lambda_1 - \lambda_m, \lambda_2, , \ldots, \lambda_m) \times [k_1, k_2, \ldots, k_{m-1}, k_1 + k_m]$, to get $T_1(\lambda) \in \triangle_1$ will mean that

$$\lambda_1 - \lambda_m > \lambda_2 + \lambda_m$$

which is the same as $\lambda_1 > \lambda_2 + 2\lambda_m$, and we are done.

And as before, the case for $\lambda \in \triangle_{11} \cap \mathcal{P}_2$ is similar. $\square$

The above proof was done in detail just to show how straightforward it is to define explicit cylinder sets.

Then we have

Proposition 9.

$$
\begin{aligned}
T_0(\triangle_{00}) &= \triangle_0 \cap M_0 \\
&= \{\lambda \in \mathcal{P} : \lambda_1 < \lambda_2 + \lambda_m, k_1 > k_m\} \\
T_0(\triangle_{01}) &= \triangle_1 \cap M_0 \\
&= \{\lambda \in \mathcal{P} : \lambda_1 > \lambda_2 + \lambda_m, k_1 > k_m\} \\
T_1(\triangle_{10}) &= \triangle_0 \cap M_1 \\
&= \{\lambda \in \mathcal{P} : \lambda_1 < \lambda_2 + \lambda_m, k_1 < k_m\} \\
T_1(\triangle_{11}) &= \triangle_1 \cap M_1 \\
&= \{\lambda \in \mathcal{P} : \lambda_1 > \lambda_2 + \lambda_m, k_1 < k_m\}
\end{aligned}
$$

Putting all of this together, gives us

Theorem 10.

1. *Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ and $2\lambda_2 < \lambda_1 + \lambda_3$ as into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ and $k_1 > k_m$, i.e.*

$$
p_{\triangle_{00}}(n) = p_{T_0(\triangle_{00})}(n).
$$

2. *Every number has as many integer partitions into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ and $2\lambda_2 > \lambda_1 + \lambda_3$ as into partitions with $\lambda_1 > \lambda_2 + \lambda_m$ and $k_1 > k_m$, i.e.*

$$
p_{\triangle_{01}}(n) = p_{T_0(\triangle_{01})}(n).
$$

3. *Every number has as many integer partitions into partitions with $\lambda_2 + \lambda_m < \lambda_1 < \lambda_2 + 2\lambda_m$ as into partitions with $\lambda_1 < \lambda_2 + \lambda_m$ and $k_1 < k_m$. i.e.*

$$
p_{\triangle_{10}}(n) = p_{T_1(\triangle_{10})}(n).
$$

4. *Every number has as many integer partitions into partitions with $\lambda_1 > \lambda_2 + 2\lambda_m$ as into partitions with $\lambda_1 > \lambda_2 + \lambda_m$ and $k_1 < k_m$. i.e.*

$$
p_{\triangle_{11}}(n) = p_{T_1(\triangle_{11})}(n).
$$

We can continue applying T to our sets, getting more and more new partition identities. We have

Proposition 11.

$$
T_0(T_0(\triangle_{00})) = \{\lambda \in \mathcal{P}_{\geqslant 3} : k_{m-1} < k_m < k_1\}
$$

$$\cup\{\lambda \in \mathcal{P}_2 : 2k_2 > k_1 > k_2\}$$
$$T_1(T_0(\triangle_{01})) = \{\lambda \in \mathcal{P} : k_1 < k_m < 2k_1\}$$
$$\cup\{\lambda \in \mathcal{P}_2 : k_1 < k_2 < 2k_1\}$$
$$T_0(T_1(\triangle_{10})) = \{\lambda \in \mathcal{P} : k_m < k_{m-1}, k_m < k_1\}$$
$$\cup\{\lambda \in \mathcal{P}_2 : k_1 > 2k_2\}$$
$$T_1(T_1(\triangle_{11})) = \{\lambda \in \mathcal{P} : 2k_1 < k_m\}$$

Note that the sets $\triangle_{ij}$ are defined solely in terms of the parts while the images $T_i(T_j(\triangle_{ij}))$ can be defined solely in terms of the multiplicities. This pattern continues, meaning that each cylinder set $\triangle_{(i_1,i_2,\dots,i_d)}$ is defined solely in terms of the parts n_i while the description of its image under d iterations of T is solely in terms of the multiplicities. (It strikes us that it would be enjoyable to find formulas for each of these sets in a straightforward manner.)

These are calculations.

We start with setting $\mu = (\mu_1, \dots, \mu_m) \times [l_1, \dots, l_m] \in \triangle_{00} \cap \mathcal{P}_{\geqslant 3}$. Set

$$T_0(T_0(\mu)) = \lambda = (\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m].$$

Then we have

$$
\begin{aligned}
T_0(T_0(\mu)) &= T_0(\mu_2, \mu_3, \dots, \mu_m, \mu_1 - \mu_2) \times [l_1 + l_2, l_3, \dots, l_m, l_1]) \\
&= (\mu_3, \dots, \mu_m, \mu_1 - \mu_2, \mu_2 - \mu_3) \times [l_1 + l_2 + l_3, l_4, \dots, l_m, l_1, l_1 + l_2] \\
&= (\lambda_1, \dots, \lambda_m) \times [k_1, \dots, k_m].
\end{aligned}
$$

We have

$$k_{m-1} = l_1 < l_1 + l_2 = k_m$$

and

$$k_m = l_1 + l_2 < l_1 + l_2 + l_3 = k_1,$$

and we have our description for $T_0(T_0(\triangle_{00} \cap \mathcal{P}_{\geqslant 3}))$.

Now let $\mu = (\mu_1, \mu_2) \times [l_1, l_2] \in \triangle_{00} \cap \mathcal{P}_2$. Following along the lines of the above, set

$$T_0(T_0(\mu)) = \lambda = (\lambda_1, \lambda_2) \times [k_1, k_2].$$

Then

$$
\begin{aligned}
T_0(T_0(\mu)) &= T_0(\mu_2, \mu_1 - \mu_2) \times [l_1 + l_2, l_1]) \\
&= (\mu_1 - \mu_2, 2\mu_2 - \mu_1) \times [2l_1 + l_2, l_1 + l_2] \\
&= (\lambda_1, \lambda_2) \times [k_1, k_2].
\end{aligned}
$$

Then we have

$$k_2 = l_1 + l_2 < 2l_1 + l_2 = k_1 < 2l_1 + 2l_2 = 2k_2$$

as desired.

