Spectral Extremal Problem on Disjoint Color-Critical Graphs

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Abstract

For a given graph F, we say that a graph G is F*-free* if it does not contain F as a subgraph. A graph is *color-critical* if it contains an edge whose deletion reduces its chromatic number. Let $K_r^+(a_1, a_2, \ldots, a_r)$ be the graph obtained from complete r-partite graph with parts of sizes $a_1 \geqslant 2, a_2, \ldots, a_r$, by adding an edge to the first part. In this paper, we focus on the spectral extrema of disjoint color-critical graphs. For fixed $t, a_1, \ldots, a_r$ ($r \geqslant 2$) and large enough n, we characterize the unique n-vertex $tK_r^+(a_1, \ldots, a_r)$-free graph having the largest spectral radius. Moreover, let $F_1, \ldots, F_t$ be t disjoint color-critical graphs with the same chromatic number. We identify the unique n-vertex $\bigcup_{i=1}^t F_i$-free graph having the largest spectral radius for sufficiently large n. Consequently, we generalize the main results obtained by Ni, Wang and Kang [Electron. J. Combin. 30 (1) (2023), No. 1.20] and by Fang, Zhai and Lin [arXiv:2302.03229v2].

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1 Introduction

In this paper, all the graphs that we consider are simple and undirected. Let $G = (V(G), E(G))$ be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The number $n(G) := |V(G)|$ and $e(G) := |E(G)|$ are called the *order* and *size* of G, respectively. Unless otherwise stated, we follow the traditional notation and terminology (see for example [2, 5]).

For two vertex disjoint graphs G and H, the *union* of G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. In particular, we write tG the vertex-disjoint union of t copies of G. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding edges joining every vertex of G to every vertex of H. Let

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$K_r(a_1, a_2, \dots, a_r)$ be the complete r-partite graph with parts of size $a_1, a_2, \dots, a_r$. An *r-partite Turán graph* $T_{n,r}$ is a complete r-partite graph $K_r(n_1, n_2, \dots, n_r)$ where $\sum_{i=1}^{r} n_i = n$ and $\lfloor \frac{n}{r} \rfloor \leqslant n_i \leqslant \lceil \frac{n}{r} \rceil$ for all $1 \leqslant i \leqslant r$.

For a given graph F, we say that a graph G is *F-free* if it does not contain F as a subgraph. The maximum size of an F-free graph of order n is known as the *Turán number of F*, and it is usually denoted by $\mathrm{ex}(n, F)$. An F-free graph is said to be *extremal with respect to* $\mathrm{ex}(n, F)$, if it has n vertices and $\mathrm{ex}(n, F)$ edges. Let $\mathrm{Ex}(n, F)$ denote the set of all extremal graphs with respect to $\mathrm{ex}(n, F)$. The problem of determining $\mathrm{ex}(n, F)$ is usually called the *Turán-type extremal problem.* The well-known Turán Theorem shows that $T_{n,r}$ is the extremal graph corresponding to $\mathrm{ex}(n, K_{r+1})$. Later, Moon [22] and Simonovits [31] showed that $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph corresponding to $\mathrm{ex}(n, tK_{r+1})$ for sufficiently large n. Recently, Fang et al. [15] determined the unique extremal graph with respect to $\mathrm{ex}(n, tC_{2r+1})$. For more advances on this topic, we refer the readers to [4, 13, 16, 18, 33].

A graph is said to be properly coloured if each vertex is coloured so that adjacent vertices have different colours. If G can be properly coloured by k colours, then we say G is *k-colourable.* The *chromatic number* $\chi(G)$ is k if G is k-colourable and not $(k-1)$-colourable. We say that $e \in E(G)$ is a *color-critical edge* of G if $\chi(G - e) < \chi(G)$. A graph G is *color-critical* if G contains a color-critical edge. It is easy to see that both the complete graph K_{r+1} and odd cycle C_{2r+1} are color-critical graphs. The following result was obtained by Simonovits [32].

