Various Bounds on the Minimum Number of Arcs in a k-Dicritical Digraph

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Abstract

The dichromatic number $\vec{\chi}(G)$ of a digraph G is the least integer k such that G can be partitioned into k acyclic digraphs. A digraph is k-dicritical if $\vec{\chi}(G) = k$ and each proper subgraph H of G satisfies $\vec{\chi}(H) \leqslant k - 1$.

We prove various bounds on the minimum number of arcs in a k-dicritical digraph, a structural result on k-dicritical digraphs and a result on list-dicolouring. We characterise 3-dicritical digraphs G with $(k-1)|V(G)|+1$ arcs. For $k \geqslant 4$, we characterise k-dicritical digraphs G on at least $k+1$ vertices and with $(k-1)|V(G)|+k-3$ arcs, generalising a result of Dirac. We prove that, for $k \geqslant 5$, every k-dicritical digraph G has at least $(k - \frac{1}{2} - \frac{1}{k-1})|V(G)| - k(\frac{1}{2} - \frac{1}{k-1})$ arcs, which is the best known lower bound. We prove that the number of connected components induced by the vertices of degree $2(k-1)$ of a k-dicritical digraph is at most the number of connected components in the rest of the digraph, generalising a result of Stiebitz. Finally, we generalise a theorem of Thomassen on list-chromatic number of undirected graphs to list-dichromatic number of digraphs.

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1 Introduction and results

A *colouring* of a directed graph (shortly digraph) G is a partition of the set of vertices of G into independent subsets and the *chromatic number* $\chi(G)$ of G is the minimum size of such a partition. This is a very natural generalisation of the notion of colouring of graphs, but not a very suitable one since it does not take into account the orientation of the arcs. Neumann-Lara introduced in 1982 [NL82] the notion of dicolouring of digraphs, which is another natural generalisation of the concept of colouring of graphs. It is more suitable than the previous one since it takes into account the orientation of the arcs. A *dicolouring* of a digraph G is a partition of the set of vertices of G inducing acyclic digraphs, and the dichromatic number $\vec{\chi}(G)$ of G is the minimum size of such a partition. This is indeed a generalisation as, with the correspondence between graphs and *symmetric* *digraphs* (that is digraphs obtained from undirected graphs by replacing each edge by a digon, where a *digon* is a pair of anti-parallel arcs), we have, for every symmetric digraph G, $\chi(G) = \vec{\chi}(G)$.

We study minimal obstructions to dicolourability. A digraph G is *dicritical* if, for every proper subdigraph H of G, $\vec{\chi}(H) < \vec{\chi}(G)$. We also say that G is k-*dicritical* when G is dicritical and $\vec{\chi}(G) = k$. Observe that any digraph G contains a $\vec{\chi}(G)$-dicritical subdigraph. This means that many problems on the dichromatic number of digraphs reduce to problem on dicritical digraphs, whose structure is more restricted. We are interested in their sparsity: we aim at computing the minimum number of arcs in a k-dicritical digraph on n vertices. Lemma 15 shows that this value is well defined for $n \geqslant k \geqslant 2$.

It is well known that every vertex in a k-critical (undirected) graph has degree at least $k-1$, and hence a k-critical graph G has at least $\frac{1}{2}(k-1)|V(G)|$ edges. Brooks' theorem implies a simple characterisation of graphs G with exactly $\frac{1}{2}(k-1)|V(G)|$ edges.

Theorem 1 (Brooks [Bro41]). *Let G be a connected graph. Then $\chi(G) \leqslant \Delta(G) + 1$ and equality holds if and only if G is an odd cycle or a complete graph.*

Similarly, it is well known (see Lemma 12(1)) that every vertex in a k-dicritical digraph has degree at least $2(k-1)$ and hence a k-dicritical digraph has at least $(k-1)|V(G)|$ arcs. Brooks' theorem was generalised in [Moh10] (see also [AA22]) to digraphs, and implies a simple characterisation of the k-dicritical digraphs G with exactly $(k-1)|V(G)|$ arcs. For G a digraph, let $\Delta_{max}(G)$ be the maximum over the vertices of G of the maximum of their in-degree and their out-degree.

Theorem 2 (Theorem 2.3 in [Moh10]). *Let G be a connected digraph. Then $\vec{\chi}(G) \leqslant \Delta_{max}(G) + 1$ and equality holds if and only if G is a directed cycle, a symmetric cycle of odd length or a symmetric complete digraph on at least 4 vertices.*

In 1957, Dirac went one step further and proved the following.

Theorem 3 (Dirac [Dir57]). *Let $k \geqslant 4$ and G a k-critical graph. If G is not K_k, then*

$$2|E(G)| \geqslant (k-1)|V(G)| + k - 3.$$

We generalise this theorem to digraphs:

Theorem 4. *Let $k \geqslant 4$ and G a k-dicritical digraph. If G is not $\overleftrightarrow{K}_k$, then*

$$|A(G)| \geqslant (k-1)|V(G)| + k - 3.$$

Dirac later identified the graphs for which the bound is tight (whose set we denote $\mathcal{D}_k$, see Section 4.2 for a definition) and improved his bound. Recall that the Kronecker symbol $\delta_{i,j}$ is equal to 1 if $i = j$ and 0 otherwise.

Theorem 5 (Dirac [Dir74])**.** *Let $k \geqslant 4$ and let G a k-critical graph. If G is neither K_k nor in $\mathcal{D}_k$, then*

$$2|E(G)| \geqslant (k-1)|V(G)| + (k-1-\delta_{k,4}).$$

It turns out that our bound is also tight exactly for the digraphs in $\mathcal{D}_k$ (via the identification between graphs and symmetric digraphs):

Theorem 6. *Let $k \geqslant 4$ and G be a k-dicritical digraph. If G is neither $\overleftrightarrow{K}_k$ nor $\mathcal{D}_k$, then:*

$$|A(G)| \geqslant (k-1)|V(G)| + (k-2).$$

The perspicacious reader will notice that our bound is weaker than Dirac's when $k \geqslant 5$. Yet our bound is tight for some digraphs (which are thus not symmetric, see Section 4.2).

It is well known that the only 3-critical graphs are odd cycles, which is the reason why Dirac's two mentioned results deal with $k \geqslant 4$. However, 3-dicritical digraphs are not as simple, as witnessed by the fact that deciding if a digraph is 2-dicolourable is NP-complete [BFJ+04]. We prove the following, where $\mathcal{D}'_3$ is a class of 3-dicritical digraphs defined in Section 4.3:

Theorem 7. *Let G be a 3-dicritical digraph. If G is not a symmetric cycle of odd length, then*

$$|A(G)| = 2|V(G)| + 1$$

if and only if $G \in \mathcal{D}'_3$, and otherwise

$$|A(G)| \geqslant 2|V(G)| + 2$$

Gallai was the first [Gal63b] to find a lower bound with a better slope than $\frac{1}{2}(k-1)$. His result was improved by Krivelevich [Kri97] using Gallai's method together with a result of Stiebitz [Sti82] that we were able to generalise to digraphs:

Theorem 8. *Let $k \geqslant 3$, G a k-dicritical digraph such that $S = \{x \in G, d(x) \leqslant 2(k-1)\}$. Then the number of connected components of $G - S$ is at most the number of connected components of $G[S]$.*

Gallai's method works on digraphs, but we obtained better bounds through other means.

In the undirected case, Kostochka and Yancey [KY14a] obtained a closed form for the minimum number of edges of a k-critical graph on n vertices in an infinite set of cases:

Theorem 9 (Theorem 4 in [KY14a])**.**

$$|E(G)| \geqslant \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil.$$

This bound is exact for $k = 4$ and $n \geqslant 6$ and for $k \geqslant 5$ and $n \equiv 1$ (mod $k-1$).

Unfortunately we were not able to obtain a comparable result. Still, adapting their method, we were able to get the following, which is the best known lower bound on the minimum number of arcs in a k-dicritical digraphs when $k \geqslant 5$.

Theorem 10. *Let $k \geqslant 5$ and G a k-dicritical digraph. Then*

$$|A(G)| \geqslant (k - \frac{1}{2} - \frac{1}{k-1})|V(G)| - k(\frac{1}{2} - \frac{1}{k-1})$$

The way the proof works makes it easy to identify the two arguments that do not allow us to get a better result. It is to be noted that our proof works for $k = 4$, but in this case a better bound is already known.

Theorem 11 (Theorem 1 in [KS20])**.** *Let G be a 4-dicritical digraph with $|V(G)| \geqslant 4$ and $|V(G)| \neq 5$. Then*

$$|A(G)| \geqslant \left\lceil \frac{10|V(G)| - 4}{3} \right\rceil$$

This bound is tight when $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$

Our last result, Theorem 76, has a slightly different flavor than the rest since it deals with list dicolouring. It necessitates a few more technical definitions to be introduced, so we postpone its description to Section 7 so as not to make this section too heavy.

Organisation of the paper

Section 2 is dedicated to notations and Section 3 to some basic results that will be needed all along the proofs. Section 4 is dedicated to the proofs of Theorems 4, 6 and 7, see respectively subsections 4.1, 4.2 and 4.3. Section 5 is dedicated to the proof of Theorem 10, Section 6 to the proof of Theorem 8. Finally, Section 7 is dedicated to our result on list dicolouring and Section 8 to the conclusion.

2 Notations

2.1 Generalities

We define $\mathbb{N} = \{0, 1, \dots\}$. For $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$ and $\mathfrak{S}_n$ the set of permutations of $[n]$. Set union will be denoted by $+$ and indexed set union with $\bigcup$. Set difference will be denoted by $-$. Excluding a bound of an interval will be denoted by a bracket facing outwards, e.g. $[0, 1[= \{x \in \mathbb{R}, 0 \leqslant x < 1\}$. For E a set and $S \subseteq E$, we denote $\mathbb{1}_S$ the indicator function of S.

2.2 Digraphs

A (*simple*) *digraph* G is a pair $(V(G), A(G))$ where $V(G)$ is the vertex set and is finite, and $A(G) \subseteq \{(u, v) \in V(G)^2, u \neq v\}$ is the set of arcs of G.The *order* of G is $|V(G)|$. We only ever need to consider digraphs up to isomorphism and hence write $G = G'$ whenever G and G' are isomorphic.

For $X, Y \subset V(G)$, we let $A_G(X,Y) = A(G) \cap (X \times Y)$ and $\overleftrightarrow{A}_G(X,Y) = A_G(X,Y) + A_G(Y,X)$. A *subdigraph* of G is a digraph G' with $V(G') \subseteq V(G)$ and $A(G') \subseteq A(G)$. For $X \subset V(G)$, the subdigraph of G *induced* by X is $G[X] = (X, A(G) \cap X^2)$. For $X \subset V(G)$, we let $G - X = G[V(G) - X]$. For $B \subset \{(u,v) \in V(G)^2, u \neq v\}$, we let $G \cup B = (V(G), A(G) + B)$ and $G \setminus B = (V(G), A(G) - B)$. For X disjoint from $V(G)$, we let $G + X = (V(G) + X, A(G))$.

If both X and $V(G)$ are contained in $V(G')$ for some introduced digraph G', we let $G + X = (V(G) + X, A(G) + \overleftrightarrow{A}_{G'}(V(G), X))$. We denote $\subseteq$ the subdigraph relation, i.e. $G \subseteq H$ whenever $V(G) \subseteq V(H)$ and $A(G) \subseteq A(H)$.

We say that G is *symmetric* when, for any $(u,v) \in A(G)$, $(v,u) \in A(G)$.

2.3 Arcs, walks, neighbours, blocks and connectivity

Let G be a digraph.

A *digon* of G is a pair of arcs of the form $\{(u,v),(v,u)\}$. We define $A^s(G) = \{(u,v) \in A(G) \mid (v,u) \notin A(G)\}$ the set of *simple arcs* of G.

A *weak walk* in G is an alternating sequence $P = (x_1, a_1, x_2, \ldots, a_{n-1}, x_n)$ of vertices and arcs of G, such that, for $i \in [n-1], a_i \in \{(x_i, x_{i+1}), (x_{i+1}, x_i)\}$, we write $V(P) = \{x_1, \ldots, x_n\}$ and we say that it is a weak walk from x_1 to x_n. It is a *walk* when, for $i \in [n-1], a_i = (x_i, x_{i+1})$. A *(weak) cycle* is a (weak) walk from a vertex to itself.

For $X_1, \ldots, X_n \subseteq V(G)$, the word $X_1 \ldots X_n$ denotes $X_1 \times \cdots \times X_n$. In particular, noticing that giving a walk is the same as giving a sequence of vertices, we denote walks (and cycles) in G as words over $V(G)$, e.g. for $u, v, w \in V(G)$, uvw denotes the walk $(u, (u,v), v, (v,w), w)$. We also write uv for an arc (u,v) and we use walks in place of sets of arcs. For instance, we write $x_1 \ldots x_n$ for a walk $(x_1, x_1x_2, x_2, \ldots, x_{n-1}x_n, x_n)$, and $G \setminus x_1 \ldots x_n$ to denote $G \setminus \{x_1x_2, \ldots, x_{n-1}x_n\}$.

For $X \subseteq V(G)$, we let $N^+(X) = \{u \in V(G) - X, A(X,u) \neq \varnothing\}$ the *out-neighbourhood* of X, $N^-(X) = \{u \in V(G) - X, A(u,X) \neq \varnothing\}$ the *in-neighbourhood* of X, $N(X) = N^+(X) + N^-(X)$ the *neighbourhood* of X, $N^+[X] = N^+(X) + X$ the *closed out-neighbourhood* of X, $N^-[X] = N^-(X) + X$ the *closed in-neighbourhood* of X and $N[X] = N(X) + X$ the *closed neighbourhood* of X. We also define $N^d(X) = N^+(X) \cap N^-(X)$, $N^s(X) = N(X) \setminus N^d(X)$, $N^{s+}(X) = N^s(X) \cap N^+(X)$ and $N^{s-}(X) = N^s(X) \cap N^-(X)$.

For $x \in V(G)$, we define $d^+(x) = |N^+(x)|$, $d^-(x) = |N^-(x)|$, $d(x) = d^+(x) + d^-(x)$, $d_{min}(x) = \min(d^+(x), d^-(x))$ and $d_{max}(x) = \max(d^+(x), d^-(x))$, respectively the *out-degree*, *in-degree*, *degree*, *min-degree* and *max-degree* of x in G.

G is *connected* if, for any $x, y \in V(G)$, there is a weak walk from x to y. A *connected component* of G is a maximal set of vertices X such that $G[X]$ is connected. We denote $\pi_0(G)$ the set of connected components of G. G is *strongly connected* if and there exists a walk from u to v for every distinct pair of vertices u, v. Note that we consider the empty set to be connected, which is not standard, but it simplifies the inductions in the proofs of Section 6.

An *arc-cut* of G is a set $A \subseteq A(G)$ of arcs such that $G \setminus A$ is not strongly connected. We say that G is *k-arc-connected* when every arc-cut of G has size at least k.

A digraph G is *non-separable* if it is connected and $G - v$ is connected for all $v \in V(G)$. Such a vertex is called a *separating vertex* of G. A *block* of a digraph G is a subdigraph which is non-separable and is maximal with respect to this property. A block B is a *leaf block* if at most one vertex of B is a separating vertex of G, the other blocks are *internal blocks*. Observe that if a digraph G is non-separable, then G itself is a leaf block. Note also that any two distinct blocks of a digraph have at most one vertex in common, and such a common vertex is always a separating vertex of the digraph.

A *directed Gallai tree* is a digraph whose blocks are either an arc, or a cycle, or a symmetric odd cycle, or a symmetric complete digraph. A *directed Gallai forest* is a digraph whose connected components are directed Gallai tree.

2.4 Basic classes of digraphs and operations on digraphs

We say that a digraph G is *complete* when $A(G) = \{uv \in V(G), u, v \in V(G)\}$ and we denote by $\overleftrightarrow{K}_n$ the complete digraph on n vertices. For $n \geqslant 2$, $\vec{P}_n = ([n], \{(i, i+1), i \in [n-1]\})$ is the *path* with n vertices, $n = \vec{P}_n \cup \{(i+1, i), i \in [n-1]\}$ is the *symmetric path* with n vertices, $\vec{C}_n = (\mathbb{Z}/n\mathbb{Z}, \{(i, i+1), i \in \mathbb{Z}/n\mathbb{Z})$ is the *cycle* on n vertices and $\overleftrightarrow{C}_n = \vec{C}_n \cup \{(i+1, i), \in \mathbb{Z}/n\mathbb{Z}\}$ is the *symmetric cycle* on n vertices. A *clique* of a digraph G is a set of vertices inducing a complete digraph.

For G a digraph and $X_1, \dots, X_n$ pairwise disjoint non-empty subsets of $V(G)$, $G/(X_i, i \in [n])$ denotes the digraph obtained from G by merging all vertices in X_i, for $i \in [n]$. Formally, let, for $u \in V(G) - \bigcup\limits_{i \in [n]} X_i, p(u) = u$ and, for $i \in [n]$ and $u \in X_i, p(u) = X_i$. Then $G/(X_i, i \in [n]) = (p(V(G)), \{(p(u), p(v)), (u, v) \in A(G)\})$. p is called the *canonical projection*. When $n = 1$, we write $G/X = G/(X)$. When $X = \{x, y\}$, we denote by $x \star y$ the new vertex resulting from the merging of x and y.

If G is a digraph and $G' = (G'_u)_{u \in V(G)}$ is a family of digraphs indexed by the vertices of G, the *substitution* $G(G')$ of G' in G is the digraph obtained from G by replacing every vertex by the corresponding digraph. Formally, considering the $V(G'_u), u \in V(G)$ pairwise disjoint, we have $G(G') = (\bigcup\limits_{u \in V(G)} V(G'_u), \bigcup\limits_{u \in V(G)} A(G'_u) + \bigcup\limits_{(u,v) \in A(G)} V(G'_u)V(G'_v))$. Considering an indexing $u : [n] \to V(G)$ of the vertices of G, we write $G(G') = G(G'_{u_1}, \dots, G'_{u_n})$.

2.5 Dicolouring and greedy dicolouring

Given a digraph G and $X \subseteq V(G)$, we say that X is acylic (in G) when $G[X]$ is acylic.

$\phi : V(G) \to \mathbb{N}$ is a *dicolouring* of G if, for $n \in \mathbb{N}, \phi^{-1}(n)$ is acyclic, i.e. has no cycle. A *a k-dicolouring* is a dicolouring with colours in $[k]$. The *dichromatic number* of G is

$$\vec{\chi}(G) = \min\{n \in \mathbb{N}, \exists \phi : V(G) \to [n] \text{ dicolouring of } G\}$$

We say that G is *dicritical* when for every proper subdigraph H of G, $\vec{\chi}(H) < \vec{\chi}(G)$. For $k \in \mathbb{N}$, we say that G is k-dicritical if furthermore $\vec{\chi}(G) = k$.

Let G be a digraph, $X \subseteq V(G)$, $(u_1, \ldots, u_n)$ an ordering of the vertices in $G - X$ and $\phi : X \to \mathbb{N}$ a dicolouring of $G[X]$. *Extending greedily* ϕ to G (with respect to the considered ordering) means colouring iteratively $u_1, \ldots, u_n$ so that, for $1 \leqslant i \leqslant n, \phi(u_i) = \min(\mathbb{N} - \phi(N^-(u_i) \cap (X + u_1 + \cdots + u_{i-1})) \cap \phi(N^+(u_i) \cap (X + u_1 + \cdots + u_{i-1})))$, i.e. we colour a vertex with the smallest integer that does not appear both in its in-neighbourhood and its out-neighbourhood. When $X = \varnothing$, we say that we colour G greedily.

2.6 Directional duality

Any universal statement about digraphs raises a dual statement by exchanging the $+$ and $-$ superscripts, both statements being simultaneously true. In particular, a digraph G is k-dicritical if and only if the digraph obtained from G by reversing the orientation of each arc is, making directional duality often useful in this context. It is out of our scope to give a formal meaning to this so we will use it as an ad hoc principle.