Now let $\mu = (\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m] \in \triangle_{01} \cap \mathcal{P}_{\geqslant 3}$. We have

$$
\begin{aligned}
T_1(T_0(\mu)) &= T_0(\mu_2, \mu_3, \ldots, \mu_m, \mu_1 - \mu_2) \times [l_1 + l_2, l_3, \ldots, l_m, l_1]) \\
&= (2\mu_2 - \mu_1, \mu_3, \ldots, \mu_m, \mu_1 - \mu_2) \times [l_1 + l_2, l_3, \ldots, l_m, 2l_1 + l_2] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
\end{aligned}
$$

We have

$$k_1 = l_1 + l_2 < k_m = 2l_1 + l_2 < 2l_1 + 2l_2 = 2k_1,$$

as we want.

The case when $\mu = (\mu_1, \mu_2) \times [l_1, l_2] \in \triangle_{01} \cap \mathcal{P}_2$ is similar.

On to the third type of cylinder set. Now we let $\mu = (\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m] \in \triangle_{10} \cap \mathcal{P}_{\geqslant 3}$. We have

$$
\begin{aligned}
T_0(T_1(\mu)) &= T_0(\mu_1 - \mu_m, \mu_2, \ldots, \mu_m) \times [l_1, l_2, \ldots, l_{m-1}, l_1 + l_m]) \\
&= (\mu_2, \ldots, \mu_m, \mu_1 - \mu_2 - \mu_m) \times [l_1 + l_2, l_3, \ldots, l_1 + l_m, l_1] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
\end{aligned}
$$

which gives us

$$k_m = l_1 < l_1 + l_2 = k_1, \ k_m = l_1 < l_1 + l_m = k_{m-1}.$$

The case when $\mu = (\mu_1, \mu_2) \times [l_1, l_2] \in \triangle_{10} \cap \mathcal{P}_2$ is similar.

Now for the last condition. Let $\mu = (\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m] \in \triangle_{11}$. Then

$$
\begin{aligned}
T_1(T_1(\mu)) &= T_1((\mu_1 - \mu_m, \mu_2, \ldots, \mu_m) \times [l_1, l_2, \ldots, l_{m-1}, l_1 + l_m]) \\
&= (\mu_1 - 2\mu_m, \mu_2, \ldots, \mu_m) \times [l_1, l_2, \ldots, l_{m-1}, 2l_1 + l_m] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
\end{aligned}
$$

which gives us

$$2k_1 = 2l_1 < 2l_1 + l_m = k_m$$

using critically that $l_m \geqslant 1$, giving us our last equality.

Thus we have

Theorem 12.

1. $p_{\triangle_{00}}(n) = p_{T_0(T_0(\triangle_{00}))}(n)$.

2. $p_{\triangle_{01}}(n) = p_{T_1(T_0(\triangle_{01}))}(n)$.

3. $p_{\triangle_{10}}(n) = p_{T_0(T_1(\triangle_{10}))}(n)$.

4. $p_{\triangle_{11}}(n) = p_{T(T_1(\triangle_{11}))}(n)$.

Let us work out the example of $\triangle_{01}(11)$. In the appendix we list all partitions of the number 11. We see that

$$\triangle_{01} = \{(6,5) \times [1,1], (5,4,2) \times [1,1,1], (4,3) \times [2,1]\}$$

As each partition is in $\triangle_0$, to apply the map T we must apply T_0:

$$(6,5) \times [1,1] \xrightarrow{T_0} (5,1) \times [2,1]$$
$$(5,4,2) \times [1,1,1] \xrightarrow{T_0} (4,2,1) \times [2,1,1]$$
$$(4,3) \times [2,1] \xrightarrow{T_0} (3,1) \times [3,2]$$

Then we get

$$T(\triangle_{01}) = \{(5,1) \times [2,1], (4,2,1) \times [2,1,1], (3,1) \times [3,2]\},$$

as desired. Each of these partitions is in $\triangle_1$, and hence we must apply T_1:

$$(5,1) \times [2,1] \xrightarrow{T_1} (4,1) \times [2,3]$$
$$(4,2,1) \times [2,1,1] \xrightarrow{T_1} (3,2,1) \times [2,1,3]$$
$$(3,1) \times [3,2] \xrightarrow{T_1} (2,1) \times [3,5]$$

Thus by calculation we have

$$T(T(\triangle_{01})) = \{(4,1) \times [2,3], (3,2,1) \times [2,1,3], (2,1) \times [3,5]\}.$$

There are as many partition identities as there are cylinder sets.
We will just look at one example.
Set

$$\triangle_d^G = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P} : \lambda_1 - \lambda_2 - d\lambda_m > 0 > \lambda_1 - \lambda_2 - (d+1)\lambda_m\}.$$

The G stands for Gauss. When $m = 2$, these sets are the natural domains for the traditional Gauss map for continued fractions. The set $\triangle_d^G$ is the cylinder set for d ones followed by a zero and for $m = 3$ are the cylinder sets in [10].