Theorem 1 ([32])**.** *Let $F_1, \dots, F_t$ be t disjoint color-critical graphs with $\chi(F_i) = r+1$ $(r \geqslant 2)$. Then $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}(n, \bigcup_{i=1}^{t} F_i)$ for sufficiently large n.*

Analog to the Turán type problem, Nikiforov [28] proposed the *spectral Turán type problem*: Given a graph H, what's the maximal spectral radius of an F-free graph with order n? We denote the maximal spectral radius of an n-vertex F-free graph by $\mathrm{ex}_{sp}(n, F)$. An F-free graph on n vertices with maximum spectral radius is called *an extremal graph with respect to* $\mathrm{ex}_{sp}(n, F)$. Let $\mathrm{Ex}_{sp}(n, F)$ denote the set of all extremal graphs with respect to $\mathrm{ex}_{sp}(n, F)$. This problem was also studied extensively: see complete graph [3, 34], odd cycle [24], 4-cycle [15, 23, 36], book graph [35], odd wheel graph [10], theta graph [35], complete bipartite graph [1, 27], path [28], friendship graph [11] and some nice surveys [8, 17, 19, 20, 21, 29]. Clearly, book graph, theta graph, complete graph and odd cycle are all color-critical graphs. The following spectral version of Turán type problem involving color-critical graphs was obtained by Nikiforov (see [25, Theorem 2]).

Theorem 2 ([25])**.** *If F is a color-critical graph with $\chi(F) = r + 1$ $(r \geqslant 2)$, then $T_{n,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, F)$.*

In this article, we focus on studying $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} F_i)$ for some given graphs $F_1, \dots, F_t$. This topic was studied for some special cases: matching [14], complete graph [30], star [7], path [6], odd cycle [15], even cycle [15]. Let $K_r^+(a_1, a_2, \dots, a_r)$ be the graph obtained from complete r-partite graph with parts of sizes $a_1 \geqslant 2, a_2, \dots, a_r$, by adding an edge to the first part. We determine the extremal graph with respect to $\mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \dots, a_r))$.

Theorem 3. *Given some positive integers $t, a_1, \dots, a_r$ with $a_1, r \geqslant 2$. Then $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \dots, a_r))$ for sufficiently large n.*

Given two positive integers $t \geqslant 1, r \geqslant 2$. Let $K_r^+(a_{11}, a_{12}, \dots, a_{1r}), \dots, K_r^+(a_{t1}, a_{t2}, \dots, a_{tr})$ be t disjoint graphs. Then for all $1 \leqslant i \leqslant t$, there exist some integers $a_1, \dots, a_r$ such that $K_r^+(a_{i1}, a_{i2}, \dots, a_{ir})$ is a subgraph of $K_r^+(a_1, a_2, \dots, a_r)$. Hence, $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} K_r^+(a_{i1}, a_{i2}, \dots, a_{ir})) \leqslant \mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \dots, a_r))$. Note that $K_r^+(a_1, a_2, \dots, a_r)$ is a color-critical graph with chromatic number $r+1$. By Theorem 1, $K_{t-1} \vee T_{n-t+1,r}$ is $\bigcup_{i=1}^{t} K_r^+(a_{i1}, a_{i2}, \dots, a_{ir})$-free. Consequently, the next result follows immediately from Theorem 3.

Corollary 4. *Let $K_r^+(a_{11}, a_{12}, \dots, a_{1r}), \dots, K_r^+(a_{t1}, a_{t2}, \dots, a_{tr})$ be t disjoint graphs, where $t \geqslant 1, r \geqslant 2$ are two positive integers. For sufficiently large n, $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} K_r^+(a_{i1}, a_{i2}, \dots, a_{ir}))$.*

Clearly, for all color-critical graphs F with $\chi(F) = r+1$, there exist some integers $a'_1, \dots, a'_r$ such that F is a subgraph of $K_r^+(a'_1, a'_2, \dots, a'_r)$. By a similar discussion as that of Corollary 4, we obtain the following result, which is the spectral version of Theorem 1.

