Balanced Nontransitive Dice: Existence and Probability

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Abstract

A triple (A, B, C) of dice is called *nontransitive* if each of $P(A < B)$, $P(B < C)$, and $P(C < A)$ is greater than $\frac{1}{2}$ and called *balanced* if $P(A < B) = P(B < C) = P(C < A)$. From the result of Trybuła, it is known that $P(A < B)$ is less than $\frac{-1+\sqrt{5}}{2}$, the golden ratio, for every balanced nontransitive triple (A, B, C) of dice. Schaefer asked whether this upper bound is tight, and Hur and Kim conjectured that the upper bound can be reduced to $\frac{1}{2} + \frac{1}{9}$. In this paper, we characterize all possible probabilities $P(A < B)$ for balanced nontransitive triples (A, B, C) of dice. Precisely, we prove that, for every rational $\frac{1}{2} < q < \frac{-1+\sqrt{5}}{2}$, there exists a balanced nontransitive triple (A, B, C) of dice with $P(A < B) = q$, which disproves Hur and Kim's conjecture and answers Schaefer's question.

We also characterize all triples (m, n, ℓ) of positive integers such that there exists a balanced nontransitive triple (A, B, C) of dice, where A, B, and C are m-sided, n-sided, and ℓ-sided dice, respectively. This generalizes Schaefer and Schweig's result showing the existence of a balanced nontransitive triple of n-sided dice for every $n \geqslant 3$.

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1 Introduction

For two dice A and B, we write $P(A < B)$ as the probability that B rolls higher than A.

For the three dice A, B, and C shown in Figure 1, we can verify that each of $P(A < B)$, $P(B < C)$, and $P(C < A)$ is greater than $1/2$. This phenomenon of nontransitive dice was introduced by Gardner [4] and was further studied in [1, 2, 3, 7, 9, 10].

Schaefer and Schweig [12] studied a nontransitive triple (A, B, C) of dice with an additional condition: $P(A < B) = P(B < C) = P(C < A)$. Such a triple is called *balanced*.

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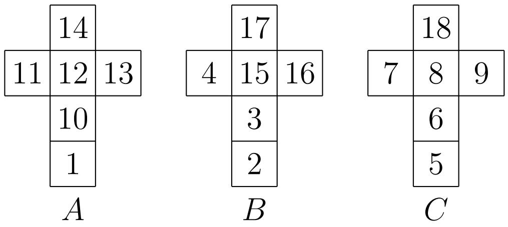

Figure 1: A nontransitive triple (A, B, C) of dice.

Since the existence of a balanced nontransitive triple of n-sided dice was proved [12], a balanced nontransitive triple of dice has been studied in various ways [5, 6, 8, 11, 13]. However all those studies have only considered dice with the same number of sides.

Our goal in this paper is to study a balanced nontransitive triple of dice, each of which does not necessarily have the same number of sides. For our purposes, we define a triple of dice as a triple (A, B, C) of pairwise disjoint sets A, B, C with $A \cup B \cup C = \{1, 2, \dots, |A| + |B| + |C|\}$. For $X = A, B, C$, we regard die X as a fair die labeled with the elements of the set X.

Definition 1. A triple (A, B, C) of dice is *nontransitive* if each of $P(A < B)$, $P(B < C)$, and $P(C < A)$ is greater than $\frac{1}{2}$.

Definition 2. A triple (A, B, C) of dice is *balanced* if $P(A < B) = P(B < C) = P(C < A)$.

An (m, n, ℓ)-*triple of dice* is a triple (A, B, C) of dice with $|A| = m$, $|B| = n$, and $|C| = \ell$. We simply write (n)-triple of dice for an (n, n, n)-triple of dice.

Example 3. In Figure 1, (A, B, C) is a nontransitive (6)-triple of dice.

Example 4. If $A = \{1, 2, 7, 8, 10\}$, $B = \{3, 4, 5, 9, 11\}$, and $C = \{6\}$, then (A, B, C) is a nontransitive balanced $(5, 5, 1)$-triple of dice because $P(A < B) = P(B < C) = P(C < A) = \frac{3}{5} > \frac{1}{2}$.