3 Tools

This section is dedicated to basic results that are used all along the proofs.

3.1 Basic properties of k-dicritical digraphs

We start with a trivial lower bound on the minimum degree of a vertex in a dicritical digraph. This result will be used so often that we will not refer to it when using it.

Lemma 12. *Let G be a digraph.*

1. *Let $x \in V(G)$ such that $\vec{\chi}(G - x) < \vec{\chi}(G)$. Then, for any $\vec{\chi}(G - x)$-dicolouring ϕ of $G - x$ and $c \in \phi(G - x)$, there is a walk from $N^+(x)$ to $N^-(x)$ in $\phi^{-1}(c)$. In particular, $d_{min}(x) \geqslant \vec{\chi}(G) - 1$.*

2. *Let $S \subseteq G$ acyclic. Then $\vec{\chi}(G/S) \geqslant \vec{\chi}(G)$*

Proof. To prove 1, assume towards a contradiction and by directional duality that we have such a ϕ and c such that there is no walk from $N^+(x)$ to $N^-(x)$ in $\phi^{-1}(c)$. Then we extend ϕ to G by setting $\phi(x) = c$.

We now prove 2. Let $p : V(G) \to V(G/S)$ be the canonical projection. Let $\phi : V(G/S) \to [\vec{\chi}(G) - 1]$. Let $\psi = \phi \circ p$. Since $\vec{\chi}(G) \geqslant k$, ψ is not a dicolouring of G. Hence we have a monochromatic cycle $C \subseteq G$. Since S is acyclic, $C \nsubseteq S$. Then, projecting C onto G/S yields a monochromatic cycle in G/S. Hence G/S is not $(\vec{\chi}(G)-1)$-dicolourable. □

Lemma 13. *Let G be a k-dicritical digraph.*

1. *Every arc is contained in an induced cycle.*

2. *For every $x \in V(G)$, $N^{s+}(x) = \varnothing \Leftrightarrow N^{s-}(x) = \emptyset$.*

Proof. Let $a \in A(G)$. If a is not contained in an induced cycle, then a $(k-1)$-dicolouring of $G \setminus a$ is a $(k-1)$ dicolouring of G.

Assume one of $N^{s+}(x)$ or $N^{s-}(x)$ is not empty. By directional duality, we may consider $y \in N^{s-}(x)$. By the first point, yx is contained in an induced cycle. The vertex following x in this cycle is in $N^{s+}(x)$. □

3.2 Basic constructions of k-dicritical digraphs

We now give a simple construction of digraphs with high dichromatic number that will be useful shortly.

The *Dirac join* of two digraphs G_1 and G_2 is $\overleftrightarrow{K}_2(\overleftrightarrow{G}_1, G_2))$.

Theorem 14 ([BBSS20]). *Given two digraph G_1 and G_2, we have $\vec{\chi}(\overleftrightarrow{K}_2(G_1, G_2)) = \vec{\chi}(G_1) + \vec{\chi}(G_2)$, and $\vec{\chi}(\overleftrightarrow{K}_2(G_1, G_2))$ is dicritical if and only G_1 and G_2 are dicritical.*

We also know how to construct easily dicritical digraphs of any reasonable order.

Lemma 15. *Let $n \geqslant k \geqslant 2$. There exists a k-dicritical digraph with order n.*

Proof. $\overleftrightarrow{K}_2(\overleftrightarrow{K}_{k-2}, \vec{C}_{n+2-k})$ is k-dicritical and has n vertices. □

In the symmetric case, it is known that there is no dicritical digraph G with $\vec{\chi}(G)+1$ vertices. This is not the case for digraphs.

Lemma 16. *Let $k \geqslant 2$. The only k-dicritical digraph with $k+1$ vertices is $\overleftrightarrow{K}_2(\overleftrightarrow{K}_{k-2}, \vec{C}_3)$.*

Proof. Let G be a k-dicritical digraph with $k+1$ vertices. Let $x, y \in V(G)$ such that $xy \notin G$. Let $H = G - x - y$. G is $(\vec{\chi}(H)+1)$-dicolourable (give the same colour to x and y) and hence $\vec{\chi}(H) \geqslant k-1$. Since $|V(H)| = k-1$, we obtain $H = \overleftrightarrow{K}_{k-1}$. Now, since $G \neq \overleftrightarrow{K}_{k+1}$, we have $x, y \in V(G)$ such that $xy \notin G$. If $yx \notin G$, since x and y have in- and out-degree at least $k-1$, we obtain $G = \overleftrightarrow{K}_2(G - x - y, \{x, y\})$. Since G is k-dicritical, we have a $(k-1)$-dicolouring ϕ of $G - x$. Set $\phi(y) = \phi(x)$ to obtain a $(k-1)$-dicolouring of G, a contradiction. Hence $y \in N^{s-}(x)$ and then by Lemma 13(2) $N^{s+}(x) \neq \varnothing$. Let $z \in N^{s+}(x)$. We proved that $G - x - y = \overleftrightarrow{K}_{k-1} = G - x - z$. If $yzy \in G$, then $G - x = \overleftrightarrow{K}_k$ and G is not k-dicritical. Hence $G - y - z = \overleftrightarrow{K}_{k-1}$. In other words, $G = \overleftrightarrow{K}_2(\overleftrightarrow{K}_{k-2}, G[\{x, y, z\}])$. By Theorem 14, $G[\{x, y, z\}]$ is 2-dicritical and hence a cycle, which concludes the proof. □

3.3 Directed Gallai Theorem and directed Gallai forest

The following theorem is used several times as a tool along the paper, and Section 7 is dedicated to a slight generalisation of it.

Theorem 17 (Theorem 15 in [BBSS20]). *If G is a k-dicritical digraph, then the subdigraph induced by vertices of degree $2(k-1)$ is a directed Gallai forest.*

3.4 Arc-connectivity

Recall that an *arc-cut* of G is a set $A \subseteq A(G)$ of arcs such that $G \setminus A$ is not strongly connected. We say that G is k*-arc-connected* when every arc-cut of G has size at least k. It is well known that a digraph G with $|V(G)| \geqslant 2$ is k-arc-connected if and only if for every partition (V_0, V_1) of $V(G)$, we have $|A_G(V_0, V_1)| \geqslant k$.

There are two technical results about arc-connectivity that will be useful later on: a lower bound on the size of an arc-cut of a k-dicritical digraph and a constraint on the dicolouring of digraphs with a small arc-cut.

The next lemma is a generalisation of a classic result on undirected graphs due to Gallai (unpublished) which was also generalised to hypergraphs in [SST18] (Theorem 12). We could not find any reference for the digraph case.

Lemma 18. *Let $k \geqslant 2$, G a k-dicritical digraph and (V_0, V_1) a partition of $V(G)$ such that $|A(V_0, V_1)| \leqslant k-1$. Let $V_0^* = N^-(V_1)$ and $V_1^* = N^+(V_0)$. Then there is $i \in \{0,1\}$ such that, for any $(k-1)$-dicolouring ϕ_i of $G[V_i]$, $|\phi_i(V_i^*)| = 1$ and, for any $(k-1)$-dicolouring ϕ_{1-i} of V_{1-i}, $|\phi_{1-i}(V_{1-i}^*)| = k-1$.*

Proof. Let, for $i \in \{0,1\}$, ϕ_i be a $(k-1)$-dicolouring of $G[V_i]$. Let G^* be the graph on $\bigsqcup_{i\in\{0,1\}} \phi_i(V_i^*)$ such that, for $i \in \{0,1\}$, $G^*[\phi_i(V_i^*)]$ is complete and, for $c_0 \in \phi_0(V_0^*)$ and $c_1 \in \phi_1(V_1^*)$, $c_0c_1 \in G^*$ if and only if there exists, for $i \in \{0,1\}$, $x_i \in \phi_i^{-1}(\{c_i\})$ such that $x_0x_1 \in G$. Since G is not k-dicolourable, G^* is not k-colourable. Since $\overline{G^*}$ is bipartite, it is perfect and hence, by the perfect graph theorem, G^* is perfect. Thus there is $X \subseteq G^*$ such that $G^*[X] = K_k$. Since, for $i \in \{0,1\}$, $|\phi_i(V_i^*)| \leqslant k-1$, $X \cap \phi_i(V_i^*) \neq \varnothing$. Since $|E_{G^*}(\phi_0(V_0^*), \phi_1(V_1^*))| \leqslant |A(V_0, V_1)| \leqslant k-1$ and, for $i \in \{0,1\}$ and $c \in \phi_i(V_i^*)$, $\phi_{1-i}(V_{1-i}^*) \cap N(c) \neq \varnothing$, $\{|\phi_i(V_i^*)|, i \in \{0,1\}\} = \{1, k-1\}$. This is true for any choice of $\phi_i, i \in \{0,1\}$, so generalising independently in ϕ_0 and ϕ_1 yields the result. □

The above lemma implies the following, which was already proved by Neumann-Lara in [NL82] (Theorem 5).

Corollary 19. *Let $k \geqslant 2$ and G be a k-dicritical digraph. Then G is $(k-1)$-arc-connected.*

4 Dirac-type bounds

Let G be a k-dicritical digraph. Every vertex of G has degree at least $2(k-1)$, yielding, by the handshake lemma, $|A(G)| = \frac{1}{2}\sum_{u\in V(G)} d(u) \geqslant (k-1)|V(G)|$. This leads us to define the excess of u: $\varepsilon_k(u) = d(u) - 2(k-1)$, the excess of $X \subseteq V(G)$: $\varepsilon_k(X) = \sum_{u\in X} \varepsilon_k(u)$ and the excess of G: $\varepsilon_k(G) = \varepsilon_k(V(G)) = 2|A(G)| - 2(k-1)|V(G)|$. When it is clear from the context, we write ε instead of ε_k.

4.1 Dirac's Theorem

We now prove Theorem 4, that we restate here for convenience.

Theorem 20. *Let $n > k \geqslant 4$ and G an n-vertex k-dicritical digraph. Then*

$$|A(G)| \geqslant (k-1)|V(G)| + k - 3.$$

In other words: $\varepsilon(G) \geqslant 2(k-3)$.

Proof. Consider a digraph G with $|V(G)| > k$ minimal such that $\varepsilon(G) < 2(k-3)$.

Claim 21. *G does not contain $\overleftrightarrow{K}_k$ minus one arc as a subdigraph.*

Proof of claim. Assume we have $W \subseteq V(G)$ and $x, y \in W$ such that $G[W] + xy = \overleftrightarrow{K}_k$. Since G is k-dicritical and $yx \in A(G)$, $G \setminus yx$ admits a $(k-1)$-dicolouring ϕ. Since G is not $(k-1)$-dicolourable, $\phi(x) = \phi(y)$ and there is a monochromatic walk in $G - yx$ from x to y of colour $\phi(x)$. Since $xy \notin A(G)$, this walk has length at least 2. Now, for each $u \in W - \{x, y\}$, define ψ_u from ϕ by exchanging the colour of u and the colour of x and y; formally: $\psi_u(u) = \phi(x)$, $\psi_u(x) = \psi_u(y) = \phi(u)$ and $\psi_u(v) = \phi(v)$ for every $v \in V(G) - \{x, y, u\}$. Since ψ_u is not a dicolouring of G, either there is a cycle of colour $\psi_u(x) = \phi(u)$ going through x or y (or both) and we set $\delta_u = 1$, or there is a cycle of colour $\psi_u(u) = \phi(x)$ going through u (which is disjoint from $W - u$) and we set $\delta_u = 0$.

Observe that if $\delta_u = 0$, then $\varepsilon(u) \geqslant 2$. Assume $\delta_u = 0$ for c vertices. Observe that:

$$\varepsilon(W - \{x, y\}) \geqslant 2c$$

and,

$$\varepsilon(x) + \varepsilon(y) \geqslant 2 \sum_{u \in W - \{x,y\}} \delta_u = 2(k - 2 - c)$$

Hence, $\varepsilon(G) \geqslant 2k - 4$, a contradiction. ◊

Note that $\overleftrightarrow{K}_k \nsubseteq G$, since $G \neq \overleftrightarrow{K}_k$ and G is k-dicritical.

Claim 22. *Let $x \neq y \in V(G)$ such that $xy \notin A(G)$ and $G/\{x, y\}$ is not k-dicritical. Let $G^* \subseteq G/\{x, y\}$ be k-dicritical and $U = V(G) - V(G^*) - x - y$. If $U \neq \varnothing$, then $G^* = \overleftrightarrow{K}_k$.*

Proof of claim. Assume towards a contradiction that $G^* \neq \overleftrightarrow{K}_k$.

By minimality of G, it suffices to show $\varepsilon(G^*) \leqslant \varepsilon(G)$. We have $\varnothing \subsetneq U \subsetneq V(G)$. Hence, by Corollary 19, G is $(k-1)$-arc-connected, so $|A(U, V(G) - U)| \geqslant k - 1$ and $|A(V(G) - U, U)| \geqslant k - 1$. We have:

$$\begin{aligned}
\varepsilon(G) - \varepsilon(G^*) &= 2(|A(G)| - |A(G^*)|) - (2k-2)(|V(G)| - |V(G^*)|) \\
&= \sum_{u \in U} d_G(u) + |A(U, V(G) - U)| + |A(V(G) - U, U)| \\
&\qquad + 2(|A(V(G) - U)| - |A(G^*)|) - (2k-2)(|U| + 1) \\
&\geqslant (2k-2)|U| + (2k-2) - (2k-2)(|U| + 1) \\
&= 0
\end{aligned}$$

$\Diamond$

Claim 23. *G contains* $\overleftrightarrow{K}_{k-1}$ *as a subdigraph.*

Proof of claim. We have $x \in V(G)$ such that $d(x) \leqslant 2k-1$ (otherwise $\varepsilon(G) \geqslant 2|V(G)| \geqslant 2(k+1)$). By directional duality, we may assume $d^+(x) = k-1$. If $N^+(x)$ is a clique, then $G[N^+(x)] = \overleftrightarrow{K}_{k-1}$ and we are done. Otherwise we have $y, z \in N^+(x)$ such that $yz \notin A(G)$. Since $d^+_{G/\{y,z\}}(x) < k-1$, $G/\{y,z\}$ is not k-dicritical and x is not in any k-dicritical subdigraph of $G/\{y,z\}$, so claim 22 yields a copy of $\overleftrightarrow{K}_{k-1}$ in G. $\Diamond$

Let $W \subseteq V(G)$ such that $G[W] = \overleftrightarrow{K}_{k-1}$. We have $x \in W$ such that $d(x) \leqslant 2k-1$ (otherwise, $\varepsilon(G) \geqslant \varepsilon(W) \geqslant 2k-2$). Observe that $|N^+(x) - W| = 1$ or $|N^-(x) - W| = 1$. Let $y \in N(x) - W$ with $y \in N^+(x)$ whenever $|N^+(x) - W| = 1$ and $y \in N^-(x)$ otherwise. We choose such a triplet $(W,\ x,\ y)$ so as to maximise the number of arcs between x and y (i.e. we choose $y \in N^d(x)$ when possible) and, subject to that, maximise the cardinality of $W_y = W \cap N^d(y)$. Let $z \in W - (N^d(y) + x)$ with minimum degree (such a z exists by Claim 21).

By lemma 12(2), $\vec{\chi}(G/\{y,z\}) \geqslant k$. Let G^* be a k-dicritical subdigraph of $G/\{y,z\}$ and $U_W = W - (V(G^*) + z)$.

Claim 24. $U_W = W - z$ *and* $G^* = \overleftrightarrow{K}_k$.

Proof of claim. We first show $x \in U_W$. If $d(x) = 2k-2$ or $y \in N^d(x)$, since $z \in N^d(x)$, $d_{G/\{y,z\}}(x) \leqslant 2k-3$ and thus $x \notin V(G^*)$. Otherwise we have $d(x) = 2k-1$ and $y \notin N^d(x)$. Observe that in this case $|N^s(x)| = 3$.

We may assume by directional duality that $|N^+(x) - W| = 1$ and hence $y \in N^+(x)$. Then we have $N^{s+}_{G/\{y,z\}}(x) = \varnothing$. If $x \in G^*$, by lemma 13(2), $N^{s-}_{G^*}(x) = \varnothing$ and hence $d_{G^*}(x) \leqslant d_G(x) - 3 < 2(k-1)$, a contradiction. So $x \notin G^*$, i.e. $x \in U_W$ and, by Claim 22, $G^* = \overleftrightarrow{K}_k$.

Assume towards a contradiction $U_W \subsetneq W - z$. Then $1 \leqslant |U_W| \leqslant k-3$. Moreover, observe that for every $u \in W - (U_W + z)$, $d_G(u) \geqslant 2|G^* - u| + 2|U_W| = 2k - 2 + 2|U_w|$. Hence:

$$\begin{aligned}
\varepsilon(G) &\geqslant \varepsilon(W - (U_W + z)) \\
&= \sum_{u \in W - (U_W + z)} (d_G(u) - (2k-2)) \\
&\geqslant \sum_{u \in W - (U_W + z)} (2|U_W|) \\
&= 2|W - (U_W + z)||U_W| \\
&= 2(k - 2 - |U_W|)|U_W| \\
&\geqslant 2(k-3) \qquad \text{(by concavity of } x \mapsto (k-2-x)x\text{)},
\end{aligned}$$

a contradiction. $\Diamond$

Let $R = V(G^*) - y \star z$. By Claim 24, $G[R] = \overleftrightarrow{K}_{k-1}$. Let $R_y = R \cap N^d(y)$. The situation is depicted in Figure 1.

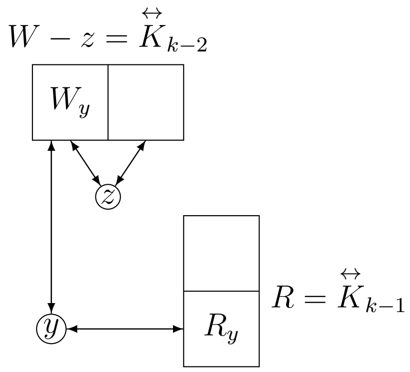

Figure 1: This figure describes the situation at the end of claim 24. $G/\{y,z\}[R \cup y \star z] = \overleftrightarrow{K}_{k-1}$.

Claim 25. *$\varepsilon(z) \geqslant k - 2 - |R_y|$. Moreover, if $|R_y| \leqslant k - 3$, equality holds only if all the arcs between z and $R - R_y$ have the same orientation.*

Proof of claim. We have $W - z \subseteq N^d(z)$ and since $G^* = \overleftrightarrow{K}_k$, we have $R - R_y \subseteq N(z)$. Let $s \in R - R_y$ (such an s exists, otherwise $G[R + y] = \overleftrightarrow{K}_k$). We may assume without loss of generality that $s \in N^+(z)$. Since G is k-dicritical, we have $\phi : G \setminus zs \to [k-1]$ a dicolouring. ϕ is not a dicolouring of G, so there is a monochromatic walk from s to z. Since W is a clique, z is the only vertex in W on the walk. Observe that s is the only element of $R - R_y$ with colour $\phi(s)$, and thus the last but one vertex on the walk is either s or not in $R - R_y$.

Observe moreover that, by Claim 24, $W - z$ is disjoint from R. Altogether, we get that $d(z) \geqslant 2(|W| - 1) + |R - R_y| + 1 \geqslant 2(k-2) + (k-1) - |R_y| + 1 = 3k - 4 - |R_y|$, and thus:

$$\varepsilon(z) = d(z) - (2k - 2) \geqslant k - 2 - |R_y|$$

Assume now we have $\varepsilon(z) = k - 2 - |R_y|$, $|R_y| \leqslant k - 3$, and for a contradiction $s_+ \in N^+(z) \cap (R - R_y)$ and $s_- \in N^-(z) \cap (R - R_y)$. Since $|R - R_y| \geqslant 2$, we may assume $s_+ \neq s_-$. As previously, $G \setminus zs_+$ admits a $(k-1)$ dicolouring, implying that either $s_+ \in N^d(z)$ or $N^-(z) - W - R_y \neq \varnothing$. Similarly, either $s_- \in N^d(z)$ or $N^+(z) - W - R_y \neq \varnothing$, which yields $\varepsilon(z) \geqslant k - 1 - |R_y|$. $\Diamond$

Recall that $W_y = N^d(y) \cap W$.