Theorem 5. *Let $F_1, \dots, F_t$ be t disjoint color-critical graphs satisfying $\chi(F_i) = r+1$, where $i = 1, \dots, t$, and $r \geqslant 2$ are positive integers. Then $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} F_i)$ for sufficiently large n.*

The remainder of this paper is organized as follows: In the next section, we give some notations, definitions and some important known results. In Section 3, we give the proofs of Theorems 3 and 5. In the last section, some concluding remarks are given.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is an $n \times n$ 0-1 matrix $A(G) = [a_{ij}]$ with $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The *spectral radius*, $\rho(G)$, of G has the maximum absolute value among all eigenvalues of $A(G)$. A subset M of $E(G)$ is called a *matching* if any two members of M are not adjacent in G. The *matching number* $\mu(G)$ is the maximal size of a matching in G. In a graph G, for a vertex subset $S \subseteq V(G)$, we denote the set of neighbours (resp. non-neighbors) of a vertex u in S by $N_S(u)$ (resp. $\overline{N}_S(u)$) and let $d_S(u) = |N_S(u)|$. If $S = V(G)$, for convenience, we denote $N_G(u) = N_{V(G)}(u)$

and $d_G(u) = d_{V(G)}(u)$. In particular, $\Delta(G) = \max\{d_G(u) | u \in V(G)\}$. The subgraph induced by S is denoted by $G[S]$. Define $E_G(A,B) = \{uv \in E(G) : u \in A, v \in B\}$, and let $e_G(A,B) = |E_G(A,B)|$. Let $G - v, G - uv$ denote the graph obtained from G by deleting a vertex $v \in V(G)$, or an edge $uv \in E(G)$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E(G)$.

In 2009, Nikiforov [26] obtained the following stability theorem.

Lemma 6 ([26]). *Let G be an n-vertex graph. For $r \geqslant 2$, $\frac{1}{\ln n} < c < r^{-8(r+21)(r+1)}$ and $0 < \varepsilon < 2^{-36} r^{-24}$, if $\rho(G) > \left(1 - \frac{1}{r} - \varepsilon\right) n$, then one of the following holds:*

- (i) *G contains a complete $r+1$ partite graph: $K_{r+1}(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$;*
- (ii) *G differs from $T_{n,r}$ in fewer than $(\varepsilon^{\frac{1}{4}} + c^{\frac{1}{8r+8}}) n^2$ edges.*

From Lemma 6, Desai et al. [12] obtained the following stability result.

Lemma 7 ([12]). *Let F be a graph with chromatic number $\chi(F) = r + 1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that if G is an F-free graph on $n \geqslant n_0$ vertices with $\rho(G) \geqslant \left(1 - \frac{1}{r} - \delta\right) n$, then G can be obtained from $T_{n,r}$ by adding and deleting at most εn^2 edges.*

Given two integers μ and Δ, let $f(\mu, \Delta) = \max\{e(G) \mid \mu(G) \leqslant \mu, \Delta(G) \leqslant \Delta\}$. In 1976, Chvátal and Hanson [9] obtained the following result.

Lemma 8 ([9]). *For every two integers $\mu \geqslant 1$ and $\Delta \geqslant 1$, we have*

$$f(\mu, \Delta) = \Delta\mu + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\mu}{\lceil \frac{\Delta}{2} \rceil} \right\rfloor \leqslant \mu(\Delta + 1).$$

The next lemma was given in [11].

Lemma 9 ([11]). *Let $V_1, \ldots, V_n$ be n finite sets. Then $|V_1 \cap \cdots \cap V_n| \geqslant \sum_{i=1}^{n} |V_i| - (n - 1)|\bigcup\limits_{i=1}^{n} V_i|$.*

The following lemma gives us upper and lower bounds on $e(T_{n,r})$.

Lemma 10. $\frac{(r-1)n^2}{2r} - \frac{r}{8} \leqslant e(T_{n,r}) \leqslant \frac{(r-1)n^2}{2r}$.