The following result implies that $P(A < B)$ is less than $\frac{-1+\sqrt{5}}{2}$, the golden ratio, for every balanced nontransitive triple (A, B, C) of dice.

Corollary 5 (Trybuła [14])**.** *Let X, Y, Z be independent random variables. If $P(X < Y) = P(Y < Z) = P(Z < X)$, then $P(X < Y) < \frac{-1+\sqrt{5}}{2}$.*

In [11], Schaefer asked whether this upper bound is sharp, and Hur and Kim [5] conjectured that the upper bound can be reduced to $\frac{1}{2} + \frac{1}{9}$. Our first result answers Schaefer's question and disproves Hur and Kim's conjecture. Specifically, we classified all possible probabilities as follows:

Theorem 6. *For every rational $\frac{1}{2} < q < \frac{-1+\sqrt{5}}{2}$, there exist a positive integer n and a balanced nontransitive (n)-triple of dice (A, B, C) such that*

$$P(A < B) = P(B < C) = P(C < A) = q.$$

In [12], Schaefer and Schweig showed the existence of a balanced nontransitive (n)-triple of dice for every $n \geqslant 3$. Our second theorem generalizes this theorem; that is, we completely classify all triples (m, n, ℓ) of positive integers such that there exists a balanced nontransitive (m, n, ℓ)-triple of dice. For triples (a, b, c) and (x, y, z) of positive integers, if there exists a permutation (x', y', z') of (x, y, z) such that $a \mid x'$, $b \mid y'$, and $c \mid z'$, then we say (a, b, c) is a *divisor* of (x, y, z) or that (x, y, z) is a *multiple* of (a, b, c). For example, $(6, 3, 2)$ is a divisor of $(4, 9, 6)$ because $6|6$, $3|9$, and $2|4$.

Theorem 7. *Let m, n, and ℓ be positive integers. Then, there exists a balanced nontransitive (m, n, ℓ)-triple of dice (A, B, C) if and only if (m, n, ℓ) is divisible by one of the following:*

(i) $(5, 5, 1)$, $(7, 7, 1)$,

(ii) $(k, k, 1)$ *for* $k \geqslant 9$,

(iii) $(3, 3, 3)$, $(4, 4, 4)$, $(6, 6, 2)$, $(8, 8, 2)$, $(12, 4, 3)$.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and preliminaries. In Section 3, we prove Theorem 6. Finally, in Section 4, we prove Theorem 7.

2 Notations and Preliminaries

A *word* σ is a finite sequence each letter of which is either A, B, or C. We denote by $|\sigma|$ the length of σ, i.e., the number of letters in σ, and for $X \in \{A, B, C\}$, we denote by $|X|_\sigma$ the number of the letters X in σ. Moreover, we let $\sigma(i)$ be the i-th letter of σ.

We can associate a triple of dice with a word as follows:

Definition 8. For a triple $D = (A, B, C)$ of dice, let $\sigma = \sigma(D)$ be the word of length $|A| + |B| + |C|$ whose i-th letter is the die including i as a label.

Example 9. For a triple $D = (A, B, C)$ of dice in Figure 1,

$$\sigma(D) = ABBBCCCCCAAAAABBBC.$$

Conversely, for a word σ, let $D(\sigma)$ be the triple of dice corresponding to σ. From Definition 8, we can clearly see that $D(\sigma)$ is uniquely determined. Using this one-to-one correspondence, we can consider a triple of dice as its associated word and vice versa.

For a word σ and distinct $X, Y \in \{A, B, C\}$, let $N_\sigma(X < Y)$ be the number of pairs (i, j) of $1 \leqslant i < j \leqslant |\sigma|$ such that $\sigma(i) = X$ and $\sigma(j) = Y$ and let $P_\sigma(X < Y) = \frac{N_\sigma(X<Y)}{|X|_\sigma |Y|_\sigma}$. Clearly, $P_\sigma(X < Y) = P_{D(\sigma)}(X < Y)$.