Claim 26. *$\varepsilon(\{y,z\}) \geqslant 2|W_y| - 2$ and equality holds only if $N^s(y) \setminus R = \emptyset$.*

Proof of claim. Since $G^* = \overleftrightarrow{K}_k$, $|\overleftrightarrow{A}(\{y,z\}, R)| \geqslant 2|R|$. Hence:

$$\begin{aligned} \varepsilon(\{y,z\}) &\geqslant 2|R| + 2|W_y| + 2(|W| - 1) - 4(k-1) \\ &= 2|W_y| - 2 \end{aligned}$$

If $N^s(y) \setminus R \neq \emptyset$, one arc incident to y is not accounted for in the previous minoration. $\lozenge$

Claim 27. *There is $x' \in R_y$ such that $d(x') \leqslant 2k-1$.*

Proof of claim. Otherwise, $\varepsilon(R_y) \geqslant 2|R_y|$. Recall that z has minimum degree among vertices of $W - W_y$. We distinguish two cases:

- If $x \in W_y$ (with $w = |W_y|, s = |R_y| \in [\![0, k-2]\!]$):

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & \varepsilon(\{y,z\}) + \varepsilon(W - (W_y + z)) + \varepsilon(R_y) \\ & \geqslant & 2w - 2 + (k-2-w)(k-2-s) + 2s \quad \text{(using Claims 25 and 26)} \\ & = & ws - (k-4)(w+s) + (k-2)^2 - 2 \\ & = & \frac{1}{4}((w+s)^2 - (w-s)^2) - (k-4)(w+s) + (k-2)^2 - 2 \end{array}$$

  Let $f(w,s)$ be this last expression. Since, for fixed $w+s$, $f(w,s)$ is decreasing in $|w-s|$ and symmetric in w and s, we consider $w', s' \in [\![0, k-2]\!]$ such that $w' + s' = w + s$ and $w' \in \{0, k-2\}$ and have:

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & f(w', s') \\ & \geqslant & \min(-(k-4)s' + (k-2)^2 - 2, \\ & & \qquad (k-2)s' - (k-4)(k-2+s') + (k-2)^2 - 2) \\ & \geqslant & \min((k-2)^2 - (k-2)(k-4) - 2, \\ & & \qquad (k-2)^2 - (k-2)(k-4) - 2) \\ & = & 2(k-3) \end{array}$$

- Otherwise, $x \notin W_y$, that is $y \notin N^d(x)$. Recall that we chose (W, x, y) so as to maximise the number of arcs between x and y. Let $u \in W_y \cup R_y$. If $d(u) \leqslant 2k-1$, then either (W, u, y) or (R, u, y) contradicts the choice of (W, x, y). Hence $d(u) \geqslant 2k$.

  We have $|W_y|, |R_y| \leqslant k-4$ (otherwise $\varepsilon(W_y) \geqslant 2(k-3)$ (resp. $\varepsilon(R_y) \geqslant 2(k-3)$)). Then (with $w = |W_y|, s = |R_y| \in [\![0, k-4]\!]$):

$$\begin{array}{rcl} \varepsilon(G) & = & \varepsilon(W - (W_y + x)) + \varepsilon(R_y) \\ & \geqslant & (k-2-w)(k-2-s) + 2s \qquad \text{(using Claim 25)} \\ & = & \frac{1}{4}((w+s)^2 - (w-s)^2) - (k-2)(w+s) + (k-2)^2 + 2s \end{array}$$

  This last expression is minimised when $s \leqslant w$ (otherwise exchange w and s) and when, for fixed $w+s$, $|w-s|$ is maximised, hence when $s = 0$ or $w = k-4$. Thus we have:

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & \min((k-2)(k-2-w),\ 2(k-2-s) + 2s) \\ & \geqslant & 2(k-2) \end{array}$$

$\lozenge$

Let $x' \in R_y$ with $d(x') \leqslant 2k-1$. Since $x' \in N^d(y)$ and we chose (W, x, y) so as to maximise the number of arcs between x and y, $x \in W_y$ (otherwise (R, x', y) contradicts the choice of (W, x, y)).

Since we chose (W, x, y) so as to maximise $|W_y|$, we have $|R_y| \leqslant |W_y|$ (otherwise (R, x', y) contradicts the choice of (W, x, y)). Also recall that z has minimum degree in $W - W_y$. Then:

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & \varepsilon(\{y,z\}) + \varepsilon(W - (W_y + z)) \\ & \geqslant & 2|W_y| - 2 + |W - (W_y + z)|\varepsilon(z) \quad \text{(using Claims 26)} \\ & \geqslant & 2|W_y| - 2 + (k - 2 - |W_y|)(k - 2 - |R_y|) \quad \text{(using Claims 25)} \\ & \geqslant & 2|W_y| - 2 + (k - 2 - |W_y|)^2 \\ & = & (|W_y| - (k-3))^2 + 2k - 7 \end{array}$$

Since $\varepsilon(G) < 2k - 6$, each inequality above is an equality, so $|W_y| = k - 3$ and then $|R_y| = k - 3$ and equality condition in Claims 25 and 26 hold. Without loss of generality, we may assume $R - R_y \subseteq N^{s+}(z)$. Since $G^* = \overleftrightarrow{K}_k$, we have $R - R_y \subseteq N^{s-}(y)$. But since G is k-dicritical, by lemma 13(2), $N^{s+}(y) \neq \varnothing$ and hence $N^{s+}(y) \setminus R \neq \varnothing$. This contradicts the equality condition in Claim 26. $\square$

4.2 Refined Dirac's bounds

The goal of this section is to prove Theorem 6, that we restate below, together with, as promised in the introduction, the digraph witnessing that the bound is tight.

First, as announced in the introduction, we have to define the set of digraphs $\mathcal{D}_k$.

Definition 28. Let $\mathcal{D}_3 = \{\overleftrightarrow{C}_{2n+1}, n \in \mathbb{N}\}$ and, for $k \geqslant 4$ (see Figure 2),

$$\mathcal{D}_k = \{\overleftrightarrow{C}_5(\overleftrightarrow{K}_{k-2}, \overleftrightarrow{K}_1, \overleftrightarrow{K}_n, \overleftrightarrow{K}_{k-1-n}, \overleftrightarrow{K}_1), 1 \leqslant n \leqslant k-2\}$$

It is clear that, for every $k \geqslant 4$ and $G \in \mathcal{D}_k$, with $a, b \in V(G)$ defined as in Figure 2, $\varepsilon(\{a, b\}) = 2(k-3)$ and the other vertices have excess 0, thus $\varepsilon(G) = 2(k-3)$.

Observe that all digraphs in $\mathcal{D}_k$, $k \geqslant 3$ are symmetric and k-dicritical. These digraphs are the same as the tight graphs characterised by Dirac in [Dir74] (see Theorem 5).

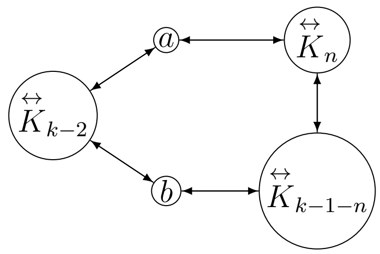

Figure 2: Digraphs in $\mathcal{D}_k$, $k \geqslant 4$.

Theorem 29. *Let $k \geqslant 4$ and G be a k-dicritical digraph such that $G \neq \overleftrightarrow{K}_k$ and $G \notin \mathcal{D}_k$. Then:*

$$|A(G)| \geqslant (k-1)|V(G)| + (k-2).$$

Equivalently: $\varepsilon(G) \geqslant 2(k-2)$.
Moreover, the bound is tight for $\overleftrightarrow{K}_2(\overleftrightarrow{K}_{k-2}, \vec{C}_3)$.

Proof. Assume the theorem is false. Let k be minimal such that the theorem does not hold for k.

Let G be a counterexample of minimal order. Since $\varepsilon(G)$ is even, $\varepsilon(G) \leqslant 2(k-3)$, so by Theorem 20, $\varepsilon(G) = 2(k-3)$. By Theorem 4, we may assume that G is not symmetric.

Let $S = \{u \in V(G) | d(u) = 2(k-1)\} = \{u \in V(G) | \varepsilon(u) = 0\}$. By Theorem 17, S induces a directed Gallai-forest.

Claim 30. $k \geqslant 5$.

Proof of claim. Assume $k = 4$. We have $\varepsilon(G) = 2$. Observe that in this case a block of S is either an arc, or a cycle, or a symmetric odd cycle. Moreover, all vertices of G have degree 6 (and thus are in S), except for exactly one vertex of degree 8 or for exactly two vertices of degree 7.

First consider the case where there is a vertex of degree 8, say u. Any non-separating vertex of $G[S]$ is in a symmetric odd cycle, since if it were in any other type of block, it would have more than two arcs incident with u. This implies that each non-separating vertex of S is linked to u via a digon and that each leaf block of $G[S]$ is a symmetric odd cycle, each of them containing at least 2 non-separating vertices. If $G[S]$ has at most 3 non-separating vertices, we have $G[S] = \overleftrightarrow{K}_3$ and then $G = \overleftrightarrow{K}_4$, a contradiction. Hence, there are 4 of them and thus $N^s(u) = \varnothing$. If $G[S]$ has only one block, then this block is $\overleftrightarrow{C}_4$, a contradiction. Hence $G[S]$ has exactly two leaf blocks, which are $\overleftrightarrow{K}_3$. Since G is not symmetric, $G[S]$ contains a cycle of length at least 3, which either leads to another non separating vertex in S or another leaf block, a contradiction.

Now consider the case where there are two vertices of degree 7, say u and v. In particular there are at most 14 arcs between $\{u, v\}$ and S.

Let B be a block of $G[S]$ and let $x \in V(B)$ be a non-separating vertex of $G[S]$. If B is an arc, then $d_G(x) \leqslant 5$, a contradiction. If B is a cycle, then x is linked to both u and v via a digon. Assume now that B is a symmetric odd cycle. Then x is incident with two arcs incident with $\{u, v\}$. Let us prove that there is a digon linking x and $\{u, v\}$. Assume towards a contradiction and without loss of generality that $\{ux, xv\} \subseteq A(G)$. Let H be obtained from G by removing the arcs ux and xv and adding the arc uv. Since x is incident with no other simple arc than ux and xv, an induced cycle of G that is not a cycle of H contains uxv, and thus any dicolouring of H is a dicolouring of G. Hence $\vec{\chi}(H) \geqslant 4$ and H contains a 4-dicritical subdigraph H^*. Since each vertex of H^* has degree at least 6, $x \notin V(H^*)$ and consequently no vertex of B is in $V(H^*)$ (by immediate induction). Since there are at least 4 arcs between $V(B)$ and $\{u, v\}$, $d_{H^*}(u) + d_{H^*}(v) \leqslant 14 - 4 = 10$. Hence u or v is not in H^* and $H^* \subsetneq G$, a contradiction.

To summarize, we get that a leaf block of $G[S]$ is either a cycle, and each of its non-separating vertex is linked to both u and v via a digon, or is a symmetric odd cycle, and each of its (at least 2) non-separating vertex is linked to one of u or v via a digon.

Moreover, there is no simple arc between a given non-separating vertex of $G[S]$ and $\{u, v\}$. In particular, there are at least two digons and no simple arc between the non-separating vertices of a given leaf block and $\{u, v\}$.

For $x \in \{u, v\}$, since $d_G(x) = 7$ is odd, $N^s(x) \neq \varnothing$, then by Lemma 13(2), $|N^s(x)| \geqslant 2$ and then since $d_G(x)$ is odd, $|N^s(x)| \geqslant 3$, and thus $|N^d(x)| \leqslant 2$.

This implies that $G[S]$ has at least one internal block. And since there are at least two digons between the non-separating vertices of a given leaf block and $\{u, v\}$, we get that $G[S]$ has exactly two leaves blocks B_1 and B_2, $N^s(u) = N^s(v) = 3$ and $N^d(u) = N^d(v) = 2$, B_1 and B_2 are either $\overleftrightarrow{K}_2$ or $\overleftrightarrow{K}_3$ and the only digons between $\{u, v\}$ and S are incident with the non-separating vertices of $G[S]$, which are all in B_1 or B_2.

Assume that $G[S]$ is a symmetric digraph. Then $G[S]$ consists in B_1 and B_2 and a symmetric path P linking B_1 and B_2. Each interior vertex of P is incident to both u and v via simple arcs. Let $H = G \setminus A^s(G) \cup uvu$. Every induced cycle of length at least 3 in G contains both u and v, hence $\vec{\chi}(H) \geqslant 4$. Let H^* be a 4-dicritical subdigraph of H. Since $|N^s(u)| = 3$, we have a separating vertex s of $G[S]$ incident to u in G. Every vertex of H^* has degree at least 6, hence $s \notin V(H^*)$. Consequently, since $G[S]$ is connected, $V(H^*) \cap S = \varnothing$, i.e. $V(H^*) \subseteq \{u, v\}$, a contradiction.

So we may assume that one of the internal block is an arc, say xy. If one of x or y, say x, is not incident with a $\overleftrightarrow{K}_3$, then $d_{G[S]}(x) \leqslant 3$, and since there is no digon between x and $\{u, v\}$, $d_G(x) \leqslant 5$, a contradiction. So both x and y are incident with a $\overleftrightarrow{K}_3$, and thus $G[S]$ is made of two $\overleftrightarrow{K}_3$ linked by an arc, namely xy. But in this case there are at most 10 arcs between S and $\{u, v\}$ and thus u and v are linked by a digon, a contradiction. $\lozenge$

Let

$$R \in \underset{R \subseteq V(G) \text{ acyclic}}{\operatorname{argmax}} \quad (\varepsilon(R), |R|)$$

Note that $\varepsilon(R) \geqslant 1$. Note also that, by maximality of $|R|$, every vertex in $V(G) - R$ has at least one in- and one out-neighbour in R.

Claim 31. $\varepsilon(R) \geqslant 2$.

Proof of claim. Assume $\varepsilon(R) = 1$. By definition of R, for every $u \in V(G)$, $\varepsilon(u) \leqslant 1$. $\varepsilon^{-1}(1)$ is a clique, because otherwise we would find an acyclic induced subdigraph of G with excess at least 2. Furthermore, $|\varepsilon^{-1}(1)| = \varepsilon(G) = 2(k-3)$. Then, since $\overleftrightarrow{K}_k \nsubseteq G$ we have $2(k-3) \leqslant k-1$ and thus $k = 5$ and $\varepsilon^{-1}(1) = \overleftrightarrow{K}_4$. Let $u \in \varepsilon^{-1}(1)$. Since $d(u) = 2k - 1 = 9$ is odd, $N^s(u) \neq \varnothing$ and hence by Lemma 13(2), $|N^s(u)| \geqslant 2$. Since $d(u)$ is odd, $|N^s(u)| \geqslant 3$ and thus $|N^s(u)| = 3$. In particular, there is no digon between $\varepsilon^{-1}(1)$ and S.

Since $k = 5$, every block of $G[S]$ is an arc, a cycle, a symmetric odd cycle or a $\overleftrightarrow{K}_4$. Let $u \in S$ be a non-separating vertex of $G[S]$. Since u has degree at most 6 in $G[S]$, there are at least two simple arcs between u and $\varepsilon^{-1}(1)$.

Besides, each arc between u and $\varepsilon^{-1}(1)$ is in an induced cycle (because G is dicritical), and thus u is incident with a simple arc in $G[S]$. Then, the block of $G[S]$ containing u is an arc or a cycle and thus there are at least 6 arcs between u and $\varepsilon^{-1}(1)$, which is impossible. $\Diamond$

Claim 32. $\overleftrightarrow{K}_{k-1} \subseteq G - R$.

Proof of claim. Since R is acyclic, $\vec{\chi}(G-R) \geqslant k-1$. Let $G^* \subseteq G-R$ be $(k-1)$-dicritical. We may assume $G^* \neq \overleftrightarrow{K}_{k-1}$.

We have $2(k-3) = \varepsilon_k(G) = \varepsilon_k(V(G)-V(G^*)) + \varepsilon_k(V(G^*))$. By claim 31, $\varepsilon_k(V(G) - V(G^*)) \geqslant 2$. By maximality of $|R|$, each vertex $u \in V(G) - R$ (and thus each vertex in $V(G^*)$) has at least an in- and an out-neighbour in R. Hence

$$\varepsilon_k(V(G^*)) \geqslant \varepsilon_{k-1}(G[V(G^*)]) = \varepsilon_{k-1}(G^*) + 2|A(G[V(G^*)]) - A(G^*)|$$

By Theorem 20, $\varepsilon_{k-1}(G^*) \geqslant 2(k-4)$. Altogether, we get:

$$\begin{aligned} 2(k-3) &= \varepsilon_k(G) \\ &= \varepsilon_k(V(G)-V(G^*)) + \varepsilon_k(V(G^*)) \\ &\geqslant 2 + \varepsilon_{k-1}(G[V(G^*)]) \\ &\geqslant 2 + \varepsilon_{k-1}(G^*) + 2|A(G[V(G^*)]) - A(G^*)| \\ &\geqslant 2 + 2(k-4) + 2|A(G[V(G^*)]) - A(G^*)| \\ &= 2(k-3) + 2|A(G[V(G^*)]) - A(G^*)| \\ &\geqslant 2(k-3) \end{aligned}$$

Every inequality is an equality, that is:

- $\varepsilon_k(V(G)-V(G^*)) = 2$, and thus $\varepsilon_k(R) = 2$ by claim 31.
- $\varepsilon_k(V(G^*)) = \varepsilon_{k-1}(G[V(G^*)])$, which implies that for every $x \in V(G^*)$, $|\overleftrightarrow{A}(x, V(G) - V(G^*))| = |\overleftrightarrow{A}(x,R)| = 2$,
- $|A(G[V(G^*)]) - A(G^*)| = 0$, that is G^* is an induced subdigraph of G, and
- $\varepsilon_{k-1}(G^*) = 2(k-4)$, which implies, by minimality of k, that $G^* \in \mathcal{D}_{k-1}$,

Let a and b be the vertices of G^* defined as in Figure 2 (replacing k by $k-1$). Since $|\overleftrightarrow{A}(a, V(G)-V(G^*))| = |\overleftrightarrow{A}(b, V(G)-V(G^*))| = 2$ and a and b are non-adjacent, by maximality of R we have $\varepsilon(R) \geqslant \varepsilon_G(\{a,b\}) = \varepsilon_{G^*,k-1}(\{a,b\}) = 2(k-4)$ and since $\varepsilon(R) = 2$, we obtain $k \leqslant 5$, and thus $k = 5$ by claim 30.

Hence $G^* \in \mathcal{D}_4$. Observe that $\mathcal{D}_4$ contains a single digraph, depicted in Figure 3. Let $x \in V(G^*)$ as in Figure 3. Since x has (exactly) one in- and one out-neighbour in R, $d_G(x) = 10$, and thus $\varepsilon(x) = 2$.

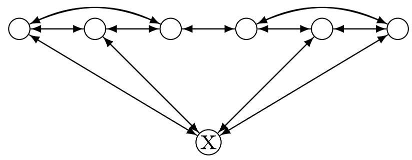

Figure 3: $G^* = \overleftrightarrow{C}_5(\overleftrightarrow{K}_2, \overleftrightarrow{K}_1, \overleftrightarrow{K}_1, \overleftrightarrow{K}_2, \overleftrightarrow{K}_1)$.

Since $\varepsilon(R) = 2$, by maximality of $\varepsilon(R)$, x is linked by a digon to every vertex with non-zero excess. Moreover, since $\overleftrightarrow{A}(x, R) = 2$, there is only one vertex in R with non-zero excess, say y, and thus $\varepsilon(y) = 2$.