Lemma 13. *For all $0 \leqslant n \leqslant d$, the map T_1^n is a one-to-one onto function from $\triangle_d^G$ to*

$$\triangle_{d-n}^G \cap \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : nk_1 < k_m\}.$$

Proof. Similar to before, set $\mu = (\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m] \in \triangle_d^G$. Set

$$T_1^n(\mu) = (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].$$

We have

$$
\begin{aligned}
T_1^n(\mu) &= T_1^n((\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m]) \\
&= (\mu_1 - n\mu_m, \mu_2, \ldots, \mu_m) \times [l_1, \ldots, nl_1 + l_m] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
\end{aligned}
$$

We have

$$
\begin{aligned}
\lambda_1 - \lambda_2 - (d-n)\lambda_m &= \mu_1 - \mu_2 - d\mu_m \\
&> 0 \\
&> \mu_1 - \mu_2 - d\mu_m - \mu_m \\
&= \lambda_1 - \lambda_2 - (d-n+1)\lambda_m
\end{aligned}
$$

For the multiplicities, we have

$$
\begin{aligned}
nk_1 &= nl_1 \\
&< nl_1 + l_m \\
&= k_m
\end{aligned}
$$

The one-to-oneness comes from that T_1 is one-to-one. It can be checked that each of the maps has a well-defined inverse, giving us that the iterative map T_1^n must also be onto. $\qquad\square$

This give us in particular that the image of $\triangle_d^G$ under T_1^d is in $\triangle_0^G = \triangle_0$.

Lemma 14. *The map $T_0 \circ T_1^d$ is a one-to-one onto function from $\triangle_d^G$ to*

$$
\{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] : dk_m < k_{m-1}, k_m < k_1\}.
$$

Proof. This will be similar to the previous lemma. Set $\mu = (\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m] \in \triangle_d^G$. Set

$$
T_0 \circ T_1^d(\mu) = (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
$$

Then

$$
\begin{aligned}
T_0 \circ T_1^d(\mu) &= T_0 \circ T_1^d((\mu_1, \ldots, \mu_m) \times [l_1, \ldots, l_m]) \\
&= T_0((\mu_1 - n\mu_m, \mu_2, \ldots, \mu_m) \times [l_1, \ldots, dl_1 + l_m]) \\
&= (\mu_2, \ldots, \mu_m, \mu_1 - \mu_2 - d\mu_m) \times [l_1 + l_2, l_3 \ldots, dl_1 + l_m, l_1] \\
&= (\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m].
\end{aligned}
$$

which gives us our result.

And as before, the one-to-oneness comes from that T_1 and T_0 are one-to-one. And it can again be checked that each of the maps has a well-defined inverse, giving us that the iterative map $T_0 \circ T_1^d$ must also be onto. $\qquad\square$

These two lemmas give us the following partition identity.

Theorem 15. *Every number has as many integer partitions into partitions with $\lambda_1 - \lambda_2 - d\lambda_m > 0 > \lambda_1 - \lambda_2 - (d+1)\lambda_m$ as into partitions, for any $0 \leqslant p \leqslant d$, $\lambda_1 - \lambda_2 - (d-p)\lambda_m > 0 > \lambda_1 - \lambda_2 - (d-p+1)\lambda_m$ and $pk_1 < k_m$, and as many partitions with $dk_m < k_{m-1}, k_m < k_1$, i.e.*

$$p_{\triangle_d^G}(n) = p_{T_1^p(\triangle_d^G)}(n) = p_{T_0 \circ T_1^p(\triangle_d^G)}(n).$$

4.3 Extending distinctness of parts via T

The map T from partitions to partitions allows us to start with any already known partition identity, apply the map T and see what happens.

We will do this for the set $\mathcal{D}$ of partitions with distinct parts (where all the $k_i = 1$) in this subsection and then for the set $\mathcal{O}$ of partitions with all the parts are odd (where each λ_i is an odd number) in the next subsection.

Set

$$
\begin{aligned}
\mathcal{E}_0 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P}_{\geqslant 2} : k_1 = 2, k_i = 1 \text{ if } i \geqslant 2\} \\
\mathcal{E}_1 &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P}_{\geqslant 2} : k_m = 2, k_i = 1 \text{ if } i < m\} \\
\mathcal{E}_D &= \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P}_{\geqslant 2} : k_1 = k_m = 2, k_i = 1 \text{ if } 1 < i < m\},
\end{aligned}
$$

Then

Theorem 16.

$$p_{\mathcal{D}}(n) = 1 + p_{\mathcal{E}_0}(n) + p_{\mathcal{E}_1}(n) + p_{\mathcal{E}_D}(n) + \lfloor (3/n)\lfloor n/3 \rfloor \rfloor.$$

The last term $\lfloor (3/n)\lfloor n/3 \rfloor \rfloor$ term is giving us an additional 1 for whenever n is divisible by 3

Proof. We know that

$$\mathcal{D} = \{(n) \times [1] : n = 1, 2, 3, \ldots\} \cup (\mathcal{D} \cap \triangle_0) \cup (\mathcal{D} \cap \triangle_1) \cup (\mathcal{D} \cap \triangle_D).$$

The first 1 in the desired equation is to account for the $\{(n) \times [1]\}$ term.

We will first show that T_0 is a one-to-one onto map from $\mathcal{D} \cap \triangle_0$ to $\mathcal{E}_0$. Let

$$\lambda = (\lambda_1, \ldots, \lambda_m) \times [1, \ldots, 1].$$

Then $T_0(\lambda)$ is

$$(\lambda_2, \ldots, \lambda_m, \lambda_1 - \lambda_2] \times [2, 1, \ldots, 1]$$

as desired. To show ontoness, we simply have to start with an element of the form

$$(\lambda_1, \ldots, \lambda_m,] \times [2, 1, \ldots, 1]$$

and apply T_0^{-1}, getting

$$(\lambda_1 + \lambda_m, \lambda_1, \ldots, \lambda_{m-1}) \times [1, \ldots, 1]$$

which is indeed in $\mathcal{D} \cap \triangle_0$

The argument that T_1 is an onto map from $\mathcal{D} \cap \triangle_1$ to $\mathcal{E}_1$ is similar.

Both of these use that T_0 and T_1 are always one-to-one maps.

Now to show that T_D is a one-to-one onto map from $\mathcal{D} \cap \triangle_D \cap \mathcal{P}_{\geqslant 3}$ to $\mathcal{E}_D$. In general, T_D is not one-to-one, but it will be in this case.

Let us start with $\lambda = (\lambda_1, \ldots, \lambda_m) \times [1, \ldots, 1] \in \mathcal{D} \cap \triangle_D$. Then $T_D(\lambda)$ is

$$(\lambda_2, \ldots, \lambda_m) \times [2, 1 \ldots, 1, 2].$$

One-to-oneness follows from Proposition 5, as as does ontoness.