Example 10. For $\sigma = \sigma(D)$ in Example 9, $N_\sigma(A < B) = 21$ and $P_\sigma(A < B) = \frac{21}{36} = \frac{7}{12}$.

We can naturally define a nontransitive word and a balanced word as follows:

Definition 11. A word σ is *nontransitive* if each of $P_\sigma(A < B)$, $P_\sigma(B < C)$, and $P_\sigma(C < A)$ exceeds $\frac{1}{2}$.

Definition 12. A word σ is *balanced* if $P_\sigma(A < B) = P_\sigma(B < C) = P_\sigma(C < A)$.

An (m, n, ℓ)-triple of dice D is associated with a word σ with $|A|_\sigma = m$, $|B|_\sigma = n$ and $|C|_\sigma = \ell$. Such a word is called an (m, n, ℓ)-word.

Definition 13. An (m, n, ℓ)*-word* is a word σ with $|A|_\sigma = m$, $|B|_\sigma = n$, and $|C|_\sigma = \ell$. We simply write an (m)*-word* for an (m, m, m)-word.

Using the above terminologies, we can rephrase Schaefer and Schweig's theorem as follows:

Theorem 14 (Schaefer and Schweig [12])**.** *For every $m \geqslant 3$, there is a balanced nontransitive (m)-word.*

3 Possible Probability

For a balanced nontransitive word σ, let $P(\sigma) = P_\sigma(A < B)$. In this section, we classify all possible probabilities of $P(\sigma)$, which implies Theorem 6.

Note that $P(\sigma)$ is a positive rational. Furthermore, by the fact that σ is nontransitive and by Corollary 5, we have $\frac{1}{2} < P(\sigma) < \frac{-1+\sqrt{5}}{2}$. We prove that, for each rational $\frac{1}{2} < q < \frac{-1+\sqrt{5}}{2}$, there exists a balanced nontransitive (m)-word σ with $P(\sigma) = q$.

A word σ is called *central* if the letters C are placed consecutively; that is, there are positive integers i and j with $1 \leqslant i < j \leqslant |\sigma|$ such that $\sigma(k) = C$ if and only if $i \leqslant k \leqslant j$. In this case, we define the type of σ as a pair (a, b) of nonnegative integers a and b, where a (respectively b) is the number of the letters A (respectively B) appearing in the first $(i - 1)$ letters in σ.

Example 15. The word $\sigma = AABACCBBA$ is central and its type is $(3, 1)$.

Lemma 16. *Let m, n, and ℓ be positive integers. If a and b are nonnegative integers with $a \leqslant m$ and $b \leqslant n$, then for every integer s with $a(n-b) \leqslant s \leqslant an + (m-a)(n-b)$, there is a central (m, n, ℓ)-word σ of type (a, b) such that $N_\sigma(A < B) = s$.*

Proof. Let σ be a central (m, n, ℓ)-word of type (a, b) with $\sigma(i) = B$ for $1 \leqslant i \leqslant b$ and $a + b + \ell + 1 \leqslant i \leqslant a + \ell + n$, i.e.,

$$\sigma = \underbrace{BB\ldots B}_{b}\,\underbrace{AA\ldots A}_{a}\,\underbrace{CC\ldots C}_{\ell}\,\underbrace{BB\ldots B}_{n-b}\,\underbrace{AA\ldots A}_{m-a}.$$

We consider a sequence $\sigma_0, \sigma_1, \ldots, \sigma_k$ of central (m, n, ℓ)-words defined as follows.

- $\sigma_0 = \sigma$,

- for each $i = 1, 2, \ldots, k$, σ_i is obtained from σ_{i-1} by switching one BA in σ_{i-1} to AB, and
- there is no BA in σ_k.