Since $\varepsilon(G) = 2(k-3) = 4 = \varepsilon(\{x, y\})$, every vertex in $V(G) - \{x, y\}$ has excess 0, i.e. $S = V(G) - \{x, y\}$. In particular, $G^* - x$ is an induced subdigraph of $G[S]$.

Observe that for each vertex u in S, $d_G(u) = 2(k-1) = 8$ and there are at most 4 arcs between u and $\{x, y\}$, so $d_{G[S]}(u) \geqslant 4$. This implies that leaf blocks of $G[S]$ are neither $\vec{P}_2$, nor $\overleftrightarrow{K}_2$, nor $\vec{C}_n$. Hence, each leaf block of $G[S]$ is either $\overleftrightarrow{C}_{2n+1}$ for some $n \geqslant 1$ or $\overleftrightarrow{K}_4$.

Since $d_G(x) = d_G(y) = 10$ and x and y are linked by a digon, there are 16 arcs between S and $\{x, y\}$, 8 between y and S, and 8 between x and S that are already known (see Figure 3).

Observe that the number of arcs between the non-separating vertices of a $\overleftrightarrow{C}_{2n+1}$ leaf block of $G[S]$ and $\{x, y\}$ is $8n$. Moreover, since $G^* - x$ is a subdigraph of $G[S]$, $G[S]$ is not $\overleftrightarrow{C}_5$. Finally, the number of arcs between the non-separating vertices of a $\overleftrightarrow{K}_4$ leaf block of $G[S]$ and $\{x, y\}$ is 6. Hence, $G[S]$ has at most two leaf blocks and these blocks are either $\overleftrightarrow{K}_3$ or $\overleftrightarrow{K}_4$.

Since G is not symmetric and is dicritical, G contains an induced cycle of length at least 3. So $G[S]$ is not symmetric. Hence one of the block of $G[S]$, say B, with vertices in $V(G^*)$ is not a leaf block. B contains one of the $\overleftrightarrow{K}_3$ of G^* and there are 4 arcs between $V(B)$ and x.

Hence the leaf blocks of $G[S]$ are $\overleftrightarrow{K}_4$ blocks, there is no arc between a separating vertex of S and $\{x, y\}$ and the non-separating vertices of $G[S]$ are either in B or in a leaf block of $G[S]$. If $G[S]$ contains a $\vec{P}_2$ block uv then, since $G[S]$ has exactly two leaf blocks, u and v are in exactly two blocks of $G[S]$ and hence $d_{G[S]}(u) \leqslant 7$. There is an arc between u and $\{x, y\}$, a contradiction. If $G[S]$ has a $\vec{C}_n$ block with $n \geqslant 3$, then this block contains a non-separating vertex of $G[S]$, a contradiction. Hence $G[S]$ is symmetric, a contradiction. $\lozenge$

Let $C = \{x_1, \ldots, x_{k-1}\} \subseteq V(G) - R$ such that $G[C] = \overleftrightarrow{K}_{k-1}$ and $d(x_1) \leqslant \cdots \leqslant d(x_{k-1})$. Let $S' = \{u \in V(G) | d(u) \leqslant 2k-1\} = \{u \in V(G) | \varepsilon(u) \leqslant 1\}$.

Claim 33. *For $x_i \in C$, $\varepsilon(x_i) \leqslant \frac{2(k-3)}{k-i+1}$. Thus, $x_1, x_2, x_3 \in S'$.*

Proof of claim. Due to the ordering on the vertices in C, we have $\varepsilon(G) \geqslant \varepsilon(R) + \varepsilon(C) \geqslant$

$\varepsilon(x_i) + (k-i)\varepsilon(x_i)$. Hence $\varepsilon(x_i) \leqslant \frac{2(k-3)}{k-i+1}$. $\Diamond$

Observe that, since every vertex has in- and out-degree at least $k-1$, each vertex in C has at least one in- and one out-neighbour in $V(G) - C$.

Claim 34. *Let $y \in V(G) - C$ such that there is $x \in C \cap S'$ with $d^-(x) \leqslant d^+(x)$ and $y \in N^-(x)$ or $d^+(x) \leqslant d^-(x)$ and $y \in N^+(x)$. Then for any $(k-1)$-dicolouring ϕ of $G - C$ and $x' \in C$, there is a (possibly empty) monochromatic walk in $G - C$ from $N^+(x') - C$ to $N^-(x') - C$ with colour $\phi(y)$.*

Proof of claim. Let $x \in C \cap S'$ satisfying the hypothesis of the claim. By directional duality, we may assume $d^-(x) \leqslant d^+(x)$ and $y \in N^-(x)$.

We first show the claim in the case $x' \neq x$. Assume towards a contradiction that we have ϕ a $(k-1)$-dicolouring of $G - C$ such that there is no monochromatic walk in $G - C$ from $N^+(x') - C$ to $N^-(x') - C$ with colour $\phi(y)$. Set $\phi(x') = \phi(y)$. We want to colour greedily vertices in $C - \{x, x'\}$ from x_{k-1} to x_1. To prove this uses only colours in $[k-1]$ we show that, when trying to colour a vertex, it has at most $k-2$ coloured in- or out-neighbours. Let $4 \leqslant i \leqslant k-1$. When colouring x_i, $\{x, x_1, \ldots, x_{i-1}\} - \{x'\}$ is uncoloured and contains at least $i-2$ vertices. Then:

$$\begin{aligned}
d_{min}(x_i) - (i-2) &\leqslant \tfrac{d(x_i)}{2} - (i-2) \\
&= k - 1 + \tfrac{\varepsilon(x_i)}{2} - (i-2) \\
&\leqslant k - 1 + \tfrac{k-3}{k-i+1} - (i-2) && \text{by claim 33} \\
&= k - 1 + \tfrac{1}{k-i+1}(k - 3 - (k-i+1)(i-2)) \\
&\leqslant k - 1 + \tfrac{1}{k-i+1}(k - 3 - 2(k-3)) && \text{by convexity and } 4 \leqslant i \leqslant k-1 \\
&< k - 1
\end{aligned}$$

Hence we can dicolour greedily $\{x_4, \ldots, x_{k-1}\} - x$. Now, for each $u \in \{x_1, x_2, x_3\} - x$, $d(u) \leqslant 2k-1$ by claim 33, and u is connected to x (that is uncoloured) by a digon. Hence we can greedily colour u. It remains to colour x. We have $x \in S'$ and $d^-(x) \leqslant d^+(x)$. Hence $d^-(x) \leqslant k-1$. Since $y \in N^-(x)$, x has two in-neighbours with the same colour (namely y and x'), so we can colour x with a colour from $[k-1]$. We obtain a $(k-1)$-dicolouring of G, a contradiction.

If $x' = x$, we apply the claim to $x'' \in \{x_1, x_2, x_3\} - x$ and $y' \in N(x'') - C$ with colour $\phi(y)$ (which exists by the claim applied to x, y and x'') and x to obtain the result. $\Diamond$

Claim 35. *Let $a \neq b \in V(G) - C$. There exists a $(k-1)$-dicolouring of $G - C$ that gives different colours to a and b.*

Proof of claim. Assume not. Then $\vec{\chi}(G - C \cup aba) \geqslant k$. Let $G^* \subseteq (G - C \cup aba)$ be k-dicritical.

We have:

$$
\begin{array}{rcl}
2(k-3) & = & \varepsilon(G) \\
& \geqslant & \varepsilon(V(G^*)) \\
& = & \varepsilon(G^*) - 2|A(G^*) - A(G)| \\
& & + |\overleftrightarrow{A}(V(G^*), V(G) - V(G^*)| + 2|A(G[V(G^*)]) - A(G^*)| \\
& \geqslant & 2(k-1) - 4 \qquad \text{by Corollary 19 and } A(G^*) - A(G) \subseteq aba \\
& = & 2(k-3)
\end{array}
$$

Every inequality is an equality, so in particular we have, $\varepsilon(G^*) = 0$, i.e. $G^* = \overleftrightarrow{K}_k$ by Theorem 20, and $|\overleftrightarrow{A}(V(G^*), V(G) - V(G^*))| = 2(k-1)$ by Corollary 19. Since, for $x \in \{a, b\}$, $d_{G[V(G^*)]}(x) = 2(k-1) - 2$, we have $a, b \in N(V(G) - V(G^*))$. Since any $(k-1)$-dicolouring of $G[V(G^*)]$ gives the same colour to a and b, by Lemma 18, any $(k-1)$-dicolouring of $G[V(G^*)]$ gives the same colour to every vertex in $N(V(G)-V(G^*))$, and thus $N(V(G) - V(G^*)) = \{a, b\}$.

Let $H = G - (V(G^*) - a - b)$. Observe that since G is not $(k-1)$-dicolourable, every $(k-1)$-dicolouring of H gives different colours to a and b. Hence $\vec{\chi}(H/\{a, b\}) \geqslant k$, i.e. $H/\{a, b\}$ contains a k-dicritical digraph H^*. If $H^* \neq \overleftrightarrow{K}_k$, then using Theorem 20,

$$
\begin{array}{rcl}
\varepsilon(G) & \geqslant & \varepsilon(V(H^*) - a \star b + a + b) \\
& \geqslant & \varepsilon(H^*) + |\overleftrightarrow{A}(\{a, b\}, V(G^*) - a - b)| - 2(k-1) \\
& \geqslant & 2(k-3) + 4(k-2) - 2(k-1) \\
& \geqslant & 2(k-2)
\end{array}
$$

Hence $H^* = \overleftrightarrow{K}_k$.

Besides,

$$
\begin{array}{rcl}
\varepsilon(G) & \geqslant & \varepsilon(a, b) \\
& \geqslant & 4(k-2) - 4(k-1) + d_{H^*}(a \star b) + |\overleftrightarrow{A}(a, V(H^*)) \cap \overleftrightarrow{A}(b, V(H^*)| \\
& & +|\overleftrightarrow{A}(\{a, b\}, V(G) - V(G^*) - V(H^*)|
\end{array}
$$

Since $d_{H^*}(a \star b) \geqslant 2(k-1)$ and $\varepsilon(G) = 2(k-3)$, we obtain $\overleftrightarrow{A}(a, V(H^*)) \cap \overleftrightarrow{A}(b, V(H^*)) = \varnothing$ and $\overleftrightarrow{A}(\{a, b\}, V(G) - V(G^*) - V(H^*)) = \varnothing$. We conclude $G \in \mathcal{D}_k$, a contradiction. $\Diamond$

Let $y \in N(C \cap S') - C$ satisfying the hypothesis of claim 34 (which exists since every vertex in C has an in- and an out-neighbour in $V(G) - C$ and $C \cap S' \neq \varnothing$). If $C \cap S \neq \varnothing$, we choose y to be adjacent to a vertex in $C \cap S$. Up to re-indexing the element of C, we may assume that, among the elements of $C \cap S' - S$, the digonal neighbours of y come first, then those that are not adjacent to y and the simple neighbours of y come last. Let $1 \leqslant i' \leqslant k-1$ be minimal such that $y \notin N^d(x_{i'})$ (such an i' exists since $G \neq \overleftrightarrow{K}_k$).

Claim 36. $\varepsilon(y) \geqslant |\overleftrightarrow{A}(y, C)| - \varepsilon(x_{i'}) - 2$.

Proof of claim. By claim 34, $G-C \cup (N^-(x_{i'})-C)y(N^+(x_{i'})-C)$ is not $(k-1)$-dicolourable and hence contains a k-dicritical digraph G^*. Since G is k-dicritical, $y \in G^*$. Then:

$$\begin{array}{rcl} d(y) & = & |\overleftrightarrow{A}(y,C)| + d_{G-C}(y) \\ & \geqslant & |\overleftrightarrow{A}(y,C)| + d_{G^*}(y) - |N^-(x_{i'}) - C| - |N^+(x_{i'}) - C| \\ & \geqslant & |\overleftrightarrow{A}(y,C)| + 2(k-1) - (\varepsilon(x_{i'}) + 2). \end{array}$$

◊

Claim 37. *$\varepsilon(x_{i'}) = 1$, $C \cap S \subseteq N^d(y)$ and $C \cap S' \subseteq N(y)$.*

Proof of claim. By claim 36 and definition of i, $\varepsilon(y) \geqslant 2(i-2) - \varepsilon(x_i)$. Now, assume $\varepsilon(x_{i'}) \geqslant 2$. Since $x_{i'} \notin R$ and $\{x_{i'}, y\}$ is acyclic, there is $z \in G - C - y$ with $\varepsilon(z) \geqslant 1$. We have:

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & \varepsilon(y) + \varepsilon(C) + \varepsilon(z) \\ & \geqslant & 2(i-2) - \varepsilon(x_{i'}) + (k-i)\varepsilon(x_{i'}) + 1 \\ & \geqslant & 2(k-3) + 1, \end{array}$$

a contradiction.

Let $x \in C \cap S$ and assume $y \notin N^d(x)$. Let $z \in N(x) - C - y$. By claim 35, we have ϕ a $(k-1)$-dicolouring of $G - C$ such that $\phi(y) \neq \phi(z)$, which contradicts claim 34.

As a consequence, by the definition of i, $\varepsilon(x_{i'}) = 1$. Now, assume y and $x_{i'}$ are not adjacent. Then by claims 35 and 34, $x_{i'}$ has at least two in- and out-neighbours in $V(G) - C$, hence $\varepsilon(x_{i'}) \geqslant 2$, a contradiction. Thus, by the choice of the ordering on the vertices in $C \cap S'$ and the definition of i, y is adjacent to every vertex in $C \cap S'$. ◊

By claims 36 and 37, we have $\varepsilon(y) \geqslant 2|S \cap C| + |(S' - S) \cap C| - 3$. Hence:

$$\begin{array}{rcl} \varepsilon(G) & \geqslant & \varepsilon(y) + \varepsilon(C) \\ & \geqslant & 2|S \cap C| + |(S' - S) \cap C| - 3 + |(S' - S) \cap C| + 2|C - S'| \\ & = & 2|C| - 3 \\ & = & 2(k-1) - 3 \\ & > & 2(k-3), \end{array}$$

a contradiction. □

4.3 Refined Dirac-type bounds for $k = 3$

The goal of this section is to prove Theorem 7 that we restate below for convenience. We first need to define the set of digraphs $\mathcal{D}'_3$ mentioned in the introduction.

Definition 38. An *extended wheel* is a digraph made of a vertex x and a triangle $abca$ together with three symmetric paths with lengths of same parity, linking x with a, b and c respectively, and such that the three paths have only x in common. One of the paths can be of length 0, that is x is equal to one of a, b, c, and the two other paths have even length.

Let $\mathcal{D}'_3$ be the set of digraphs containing extended wheels and the all digraphs obtained from the digraph pictured in Figure 4 by replacing any digon by an odd symmetric path.

It is easy to check that digraphs in $\mathcal{D}'_3$ are 3-dicritical, and have excess 2.

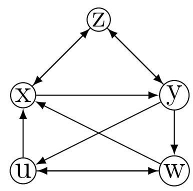

Figure 4: The digraph appearing in the definition of $\mathcal{D}'_3$.

We will also use the following theorem.

Theorem 39. *[ABHR22] If G is a 3-dicritical oriented graph, then*

$$|A(G)| \geqslant \frac{7|V(G)|+2}{3}$$

Theorem 40. *Let G be a 3-dicritical digraph which is not a symmetric odd cycle. Then $\varepsilon(G) = 2$ if and only if $G \in \mathcal{D}'_3$, and otherwise $\varepsilon(G) \geqslant 4$.*

Proof. Assume we have a counterexample G of minimal order.

By Brooks' Theorem, $\varepsilon(G) \geqslant 1$, and since $\varepsilon(G)$ is even, $\varepsilon(G) = 2$. Thus, either G contains a vertex with excess 2 or two vertices with excess 1. As usual, let $S = \varepsilon^{-1}(0)$. By Theorem 17, $G[S]$ is a directed Gallai forest. Note that, since odd symmetric cycles are 3-dicritical, the blocks of $G[S]$ are either $\overleftrightarrow{K}_1$, $\vec{P}_2$, or cycles. This implies in particular that a non-separating vertex of $G[S]$ is incident with at least 2 arcs incident with vertices in $V(G) - S$. These facts are constantly used during the proof.

Claim 41. *G has at least one digon.*

Proof of claim. Assume towards a contradiction that G has no digon, i.e. G is an oriented graph. By Theorem 39, $|A(G)| \geqslant \frac{7|V(G)|+2}{3}$. Moreover, since $\varepsilon(G) = 2$, we have $|A(G)| = 2|V(G)| + 2$. We then have $|V(G)| \leqslant 4$ which is clearly impossible. ♢

Claim 42. *Let P be a $\overleftrightarrow{P}_4$ in G. Then the interior vertices of P are not both in S.*

Proof of claim. We proceed by contradiction. Assume for contradiction that G contains a $\overleftrightarrow{P}_4$ on vertices a, b, c, d such that b and c are its interior vertices and are in S. Let $H = G - \{b, c\} + ada$.

Since $d_G(b) = d_G(c) = 4$, $d_H(a) = d_G(a)$ and $d_G(d) = d_H(d)$, we have $\varepsilon(G) = \varepsilon(H)$.

Assume that we have a 2-dicolouring ϕ of H. Then, by giving colour $\phi(a)$ to c, colour $\phi(d)$ to b, and colour $\phi(v)$ to every $v \in V(G) - \{b, c\}$, we obtain a 2-dicolouring of G, a contradiction. So $\vec{\chi}(H) = 3$.

Let e be an arc of H. If $e \notin \{ad, da\}$, then $e \in A(G)$, $G - e$ is 2-dicolourable, and any 2-dicolouring of $G - e$ gives distinct colours to a and d, so $H - e$ is also 2-dicolourable. If $e \in \{ad, da\}$, then a 2-dicolouring of $G - \{b, c\}$ gives distinct colours to a and d (otherwise we can easily extend it to a 2-dicolouring of G), and thus is a 2-dicolouring of $H - e$. Since H has no isolated vertex, H is 3-dicritical.

Finally, H is not in $\mathcal{D}'_3$, for otherwise G is too, contradicting the minimality of G. $\Diamond$

Claim 43. $\forall x \in V(G), \varepsilon(x) \leqslant 1$.

Proof of claim. We assume towards a contradiction that there is $x \in V(G)$ such that $\varepsilon(x) = 2$, i.e. $d(x) = 6$. Since $\varepsilon(G) = 2$, we have $V(G) - x = S$.

For every $s \in S$, $d_{G[S]}(s) \geqslant 2$ (because s is incident with at most 2 arcs incident with x, and has degree 4 in G). This implies that no connected component of $G[S]$ is a $\overleftrightarrow{K}_1$ or a $\vec{P}_2$ and no leaf block of $G[S]$ is a $\vec{P}_2$. In particular the leaf blocks of $G[S]$ are cycles.

If a connected component of $G[S]$ is a $\overleftrightarrow{K}_2$, then it forms a $\overleftrightarrow{K}_3$ with x, a contradiction. If a connected component of $G[S]$ is a $\vec{C}_3$, then it forms an extended wheel with x, a contradiction. If a connected component of $G[S]$ is a cycle of length at least 4, then x is linked by a digon to each of its vertices, implying that $d(x) \geqslant 8$, a contradiction. So the connected components of $G[S]$ have at least two leaf blocks.

A leaf block $\vec{C}_n$, $n \geqslant 2$ has $n-1$ non-separating vertices, each of them being connected to x via a digon. Thus, $G[S]$ has at most 3 non-separating vertices, and its leaf blocks are either $\overleftrightarrow{K}_2$ or $\vec{C}_3$. More precisely, $G[S]$ is connected and its leaf blocks are either three $\overleftrightarrow{K}_2$, or two $\overleftrightarrow{K}_2$, or one $\overleftrightarrow{K}_2$ and one $\vec{C}_3$.