Finally, turn to T_D acting on $\mathcal{D} \cap \triangle_D \cap \mathcal{P}_2$. An element in this set has the form $(2\lambda, \lambda) \times [1, 1]$, and hence only deals with partitions of numbers that are divisible by 3. By definition $T_D((2\lambda, \lambda) \times [1, 1]) = (\lambda) \times [3]$. This is what gives us the needed additional 1 is n if divisible by 3, and hence the final term of $\lfloor (3/n)\lfloor 3/n \rfloor \rfloor$. $\qquad \square$

For an example, we will see how T is a one-to-one onto map from $\mathcal{D}(11) \cap \triangle_0$ to $\mathcal{E}_0(11)$. From the appendix, we have

$$\mathcal{D}(11) \cap \triangle_0 = \{(7, 4) \times [1, 1], (6, 5) \times [1, 1], ((5, 4, 2) \times [1, 1, 1]\}$$

and

$$\mathcal{E}_0(11) = \{(4, 3) \times [2, 1], (4, 3) \times [2, 1], (4, 2, 1) \times [2, 1, 1]\}.$$

Applying T:

$$(7, 4) \times [1, 1] \xrightarrow{T_0} (4, 3) \times [2, 1]$$
$$(6, 5) \times [1, 1] \xrightarrow{T_0} (5, 1) \times [2, 1]$$
$$((5, 4, 2) \times [1, 1, 1] \xrightarrow{T_0} (4, 2, 1) \times [2, 1, 1]$$

giving us our result.

4.4 Extending oddness of parts via T

Using that $\mathcal{O}$ is the set of all partitions all of whose parts are odd and setting

$$\mathcal{F}_0 = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P}_{\geqslant 2} : \lambda_m \text{ even}, \lambda_i \text{ odd if } i < m, k_1 > k_m\}$$
$$\mathcal{F}_1 = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P}_{\geqslant 2} : \lambda_1 \text{ even}, \lambda_i \text{ odd if } i > 1, k_1 < k_m\}$$

then we have

Theorem 17.

$$p_{\mathcal{O}}(n) = (\text{number of odd factors of } n) + p_{\mathcal{F}_0}(n) + p_{\mathcal{F}_1}(n).$$

Proof. By now the path is clear. First, if $2k+1$ divides n, then one of the odd partitions would be $(2k+1) \times [n/(2k+1)]$. Each of these has dimension one, and hence the triangle map would not help us with this particular odd partition. This is why we need to explicitly add in the "number of odd factors" of n.

Next, we split the partitions of odd parts $\mathcal{O}$ of dimension at least two into the three disjoint sets

$$\mathcal{O} \cap \mathcal{P}_{\geqslant 2} = (\mathcal{O} \cap \triangle_0) \cup (\mathcal{O} \cap \triangle_1) \cup (\mathcal{O} \cap \triangle_D)$$

and find a clean description of the image of each of these sets under the map T. In this case, the last set $\mathcal{O} \cap \triangle_D$ is empty, since we cannot have $\lambda_1 = \lambda_2 + \lambda_m$ with all three of $\lambda_1, \lambda_2, \lambda_m$ being odd numbers.

Let $\lambda \in \mathcal{O} \cap \triangle_0$. Then

$$
\begin{aligned}
T_0(\lambda) &= T_0((\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]) \\
&= (\lambda_2, \ldots, \lambda_m, \lambda_1 - \lambda_2] \times [k_1 + k_2, k_3, \ldots, k_m, k_1]
\end{aligned}
$$

a partition all of whose parts are odd, save the smallest, which is indeed even. Further, the multiplicity of the largest part $k_1 + k_2$, is clearly strictly greater than the multiplicity of the smallest part, k_1. We know that T_0 is one-to-one. We need to show that if we start with a partition λ all of whose parts are odd, save the smallest, that T_0^{-1} is in $\mathcal{O}$.

Since the multiplicity of the largest part is strictly greater than the multiplicity of the smallest part, we know that we must act on the partition by T_0^{-1} :

$$
\begin{aligned}
T_0^{-1}(\lambda) &= T_0^{-1}((\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]) \\
&= (\lambda_1 + \lambda_m, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}) \times [k_m, k_1 - k_m, k_2, k + 3, \ldots, k_{m-1}]
\end{aligned}
$$

all of whose parts are odd.

The argument that T_1 is a one-to-one and onto map from $\mathcal{O} \cap \triangle_1$ to the set of all partitions all of whose parts are odd, save the largest part which must be even and with the multiplicity of the largest part strictly smaller than the multiplicity of the smallest part, is similar. $\square$

Putting the two theorems from the last two subsections together, along with Euler's original identity, we have

Theorem 18. *The number of ways of partitioning any positive integer n from each of the following three sets are all equal.*

1. *All parts are distinct*

2. *All parts are odd*

3. $\{(n)\times[1]\}$ *union the set where all parts are odd, save the smallest part which must be even and with the multiplicity of the largest part strictly greater than the multiplicity of the smallest part, union the set where all parts are all odd, save the largest part which must be even and with the multiplicity of the largest part strictly smaller than the multiplicity of the smallest part.*

Thus

$$
\begin{aligned}
p_{\mathcal{D}}(n) &= p_{\mathcal{O}}(n) \\
&= 1 + p_{\mathcal{E}_0}(n) + p_{\mathcal{E}_1}(n) + p_{\mathcal{E}_D}(n) + \lfloor (3/n)\lfloor n/3 \rfloor \rfloor \\
&= (number\ of\ odd\ factors\ of\ n) + p_{\mathcal{F}_0}(n) + p_{\mathcal{F}_1}(n).
\end{aligned}
$$

5 A Summary of Relevant Sets

We list in a table most of the various sets that we have cared about in this paper. For all, we use that the λ_i make up the parts and the k_i the multiplicities. For all m, we have $\lambda_1 > \cdots > \lambda_m > 0$ and $k_i > 0$ for $i = 1, \ldots, m$.