For example, if $\sigma = BBACBAA$, then

$$\sigma_0 = BBACBAA, \quad \sigma_1 = BABCBAA, \quad \sigma_2 = ABBCBAA,$$
$$\sigma_3 = ABBCABA, \quad \sigma_4 = ABBCAAB.$$

From the definition of the sequence, k is indeed fixed as $ab + (m-a)(n-b)$. Note that, for each $i = 1, 2, \ldots, k$,

$$N_{\sigma_i}(A < B) = N_{\sigma_{i-1}}(A < B) + 1$$

and

$$N_{\sigma_0}(A < B) = a(n-b), \quad N_{\sigma_k}(A < B) = an + (m-a)(n-b).$$

Thus, for each $a(n-b) \leqslant s \leqslant an + (m-a)(n-b)$, $\sigma_{s-a(n-b)}$ is a central (m, n, ℓ)-word of type (a, b) with $N_{\sigma_{s-a(n-b)}}(A < B) = s$. This completes the proof. □

Now, we are ready to prove Theorem 6.

Proof of Theorem 6. We prove that, for every rational $\frac{1}{2} < q < \frac{-1+\sqrt{5}}{2}$, there exists a central balanced nontransitive (m)-word σ with $P(\sigma) = q$.

We choose a sufficiently large positive integer m, such that qm is an integer. Because $\frac{1}{2} < q < \frac{-1+\sqrt{5}}{2}$, it follows that

$$(1-q)^2 m^2 \leqslant qm^2 \leqslant (1-q)m^2 + (1-q)qm^2.$$

Thus, from Lemma 16 with $a = (1-q)m$ and $b = qm$, there exists a central (m)-word σ of type $((1-q)m, qm)$ such that $N_\sigma(A < B) = qm^2$. Clearly, $N_\sigma(B < C) = qm^2$ and $N_\sigma(C < A) = qm^2$. Hence, σ is a balanced nontransitive (m)-word with $P(\sigma) = q$. This completes the proof. □

Note that every $(m, n, 1)$-word is central. In the following, we classify all pairs of positive integers (m, n) such that there exists a balanced nontransitive $(m, n, 1)$-word.

Lemma 17. *Let m and n be positive integers. If* $\gcd(m, n) \neq 1, 2, 3, 4, 6, 8$*, then there exists a balanced nontransitive $(m, n, 1)$-word.*

Proof. Let $\gcd(m, n) = d\,(\neq 1, 2, 3, 4, 6, 8)$ and $k = \lceil \frac{d+1}{2} \rceil$. Let $m = dm'$ and $n = dn'$. For every $(m, n, 1)$-word σ of type $((d-k)m', kn')$, we have $P_\sigma(B < C) = P_\sigma(C < A) = \frac{k}{d}$. Therefore, it is enough to show that there exists such a word σ with $P_\sigma(A < B) = \frac{k}{d}$ or with $N_\sigma(A < B) = \frac{mnk}{d} = m'n'kd$. From Lemma 16, we know that there exists an $(m, n, 1)$-word σ of type $((d-k)m', kn')$ with $N_\sigma(A < B) = s$ for every $(d-k)^2 m'n' \leqslant s \leqslant (d^2 - k^2)m'n'$. Because $d \neq 1, 2, 3, 4, 6, 8$ and $k = \lceil \frac{d+1}{2} \rceil$, it holds that $(d-k)^2 m'n' \leqslant m'n'kd \leqslant (d^2 - k^2)m'n'$. Therefore, a balanced nontransitive $(m, n, 1)$-word exists. □

Indeed, the converse of Lemma 17 is also true; that is, there exists a balanced nontransitive $(m, n, 1)$-word if and only if $\gcd(m, n) \neq 1, 2, 3, 4, 6, 8$. This is an easy corollary of Lemma 22.

4 Existence of Balanced Nontransitive (m, n, ℓ)-words

Let σ be a balanced nontransitive (m, n, ℓ)-word. By replacing each of the letters B and C in σ with C and B, respectively, and then reversing σ, we obtain a balanced nontransitive (m, ℓ, n)-word. The same procedure using the letters A and C in σ produces an (ℓ, n, m)-word. This implies the following.