Proof. Let $n \equiv b \pmod r$, where $0 \leqslant b \leqslant r - 1$. Then we have

$$e(T_{n,r}) = \left(\frac{n-b}{r}\right)^2 \binom{r}{2} + b\left(n - 1 - \frac{n-b}{r}\right) - \binom{b}{2} = \frac{b^2 - rb + n^2 r - n^2}{2r}.$$

Let $g(x) = \frac{x^2 - rx + n^2 r - n^2}{2r}$. By a direct calculation, we have $g'(x) = \frac{2x-r}{2r}$. Thus, $g(x)$ is a monotonically decreasing function for $x \in [0, \frac{r}{2}]$ and it is a monotonically increasing function in x for $x \in [\frac{r}{2}, r-1]$. Note that $g(0) = \frac{r-1}{2r} n^2 > \frac{r-1}{2r}(n^2 - 1) = g(r-1)$. Then we have $g\left(\frac{r}{2}\right) \leqslant g(x) \leqslant g(0)$ for $x \in [0, r-1]$. Thus, $\frac{r-1}{2r} n^2 - \frac{r}{8} \leqslant e(T_{n,r}) \leqslant \frac{r-1}{2r} n^2$, as desired. □

3 The proofs of Theorems 3 and 5

In this section, we give the proof of Theorems 3 and 5. From Theorem 2, we know that Theorem 3 holds for $t=1$. In the following, we assume that $t \geqslant 2$.

Let G be in $\mathrm{Ex}_{sp}(n, tK_r^+(a_1,\dots,a_r))$ with $\sum_{i=1}^r a_i = h$. The next lemma establishes a lower bound on $\rho(G)$.

Lemma 11. $\rho(G) \geqslant \left(1-\frac{1}{r}\right)n + \frac{2t-2}{r} - \frac{(2t+r-2)^2}{4nr}$.

Proof. Recall $K_r^+(a_1,a_2,\dots,a_r)$ is a color-critical graph with chromatic number $r+1$. By Theorem 1, $K_{t-1}\vee T_{n-t+1,r}$ is $tK_r^+(a_1,\dots,a_r)$-free. Note that $G \in \mathrm{Ex}_{sp}(n, tK_r^+(a_1,\dots,a_r))$. Let $\mathbf{1}$ be the all-one vector. Then

$$\begin{aligned}
\rho(G) &\geqslant \rho\left(K_{t-1}\vee T_{n-t+1,r}\right) \\
&\geqslant \frac{\mathbf{1}^T A\left(K_{t-1}\vee T_{n-t+1,r}\right)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} \\
&= \frac{2e\left(K_{t-1}\vee T_{n-t+1,r}\right)}{n} \\
&\geqslant \frac{2}{n}\left(\binom{t-1}{2} + (t-1)(n-t+1) + \frac{r-1}{2r}(n-t+1)^2 - \frac{r}{8}\right) \\
&= \frac{r-1}{r}n + \frac{2t-2}{r} - \frac{(2t+r-2)^2}{4nr},
\end{aligned}$$

as desired. □

Lemma 12. *For a given positive constant $\xi < \frac{1}{8r^3h}$ and sufficiently large n, we have $e(G) \geqslant e\left(T_{n,r}\right) - \xi^2 n^2$. Moreover. G has a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that $\sum\limits_{1\leqslant i<j\leqslant r} e(V_i,V_j)$ attains the maximum, $\sum_{i=1}^r e(V_i) \leqslant \xi^2 n^2$ and $\left||V_i| - \frac{n}{r}\right| \leqslant 2\xi n$ for $1 \leqslant i \leqslant r$.*

Proof. Recall that $\chi(K_r^+(a_1,\dots,a_r)) = r+1$ and G is $tK_r^+(a_1,\dots,a_r)$-free. By Lemma 11, we have $\rho(G) \geqslant \left(1-\frac{1}{r}\right)n + \frac{2t-2}{r} - \frac{(2t+r-2)^2}{4nr}$. Then by Lemma 7, there exists a positive constant ξ such that $e(G) \geqslant e\left(T_{n,r}\right) - \xi^2 n^2$. Furthermore, there exists a vertex partition $V(G) = U_1 \cup \cdots \cup U_r$ with $\left\lfloor \frac{n}{r} \right\rfloor \leqslant |U_i| \leqslant \left\lceil \frac{n}{r} \right\rceil$ such that $\sum_{i=1}^r e\left(U_i\right) \leqslant \xi^2 n^2$. Choose a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that $\sum\limits_{1\leqslant i<j\leqslant r} e\left(V_i,V_j\right)$ attains the maximum. Then $\sum_{i=1}^r e\left(V_i\right) \leqslant \sum_{i=1}^r e\left(U_i\right) \leqslant \xi^2 n^2$.