sets	dim $= 2$	dim $\geqslant 3$
$\triangle_0$	$2\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m > \lambda_1$
$\triangle_1$	$2\lambda_2 < \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1$
$\triangle_D$	$2\lambda_2 = \lambda_1$	$\lambda_2 + \lambda_m = \lambda_1$
$\triangle_{00}$	$2\lambda_2 > \lambda_1, 2\lambda_1 > 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 < \lambda_1 + \lambda_3$
$\triangle_{01}$	$2\lambda_2 > \lambda_1, 2\lambda_1 < 3\lambda_2$	$\lambda_2 + \lambda_m > \lambda_1, 2\lambda_2 > \lambda_1 + \lambda_3$
$\triangle_{10}$	$2\lambda_2 < \lambda_1, 3\lambda_2 > \lambda_1$	$\lambda_2 + \lambda_m < \lambda_1, \lambda_2 + 2\lambda_m > \lambda_1$
$\triangle_{11}$	$3\lambda_2 < \lambda_1$	$\lambda_2 + 2\lambda_m < \lambda_1$
$M_0 = T_0(\triangle_0)$	$k_1 > k_2$	$k_1 > k_m$
$M_1 = T_1(\triangle_1)$	$k_1 < k_2$	$k_1 < k_m$
$T_0(\triangle_{00})$	$2\lambda_2 > \lambda_1, k_1 > k_2$	$\lambda_2 + \lambda_m > \lambda_1, k_1 > k_m$
$T_0(\triangle_{01})$	$2\lambda_2 < \lambda_1, k_1 > k_2$	$\lambda_2 + \lambda_m < \lambda_1, k_1 > k_m$
$T_1(\triangle_{10})$	$2\lambda_2 > \lambda_1, k_1 < k_2$	$\lambda_2 + \lambda_m > \lambda_1, k_1 < k_m$
$T_1(\triangle_{11})$	$2\lambda_2 < \lambda_1, k_1 < k_2$	$\lambda_2 + \lambda_m < \lambda_1, k_1 < k_m$
$T_0(T_0(\triangle_{00}))$	$2k_2 > k_1 > k_2$	$k_1 > k_m > k_{m-1}$
$T_1(T_0(\triangle_{01}))$	$2k_1 > k_2 > k_1$	$2k_1 > k_m > k_1$
$T_0(T_1(\triangle_{10}))$	$2k_2 < k_1$	$k_1 > k_m, k_{m-1} > k_m$
$T_1(T_1(\triangle_{11}))$	$2k_1 < k_2$	$k_m > 2k_1$
$\mathcal{D}$	$k_1 = k_2 = 1$	$k_1 = \cdots = k_m = 1$
$\mathcal{E}_0$	$k_1 = 2, k_2 = 1$	$k_1 = 2, k_2 \ldots = k_m = 1$
$\mathcal{E}_1$	$k_1 = 1, k_2 = 2$	$k_1, \ldots = k_{m-1}, k_m = 2$
$\mathcal{E}_D$	$k_1 = 2, k_2 = 2$	$k_1 = 2, k_2 \ldots = k_{m-1}, k_m = 2$
$\mathcal{O}$	λ_1, λ_2 odd	λ_i odd, $i = 1, \ldots, m$
$\mathcal{F}_0$	λ_1 odd, λ_2 even	λ_i odd $i = 1, \ldots, m-1, \lambda_2$ even
$\mathcal{F}_1$	λ_1 even, λ_2 odd	λ_1 even, λ_i odd $i = 2, \ldots, m$

All of these sets only make sense for partitions whose dimensions are greater than or equal to two, save for the sets $\mathcal{D}$ and $\mathcal{O}$, as we have

$$
\begin{aligned}
\mathcal{D} \cap \mathcal{P}_1 &= \{(n) \times [1] : n = 1, 2, 3, \ldots\}, \\
\mathcal{O} \cap \mathcal{P}_1 &= \{(3n) \times [k] : n, k = 1, 2, 3, \ldots\}.
\end{aligned}
$$

6 On Their Generating Functions

It is straightforward to find the generating functions for the sets defined in Section 5, as we will see. This will allow us to translate each of our partition identities into identities of the corresponding generating functions.

We will use the standard notation

$$
\begin{aligned}
(a; q)_0 &= 1 \\
(a; q)_n &= (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \\
(a; q)_\infty &= (1 - a)(1 - aq)(1 - aq^2) \cdots
\end{aligned}
$$

We started with the set $\mathcal{P}$ of all partitions $(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, $k_i > 0$ and for all i, $\lambda_i, k_i \in \mathbb{Z}$. Then it is well known that the generating function for $p_{\mathcal{P}}(n)$ is

$$
\sum_{n=0}^{\infty} p_{\mathcal{P}}(n) q^n = \frac{1}{(q : q)_\infty} = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)}.
$$

Here we are using the convention that $p_{\mathcal{P}}(0) = 1$.

We set

$$
\mathcal{P}_N = \{(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \in \mathcal{P} : m = N\},
$$

all the partitions of dimension N and let $\mathcal{P}_{\geqslant N}$ be the set of all partitions with dimension at least N. Then we have the generating function for partitions of dimension one being

$$
\sum_{n=0}^{\infty} p_{\mathcal{P}_1}(n) q^n = \sum_{n=1}^{\infty} d(n) q^n = \sum_{m=1}^{\infty} \frac{q^m}{1 - q^m},
$$

where $d(n)$ is the divisor function of n, meaning that it is the number of divisors of n, including 1 and the number n itself.

More generally, we have

$$
\sum_{n=0}^{\infty} p_{\mathcal{P}_N}(n) q^n = \sum_{\lambda_1 > \cdots > \lambda_N > 0} \frac{q^{\lambda_1}}{(1 - q^{\lambda_1})} \cdots \frac{q^{\lambda_N}}{(1 - q^{\lambda_N})}.
$$

We know that

$$
\mathcal{P}(n) = \{(m) \times [k] : mk = n\} \cup \triangle_0(n) \cup \triangle_1(n) \cup \triangle_D(n)
$$

where all of these sets are disjoint from each other.