Assume first G has two leaf blocks, one $\overleftrightarrow{K}_2$ and one $\vec{C}_3$. Since $d_G(x) \leqslant 6$, all the arcs between x and S are incident to a non-separating vertex of one of the leaf blocks, and hence every internal block of $G[S]$ is a $\overleftrightarrow{K}_2$. Since G is not an extended wheel, the path of digons between x and the separating vertex of the $\vec{C}_3$ leaf block of $G[S]$ has even length, so G is 2-dicolourable, a contradiction.

Assume now that $G[S]$ has three $\overleftrightarrow{K}_2$ leaf blocks $\{a_1, b_1\}$, $\{a_2, b_2\}$ and $\{a_3, b_3\}$ such that, for $i = 1, 2, 3$, a_i is a separating vertex of $G[S]$ and b_i is linked by a digon to x. Since G is 3-dicritical, for every 2-dicolouring ϕ of $G - x$, we have $\{\phi(b_1), \phi(b_2), \phi(b_3)\} = \{1, 2\}$, and thus $\{\phi(a_1), \phi(a_2), \phi(a_3)\} = \{1, 2\}$, and no proper subdigraph of $G - x$ has this property. Hence, the digraph H obtained from G by deleting b_1, b_2, b_3 and adding digons between x and a_i for $i = 1, 2, 3$ is 3-dicritical. Moreover, since $\varepsilon(\{b_1, b_2, b_3\}) = 0$ and, for $u \in V(G) - \{b_1, b_2, b_3\}, d_G(u) = d_H(u)$, we have $\varepsilon(H) = \varepsilon(G) = 2$, a contradiction to the minimality of G.

Finally, assume that $G[S]$ has two $\overleftrightarrow{K}_2$ leaf blocks, say $\{a, b\}$ and $\{c, d\}$ where b and c are separating vertices of $G[S]$. Then a and d are linked to x via a digon. By claim 42, b is not linked to a vertex of $S - \{a\}$ by a digon, and similarly, c is not linked to a vertex of $S - \{d\}$ by a digon. Hence, since $d(x) = 6$, we get that b and c are linked by an arc, as well as b and x, and c and x, and this gives us a full description of G up to the orientation of the three simple arcs. If $G[\{b,c,x\}] = \vec{C}_3$, then G is an extended wheel (in which one of the symmetric paths has length 0), and otherwise G is 2-dicolourable. A contradiction in both cases. $\lozenge$

From the previous claim, we get that G has two vertices, say x and y, with excess 1 (i.e. degree 5), and the other vertices have excess 0, that is $S = V(G) - \{x, y\}$. For $u \in \{x, y\}$, since $d(u) = 5$ is odd, $N^s(u) \neq \varnothing$ and hence, by Lemma 13(2), $|N^s(u)| \geqslant 2$ and then, since $d(u)$ is odd, $|N^s(u)| \geqslant 3$. In particular, x and y are incident with at most one digon.

Claim 44. *Let u be a non-separating vertex of $G[S]$ in a $\overleftrightarrow{K}_2$ block. Then $N^s(u) = \varnothing$, and thus u is linked to (exactly) one of x, y by a digon.*

Proof of claim. Assume not. Then $N^s(u) = \{x, y\}$. Since the arc between u and x is contained in an induced cycle, we may assume that $yu, ux \in A(G)$, and any induced cycle containing yu or ux contains both yu and ux.

This implies that $H = G \setminus \{yu, ux\} \cup yx$ is not 2-dicolourable (for otherwise G is too) and hence contains a 3-dicritical digraph H^*. Observe that u has degree 2 in H, so $u \notin V(H^*)$ and by immediate induction, denoting S_u the connected component of $G[S]$ containing u, we have that $S_u \cap V(H^*) = \varnothing$. Since $|S_u| \geqslant 2$, S_u contains at least one other non-separating vertex of $G[S]$, say w, and w is incident with two arcs incident with $\{x, y\}$. This implies that $d_{H^*}(x) + d_{H^*}(y) \leqslant 10 + 2 - 4 = 8$. Since x and y are in $V(H^*)$ (for otherwise H^* is a subdigraph of G), the inequality is an equality, which implies firstly that all vertices of H^* have degree 4 in H^*, and thus H^* is a symmetric odd cycle by Theorem 2, and secondly that $G[S_u]$ has exactly two non-separating vertices, i.e. $G[S_u]$ is a symmetric path with extremities u and w.

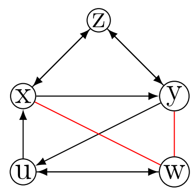

Figure 5: The digraph at the end of the proof of claim 44. We don't know the orientation of the two red arcs, and there might be a symmetric path of length 2 linking u and w instead of a digon.

By claim 42, we have $H^* = \overleftrightarrow{K}_3$. Since x and y are incident to at most one digon, we have $N^s(w) = \{x, y\}$. Besides, $xy \in A(G)$ and hence, $V(G) = S_u \cup V(H^*)$. By claim 42, we have $|S_u| \leqslant 3$. If $|S_u| = 3$, colouring x, y and the vertex in $S_u - u - w$ with colour 1 and the other vertices with colour 2 yields a 2-dicolouring of G, a contradiction. Hence $S_u = \{u, w\}$. If $ywx \subseteq G$, then $G \in \mathcal{D}_3'$, a contradiction. Hence, by Lemma 13(2), $xwy \subseteq G$. Every induced cycle containing xw contains yux and hence has a chord (namely xy), a contradiction. Hence G is not dicritical, a contradiction. $\lozenge$

Claim 45. *$G[S]$ has no $\overleftrightarrow{K}_1$-block.*

Proof of claim. Assume $G[S]$ contains a $\overleftrightarrow{K}_1$ block $\{u\}$. Then u is connected to x and y by digons. So, there is no digon between $\{x, y\}$ and $S - u$. By claim 44, the leaf blocks of $G[S]$ are cycle of length at least 3. Since there are at most 6 arcs between $\{x, y\}$ and $S - u$, $G[S - u] = \vec{C}_3$. Since x has degree 5, it cannot be adjacent with y and with the three vertices of the $\vec{C}_3$. So there exists $v \in S - u$ such that x is not adjacent with either y or v. Hence $\{x, y, v\}$ is acyclic. Now, colouring $\{x, y, v\}$ with colour 1 and the other vertices with colour 2 yields a 2-dicolouring of G, a contradiction. ♢

Claim 46. *$G[S]$ has no $\overleftrightarrow{K}_2$ leaf block.*

Proof of claim. Assume towards a contradiction that $G[S]$ contains a $\overleftrightarrow{K}_2$ leaf block, say $\{u, v\}$, with u non-separating in $G[S]$. By claim 44, we may assume without loss of generality that there is a digon between x and u.

Assume there is $w \in N^d(v) - u$. Then $w \neq x$ because G has no $\overleftrightarrow{K}_3$. Hence $\{x, u, v, w\}$ contradicts claim 42. So $|N^d(v)| = 1$ and thus $|N^s(v)| = 2$. By claim 44 v is separating in $G[S]$.

Write $N^s(v) = \{a, b\}$, with $b \in S$. By Lemma 13(2) and directional duality, we may assume that $bv, va \in A(G)$, and we get that every induced cycle containing bv or va contains both bv and va. This implies that $H = G \setminus bva \cup ba$ is not 2-dicolourable, for otherwise so is G. So $\vec{\chi}(H) = 3$. Let H^* be a 3-dicritical subdigraph of H. Note that every vertex in $V(H^*)$ has degree at least 4 in H^*. Hence $v \notin V(H^*)$, which implies $u \notin V(H^*)$. Since $\varepsilon_G(x) = 1$ and $u \in N^d(x)$, if $x \in V(H^*)$, then $\varepsilon_{H^*}(x) < 0$, which is impossible. Hence $x \notin V(H^*)$. As $d_H \leqslant d_G$, we obtain $\varepsilon_G(V(H^*)) \leqslant \varepsilon_G(y) \leqslant 1$. Then, since G is 2-arc-connected, $|\overleftrightarrow{A}_G(V(H^*), V(G) - V(H^*))| \geqslant 4$. As we added exactly one arc when constructing H, we obtain $\varepsilon(H^*) \leqslant \varepsilon_G(V(H^*)) + 2 - 4 < 0$, a contradiction. ♢

Claim 47. *$G[S]$ has no $\vec{P}_2$ leaf block.*

Proof of claim. Assume there is a $\vec{P}_2$ leaf block in $G[S]$, say $\{u, v\}$ with u non-separating in $G[S]$. We may assume without loss of generality that there is a digon between u and x, and a simple arc between u and y. Moreover, since the arc between u and v is in an induced cycle, we may assume that $vuy \subseteq G$ and we get that all induced cycle going through vu goes through uy. This implies that $H = G \setminus vu \cup yu$ is not 2-dicolourable and hence contains a 3-dicritical digraph H^*. Let S_v be the connected component of v in $G[S]$. Every vertex in H^* has degree at least 4 in H^*, and v has degree 3 in H, so $v \notin V(H^*)$ and an immediate induction shows that $V(S_v) \cap V(G^*) = \{u\}$. In G, $S_v - u$ contains a non-separating vertex of $G[S]$, which is incident with (at least) two arcs incident with $\{x, y\}$. So $d_{H^*}(x) + d_{H^*}(y) \leqslant 10 - 2 + 1 = 9$. Hence $\varepsilon(H^*) \leqslant 1$. Since $\varepsilon(H^*)$ is even, $\varepsilon(H^*) = 0$ and thus H^* is a symmetric odd cycle. Now, since $u \in V(H^*)$ (for otherwise H^* is a subgraph of G), we get that $x \in V(H^*)$. So x is incident with two digons in H^* and thus in G, a contradiction. ♢

Claim 48. *$G[S]$ has exactly two leaf blocks, which are cycles of length at least 3. Moreover, there are at least 8 arcs between the non-separating vertices of $G[S]$ and $\{x, y\}$.*

Proof of claim. By claims 45, 46 and 47, every leaf block of $G[S]$ is a cycle of length at least 3. For each such block B, we have $|\overleftrightarrow{A}(B, \{x, y\})| \geqslant 4$ and since $d(x) + d(y) = 10$, there are at most two of them.

Assume towards a contradiction that $G[S]$ has only one leaf block. Then $G[S]$ has only one block which is a cycle of length at least 3.

Assume first that there is no arc between x and y. Then $G[S] = \vec{C}_5$. Since $|N^d(x)| \leqslant 1$ and $|N^d(y)| \leqslant 1$, we have $s \in S - N^d(x) - N^d(y)$. Then, colouring x, y, s with colour 1 and all other vertices with colour 2 yields a 2-dicolouring of G, a contradiction.

Assume now that there is a simple arc between x and y, say $xy \in A(G)$. Then $G[S] = \vec{C}_4$, say $G[S] = s_1s_2s_3s_4s_1$. By claim 41, G contains a digon. Assume without loss of generality that there is a digon between x and s_1. So x is non-adjacent with one of the vertices s_i of S, $i \neq 1$. If there is no digon between y and s_i, then colouring x, y, s_i with colour 1, and the other vertices with colour 2 yields a 2-dicolouring of G, a contradiction. So there is a digon between y and s_i. Hence y is non-adjacent to some vertex in S. Let $s_j \in S$ with $j \neq 1$ and $j \neq i$ such that y is non-adjacent to a vertex in $S - s_i - s_j$ (which exists since $|S| = 4$). Then, colouring x, s_i and s_j with colour 1 and the other vertices with colour 2 yields a 2-dicolouring of G, a contradiction.

Finally, assume that there is a digon between x and y. Then $G[S]$ is a cycle of length 3 and there is no digon between S and $\{x, y\}$ (because $|N^d(x)| \leqslant 1$ and $|N^d(y)| \leqslant 1$)). By Lemma 13(2), x has both an in- and an out-neighbour in S. By directional duality, we may assume $|N^{s+}(x)| = 2$. Now, colouring $N^+[x]$ with colour 1 and the rest of the vertices with colour 2 yields a 2-dicolouring of G, a contradiction.

Hence, $G[S]$ has exactly two leaf blocks, which are cycles of length at least 3. Each of these leaf blocs have at least two non-separating vertices, and each of these vertices are incident with two arcs incident with $\{x, y\}$. So the second part of the statement holds. ♢

Claim 49. *There is no digon between S and $\{x, y\}$.*

Proof of claim. Assume there is such a digon. Without loss of generality, assume there exists $u \in N^d(x) \cap S$.

If u is separating in $G[S]$, then u is in two $\vec{P}_2$ blocks and hence, by claim 47, its neighbours in S are separating in $G[S]$ and are each incident with at least one arc incident with $\{x, y\}$. Hence there are at most 6 arcs between the non-separating vertices of $G[S]$ and $\{x, y\}$, which is impossible by claim 48.

Hence u is non-separating in $G[S]$. Let B be the block of $G[S]$ containing u. By claim 46, B is not a $\overleftrightarrow{K}_2$ block, so B is a cycle of length at least 3. Let $u^- \in N^{s-}(u)$ and $u^+ \in N^{s+}(u)$. Since the only induced cycle going through uu^+ or u^-u is B, $H = G \setminus uu^+ \cup uu^-$ is not 2-dicolourable and hence contains a 3-dicritical digraph H^*. Since every vertex in H^* has degree at least 4, an immediate induction on the walk from u^+ to u^- in B shows that $V(H^*) \cap V(B) \subseteq \{u, u^-\}$. In particular, $d_{H^*}(u^-) \leqslant 4$. Since

$u^+ \notin V(H^*)$, there is a non-separating vertex of $G[S]$ that is not in H^*. Hence, if $x, y \in V(H^*)$, then $d_{H^*}(x) + d_{H^*}(y) \leqslant 9$. In any case, $\varepsilon(G^*) \leqslant 1$. So $\varepsilon(G^*) = 0$ and thus H^* is a symmetric odd cycle. We have $u, u^- \in V(H^*)$, for otherwise H^* is a subdigraph of G. Since $d_{H^*}(u) = 4$, $x \in V(H^*)$ and thus x is incident with two digons, a contradiction. ◊

Claim 50. *There is no digon in S.*

Proof of claim. Let P be a maximal symmetric path in $G[S]$ and let u and v its extremities. Assume towards a contradiction that P has length at least 1, i.e. $u \neq v$. If both u and v are in $\vec{P}_2$ blocks, then the extremities of these two $\vec{P}_2$ are separating vertices by claim 48, and thus each of (the four of) them is adjacent to x or y. Hence there are at most 6 arcs between the non-separating vertices of $G[S]$ and $\{x, y\}$, contradicting claim 48.

Hence we may assume that u is not in a $\vec{P}_2$ block. By maximality of P, it is not in a second $\overleftrightarrow{K}_2$ block. Hence it is in a cycle of length at least 3. Let $u^- \in N^{s-}(u)$ and $u^+ \in N^{s+}(u)$. Since an induced cycle containing uu^+ or u^-u contains both uu^+ and u^-u, $H = G \setminus u^-uu^+ \cup u^-u^+$ is not 2-dicolourable. So H contains a 3-dicritical digraph H^*. Since every vertex in H^* has degree at least 4, an immediate induction on the component of $G[S] \setminus u^-uu^+$ containing u finds a non-separating vertex of $G[S]$ which is not in H^*. So $\varepsilon(H^*) \leqslant 1$, and thus $\varepsilon(H^*) = 0$ and thus H^* is a symmetric odd cycle. If $V(H^*) \subset S$, then $G[S]$ contains a symmetric cycle minus one arc, which is impossible. Hence $V(H^*) \cap \{x, y\} \neq \varnothing$, which contradicts claim 49. ◊

By claim 41, G contains a digon. By claims 49 and 50, there is a digon between x and y. Hence there are 6 arcs between S and $\{x, y\}$, a contradiction to claim 48. □

5 Kostochka-Yancey-type bound

The main goal of this paper is to obtain the best bounds on the minimum number of arcs in dicritical digraphs with fixed order and dichromatic number. One way of doing so, is to search for such bounds as linear functions of the order and search for the best slope. We give here a nice characterisation of this quantity.

The directed Hajós join describes a way to build k-critical digraphs from any two k-dicritical digraphs, with the following properties:

Lemma 51 (Theorem 2 in [BBSS20])**.** *Let $k \geqslant 2$ and let G_1 and G_2 be k-dicritical digraphs. Then there exists a k-dicritical digraph G with $|A(G)| = |A(G_1)| + |A(G_2)| - 1$ and $|V(G)| = |V(G_1)| + |V(G_2)| - 1$*

Lemma 52. *Let $k \geqslant 2$ and, for $n \geqslant k$, $d_k(n)$ be the minimum number of arcs in a k-dicritical digraph of order n. Then:*

$$\frac{1}{n} d_k(n) \underset{n \to +\infty}{\longrightarrow} \inf_{G \text{ } k\text{-dicritical}} \frac{|A(G)| - 1}{|V(G)| - 1} \in [k-1, k - \frac{2}{k-1}]$$

Proof. Given two integers n and m, we write $n\%m$ for the rest of the euclidean division of n by m. First, by Lemma 15, d_k is well-defined. Then, by Lemma 51, we have, for $a, b \geqslant k$:

$$d_k(a+b-1) \leqslant d_k(a) + d_k(b) - 1$$

and hence, for $a \geqslant b$,

$$\begin{aligned} d_k(a) &= d_k(b + (b-1)\left\lfloor \tfrac{a-b}{b-1} \right\rfloor + (a-b)\%(b-1)) \\ &= d_k(b + (a-b)\%(b-1)) + \sum_{i=0}^{\left\lfloor \frac{a-b}{b-1} \right\rfloor - 1} (d_k(b + (a-b)\%(b-1) + (i+1)(b-1)) \\ &\qquad - d_k(b + (a-b)\%(b-1) + i(b-1))) \\ &\leqslant d_k(b + (a-b)\%(b-1)) + \left\lfloor \tfrac{a-b}{b-1} \right\rfloor (d_k(b) - 1) \end{aligned}$$

i.e.

$$\frac{1}{a} d_k(a) \leqslant \frac{d_k(b) - 1}{b-1} + O(\frac{1}{a})$$

This yields $\limsup_{n \to +\infty} \frac{1}{n} d_k(n) \leqslant \inf_{G\ k\text{-dicritical}} \frac{|A(G)|-1}{|V(G)|-1}$. But it is immediate that:

$$\inf_{G\ k\text{-dicritical}} \frac{|A(G)|-1}{|V(G)|-1} \leqslant \liminf_{n \to +\infty} \frac{1}{n} d_k(n)$$

and the result follows (the upper bound comes from Theorem 9). □

5.1 Minimum number of arcs in a k-dicritical digraph

The goal of this section is to prove Theorem 10 that we restate below for convenience, see Theorem 56.

Let G be a digraph. Two distinct vertices $u, v \in V(G)$ are *twins* in G when $N^+[u] = N^+[v]$ and $N^-[u] = N^-[v]$. In particular a pair of twins are linked by a digon.

Definition 53. Let G be a digraph, $R \subseteq G$, and $\phi : R \to [k-1]$ be a dicolouring of $G[R]$. For $i \in [k-1]$, let $X_i = \phi^{-1}(i)$. We define $Y(G, R, \phi)$ as the digraph obtained from G after contracting each X_i into a single vertex x_i, and adding a digon between x_i and x_j for every $i \neq j$.

Lemma 54. *Let G be a digraph, $R \subseteq V(G)$, and ϕ be a $(k-1)$-dicolouring of $G[R]$. If $\vec{\chi}(G) \geqslant k$, then $\vec{\chi}(Y(G, R, \phi)) \geqslant k$.*

Proof. By lemma 12(2) and because adding arcs does not decrease the dichromatic number. □

We will also need the following technical lemma.