Observation 18. *If there exists a balanced nontransitive (m, n, ℓ)-word, then for every permutation (x, y, z) of (m, n, ℓ), there exists a balanced nontransitive (x, y, z)-word.*

Now, we prove several lemmas to prove Theorem 7.

Lemma 19. *Let k be a positive integer. If there exists a balanced nontransitive (m, n, ℓ)-word σ, then there exists a balanced nontransitive $(m, n, \ell k)$-word.*

The following example will be useful to understand how we produce a balanced nontransitive $(m, n, \ell k)$-word from a balanced nontransitive (m, n, ℓ)-word.

Example 20. The $(5, 5, 1)$-word $\sigma = AABBBCAABAB$ is balanced nontransitive and has winning probability $P_\sigma(A < B) = \frac{3}{5}$. We replace the letter C in σ with k consecutive C's to obtain a $(5, 5, k)$-word

$$\sigma' = AABBB\,\underbrace{CC\ldots C}_{k}\,AABAB.$$

Then, $P_{\sigma'}(A < B) = P_{\sigma'}(B < C) = P_{\sigma'}(C < A) = \frac{3}{5}$. So, σ' is balanced nontransitive and has winning probability $\frac{3}{5}$.

Proof of Lemma 19. Let σ' be the word obtained from σ by replacing each letter C in σ with k consecutive C's. For example, if $\sigma = CBBAACACBACBABCCBA$ and $k = 2$, then

$$\sigma' = (CC)BBAA(CC)A(CC)BA(CC)BAB(CC)(CC)BA.$$

Clearly, σ' is an $(m, n, \ell k)$-word. Furthermore, $P_{\sigma'}(A < B) = P_\sigma(A < B)$, $P_{\sigma'}(B < C) = P_\sigma(B < C)$ and $P_{\sigma'}(C < A) = P_\sigma(C < A)$. That is, σ' is a balanced nontransitive $(m, n, \ell k)$-word with $P(\sigma') = P(\sigma)$. This completes the proof. $\square$

Combining Observation 18 and Lemma 19, we obtain the following:

Corollary 21. *Let m, n, and ℓ be positive integers. Suppose there exists a balanced nontransitive (m, n, ℓ)-word. Then, there exists a balanced nontransitive (x, y, z)-word for every (x, y, z) divisible by (m, n, ℓ).*

Lemma 22. *If $\frac{mn\ell}{\operatorname{lcm}(m,n,\ell)} \in \{1, 2, 3, 4, 6, 8\}$, then there are no balanced nontransitive (m, n, ℓ)-words.*

Proof. Let $L = \frac{mn\ell}{\operatorname{lcm}(m,n,\ell)}$. Suppose there exists a balanced nontransitive (m,n,ℓ)-word σ, and let $P(\sigma) = p$. By Theorem 6, we know that $\frac{1}{2} < p < \frac{-1+\sqrt{5}}{2}$.

Let $N_\sigma(A < B) = x$, $N_\sigma(B < C) = y$, and $N_\sigma(C < A) = z$. Then, $p = \frac{x}{mn} = \frac{y}{n\ell} = \frac{z}{\ell m} = \frac{N}{mn\ell}$ for some integer N divisible by $\operatorname{lcm}(m,n,\ell)$. Let $N = k \operatorname{lcm}(m,n,\ell)$. Because $\frac{1}{2} < p < \frac{-1+\sqrt{5}}{2}$, it follows that $\frac{1}{2} < k \cdot \frac{\operatorname{lcm}(m,n,\ell)}{mn\ell} = \frac{k}{L} < \frac{-1+\sqrt{5}}{2}$, and, therefore

$$\frac{L}{2} < k < \frac{-1+\sqrt{5}}{2} L.$$

However, for each $L \in \{1,2,3,4,6,8\}$, there are no integers between $\frac{L}{2}$ and $\frac{-1+\sqrt{5}}{2}L$ (see Table 1); therefore k does not exist, which yields a contradiction. This completes the proof. $\square$

L	$\frac{L}{2}$	$\frac{-1+\sqrt{5}}{2}L$
1	0.5	0.618 ...
2	1	1.236 ...
3	1.5	1.854 ...
4	2	2.471 ...
5	2.5	3.090 ...
6	3	3.708 ...
7	3.5	4.326 ...
8	4	4.944 ...