Let $\max\limits_{1\leqslant i\leqslant r} ||V_i| - \frac{n}{r}| = a$. Without loss of generality assume that $||V_1| - \frac{n}{r}| = a$. Then we have

$$\begin{aligned}
e(G) &= \sum_{1\leqslant i<j\leqslant r} e\left(V_i,V_j\right) + \sum_{i=1}^r e\left(V_i\right) \\
&\leqslant \sum_{1\leqslant i<j\leqslant r} |V_i||V_j| + \xi^2 n^2
\end{aligned}$$

$$= |V_1|(n-|V_1|) + \sum_{2\leqslant i<j\leqslant r} |V_i||V_j| + \xi^2 n^2$$

$$= |V_1|(n-|V_1|) + \frac{1}{2}\left(\left(\sum_{i=2}^{r}|V_i|\right)^2 - \sum_{i=2}^{r}|V_i|^2\right) + \xi^2 n^2.$$

By Hölder's inequality, we have $\left(\sum_{i=2}^{r}|V_i|\right)^2 \leqslant (r-1)\sum_{i=2}^{r}|V_i|^2$. Together with $||V_1|-\frac{n}{r}| = a$, we get

$$\begin{aligned} e(G) &\leqslant |V_1|(n-|V_1|) + \frac{1}{2}(n-|V_1|)^2 - \frac{1}{2(r-1)}(n-|V_1|)^2 + \xi^2 n^2 \\ &\leqslant \frac{r-1}{2r}n^2 - \frac{r}{2r-2}a^2 + \xi^2 n^2. \end{aligned} \tag{3.1}$$

Recall that

$$e(G) \geqslant e\left(T_{n,r}\right) - \xi^2 n^2 \geqslant \frac{r-1}{2r}n^2 - \frac{r}{8} - \xi^2 n^2.$$

Together with (3.1), we have

$$a \leqslant \sqrt{\frac{4(r-1)}{r}\xi^2 n^2 + \frac{r-1}{4}} \leqslant \sqrt{4\xi^2 n^2} = 2\xi n,$$

as desired. □

Lemma 13. *Let* $L = \left\{v \in V(G) | d_G(v) \leqslant \left(1 - \frac{1}{r} - 6\xi\right) n\right\}$*. Then* $|L| \leqslant \xi n$*.*

Proof. Suppose to the contrary that $|L| > \xi n$. Then there exists a subset S of L such that $|S| = \lfloor \xi n \rfloor$. We have

$$\begin{aligned} e(G-S) &\geqslant e(G) - \sum_{v\in S} d_G(v) \\ &\geqslant e\left(T_{n,r}\right) - \xi^2 n^2 - \xi n\left(1 - \frac{1}{r} - 6\xi n\right) \\ &\geqslant \left(\frac{r-1}{2r} + 5\xi^2\right)n^2 - \left(1 - \frac{1}{r}\right)\xi n - \frac{r}{8} \\ &> \left(\frac{r-1}{2r} + \frac{r-1}{2r}\xi^2 - \frac{r-1}{r}\xi\right)n^2 + \frac{(1-\xi)(t+r-2)}{r}n - \frac{(t-2)(t+r-2)}{2r} \\ &= \frac{r-1}{2r}(n - \xi n + 1 - t + 1)^2 + (t-1)(n - \xi n + 1 - t + 1) + \binom{t-1}{2} \\ &\geqslant e(K_{t-1} \vee T_{n-\lfloor \xi n\rfloor - t+1,r}). \end{aligned}$$

Recall $K_r^+(a_1, a_2, \ldots, a_r)$ is a color-critical graph with chromatic number $r+1$. By Theorem 1, we have $ex(n - \lfloor \xi n \rfloor, tK_r^+(a_1, \ldots, a_r)) = e(K_{t-1} \vee T_{n-\lfloor \xi n\rfloor - t+1,r})$. Note that $|V(G-S)| = n - \lfloor \xi n \rfloor$. Then $G-S$ contains a copy of $tK_r^+(a_1, \ldots, a_r)$. So, G contains a copy of $tK_r^+(a_1, \ldots, a_r)$, a contradiction. □