Lemma 55 (Lemma 17 in [KY14a]). *Let $k \geqslant 3$, $R_* = \{u_1, \dots, u_s\}$ be a set, and $\omega : R_* \to \mathbb{N}^*$ such that $\omega(u_1) + \cdots + \omega(u_s) \geqslant k-1$. Then for each $1 \leqslant i \leqslant (k-1)/2$, there exists a graph H with $V(H) = R_*$ and $|E(H)| = i$ such that for every independent set M in H with $|M| \geqslant 2$,*

$$\sum_{u \in R_* - M} \omega(u) \geqslant i$$

Our aim is to show the following theorem:

Theorem 56. *For every k-dicritical digraph G,*

$$|A(G)| \geqslant (k - \frac{1}{2} - \frac{1}{k-1})|V(G)| - k(\frac{1}{2} - \frac{1}{k-1})$$

Proof. Let $\varepsilon \in]0, \frac{1}{2} - \frac{1}{k-1}[$. Define the potential of a digraph G as follows:

$$\rho(G) = (k-1+\varepsilon)|V(G)| - |A(G)|$$

and for $R \subseteq V(G)$, the potential of R in G is $\rho_G(R) = \rho(G[R])$.

Let us first discuss the potential of cliques.

Claim 57. *For $i \geqslant 1$, $\rho(\overleftrightarrow{K}_i) = i(k-i+\varepsilon)$. In particular:*

- $\rho(\overleftrightarrow{K}_1) = k-1+\varepsilon$
- $\rho(\overleftrightarrow{K}_{k-1}) = (k-1)(1+\varepsilon)$
- $\rho(\overleftrightarrow{K}_k) = k\varepsilon$.

Besides, $\rho(\overleftrightarrow{K}_k) < \rho(\overleftrightarrow{K}_1) < \rho(\overleftrightarrow{K}_{k-1}) < \min_{2 \leqslant i \leqslant k-2} \rho(\overleftrightarrow{K}_i)$ (the last inequality can be seen easily using the concavity of $i \mapsto \rho(\overleftrightarrow{K}_i)$).

Note that if G is a digraph and H is a spanning proper subdigraph of G, then $\rho(G) \leqslant \rho(H)$. In particular $\rho(\overleftrightarrow{K}_{|V(G)|}) \leqslant \rho(G)$. These two easy facts are often used in the proof.

We are going to show that, for any k-dicritical digraph G, $\rho(G) \leqslant \rho(\overleftrightarrow{K}_k) = k\varepsilon$. This indeed implies the theorem because we get that $|A(G)| \geqslant 2(k-1+\varepsilon)|V(G)| - k\varepsilon$. This being true for any $\varepsilon \in]0, \frac{1}{2} - \frac{1}{k-1}[$, it also holds for $\varepsilon = \frac{1}{2} - \frac{1}{k-1}$, which gives $|A(G)| \geqslant (k - \frac{1}{2} - \frac{1}{k-1})|V(G)| - k(\frac{1}{2} - \frac{1}{k-1})$ as wanted.

We order the digraphs lexicographically on

$$\begin{aligned} G \quad \mapsto \quad & (|V(G)|, |A(G)|, |A^s(G)|, \\ & -|\{(u,v) \in V(G)^2, d(u) = d(v) = 2(k-1) \wedge u \text{ and } v \text{ are twins}\}|) \end{aligned}$$

(denoting $\preceq$ the ordering) and consider a $\preceq$-minimal counter-example G. So $\rho(G) > \rho(\overleftrightarrow{K}_k)$ and we minimise the number of vertices, then the number of arcs, then the number of simple arcs, and finally we maximise the number of twins of degree $2(k-1)$.

Let $S = \{u \in V(G), d(u) = 2(k-1)\}$.

We start the proof by a lower bound on the potential of a subset of $V(G)$.

Claim 58. *Let $R \subsetneq V(G)$. If $|R| \geqslant 2$, then $\rho_G(R) > \rho(\overleftrightarrow{K}_1) = k - 1 - \varepsilon$.*

Proof of claim. Let $R \in \underset{\substack{W \subsetneq V(G) \\ |W| \geqslant 2}}{\operatorname{argmin}} \rho_G(W)$. Towards a contradiction, we assume $\rho_G(R) \leqslant \rho(\overleftrightarrow{K}_1)$.

Since $\rho(\overleftrightarrow{K}_{|R|}) \leqslant \rho_G(R) \leqslant \rho(\overleftrightarrow{K}_1) < \min_{2 \leqslant i \leqslant k-1} \rho(\overleftrightarrow{K}_i)$, we have $|R| \geqslant k$. Since $R \subsetneq V(G)$ and G is k-dicritical, we have a dicolouring $\phi : G[R] \to [k-1]$. Let $Y = Y(G, R, \phi)$ and $X = V(Y) - V(G)$. Since $\vec{\chi}(G) = k$, by lemma 54 we have $\vec{\chi}(Y) \geqslant k$ and hence Y contains a k-dicritical subdigraph Y^*.

Since $|R| \geqslant k$, $|V(Y^*)| \leqslant |V(Y)| = |V(G)| - |R| + (k-1) < |V(G)|$, so $Y^* \prec G$ and hence $\rho(Y^*) \leqslant \rho(\overleftrightarrow{K}_k)$.

Since G is k-dicritical, $Y^* \nsubseteq G$ and hence $X \cap V(Y^*) \neq \varnothing$. So,

$$\rho(\overleftrightarrow{K}_1) \leqslant \rho(\overleftrightarrow{K}_{|V(Y^*) \cap X|}) \leqslant \rho_{Y^*}(V(Y^*) \cap X)$$

.

We have:

$$\begin{aligned}
\rho_G(V(Y^*) - X + R) &= \rho_G(V(Y^*) - X) + \rho_G(R) - |\overleftrightarrow{A}_G(V(Y^*) - X, R)| \\
&\leqslant \rho(Y^*) - \rho_{Y^*}(V(Y^*) \cap X) + \rho_G(R) \\
&\qquad + |\overleftrightarrow{A}_{Y^*}(V(Y^*) - X, V(Y^*) \cap X)| - |\overleftrightarrow{A}_G(V(Y^*) - X, R)| \\
&\leqslant \rho_G(R) + \rho(\overleftrightarrow{K}_k) - \rho(\overleftrightarrow{K}_1) \\
&< \rho_G(R)
\end{aligned}$$

Since $2 \leqslant |R| \leqslant |V(Y^*) - X + R|$, by minimality of R, $V(Y^*) - X + R = V(G)$ and thus:

$$\rho(G) \leqslant \rho(\overleftrightarrow{K}_k) + \rho_G(R) - \rho(\overleftrightarrow{K}_1) \leqslant \rho(\overleftrightarrow{K}_k),$$

a contradiction. ◊

We are now ready to obtain a much stronger lower bound.

Claim 59. *Let $R \subsetneq G$ such that $|R| \geqslant 2$. If $\rho_G(R) \leqslant \rho(\overleftrightarrow{K}_{k-1}) = (k-1)(1+\varepsilon)$, then $G[R] = \overleftrightarrow{K}_{k-1}$.*

Proof of claim. Let $R \in \underset{\substack{W \subsetneq V(G) \\ |W| \geqslant 2 \wedge G[W] \neq \overleftrightarrow{K}_{k-1}}}{\operatorname{argmin}} \rho_G(W)$. Towards a contradiction, we assume $\rho_G(R) \leqslant \rho(\overleftrightarrow{K}_{k-1})$. Since $\rho_G(R) \leqslant \rho(\overleftrightarrow{K}_{k-1}) < \min_{2 \leqslant i \leqslant k-2} \rho(\overleftrightarrow{K}_i)$ and $G[R] \neq \overleftrightarrow{K}_{k-1}$, we have $|R| \geqslant k$.

Let $i = \left\lceil \rho_G(R) - \rho(\overleftrightarrow{K}_k) \right\rceil - 1$, so that $\rho(\overleftrightarrow{K}_k) + i < \rho_G(R) \leqslant \rho(\overleftrightarrow{K}_k) + i + 1$. By claim 58, we have $k - 1 + \varepsilon = \rho(\overleftrightarrow{K}_1) < \rho_G(R) \leqslant \rho(\overleftrightarrow{K}_k) + i + 1$ and hence since $\varepsilon < \frac{1}{2} - \frac{1}{k-1}$, we have $i > k - 1 + \varepsilon - k\varepsilon - 1 > \frac{k-1}{2}$. In particular $i \geqslant 2$.

Besides, we have $\rho(\overleftrightarrow{K}_k) + i < \rho_G(R) \leqslant \rho(\overleftrightarrow{K}_{k-1})$ and hence $i < (k-1)(1+\varepsilon) - k\varepsilon = k - 1 - \varepsilon$ which gives $i \leqslant k - 2$.

Since by Corollary 19 $|\overleftrightarrow{A}(N(V(G) - R), G - R)| \geqslant 2(k-1)$, Lemma 55 with $\omega : x \in N(V(G) - R) \mapsto |\overleftrightarrow{A}(x, G - R)|$ implies the existence of a set of digons A with end vertices in $N(V(G)-R)$ of size $\left\lfloor \frac{i}{2} \right\rfloor$ such that for every $I \subseteq N(V(G)-R)$ with $|I| \geqslant 2$ and independent in the digraph $(N(V(G)-R), A)$, we have $|\overleftrightarrow{A}(N(V(G)-R)-I, V(G)-R)| \geqslant \left\lfloor \frac{i}{2} \right\rfloor$.

We show that $G[R] \cup A$ is $(k-1)$-dicolourable. If it is not the case, we have $G^* \subseteq G[R] \cup A$ k-dicritical. Then, $\rho(G^*) \geqslant \rho_G(G^*) - 2\left\lfloor \frac{i}{2} \right\rfloor \geqslant \rho_G(R) - i > \rho(\overleftrightarrow{K}_k)$, which contradicts the minimality of G.

Let $\phi : R \to [k-1]$ be a dicolouring of $G[R] \cup A$. Let $Y = Y(G, R, \phi)$ and $X = V(Y) - V(G)$. Since $\vec{\chi}(G) = k$, by lemma 54 we have $\vec{\chi}(Y) \geqslant k$ and hence Y contains a k-dicritical subdigraph Y^*. Since $|R| \geqslant k$, we have $|V(Y^*)| < |V(G)|$, that is $Y^* \prec G$. By minimality of G, $\rho(Y^*) \leqslant \rho(\overleftrightarrow{K}_k)$. Since G is k-dicritical, $Y^* \nsubseteq G$ and hence $X \cap V(Y^*) \neq \varnothing$. We have:

$$\begin{aligned}
\rho_G(Y^* - X + R) &= \rho_G(Y^* - X) + \rho_G(R) - |\overleftrightarrow{A}(Y^* - X, R)| \\
&= \rho_Y(Y^* - X) + \rho_G(R) - |\overleftrightarrow{A}(Y^* - X, R)| \\
&= \rho_Y(Y^*) - \rho_Y(Y^* \cap X) + \rho_G(R) \\
&\quad + |\overleftrightarrow{A}(Y^* - X, Y^* \cap X)| - |\overleftrightarrow{A}(Y^* - X, R)| \\
&\leqslant \rho(Y^*) - \rho_Y(Y^* \cap X) + \rho_G(R) \\
&\quad + |\overleftrightarrow{A}(Y^* - X, Y^* \cap X)| - |\overleftrightarrow{A}(Y^* - X, R)|
\end{aligned}$$

If $|Y^* \cap X| \geqslant 2$, we obtain: $\rho_G(Y^* - X + R) \leqslant \rho(\overleftrightarrow{K}_k) - \rho(\overleftrightarrow{K}_{k-1}) + \rho_G(R) \leqslant \rho(\overleftrightarrow{K}_k) < \rho(\overleftrightarrow{K}_1)$, a contradiction. Hence $|Y^* \cap X| = 1$. Then: $\rho_G(Y^* - X + R) \leqslant \rho(\overleftrightarrow{K}_k) - \rho(\overleftrightarrow{K}_1) + \rho(\overleftrightarrow{K}_k) + i + 1 - \left\lfloor \frac{i}{2} \right\rfloor$. By claim 58, we have $\rho_G(Y^* - X + R) > \rho(\overleftrightarrow{K}_k)$. We obtain $k - 1 + \varepsilon - k\varepsilon < i + 1 - \left\lfloor \frac{i}{2} \right\rfloor \leqslant i + 1 - \frac{i-1}{2} = \frac{i+3}{2} \leqslant \frac{k+1}{2}$ and hence $\varepsilon \geqslant \frac{1}{2} - \frac{1}{k-1}$, a contradiction. ◊

We are now going to show some strong structural properties of G.

Claim 60. *Let $R \subsetneq V(G)$ and A be a set of at most $k-2$ arcs with end vertices in R. Then $G[R] \cup A$ is $(k-1)$-dicolourable.*

Proof of claim. Otherwise, let $G^* \subseteq G[R] \cup A$ be k-dicritical. We have $|R| \geqslant |V(G^*)| \geqslant k$. In particular $G[R] \neq \overleftrightarrow{K}_{k-1}$, so $\rho_G(R) > \rho(\overleftrightarrow{K}_{k-1})$. Hence $\rho(G^*) \geqslant \rho_G(V(G^*)) - (k-2) > \rho(\overleftrightarrow{K}_{k-1}) - (k-2) = (k-1)(1+\varepsilon) - (k-2) = k\varepsilon + 1 - \varepsilon \geqslant \rho(\overleftrightarrow{K}_k)$, and since $G^* \prec G$, we get a contradiction with the minimality of G. ♢

Claim 61. *Let $u \in G$ with $d(u) \leqslant 2k-1$. Then $N^d(u) = \varnothing$ or $N^s(u) = \varnothing$.*

Proof of claim. We proceed by contradiction. By directional duality, we may assume $|N^+(u)| \geqslant |N^-(u)|$. Let $N^{s+}(u) = \{x_i^+, 1 \leqslant i \leqslant t\}$ and $N^{s-}(u) = \{x_i^-, 1 \leqslant i \leqslant s\}$ (with $s \leqslant t$). If $s = 0$, by lemma 13(2), $t = 0$ and $N^s(u) = \varnothing$. If $s = k-1$, then $N^d(u) = \varnothing$. So $1 \leqslant s \leqslant k-2$.

Let $H = G \setminus \{ux_i^+, 1 \leqslant i \leqslant s\} \cup uN^{s-}(u)$. Any dicolouring of H is a dicolouring of G, so $\vec{\chi}(H) \geqslant k$. Let $H^* \subseteq H$ be k-dicritical. Since $s \leqslant k-2$, we have $|uN^{s-}(u)| \leqslant k-2$ and thus, by claim 60, $V(H^*) = V(G)$. Note that $H^* \prec G$ (because we moved the arcs so as to create digons). We have $\rho(H^*) \geqslant \rho(G) > \rho(\overleftrightarrow{K}_k)$, a contradiction to the minimality of G. ♢

Claim 62. *Let $x, y \in V(G)$ such that $xy \in A(G)$, $yx \notin A(G)$, $d^+(x) = k-1$ and $d(y) \leqslant 2k-1$. Then $d^-(y) = k$. In particular, any pair of vertices in S are either non adjacent, or linked by a digon.*

Proof of claim. Assume towards a contradiction that $d^-(y) = k-1$. By claim 61, we have $z \in N^{s-}(y) - x$. Let $H = G - x - y \cup zN^{s+}(y)$.

We have $\chi(H) \geqslant k$. Otherwise, consider $\phi : H \to [k-1]$ a dicolouring. Since $d^+_{G-y}(x) < k-1$, we can extend ϕ into a $(k-1)$-dicolouring of $G - y$. Since ϕ cannot be extended into a $(k-1)$-dicolouring of G, we have $\phi(N^-(y)) = [k-1]$. Since $|N^-(y)| = k-1$, ϕ is injective on $N^-(y)$. Set $\phi(y) = \phi(z)$. Let C be a monochromatic cycle. We have $z' \in N^+(y)$ such that $zyz' \subseteq C$. Then $C \setminus zyz' \cup zz'$ is a monochromatic cycle in $G - y$, a contradiction.

Let $H^* \subseteq H$ k-dicritical. Since H^* is not a subdigraph of G, $z \in V(H^*)$ and at least one of the added arc is in $A(H^*)$. Then:

$$\begin{aligned}
\rho_G(V(H^*) + y) &= \rho(H^*) + \rho(\overleftrightarrow{K}_1) - (|A_G(V(H^*) + y)| - |A(H^*)|) \\
&\leqslant \rho(\overleftrightarrow{K}_k) + \rho(\overleftrightarrow{K}_1) - 1 \\
&= \rho(\overleftrightarrow{K}_{k-1}) + 2\varepsilon - 1 \\
&< \rho(\overleftrightarrow{K}_{k-1}).
\end{aligned}$$

Since $x \notin V(H^*)$, $V(H^*) + y \neq V(G)$ and we obtain a contradiction to Claim 59. ♢

Claim 63. *Let $X = \overleftrightarrow{K}_{k-1} \subseteq G$ and $x, y \in X \cap S$. Then x and y are twins.*

Proof of claim. By claim 61, $N(x) = N^d(x)$ and $N(y) = N^d(y)$. Let $u_x \in N(x) - X$ and $u_y \in N(y) - X$. Assume towards a contradiction that $u_x \neq u_y$. Let $H = G - x - y \cup u_x u_y u_x$. By claim 60, we have $\phi : H \to [k-1]$ a dicolouring. We have $\phi(u_x) \neq \phi(u_y)$. If $\phi(u_y) \in \phi(X - x - y)$, we take $\phi(x) \in [k-1] - \phi(X - x - y)$, otherwise we set $\phi(x) = \phi(u_y)$. In both cases y has two neighbours with the same colour and hence we can extend ϕ greedily to G, a contradiction. ♢

A set of vertices C of G is a *cluster* if $C \subseteq S$, C is a clique, each pair of vertices in C are twins, and C is maximal with these properties.

Claim 64. *Let C be a cluster of G. Then $|C| \leqslant k - 3$.*

Proof of claim. By claim 61, a cluster of size at least $k - 2$ would be at most 2 arcs away from being a $\overleftrightarrow{K}_k$, contradicting claim 60. ♢

Claim 65. *Let $x, y \in S$ such that there is a digon between x and y, x (resp. y) is in a cluster of size s (resp. t), x and y are not twins and $t \leqslant s$. Then x is in a $\overleftrightarrow{K}_{k-1}$ and $t = 1$.*

Proof of claim. By claim 61, $N^s(x) = \varnothing$. Let $G' = G - y + x'$ so that $N[x'] = N^d[x'] = N[x]$ (i.e. x and x' are twins and linked by a digon). We have $|V(G')| = |V(G)|$. Since $y \in N^d_G(x)$ and $x' \in N^d_{G'}(x')$, $d_G(y) = d_G(x) = d_{G'}(x) = d_{G'}(x')$ and hence $|A(G')| = |A(G)|$. Furthermore, $N^s_{G'}(x') = \varnothing$, so $|A^s(G')| \leqslant |A^s(G')|$. Removing y reduces the number of twins by $2(s-1)$. Subsequently, adding x' increases the number of twins by $2t$. Since $t \leqslant s$, we conclude $G' \prec G$.

Assume we have $\phi' : G' \to [k-1]$ a dicolouring. Set, for $u \in V(G) - \{x, y\}$, $\phi(u) = \phi'(u)$, then $\phi(y) \in [k-1] - \phi'(N(y) - x)$ and finally $\phi(x) \in \{\phi'(x), \phi'(x')\} - \{\phi(y)\}$. It is easy to check that ϕ is a $(k-1)$-dicolouring of G, a contradiction.

Hence $\vec{\chi}(G') \geqslant k$. Let $G^* \subseteq G'$ be k-dicritical. We have $G^* \prec G$, so $\rho(G^*) \leqslant \rho(\overleftrightarrow{K}_k)$. Besides, G is k-dicritical and hence $x' \in V(G^*)$. We have $\rho_G(V(G^*) - x') \leqslant \rho(\overleftrightarrow{K}_k) - \rho(\overleftrightarrow{K}_1) + 2(k-1) = \rho(\overleftrightarrow{K}_{k-1})$. Since $y \notin V(G^*) - x'$, by claim 59, $G^* - x' = \overleftrightarrow{K}_{k-1}$. Finally, since $x' \in V(G^*)$, G^* is k-dicritical and $d(x') = 2(k-1)$, we have $x \in N[x'] = V(G^*)$. Hence x is in a $(k-1)$-clique in G.