Table 1: There are no integers between $\frac{L}{2}$ and $\frac{-1+\sqrt{5}}{2}L$.

Now, we are ready to prove Theorem 7.

Proof of Theorem 7. We first prove the 'if' part.

By Lemma 17, there exists a balanced nontransitive $(k,k,1)$-word if $k = 5, 7$ or $k \geqslant 9$. By Theorem 14, we know that a balanced nontransitive $(3,3,3)$-word and $(4,4,4)$-word exist. For other cases, i.e., $(m,n,\ell) = (6,6,2), (8,8,2), (12,4,3)$, we construct a balanced nontransitive (m,n,ℓ)-word as follows:

- $(m,n,\ell) = (6,6,2)$: $BCAAAABBBBABCA$ is a balanced nontransitive $(6,6,2)$-word with winning probability $\frac{7}{12}$.
- $(m,n,\ell) = (8,8,2)$: $BCAAAAABBBBBBABACA$ is a balanced nontransitive $(8,8,2)$-word with winning probability $\frac{9}{16}$.
- $(m,n,\ell) = (12,4,3)$: $CAAAAAAABBBCBACAAAA$ is a balanced nontransitive $(12,4,3)$-word with winning probability $\frac{7}{12}$.

Hence, by Corollary 21, if (m,n,ℓ) is a multiple of one in the list of Theorem 7, then there is a balanced nontransitive (m,n,ℓ)-word. This proves the 'if' part.

Next, we prove the 'only if' part. Suppose there exists a balanced nontransitive (m,n,ℓ)-word for some (m,n,ℓ) that is not divisible by any in the list of Theorem 7. We observe that each of $\gcd(m,n)$, $\gcd(n,\ell)$, and $\gcd(\ell,m)$ is either $1,2,3,4,6$, or 8 by Lemma 17 and Corollary 21. That is, it must be the case that

$$m = 2^{a_1}3^{b_1}x, \quad n = 2^{a_2}3^{b_2}y, \quad \ell = 2^{a_3}3^{b_3}z$$

for some nonnegative integers $a_1, a_2, a_3, b_1, b_2, b_3$ and pairwise coprime positive integers x, y, z not divisible by 2 or 3. Since (m,n,ℓ) is not divisible by $(3,3,3)$, at least one of b_1, b_2, and b_3 is 0. Without loss of generality, let $b_3 = 0$. Since $\gcd(m,n)$ is not divisible by 9, $b_1 \leqslant 1$ or $b_2 \leqslant 1$. Let $b_2 \leqslant 1$ and $b_1 \geqslant b_2$.

Let $M = \max(a_1, a_2, a_3)$. By Lemma 22, we have

$$\frac{mn\ell}{\operatorname{lcm}(m,n,\ell)} = \frac{2^{a_1+a_2+a_3} \cdot 3^{b_1+b_2+b_3} \cdot xyz}{2^M \cdot 3^{b_1} \cdot xyz} = 2^{a_1+a_2+a_3-M} \cdot 3^{b_2} \neq 1,2,3,4,6,8. \qquad (1)$$