We can calculate that

$$\sum_{n=0}^{\infty} p_{\triangle_0}(n)q^n = \sum_{m=2}^{\infty} \sum_{\left(\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 < \lambda_2 + \lambda_m}\right)} \frac{q^{\lambda_1}}{(1-q^{\lambda_1})} \cdots \frac{q^{\lambda_m}}{(1-q^{\lambda_m})}$$

$$\sum_{n=0}^{\infty} p_{\triangle_1}(n)q^n = \sum_{m=2}^{\infty} \sum_{\left(\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 > \lambda_2 + \lambda_m}\right)} \frac{q^{\lambda_1}}{(1-q^{\lambda_1})} \cdots \frac{q^{\lambda_m}}{(1-q^{\lambda_m})}$$

$$\sum_{n=0}^{\infty} p_{\triangle_D}(n)q^n = \sum_{m=2}^{\infty} \sum_{\left(\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 = \lambda_2 + \lambda_m}\right)} \frac{q^{\lambda_1}}{(1-q^{\lambda_1})} \cdots \frac{q^{\lambda_m}}{(1-q^{\lambda_m})}$$

$$\sum_{n=0}^{\infty} p_{M_0}(n)q^n = \sum_{m=2}^{\infty} \sum_{(\lambda_1 > \cdots > \lambda_m > 0)} \sum_{k_1 > k_m > 0} q^{k_1 \lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1 - \lambda_i} \right) q^{k_m \lambda_m}$$

$$\sum_{n=0}^{\infty} p_{M_1}(n)q^n = \sum_{m=2}^{\infty} \sum_{(\lambda_1 > \cdots > \lambda_m > 0)} \sum_{k_m > k_1 > 0} q^{k_1 \lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1 - \lambda_i} \right) q^{k_m \lambda_m}$$

$$\sum_{n=0}^{\infty} p_{\triangle_{00}}(n)q^n = \sum_{m=3}^{\infty} \sum_{\left(\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 < \lambda_2 + \lambda_m \\ 2\lambda_2 < \lambda_1 + \lambda_3}\right)} \left(\prod_{i=1}^{m} \frac{q^{\lambda_i}}{1 - \lambda_i} \right)$$
$$+ \sum_{\left(\substack{\lambda_1 > \lambda_2 > 0 \\ \lambda_1 < 2\lambda_2, 3\lambda_2 < 2\lambda_1}\right)} \left(\frac{q^{\lambda_1}}{1 - \lambda_1} \cdot \frac{q^{\lambda_2}}{1 - \lambda_2} \right)$$

$$\sum_{n=0}^{\infty} p_{\triangle_{01}}(n)q^n = \sum_{m=3}^{\infty} \sum_{\left(\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 < \lambda_2 + \lambda_m \\ 2\lambda_2 > \lambda_1 + \lambda_3}\right)} \left(\prod_{i=1}^{m} \frac{q^{\lambda_i}}{1 - \lambda_i} \right)$$
$$+ \sum_{\left(\substack{\lambda_1 > \lambda_2 > 0 \\ \lambda_1 < 2\lambda_2, 3\lambda_2 > 2\lambda_1}\right)} \left(\frac{q^{\lambda_1}}{1 - \lambda_1} \cdot \frac{q^{\lambda_2}}{1 - \lambda_2} \right)$$

$$\sum_{n=0}^{\infty} p_{\triangle_{10}}(n)q^n = \sum_{m=3}^{\infty} \sum_{\begin{pmatrix} \lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 > \lambda_2 + \lambda_m \\ \lambda_1 < \lambda_2 + 2\lambda_m \end{pmatrix}} \left(\prod_{i=1}^{m} \frac{q^{\lambda_i}}{1-\lambda_i} \right)$$

$$+ \sum_{\begin{pmatrix} \lambda_1 > \lambda_2 > 0 \\ 3\lambda_2 > \lambda_1 > 2\lambda_2 \end{pmatrix}} \left(\frac{q^{\lambda_1}}{1-\lambda_1} \cdot \frac{q^{\lambda_2}}{1-\lambda_2} \right)$$

$$\sum_{n=0}^{\infty} p_{\triangle_{11}}(n)q^n = \sum_{m=3}^{\infty} \sum_{\begin{pmatrix} \lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 > \lambda_2 + 2\lambda_m \end{pmatrix}} \left(\prod_{i=1}^{m} \frac{q^{\lambda_i}}{1-\lambda_i} \right)$$

$$+ \sum_{\begin{pmatrix} \lambda_1 > \lambda_2 > 0 \\ \lambda_1 > 3\lambda_2 \end{pmatrix}} \left(\frac{q^{\lambda_1}}{1-\lambda_1} \cdot \frac{q^{\lambda_2}}{1-\lambda_2} \right)$$

$$\sum_{n=0}^{\infty} p_{T_0\triangle_{00}}(n)q^n =$$

$$\sum_{m=2}^{\infty} \sum_{\begin{pmatrix} \lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 < \lambda_2 + \lambda_m \end{pmatrix}} \sum_{k_1=2}^{\infty} \sum_{\begin{pmatrix} k_i > 0, i=1,\ldots m \\ k_m < k_1 \end{pmatrix}} q^{k_1\lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1-\lambda_i} \right) q^{k_m\lambda_m}$$

$$\sum_{n=0}^{\infty} p_{T_0\triangle_{01}}(n)q^n =$$

$$\sum_{m=2}^{\infty} \sum_{\begin{pmatrix} \lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 > \lambda_2 + \lambda_m \end{pmatrix}} \sum_{k_1=2}^{\infty} \sum_{\begin{pmatrix} k_i > 0, i=1,\ldots m \\ k_m < k_1 \end{pmatrix}} q^{k_1\lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1-\lambda_i} \right) q^{k_m\lambda_m}$$