Now, $N[x] - y = \overleftrightarrow{K}_{k-1}$. If the cluster of y contains a vertex $x' \in N[x] - y$, then x' and y are twins and thus $N[x]$ is $\overleftrightarrow{K}_k$, a contradiction. So the cluster of y is disjoint from $N[x] - y$, but any vertex in the cluster of y is a neighbour of x, so the cluster of y is reduced to y, i.e. $t = 1$. ♢

Claim 66. *Let C be a cluster with $|C| \geqslant 2$.*

1. *If $\overleftrightarrow{K}_{k-1} \not\subseteq G[N[C]]$, then $\forall u \in N(C), d(u) \geqslant 2(k - 1 + |C|)$.*

2. *If there is $X \subseteq N[C]$ such that $G[X] = \overleftrightarrow{K}_{k-1}$, then $\forall u \in X - C, d(u) \geqslant 2(k-1+|C|)$.*

Proof of claim. Assume towards a contradiction that we have $u \in N(C)$ such that $d(u) < 2(k-1+|C|)$ and, if there is $X \subseteq N[C]$ such that $G[X] = \overleftrightarrow{K}_{k-1}$, $u \in X - C$.

Assume $u \in S$. For $c \in C \cap N(u)$, since $|C| \geqslant 2$, by claim 61, $u \in N^d(c)$. By claim 65, since $|C| \neq 1$, $G[C] \subseteq \overleftrightarrow{K}_{k-1}$. Then $\overleftrightarrow{K}_{k-1} \subseteq G[N[C]]$ and hence by definition of u, there is $X \subseteq N[C]$ such that $G[X] = \overleftrightarrow{K}_{k-1}$ and $u \in X - C$. Since $d(u) = 2(k-1)$, there is $c \in C \cap X$. By claim 63, u and c are twins, i.e. $u \in C$, a contradiction. So $d(u) \geqslant 2k-1$.

Let $c \in C$ and $G' = G - u + c'$ where c' is a new vertex such that $N^+[c'] = N^+[c]$ and $N^-[c'] = N^-[c]$, i.e. c and c' are twins. Assume we have ϕ' a $(k-1)$-dicolouring of G'. Set, for $x \in G - C - u$, $\phi(x) = \phi'(x)$. Then take $\phi(u) \in [k-1] - (\phi(N^+(u) - C) \cap \phi(N^-(u) - C))$ (which is not empty since $d(u) < 2(k-1+|C|)$) and then colour C with colours in $\phi'(C + c') - \phi(u)$. This is a $(k-1)$-dicolouring of G, a contradiction. Hence $\vec{\chi}(G') \geqslant k$ and G' contains a k-dicritical digraph G^*. Since $d(u) \geqslant 2k-1 > 2(k-1) = d_{G'}(c')$, $|A(G')| < |A(G)|$ and hence $G' \prec G$. Hence $\rho(G^*) \leqslant \rho(\overleftrightarrow{K}_k)$. Since $G^* \nsubseteq G$, we have $c' \in V(G^*)$. Since $d(c') = 2(k-1)$, we obtain $C \subseteq V(G^*)$. We have: $\rho_G(G^* - c') \leqslant \rho(G^*) - \rho(\overleftrightarrow{K}_1) + 2(k-1) \leqslant \rho(\overleftrightarrow{K}_{k-1})$. Hence by claim 59, $G^* - c' = \overleftrightarrow{K}_{k-1}$. We have $N[C] - u = \overleftrightarrow{K}_{k-1}$. Hence, by the choice of u, there is $X \subseteq N[C]$ such that $G[X] = \overleftrightarrow{K}_{k-1}$ and $u \in X$. Let $v \in N[C] - X$. Then $N[C] \cup uvu = \overleftrightarrow{K}_k$, A contradiction to claim 60. ◊

We are now going to obtain a contradiction using the discharging method. Let $\alpha = \frac{\varepsilon}{k-2}$. Each $u \in V(G)$ starts with charge $d(u)$. We apply the following rules (observe that any charge sent through an arc is at least α):

- Every vertex with degree at least $2k$ keeps $2(k-1+\varepsilon)$ to himself and distributes the rest equally along its arcs: it sends charge $\frac{d(u)-2(k-1+\varepsilon)}{d(u)}$ through each of its arcs. Note that this expression increases with $d(u)$ and hence is at least $\frac{1-\varepsilon}{k} \geqslant \alpha$.
- Every vertex with degree $2k-1$ and k out-neighbours (resp. k in-neighbours) sends charge α to its out-neighbours (resp. in-neighbours).
- For every $u \in S$ such that u is in a cluster of size at least 2 which is in a $(k-1)$-clique X, u sends 2α to its unique neighbour that is not in X.

The uniqueness of the neighbour of u in the last bullet is due to claim 61. Indeed, since u is in a $(k-1)$-clique, $N^s(u) = \varnothing$ and hence $|N(u)| = k-1$.

Let, for $u \in V(G)$, $w(u)$ be its resulting charge. We are going to prove that for every $u \in V(G)$, $w(u) \geqslant 2(k-1+\varepsilon)$.

- Let $u \in V(G)$ such that $d(u) \geqslant 2k$. Then by construction, $w(u) = 2(k-1+\varepsilon)$.
- Let $u \in V(G)$ such that $d(u) = 2k-1$ and $d^-(u) = k-1$. By claim 61, $N^d(u) = \varnothing$. By claim 62, for every $x \in N^-(u)$, $d^+(x) \geqslant k$. So u receives charge (at least α) through $k-1$ arcs and sends α through k arcs. Hence $w(u) \geqslant d(u) - \alpha \geqslant 2(k-1+\varepsilon)$.

- Let $u \in S$ such that u is in a cluster of size 1. So u does not send any charge. Claims 61 distinguishes two cases.
Assume first $N^d(u) = \varnothing$. Then by claim 62, for every $y \in N^-(u)$, either $d(y) \geqslant 2k$, or $d^+(y) \geqslant k$. In both cases y sends at least α to u. The same holds for the out-neighbours of u. So $w(u) = d(u) + 2(k-1)\alpha \geqslant 2(k-1+\varepsilon)$.

  Assume now $N^s(u) = \varnothing$. By claim 61, no neighbour of u has degree $2k-1$. If u is in a $(k-1)$-clique of G, by claim 63, every neighbour of u in this clique has degree at least $2k$, and hence sends charge to u. Hence $w(u) \geqslant d(u)+2(k-2)\alpha \geqslant 2(k-1+\varepsilon)$. Assume this is not the case. Let $v \in N(u)$. If $d(v) \geqslant 2k$, then v sends 2α to u. Otherwise, $v \in S$. Since u is not in a $(k-1)$-clique of G, by claim 65, v is in a cluster of size at least 2 and in a $(k-1)$-clique. Hence, by the third rule, v sends 2α to u. Thus, $w(u) = d(u) + 2(k-1)\alpha \geqslant 2(k-1+\varepsilon)$.

- Let $u \in S$ such that u is in a cluster C of size $c \geqslant 2$. Note that by claim 61, $N^s(u) = \varnothing$.
If $\overleftrightarrow{K}_{k-1} \not\subseteq G[N[C]]$, then u does not send any charge and, by claim 66 1,

  it has $k-1+c$ neighbours of degree at least $2(k-1+c) \geqslant 2k$ and hence send charge towards u by rule 1:

$$w(u) \geqslant d(u) + 2(k-c)\frac{2(k-1+c) - 2(k-1+\varepsilon)}{2(k-1+c)}$$

  Otherwise, let $X \subset N[C]$ such that $G[X] = \overleftrightarrow{K}_{k-1}$ and $u \in X$. By claim 66 2, all vertices in $X - C$ have degree at least $2(k-1+c) \geqslant 2k$ and hence send charge towards u. Finally, u sends charge to at most one vertex (its neighbour that is not in X):

$$w(u) \geqslant d(u) + 2(k-1-c)\frac{2(k-1+c) - 2(k-1+\varepsilon)}{2(k-1+c)} - 2\alpha$$

  In both cases, $w(u) \geqslant 2(k-1) + 2(c-\varepsilon)\frac{k-1-c}{k-1+c} - 2\frac{\varepsilon}{k-2}$. We have:

$$\begin{aligned} w(u) \geqslant 2(k-1+\varepsilon) \quad &\Leftrightarrow \quad (k-2)(c-\varepsilon)(k-1-c) - \varepsilon(k-1+c) \\ &\qquad -(k-2)(k-1+c)\varepsilon \geqslant 0 \\ &\Leftrightarrow \quad (2(k-1)(k-2) + k-1+c)\varepsilon \leqslant (k-2)c(k-1-c) \end{aligned}$$

  The first expression is concave in c, so by claim 64, we only have to check it for $c \in \{2, k-3\}$. Since $\varepsilon < \frac{1}{2} - \frac{1}{k-1}$, we only need to check $(2(k-1)(k-2)+k-1+c)(\frac{1}{2} - \frac{1}{k-1}) \leqslant (k-2)c(k-1-c)$. For $c = 2$, we obtain: $(k-3)(2k^2-7k+7) \geqslant 0$, which is true since the degree 2 polynomial has discriminant -7 and hence is always positive. For $c = k-3$, we obtain $(k-3)(2k^2-8k+7) \geqslant 0$, which is true since the largest root of the polynomial of degree 2 is $2 + \frac{1}{\sqrt{2}}$.

Hence $|A(G)| = \frac{1}{2}\sum_{u \in G} d(u) = \frac{1}{2}\sum_{u\in G} w(u) \geqslant (k-1+\varepsilon)|V(G)|$, i.e. $\rho(G) \leqslant 0$, a contradiction. □

6 Generalisation of a result of Stiebitz

The goal of this section is to prove Theorem 8.

Recall that $\pi_0(G)$ denotes the set of connected components of G. We are actually going to prove the following stronger statement:

Theorem 67. *Let G be a connected digraph, $k \geqslant 3$ and $X \subseteq V(G)$ such that:*

- $\forall u \in X, d(u) \leqslant 2(k-1)$.
- $\forall S \in \pi_0(G[X]), \vec{\chi}(G-S) \leqslant k-1$
- $|\pi_0(G-X)| > |\pi_0(G[X])|$

Then $\vec{\chi}(G) \leqslant k-1$.

We will need the following definition.

Definition 68. For G a digraph, $X \subseteq V(G)$ and P a partition of $\pi_0(G-X)$, we define the following (undirected) bipartite graph:

$$B(G,X,P) = (\pi_0(G[X]) + P, \{ST | S \in \pi_0(G[X]), T \in P, \overleftrightarrow{A}(S, \bigcup_{C \in T} C) \neq \varnothing\}).$$

Let B be a bipartite graph with partite sets U and V. A 2-forest of B with respect to U is a spanning forest of B in which every vertex in U has degree 2.

The following remark describes a method to extend the dicolouring of a partially dicoloured digraph that will be used a lot during the proof.

Remark 69. Let G be a digraph, $H \subseteq G$ connected, $x \in V(H)$ and ϕ a $(k-1)$-dicolouring of $G-H$. Assume that, for every $u \in V(H), d_G(u) \leqslant 2(k-1)$. Then, given the reverse ordering of a BFS of the underlying graph of H starting in x, ϕ can be be greedily extended to $G-x$ (because, when colouring $u \in V(H)$, u is incident with at most $2k-3$ arcs incident with an already coloured vertex).
Moreover, if $\phi(N^+(x)) \neq [1, k-1]$ or $\phi(N^-(x)) \neq [k-1]$, then ϕ can be extended to G.

The next Lemma is a strong version of Theorem 67 in the case where $|\pi_0(G[X])| = 1$.

Lemma 70. *Let G be a connected digraph and $X \subseteq V(G)$ such that:*

- $\forall u \in X, d(u) \leqslant 2(k-1)$
- *$G[X]$ is connected*
- *$G-X$ is disconnected*

Then, for any $(k-1)$-dicolouring ϕ of $G-X$, there is a $(k-1)$-dicolouring ψ of G so that

$$\forall C \in \pi_0(G-X), \exists \sigma \in \mathfrak{S}_{k-1}, \phi_{|C} = \sigma \circ \psi_{|C}.$$

Proof. We proceed by induction on $|X|$. The result is trivial when $X = \varnothing$. Let ϕ be a $(k-1)$-dicolouring of $G - X$. Let $x \in X$ such that $G[X - x]$ is connected (any leaf on a spanning tree of $G[X]$ suits).

Assume first that $G-(X-x)$ is disconnected. Let $S \in \pi_0(G-(X-x))$ such that $x \in S$. Since G is connected, $X - x \neq \varnothing$ and hence, since $G[X]$ is connected, $d_{G[S]}(x) < 2(k-1)$. So we can extend ϕ to $G - (X - x)$ and then apply induction on $X - x$.

Assume now that $G - (X - x)$ is connected. So, for all $S \in \pi_0(G - X), S \cap N(x) \neq \varnothing$. Let $S_0 \neq S_1 \in \pi_0(G - X)$. We can permute colours in S_0 and in S_1 so that x has neighbours in both S_0 and S_1 with the same colour, say 1. Call ψ the obtained colouring. Now, greedily extend ψ to $G - x$ as in Remark 69. We may assume that $\psi(N^+(x)) = [k-1]$ or $\psi(N^-(x)) = [k-1]$. Since $d(x) \leqslant 2(k-1)$, we have $1 \notin \psi(N(x) \cap X)$. Set $\psi(x) = 1$. We may assume that ψ is not a dicolouring of G. So we have an induced cycle C containing x. x has exactly two neighbours with colour 1, one in S_0, the other in S_1. Hence $V(C) \cap S_0 \neq \varnothing$ and $V(C) \cap S_1 \neq \varnothing$. Since $G - X$ is disconnected, $V(C) \cap (X - x) \neq \varnothing$. Let y be the last vertex of $V(C) \cap (X - x)$ to be coloured. Since the neighbours of x in C are not in X, the neighbours of y in $V(C)$ were coloured when colouring y. Since we extended ψ greedily, $\psi(y) \neq 1$, a contradiction. □

We need the following technical lemma on (undirected) bipartite graphs.

Lemma 71 (Lemma 3.6 in [Sti82])**.** *Let B be a bipartite graph with partite sets S and T, such that $|T| = |S| + 1$ and B contains a 2-forest with respect to S. There exists $s \in S$ such that for every $t, t' \in N(s)$, B contains a 2-forest with respect to S containing st and st'.*

The next lemma is again a strong version of Theorem 67 in a particular case.

Lemma 72. *Let G be a connected digraph, $X \subseteq V(G)$, $n = |\pi_0(G[X])|$ and $P = (P_0, \ldots, P_n)$ a partition of $\pi_0(G - X)$ such that:*

- *$\forall u \in X, d(u) \leqslant 2(k-1)$.*
- *$B(G, X, P)$ contains a 2-forest with respect to $\pi_0(G[X])$.*

Then, for any $(k-1)$-dicolouring ϕ of $G - X$, there is a $(k-1)$-dicolouring ψ of G so that

$$\forall 0 \leqslant i \leqslant n, \exists \sigma \in \mathfrak{S}_{k-1}, \phi_{|\bigcup_{C \in P_i} C} = \sigma \circ \psi_{|\bigcup_{C \in P_i} C}.$$

Proof. We show the claim by induction on $|X|$.

By Lemma 70, we may assume $G[X]$ disconnected. Set $B = B(G, X, P)$. Let ϕ be a $(k-1)$-dicolouring of $G - X$. By Lemma 71, we have $S \in \pi_0(G[X])$ such that, for any $0 \leqslant i \neq j \leqslant n$ such that $SP_i, SP_j \in E(B)$, B contains a 2-forest with respect to $\pi_0(G[X])$ containing SP_i and SP_j. Let s be a non-separating vertex of $G[S]$. We distinguish two cases:

Assume first that $|\{0 \leqslant i \leqslant n, \overleftrightarrow{A}(s, \bigcup_{C \in P_i} C) \neq \varnothing\}| \leqslant 1$. Since B contains a 2-forest with respect to $\pi_0(G[X])$, $d_B(S) \geqslant 2$ and thus there is a vertex in $S \setminus s$ that has a neighbour in X. In particular, $|S| \geqslant 2$. Since S is connected, $d_{G-S+s}(s) < 2(k-1)$, so we can extend greedily ϕ to $G-(X-s)$. Since s is non-separating in $G[S]$, $|\pi_0(G[X-s])| = |\pi_0(G[X])|$. If $N(s) \subseteq S$, then $\{s\}$ is a connected component of $G-X+s$, and we set $P' = (P_0 + \{s\}, P_1, \ldots, P_n)$. Otherwise, let $C \in \pi_0(G-(X-s))$ such that $s \in V(C)$. Up to reindexing P, we may assume that $C - s \subset \bigcup_{C' \in P_0} C'$ and set $P' = (\{C' \in P_0, C' \cap C = \varnothing\} + C, P_1, \ldots, P_n)$. Now, $B(G,X,P)$ is isomorphic to a spanning subdigraph of $B(G, X-s, P')$ and hence $B(G, X-s, P')$ contains a 2-forest. We conclude by induction.

Assume now that $|\{0 \leqslant i \leqslant n, \overleftrightarrow{A}(s, \bigcup_{C \in P_i} C) \neq \varnothing\}| \geqslant 2$.

Up to reindexing P, we may assume $\overleftrightarrow{A}(s, \bigcup_{C \in P_0} C) \neq \varnothing$ and $\overleftrightarrow{A}(s, \bigcup_{C \in P_1} C) \neq \varnothing$. Let $u_0 \in N(s) \cap \bigcup_{C \in P_0} C$ and $u_1 \in N(s) \cap \bigcup_{C \in P_1} C$ and $C_0, C_1 \in \pi_0(G-X)$ containing u_0 and u_1 respectively. By directional duality, we may assume $u_0 \in N^+(s)$.

Up to permuting colours in C_0 and C_1, we may assume $\phi(u_0) = \phi(u_1) = 1$. Let $G' = G \cup u_0u_1 - S$ and $X' = X - S$. Note that $C_0 + C_1$ is a connected component of G'. We set $P' = (P_0 - C_0 + P_1 - C_1 + (C_0 + C_1), P_2, \ldots, P_n)$. Note that ϕ is a dicolouring of $G' - X'$ and P' is a partition of $\pi_0(G' - X')$. As $B(G', X', P') = B(G, X, P) - S/\{P_0, P_1\}$, the 2-forest in $B(G,X,P)$ containing SP_0 and SP_1 yields a 2-forest in $B(G', X', P')$. Hence, by induction hypothesis, we may turn ϕ into a dicolouring ψ of G' with the properties of the output of the theorem.

Note that ψ is a dicolouring of $G-S$. We extend ψ to $G-s$ as in remark 69, and we may assume that $\psi(N^-(s)) = [k-1]$ and $\psi(N^+(s)) = [k-1]$.

Set $\psi(s) = 1 = \psi(u_0) = \psi(u_1)$. Since $\psi(N^+(s)) = [k-1]$ and $u_0 \in N^+(s)$, we have that $u_1 \in N^-(s)$.

We may assume that there is a monochromatic induced cycle R containing s (otherwise we are done). Observe that s has exactly two neighbours with colour 1, namely u_0 and u_1, so R contains $u_1 s u_0$. Since ψ is also a dicolouring of G' and $u_0u_1 \in A(G')$, there is no monochromatic walk from u_1 to u_0 in $G-S$. So there is a vertex $y \in V(R) \cap (V(S) - s)$. Assume y is the last vertex in $V(R) \cap (V(S) - s)$ to be coloured. Since the neighbours of s in R are not in S, the neighbours of y in R were coloured when colouring y. Since we extended ψ greedily, $\psi(y) \neq 1$, a contradiction. □

We need a second technical lemma on (undirected) bipartite graphs before concluding.