Lemma 14. *Let* $W = \bigcup\limits_{i=1}^{r} W_i$*, where* $W_i = \left\{v \in V_i \mid d_{V_i}(v) \geqslant \frac{2r}{r-1}\xi n\right\}$*. Then* $|W| \leqslant \frac{r-1}{r}\xi n$*.*

Proof. By Lemma 12 and the definition of W, we have

$$\begin{aligned}
\xi^2 n^2 &\geqslant \sum_{i=1}^{r} e\left(V_i\right) \\
&= \sum_{i=1}^{r}\left(\frac{1}{2}\sum_{v\in V_i} d_{V_i}(v)\right) \\
&\geqslant \frac{1}{2}\sum_{i=1}^{r}\sum_{v\in W_i} d_{V_i}(v) \\
&\geqslant \frac{1}{2}|W|\frac{2r}{r-1}\xi n \\
&= \frac{r}{r-1}|W|\xi n.
\end{aligned}$$

Thus, $|W| \leqslant \frac{r-1}{r}\xi n$. ☐

For any $S \subseteq V(G)$ with $|S| \leqslant th$, we define $V_i' = V_i \backslash (L \cup W \cup S)$ and let $V' = \bigcup\limits_{i=1}^{r} V_i'$. Choose a vertex $v \in V_i'$. By the definition of L and W, we have $d_G(v) > \left(1 - \frac{1}{r} - 6\xi\right) n$ and $d_{V_i}(v) < \frac{2r}{r-1}\xi n$. Then for $j \in [r]$ and $j \neq i$, we get

$$\begin{aligned}
d_{V_j'}(v) &\geqslant d_{V_j}(v) - |L| - |W| - |S| \\
&\geqslant d_G(v) - d_{V_i}(v) - (r-2)(\frac{n}{r} + 2\xi n) - |L| - |W| - |S| \\
&> (1 - \frac{1}{r} - 6\xi)n - \frac{2r}{r-1}\xi n - (n + 2r\xi n - \frac{2n}{r} - 4\xi n) - \xi n - \xi n - th \\
&> (\frac{1}{r} - \frac{2r^2 + 4r - 4}{r-1} - 1)\xi n \\
&\geqslant (\frac{1}{r} - 9\xi - 2r\xi)n.
\end{aligned}$$

Recall $V' = \bigcup_{i=1}^{r} V_i'$, we have

$$d_{V'}(v) \geqslant d_{V_j'}(v) > (\frac{1}{r} - 9\xi - 2r\xi)n. \tag{3.2}$$

Lemma 15. *For any* $S \subseteq V(G)$ *with* $|S| \leqslant th$*, if there exists an edge within* $G\left[V_i'\right]$ *for some* $i \in [r]$*, then* $G - (L \cup W \cup S)$ *contains a* $K_r^+(a_1, \ldots, a_r)$*.*

Proof. Without loss of generality, assume that v_0u_0 is an edge within $G[V_1']$. By (3.2) and $|V_2| \leqslant \frac{n}{r} + 2\xi n$, we have

$$|N_{V_2'}(u_0) \cap N_{V_2'}(v_0)| \geqslant d_{V_2'}(u_0) + d_{V_2'}(v_0) - |V_2'| \geqslant (\frac{1}{r} - 20\xi - 4r\xi)n > a_2.$$

Thus, there exist a_2 vertices, say $u_{2,1},\ldots,u_{2,a_2}$, in V_2' such that the subgraph induced by $\{v_0,u_0\}$ and $\{u_{2,1},\ldots,u_{2,a_2}\}$ contains a copy of $K_2^+(2,a_2)$. For any integer l with $2\leqslant l\leqslant r-1$, assume that there are a_l vertices in V_l', say $u_{l,1},\ldots,u_{l,a_l}$, such that the subgraph induced by $\{v_0,u_0\},\{u_{2,1},\ldots,u_{2,a_2}\},\ldots,\{u_{l,1},\ldots,u_{l,a_l}\}$ contains a copy of $K_l^+(2,a_2,\ldots,a_l)$. We next consider the common neighbors of above $2+\sum_{i=2}^{l}a_i$ vertices in V_{l+1}'. By (3.2), $\xi<\frac{1}{8r^3h}$, $|V_{l+1}|\leqslant\frac{n}{r}+2\xi n$ and $\sum_{i=2}^{l}a_i\leqslant h-2$, we have