$$\sum_{n=0}^{\infty} p_{T_1\triangle_{10}}(n)q^n =$$

$$\sum_{m=2}^{\infty} \sum_{\begin{pmatrix} \lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 < \lambda_2 + \lambda_m \end{pmatrix}} \sum_{k_m=2}^{\infty} \sum_{\begin{pmatrix} k_i > 0, i=1,\ldots m \\ k_1 < k_m \end{pmatrix}} q^{k_1\lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1-\lambda_i} \right) q^{k_m\lambda_m}$$

$$\sum_{n=0}^{\infty} p_{T_1\triangle_{11}}(n)q^n =$$

$$\sum_{m=2}^{\infty} \sum_{\substack{\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 > \lambda_2 + \lambda_m}} \sum_{k_m=2}^{\infty} \sum_{\substack{k_i > 0, i=1,\ldots m \\ k_1 < k_m}} q^{k_1 \lambda_1} \left(\prod_{i=2}^{m-1} \frac{q^{\lambda_i}}{1-\lambda_i} \right) q^{k_m \lambda_m}$$

$$\sum_{n=0}^{\infty} p_{T_0(T_0(\triangle_{00}))}(n) q^n =$$

$$\sum_{m=3}^{\infty} \sum_{\lambda_1 > \cdots > \lambda_m > 0} \sum_{\substack{k_i > 0, i=1, m-1, m \\ k_{m-1} < k_m < k_1}} q^{k_1 \lambda_1} \prod_{i=1}^{m-2} \frac{q^{\lambda_i}}{1-q^{\lambda_i}} q^{k_{m-1} \lambda_{m-1}} q^{k_m \lambda_m} +$$

$$\sum_{\lambda_1 > \lambda_2 > 0} \sum_{\substack{k_1, k_2 > 0, \\ k_2 < k_1 < 2k_2}} q^{k_1 \lambda_1} q^{k_2 \lambda_2}$$

$$\sum_{n=0}^{\infty} p_{T_1(T_0(\triangle_{01}))}(n) q^n = \sum_{m=3}^{\infty} \sum_{\lambda_1 > \cdots > \lambda_m > 0} \sum_{\substack{k_1, k_m > 0 \\ k_1 < k_m < 2k_1}} q^{k_1 \lambda_1} \prod_{i=1}^{m-1} \frac{q^{\lambda_i}}{1-q^{\lambda_i}} q^{k_m \lambda_m} +$$

$$\sum_{\lambda_1 > \lambda_2 > 0} \sum_{\substack{k_1, k_2 > 0, \\ k_1 < k_2 < 2k_1}} q^{k_1 \lambda_1} q^{k_2 \lambda_2}$$

$$\sum_{n=0}^{\infty} p_{T_0(T_1(\triangle_{10}))}(n) q^n =$$

$$\sum_{m=3}^{\infty} \sum_{\lambda_1 > \cdots > \lambda_m > 0} \sum_{\substack{k_1, k_{m-1}, k_m > 0 \\ k_m < k_1, k_m < k_{m-1}}} q^{k_1 \lambda_1} \prod_{i=1}^{m-2} \frac{q^{\lambda_i}}{1-q^{\lambda_i}} q^{k_{m-1} \lambda_{m-1}} q^{k_m \lambda_m} +$$

$$\sum_{\lambda_1 > \lambda_2 > 0} \sum_{\substack{k_1, k_2 > 0, \\ 2k_2 < k_1}} q^{k_1 \lambda_1} q^{k_2 \lambda_2}$$

$$\sum_{n=0}^{\infty} p_{T_1(T_1(\triangle_{11}))}(n) q^n = \sum_{m=3}^{\infty} \sum_{\lambda_1 > \cdots > \lambda_m > 0} \sum_{\substack{k_1, k_m > 0 \\ 2k_1 < k_m}} q^{k_1 \lambda_1} \prod_{i=1}^{m-1} \frac{q^{\lambda_i}}{1-q^{\lambda_i}} q^{k_m \lambda_m}$$

$$\sum_{n=0}^{\infty} p_D(n)q^n = \prod_{k=1}^{\infty}(1+q^k)$$

$$\sum_{n=2}^{\infty} p_{\mathcal{E}_0}(n)q^n = \sum_{m=2}^{\infty}(1+q)\cdots(1+q^{m-1})q^{2m}$$

$$\sum_{n=2}^{\infty} p_{\mathcal{E}_1}(n)q^n = \sum_{k=1}^{\infty} q^{2k}\prod_{n>k}(1+q^m)$$

$$\sum_{n=2}^{\infty} p_{\mathcal{E}_1}(n)q^n = \sum_{1 \leqslant k_1 < k_m} q^{2k_1}\prod_{n=k_1+1}^{k_m-1}(1+q^n)q^{2k_m}$$

$$\sum_{n=0}^{\infty} p_{\mathcal{O}}(n)q^n = \prod_{k=0}^{\infty}\frac{1}{1-q^{2k+1}}$$

$$\sum_{n=2}^{\infty} p_{\mathcal{F}_0}(n)q^n = \sum_{m=2}^{\infty}\sum_{k_m=1}^{\infty}\sum_{\left(\begin{array}{c}\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_m \text{ even} \\ \lambda_i \text{ odd if } i < m \\ k_1 > k_m\end{array}\right)} q^{k_1\lambda_1}\left(\prod_{i=2}^{m-1}\frac{q^{\lambda_i}}{1-q^{\lambda_i}}\right)q^{k_m\lambda_m}$$

$$\sum_{n=2}^{\infty} p_{\mathcal{F}_0}(n)q^n = \sum_{m=2}^{\infty}\sum_{k_1=1}^{\infty}\sum_{\left(\begin{array}{c}\lambda_1 > \cdots > \lambda_m > 0 \\ \lambda_1 \text{ even} \\ \lambda_i \text{ odd if } i > 1 \\ k_1 < k_m\end{array}\right)} q^{k_1\lambda_1}\left(\prod_{i=2}^{m-1}\frac{q^{\lambda_i}}{1-q^{\lambda_i}}\right)q^{k_m\lambda_m}$$