Suppose $b_2 = 1$. Then, $b_1 \geqslant b_2 = 1$. By (1), $a_1 + a_2 + a_3 - M$ is at least 2. If $a_1, a_2, a_3 \geqslant 1$, then because $b_1, b_2 \geqslant 1$, (m,n,ℓ) is a multiple of $(6,6,2)$, a contradiction. Thus, either a_1, a_2 or a_3 is 0. Since $a_1 + a_2 + a_3 - M \geqslant 2$ and $M = \max(a_1, a_2, a_3)$, two of a_1, a_2 and a_3 are at least 2. If $a_1, a_2 \geqslant 2$ and $a_3 = 0$, then (m,n,ℓ) is a multiple of $(12,12,1)$, a contradiction. If either a_1 or a_2 is 0, then $a_3 \geqslant 2$, and (m,n,ℓ) is a multiple of $(12,4,3)$, which yields a contradiction. Thus, $b_2 = 0$. Then, $a_1 + a_2 + a_3 - M \geqslant 4$ by (1). If $a_1, a_2, a_3 \geqslant 2$, then (m,n,ℓ) is a multiple of $(4,4,4)$. If either a_1, a_2 or a_3 is 0, then since $a_1 + a_2 + a_3 - M \geqslant 4$ and $M = \max(a_1, a_2, a_3)$, the other two must be at least 4. Then,(m,n,ℓ) is a multiple of $(16,16,1)$. Hence, one of a_1, a_2 and a_3 is 1, and the other two are at least 3. In this case, (m,n,ℓ) is a multiple of $(8,8,2)$, a contradiction. This completes the proof. □

5 Future Research Directions

The nontransitive balanced dice is defined for a set of three dice in Definition 1 and 2. It can be naturally extended to a set of $n(\geqslant 3)$ dice as follows:

Definition 23. For positive integer $n(\geqslant 3)$, a set $\{A_1, A_2, \ldots, A_n\}$ of dice is *nontransitive* and *balancecd* if the following holds.

$$P(A_1 < A_2) = P(A_2 < A_3) = \cdots = P(A_n < A_1) > \frac{1}{2}$$

This extension was introduced in [12, 13], and it was shown that for every $n \geqslant 3$ and $m \geqslant 3$, nontransitive balanced m-sided n dice exist in [12, 13]. It would be interesting to extend Theorem 6 and 7 to this general setting.

Problem 24. For $n \geqslant 3$, find the least upper bound of the winning probability of nontransitive balanced n dice.

Problem 25. Characterize all n-tuples $(k_1, k_2, \ldots, k_n)$ of positive integers so that there exist balanced nontransitive dice $A_1, A_2, \ldots, A_n$ where A_i has k_i-sides.

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References

[1] L. Angel and M. Davis. A direct construction of nontransitive dice sets. *Journal of Combinatorial Designs*, 25(11): 523–529, 2017.

[2] C. R. Blyth. Some probability paradoxes in choice from among random alternatives. *Journal of the American Statistical Association*, 67: 366–381, 1972.

[3] J. Buhler, R. Graham and A. Hales. Maximally nontransitive dice. *American Mathematical Monthly*, 125(5): 387–399, 2018.

[4] M. Gardner. Mathematical games. *Scientific American*, 223: 110–115, 1970.

[5] I. Hur and Y. Kim. Possible probability and irreducibility of balanced nontransitive dice. *Journal of Mathematics*, ID 6648248, 8, 2021.

[6] S. Joyce, A. Schaefer, D. West and T. Zaslavsky. Strongly connectable digraphs and non-transitive dice. *AKCE International Journal of Graphs and Combinatorics*, 17(1): 480–485, 2020.

[7] B. G. Kronenthal and L. Traldi. Tied dice. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 68: 85–96, 2009.

[8] S. Lu and S. Kijima. Is There a Strongest Die in a Set of Dice with the Same Mean Pips? *IEICE Technical Report; IEICE Tech. Rep.*, 2022.

[9] C. M. Rump. Strategies for Rolling the Efron Dice. *Mathematics Magazine*, 74(3): 212–216, 2001.

[10] R. P. Savage Jr. The paradox of nontransitive dice. *American Mathematical Monthly*, 101(5): 429–436, 1994.

[11] A. Schaefer. Balanced non-transitive dice. *Master's thesis, University of Kansas*, 2011.

[12] A. Schaefer and S. Schweig. Balanced Non-Transitive Dice. *The College Mathematics Journal*, 48(1): 10–16, 2017.

[13] A. Schaefer. Balanced Non-Transitive Dice II: Tournaments. arXiv:1706.08986, 2017.

[14] S. Trybuła. On the paradox of three random variables. *Polska Akademia Nauk. Instytut Matematyczny. Zastosowania Matematyki*, 5: 321–332, 1960/61