Lemma 73 (Lemmas 3.4 and 3.5 in [Sti82])**.** *Let B be a bipartite graph with partite sets S and T such that $|T| \geqslant |S| + 1$ and, for any $S' \in \mathcal{P}(S) - \{\varnothing, S\}$, $|\pi_0(B - S')| \leqslant |S'|$. Let $s \in S$ and $t \neq t' \in T$. Then B contains a 2-forest with respect to S which contains st and st'.*

Proof of Theorem 67. We prove the result by induction on $|X|$. By claim 70, we may assume $G[X]$ disconnected. By induction hypothesis, we may assume that, for any $P \in \mathcal{P}(\pi_0(G[X]) - \{\varnothing, \pi_0(G[X])\}$, we have $|\pi_0(G - \bigcup_{C \in P} C)| \leqslant |P|$, since if not we can apply induction on $\bigcup_{C \in P} V(C)$. Let $P = (P_0, \dots, P_{|\pi_0(G[X])|})$ be a partition of $\pi_0(G - X)$. By Lemma 73, $B(G, X, P)$ has a 2-forest. Since $\vec{\chi}(G - X) \leqslant k - 1$, by Lemma 72, $\vec{\chi}(G) \leqslant k - 1$. □

7 List-dicolouring

Let G be a digraph. A *list assignment* of G is a mapping $L : V(G) \to \mathcal{P}(C)$, where C is a set of colours. An L-*dicolouring* of G is a dicolouring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G admits an L-dicolouring, then it is L-*dicolourable.* If H is a subgraph of G, we abuse notations and write L for the restriction of L to H. Recall that, given a vertex x of a digraph, $d_{max}(x) = \max(d^+(x), d^-(x))$ and $d_{min}(x) = \min(d^+(x), d^-(x))$.

In [HM11], Mohar and Harutyunyan proved the following, generalising a fundamental result of Gallai [Gal63a].

Theorem 74 (Theorem 2.1 in [HM11])**.** *Let G be a connected digraph, and L a list-assignment for G such that $|L(v)| \geqslant d_{max}(v)$ for every $v \in V(G)$. If D is not L-dicolourable, then $d^+(v) = d^-(v)$ for every $v \in V(G)$ and every block of G is a cycle, a symmetric odd cycle, or a complete digraph.*

Observe that, in the above theorem, the blocks can not be arcs, so the output is a particular type of directed Gallai forest. Later on, Bang-Jensen et al. generalised the result of Mohar and Harutyunyan by proving Theorem 17 that we restate here for convenience.

Theorem 75 (Bang-Jensen, Bellitto, Schweser and Stiebitz [BBSS20])**.** *If G is a k-dicritical digraph, then the subdigraph induced by vertices of degree $2(k - 1)$ is a directed Gallai forest.*

Interestingly, contrary to the directed case, the undirected analogues of the two previous results both output an (undirected) Gallai forest, that is a graph whose blocks are odd (undirected) cycles or complete graphs.

The goal of this section is to generalise the result of Bang-Jensen et al. by generalising a theorem proved by Thomassen [Tho97] in the undirected case.

Theorem 76. *Let G be a connected digraph, $X \subseteq V(G)$ connected and L a list-assignment of G such that $G - X$ is L-dicolourable, G is not L-dicolourable and $\forall x \in X, |L(x)| \geqslant d_{\max}(x)$. Then $G[X]$ is a directed Gallai forest.*

The proof of Theorem 76 is almost the same as the proof of Theorem 75.

The next proposition states some easy yet important facts that will be often used during the proof.

Proposition 77. *Let G be a connected digraph, $X \subseteq V(G)$ connected and L a list-assignment of G such that $G - X$ is L-dicolourable, G is not L-dicolourable and $\forall x \in X, |L(x)| \geqslant d_{\max}(x)$.*

Then, for every $x \in X$, the following statements hold:

1. $|L(x)| = d^+(x) = d^-(x)$,

2. $G - x$ *is* L*-dicolourable.*

3. *For every* L*-dicolouring of* $G - x$*, every colour of* $L(x)$ *appears in both* $N^+(x)$ *and* $N^-(x)$*.*

4. *Given an* L*-dicolouring* ϕ *of* $G - x$ *and* $y \in X \cap N(x)$*, uncolouring* y *and colouring* x *with the colour of* y *yields an* L*-dicolouring of* $G - y$*.*

Proof. Let $x \in X$.

To prove 1, it suffices to show that $|L(x)| \leqslant d_{\min}(x)$. We prove it for any G, X, L and x by induction on $|V(G)|$. If $|V(G)| \leqslant 2$, the result is clear, so assume $|V(G)| \geqslant 3$ Assume towards a contradiction that $|L(x)| > d_{\min}(x)$. Let $G' = G - x$. We can greedily extend any L-dicolouring of G' to an L-dicolouring of G, so G' is not L-dicolourable. Hence G' has a connected component C' that is not L-dicolourable. Since $G - X$ is L-dicolourable, $C' \cap X \neq \varnothing$. Furthermore, since X is connected, we have $y \in C' \cap X \cap N(x)$. By the induction hypothesis applied to $G[C']$, $C' \cap X$ and L, we have $|L(y)| = d^+_{G[C']}(y) = d^-_{G[C']}(y)$. By directional duality, we may assume $x \in N^+(y)$. Then: $d^+_{G[C']}(y) = |L(y)| \geqslant d^+(y) \geqslant d^+_{G[C']}(y) + 1$, a contradiction. This proves the first statement.

We now prove 2. It suffices to prove that every connected component of $G - x$ is L-dicolourable. Let $C \in \pi_0(G - x)$. Let $D_1, \dots, D_n$ be the connected components of $G[C \cap X]$. We prove by induction on $i \in [\![0, n]\!]$ that $G[C - X + D_1 + \cdots + D_i]$ is L-dicolourable. Since $G - X$ is L-dicolourable, $G[C - X]$ is too. Now, let $i \in [\![0, n-1]\!]$ and assume $G[C - X + D_1 + \cdots + D_i]$ L-dicolourable. Since X is connected, we have $y \in D_{i+1} \cap N(x)$. We have $|L(y)| = d^+(y) = d^-(y) > d_{\min,G[C]}(y)$, so the first statement applied to $G[C - X + D_1 + \cdots + D_i]$, D_{i+1}, L and y yields that $G[C - X + D_1 + \cdots + D_i]$ is L-dicolourable, which concludes the proof.

Statement 3 follows easily from the fact that G is not L-dicolourable.

For the proof of 4, assume (by symmetry) that $xy \in A(G)$. It follows from the third statement that, after uncolouring y, x has no out-neighbour coloured $\phi(y)$, and thus giving colour $\phi(y)$ to x does not create a monochromatic cycle. □

In the rest of the proof, we will call the procedure that is described in Proposition 77 4 *shifting* the colour from y to x, and sometimes write briefly $y \to x$. Moreover, given G, X and L as in the statement of Proposition 77, a weak cycle $C = (v_1, a_1, v_2, \dots v_k, a_k, v_1)$ in $G[X]$ and an L-colouring of $G - v_1$, we can shift each vertex of C one after another, starting with $v_k \to v_1$ and get a new L-dicolouring of $G - v$. We say that we *clockwise shift colours around* C, see Figure 6. Starting with $v_2 \to v_1$, we say that we *counter-clockwise shift colours around* C

Lemma 78. *Let* G *be a connected digraph,* $X \subseteq V(G)$ *connected and* L *a list-assignment for* G *such that* $G - X$ *is* L*-dicolourable,* G *is not* L*-dicolourable and* $\forall x \in X, |L(x)| \geqslant d_{\max}(x)$*. Let* C *be a weak cycle in* $G[X]$ *of length* $k \geqslant 3$ *that is not a cycle. Then* $V(C)$ *is either a clique or induces an odd symmetric cycle.*

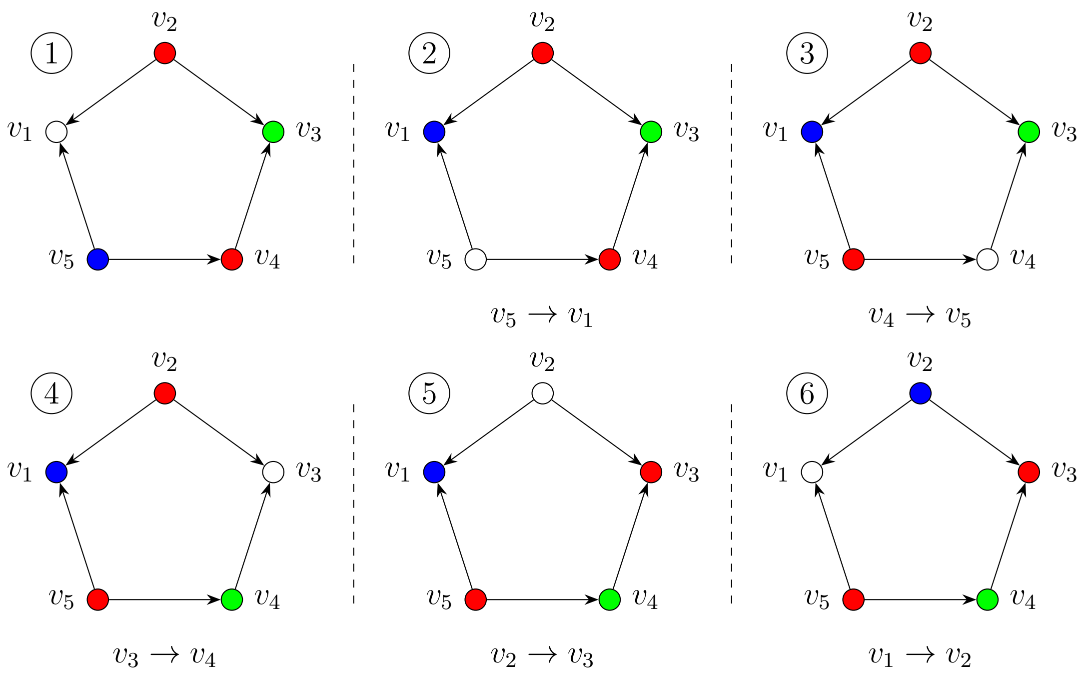

Figure 6: The white vertex denotes the uncoloured vertex during the clockwise shifting around the weak cycle.

Proof. Write $C = (v_1, a_1, v_2 \dots, v_k, a_k, v_1)$. We prove the result by induction on k. All along the proof, subscripts are taken modulo k. In particular, v_k and v_1 are considered to be consecutive vertices of C.

Claim 79. *For every $i \in [k]$ and any L-dicolouring ϕ of $G - v_i$, no two consecutive vertices of C receive the same colour. Moreover, $\phi(v_{i-1}) \neq \phi(v_{i+1})$.*

Proof of claim. Let $i \in [k]$ and let L be an L-dicolouring of $G - v_i$. Assume towards a contradiction that two consecutive vertices in C have the same colour. Since C is not a cycle of G, there exists $j \in [k]$ such that v_{j-1} and v_{j+1} are both in-neighbours of v_j or both out-neighbours of v_j. We may shift colours around C until v_j is left uncoloured and v_{j-1} and v_{j+1} have the same colour, a contradiction to Proposition 77 3. Now, if $\phi(v_{i-1}) = \phi(v_{i+1})$, we can simply shift the colour from v_{i-1} to v_i and get a contradiction with the first fact. ♢

By Proposition 77 2, we have an L-dicolouring ϕ of $G - v_1$.

First suppose that k is odd. Up to shifting colours and renaming the vertices, we may assume that $a_k = v_k v_1$ and $a_1 = v_1 v_2$. We consider two cases.

Assume first that there is an arc $a \in A(G)$ between v_1 and v_i for some $2 < i < k$. Let $C_0 = (v_1, a_1, v_2, \dots, v_i, a, v_1)$ and $C_1 = (v_1, a, v_i, a_i, v_{i+1}, \dots v_k, a_k, v_1)$. One of C_0 and C_1 is not a cycle and hence, by induction, $v_1 v_i v_1 \subseteq A(G)$. By symmetry, we may assume that C_0 is even and C_1 odd. Choosing the appropriate arc between v_1 and v_i makes C_0 acyclic and hence, by induction, $V(C_0)$ is a clique. Similarly, $V(C_1)$ induces a symmetric cycle or is a clique. For $j \in [\![2, i-1]\!]$, let $C_j = (v_1, v_j v_1, v_j, v_i v_j, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$. Since C_1 is odd, C_j is even, so by induction, $V(C_j)$ is a clique. Hence $V(C)$ is a clique.

Now, suppose there is no arc between v_1 and v_i for $i \in [\![3, k-1]\!]$. By claim 79, $\phi(v_k) \neq \phi(v_2)$.

If the (unique) out-neighbour of v_1 with colour $\phi(v_k)$ is not v_k, then we shift colours clockwise around C and get two out-neighbours of v_1 with the same colour, a contradiction to Proposition 77 3.

Thus, $v_1 v_k \in A(G)$. Similarly, $v_2 v_1 \in A(G)$. Hence, we have either $v_1 v_2 v_3 \subseteq A(G)$ or $v_3 v_2 v_1 \subseteq A(G)$, so we can repeat the argument and get a digon between v_2 and v_3. This way, we get that there is a digon between each pair of consecutive vertices of C and thus $G[C]$ is a symmetric odd cycle.

Suppose now that k is even. Up to shifting colours and renaming the vertices, we may assume that $a_k = v_k v_1$ and $a_1 = v_2 v_1$. By claim 79, $\phi(v_k) \neq \phi(v_2)$ and $|\{\phi(v_i), 2 \leqslant i \leqslant k\}| \geqslant 3$.

Let $3 \leqslant j \leqslant k-1$ such that $\phi(v_j) \notin \{\phi(v_2), \phi(v_{k-1})\}$. We shift colours around C until v_2 is coloured $\phi(v_j)$. By Proposition 77 3, $\phi(v_2)$ and $\phi(v_k)$ still appear in the in-neighbourhood of v_1 and thus we have $3 \leqslant i \leqslant k-1$ such that $v_i v_1 \in A(G)$.

Assume first that i is even. Then both the weak paths $(v_1, a_1, v_2, \ldots, v_i, v_i v_1, v_1)$ and $(v_1, v_i v_1, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$ are even and are not a cycle, so by induction, $\{v_1, v_2 \ldots, v_i\}$ and $\{v_i, v_{i+1}, \ldots, v_k\}$ are cliques. Hence $(v_1, v_1 v_3, v_3, \ldots, v_k, a_k, v_1)$ is odd, and $\{v_1, v_3, v_4 \ldots, v_k\}$ does not induce a symmetric odd cycle (because v_3 and v_k are adjacent). So, by induction, $\{v_1, v_3, v_4, \ldots, v_k\}$ is a clique. The same holds for $(v_1, a_1, v_2, \ldots, v_{i-2}, v_{i-2} v_i, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$, so $V(C)$ is a clique.

Assume now that i is odd. In particular, we have that $(v_1, a_1, v_2, \ldots, v_i, v_i v_1, v_1)$ and $(v_1, v_i v_1, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$ are odd cycles and thus, by induction, each pair of consecutive vertices of C induces a digon and $v_1 v_i v_1 \subseteq G$. If $k = 4$, then the argument of the paragraph following the assumption that k is even finds a digon between v_2 and v_4. So we may assume $k \geqslant 6$.

Assume that both $\{v_1, v_2, \ldots, v_i\}$ and $\{v_1, v_i, v_{i+1}, \ldots, v_k\}$ induce a symmetric cycle. Since $k \geqslant 6$, one of $(v_1, a_1, v_2, \ldots, v_i, v_i v_1, v_1)$ and $(v_1, v_i v_1, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$ has length at least 5. Assume without loss of generality that it is $(v_1, a_1, v_2, \ldots, v_i, v_i v_1, v_1)$ (so $i \geqslant 5$). Counter-clockwise shifting colours around $(v_1, a_1, v_2, \ldots, v_i, v_i v_1, v_1)$, and noticing that in the new L-dicolouring of $G - v_1$, the in-neighbours of v_1 have the same colours as in the previous one, we get that $\phi(v_3) = \phi(v_i)$. Now, counter-clockwise shifting colours (of ϕ) around $(v_1, v_i v_1, v_i, a_i, v_{i+1}, \ldots, v_k, a_k, v_1)$, the same argument yields $\phi(v_3) = \phi(v_{i+1})$. So $\phi(v_i) = \phi(v_{i+1})$, a contradiction to claim 79.

Hence, we may assume without loss of generality that $\{v_1, v_2, \ldots, v_i\}$ does not induce a symmetric odd cycle. In particular $i \geqslant 5$. By induction, $\{v_1, v_2, \ldots, v_i\}$ induces a clique. By applying induction to $(v_1, v_3 v_1, v_3, a_3, v_4, \ldots, v_k, a_k, v_1)$, we get that $V(C) - v_2$ is a clique. Finally, by applying induction on $(v_1, a_1, v_2, v_4 v_2, v_4, \ldots, v_k, a_k, v_1)$, we get that $V(C) - v_3$ is a clique and thus that $V(C)$ is a clique. □

Proof of Theorem 76. Let B be a block of $G[X]$. If $|B| \leqslant 3$, B is either an simple arc, a digon, a $\vec{C}_3$, or a $\overset{\leftrightarrow}{K}_3$ by Lemma 78. So we assume $|V(B)| \geqslant 4$. By Lemma 78, we may assume that B is not a cycle. So there are two vertices in $V(B)$ linked by three internally vertex-disjoint weak walks. Call P_0, P_1, P_2 these three weak walks. Two of these walks form a weak cycle that is not a cycle. Hence by Lemma 78, they form a symmetric cycle. Two of P_0, P_1 and P_2, say P_0 and P_1, form a weak cycle C of even length. One of P_0 and P_1 is symmetric, so up to choosing the arcs in it, C is not a cycle. By Lemma 78, $V(C)$ is a clique and observe that $|V(C)| \geqslant 4$. Let R be a maximal clique containing $V(C)$. We may assume $R \neq V(B)$. Let $v \in V(B) - R$. Since B is a block, there are two weak walks P and Q from v to R whose only common vertex is v. Let p and q their respective end-vertices in R. Let $w \in R$. We have $z \in R - p - q - w$. One of $vPpwqQv$ or $vPpwzqQv$ is odd, and both can be chosen undirected and none of them is an induced cycle (because p and q are adjacent). Hence, by Lemma 78, the vertices of one of them induce a clique, and thus w is linked by digon to $V(P) \cup V(Q)$. So $R \cup V(P) \cup V(Q)$ is a clique, a contradiction to the maximality of R. $\square$

8 Conclusion

For $k \geqslant 4$, let $\mathcal{F}_k = \{\overset{\leftrightarrow}{C}_5(\overset{\leftrightarrow}{K}_1, \overset{\leftrightarrow}{K}_{a_1}, \overset{\leftrightarrow}{K}_{a_2}, \overset{\leftrightarrow}{K}_{b_2}, \overset{\leftrightarrow}{K}_{b_1}) \mid a_1+a_2 = b_1+b_2 = k-1, a_2+b_2 = k-1\}$. Recall that we identify (undirected) graphs with symmetric digraphs. Kostochka and Stiebitz [KS99] proved that, for $k \geqslant 4$, k-critical graphs have excess at least $2(k-3)$ except for K_k and graphs in $\mathcal{F}_k$. It is natural to wonder if such a characterisation exists for digraphs. Anyway, k-dicritical digraphs being more complicated that their undirected counter part, we doubt it.

Our bound on the minimal number of arcs in a k-dicritical digraph on n vertices (Theorem 10) is clearly not tight. Kostochka and Stiebitz conjectured [KS20] that k-dicritical digraphs on at least $k+1$ vertices with minimum density are symmetric, i.e. are the same as in the case of undirected graphs. It is to be noted that in an other breakthrough result, Kostochka and Yancey characterised the k-critical graphs that are tight for the bound of Theorem 9.

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