Then the generating interpretation of Theorem 4 and Theorem 7 is that

$$\sum_{n=2}^{\infty} p_{\triangle_0}(n)q^n = \sum_{n=2}^{\infty} p_{\mathcal{M}_0}(n)q^n, \ \sum_{n=2}^{\infty} p_{\triangle_1}(n)q^n = \sum_{n=2}^{\infty} p_{\mathcal{M}_1}(n)q^n.$$

The generating function interpretation for Theorems 10 and 12 is

$$
\begin{array}{ccccc}
\sum_{n=2}^{\infty} p_{\triangle_{00}}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_0(\triangle_{00})}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_0(T_0(\triangle_{00}))}(n)q^n \\
\sum_{n=2}^{\infty} p_{\triangle_{01}}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_0(\triangle_{01})}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_1(T_0(\triangle_{01}))}(n)q^n \\
\sum_{n=2}^{\infty} p_{\triangle_{10}}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_1(\triangle_{10})}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_0(T_1(\triangle_{10}))}(n)q^n \\
\sum_{n=2}^{\infty} p_{\triangle_{11}}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_1(\triangle_{11})}(n)q^n &=& \sum_{n=2}^{\infty} p_{T_1(T_1(\triangle_{11}))}(n)q^n.
\end{array}
$$

Theorem 16 is now

$$\sum_{n=0}^{\infty} p_D(n)q^n = \sum_{n=0}^{\infty} q^n + \sum_{n=2}^{\infty} p_{\mathcal{E}_0}(n)q^n + \sum_{n=2}^{\infty} p_{\mathcal{E}_1}(n)q^n + \sum_{n=2}^{\infty} p_{\mathcal{E}_D}(n)q^n + \sum_{k=1}^{\infty} q^{3k}.$$

The $\sum_{n=0}^{\infty} q^n$ term is reflecting all partitions of the form $(n) \times [1]$ and the $\sum_{k=1}^{\infty} q^{3k}$ is capturing all partitions of the form $(k) \times [3]$.

Now for the generating function version of Theorem 17:

$$\sum_{n=1}^{\infty} p_{\mathcal{O}}(n)q^n = \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1-q^{2k+1}} + \sum_{n=2}^{\infty} p_{\mathcal{F}_0}(n)q^n + \sum_{n=2}^{\infty} p_{\mathcal{F}_0}(n)q^n.$$

The first term on the right is capturing all the partition with odd parts of dimension one, and hence the partitions fo the form $(2n+1) \times [k]$.

7 Conclusion

While we find the new partition identities interesting, we find the method by which they were discovered as more important, namely the recognition that the triangle map provides a map from partitions of dimension two or greater to partitions.

The space of partitions $\mathcal{P}_{\geqslant 2}$ can be naturally split into three disjoint subsets $\triangle_0$, $\triangle_1$ and $\triangle_D$. The map T is one-to-one on $\triangle_0$ and $\triangle_1$. Hence for any subset S of $\triangle_0$ and $\triangle_1$, we have $p_S(n) = p_T(S)(n)$. This provided a new systematic method for producing many new partition identities, namely find an "interesting" subset S in the space of partitions. Then study $T(S), T^2(S), T^3(S), \dots$ and $T^{-1}(S), T^{-2}(S), T^{-3}(S), \dots$. The proofs of these new identities will be easy and straightforward, as we saw in the examples given in Section 4.

There are, though, many other multi-dimensional continued fraction algorithms. Most seem not to be useful for partitions, as briefly discussed in section six of [5] for partitions in $\mathcal{P}_3$. We do not really understand why the triangle map, and a few other multi-dimensional continued fraction algorithms, can be used on partitions while most cannot. Is there an underlying geometric or dynamical reason for the triangle map to be, let us say, *partition friendly*, or is it simply a coincidence. This strikes us a hard (if not a mathematical precise) question.

8 Appendix

From Mathematica, we know that all 56 partitions of 11 are:

$$
\begin{array}{llll}
(11) \times [1] & (10,1) \times [1,1] & (9,2) \times [1,1] & (9,1) \times [1,2] \\
(8,3) \times [1,1] & (8,2,1) \times [1,1,1] & (8,1) \times [1,3] & (7,4) \times [1,1] \\
(7,3,1) \times [1,1,1] & (7,2) \times [1,2] & (7,2,1) \times [1,1,2] & (7,1) \times [1,4] \\
(6,5) \times [1,1] & (6,4,1) \times [1,1,1] & (6,3,2) \times [1,1,1] & (6,3,1) \times [1,1,2] \\
(6,2,1) \times [1,2,1] & (6,2,1) \times [1,1,3] & (6,1) \times [1,5] & (5,1) \times [2,1] \\
(5,4,2) \times [1,1,1] & (5,4,1) \times [1,1,2] & (5,3) \times [1,2] & (5,3,2,1) \times [1,1,1,1] \\
(5,3,1) \times [1,1,3] & (5,2) \times [1,3] & (5,2,1) \times [1,2,2] & (5,2,1) \times [1,1,4] \\
(5,1) \times [1,6] & (4,3) \times [2,1] & (4,2,1) \times [2,1,1] & (4,1) \times [2,3] \\
(4,3,1) \times [1,2,1] & (4,3,2) \times [1,1,2] & (4,3,2,1) \times [1,1,1,2] & (4,3,1) \times [1,1,4] \\
(4,2,1) \times [1,3,1] & (4,2,1) \times [1,2,3] & (4,2,1) \times [1,1,5] & (4,1) \times [1,7] \\
(3,2) \times [3,1] & (3,1) \times [3,2] & (3,2,1) \times [2,2,1] & (3,2,1) \times [2,1,3] \\
(3,1) \times [2,5] & (3,2) \times [1,4] & (3,2,1) \times [1,3,2] & (3,2,1) \times [1,2,4] \\
(3,2,1) \times [1,1,6] & (3,1) \times [1,8] & (2,1) \times [5,1] & (2,1) \times [4,3] \\
(2,1) \times [3,5] & (2,1) \times [2,7] & (2,1) \times [1,9] & (1) \times [11]
\end{array}
$$

We encourage readers to use this list to check all the partition identities that are given in this paper.

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