$$\begin{aligned}
&\left|N_{V_{l+1}'}(v_0)\bigcap N_{V_{l+1}'}(u_0)\bigcap\Big(\bigcap_{i\in[l]\setminus\{1\},j\in[a_i]}N_{V_{l+1}'}(u_{i,j})\Big)\right|\\
\geqslant&\Big(2+\sum_{i=2}^{l}a_i\Big)\Big(\frac{1}{r}-9\xi-2r\xi\Big)n-\Big(1+\sum_{i=2}^{l}a_i\Big)\left|V_{l+1}'\right|\\
\geqslant&\Big(2+\sum_{i=2}^{l}a_i\Big)\Big(\frac{1}{r}-9\xi-2r\xi\Big)n-\Big(1+\sum_{i=2}^{l}a_i\Big)\Big(\frac{1}{r}+2\xi\Big)n\\
>&\Big(\frac{1}{r}-13rh\xi\Big)n\\
>&a_{l+1}.
\end{aligned}$$

Then we can obtain a_{l+1} vertices $u_{l+1,1},\ldots,u_{l+1,a_{l+1}}\in V_{l+1}'$. Moreover, the subgraph induced by $\{v_0,u_0\},\{u_{2,1},\ldots,u_{2,a_2}\},\ldots,\{u_{l+1,1},\ldots,u_{l+1,a_{l+1}}\}$ contains a copy of $K_{l+1}^+(2,a_2,\ldots,a_{l+1})$ in G. Then for every $2\leqslant i\leqslant r$, there exist a_i vertices in V_i' such that $\{v_0,u_0\}$, $\{u_{2,1},\ldots,u_{2,a_2}\},\ldots,\{u_{r,1},\ldots,u_{r,a_r}\}$ induced a $K_r^+(2,a_2,\ldots,a_r)$ in G. Similarly, we get the number of common neighbors of $\{u_{2,1},\ldots,u_{2,a_2}\},\ldots,\{u_{r,1},\ldots u_{r,a_r}\}$ in $V_1'\setminus\{v_0,u_0\}$.

$$\begin{aligned}
\left|\Big(\bigcap_{i\in[r]\setminus\{1\},j\in[a_i]}N_{V_1'}(u_{i,j})\Big)\setminus\{v_0,u_0\}\right|&\geqslant\left(\sum_{i=2}^{r}a_i\right)\left(\frac{1}{r}-9\xi-2r\xi\right)n-\left(\sum_{i=2}^{r}a_i-1\right)|V_1'|-2\\
&\geqslant\left(\sum_{i=2}^{r}a_i\right)\left(\frac{1}{r}-9\xi-2r\xi\right)n\\
&\quad-\left(\sum_{i=2}^{r}a_i-1\right)\left(\frac{1}{r}+2\xi\right)n-2\\
&>\left(\frac{1}{r}-13rh\xi\right)n-2\\
&>a_1-2.
\end{aligned}$$

Let $u_{1,3},\ldots,u_{1,a_1}\in V_1'\setminus\{v_0,u_0\}$ be the common neighbors of $\{u_{2,1},\ldots,u_{2,a_2}\},\ldots,\{u_{r,1},\ldots,u_{r,a_r}\}$. Then the subgraph induced by $\{u_0,v_0,u_{1,3},\ldots,u_{1,a_1}\}\cup\{u_{i,1},\ldots,u_{i,a_i}|2\leqslant i\leqslant r\}$ contains a copy of $K_r^+(a_1,\ldots,a_r)$, i.e., $G-(L\cup W\cup S)$ contains a $K_r^+(a_1,\ldots,a_r)$. $\square$

Lemma 16. *For $i\in[r]$, we have $\Delta\left(G\left[V_i\setminus(L\cup W)\right]\right)<th$.*

Proof. Suppose that $\Delta\left(G\left[V_i\backslash(L\cup W)\right]\right)\geqslant th$. Then there exists a vertex $v_0\in V_i\backslash(L\cup W)$ such that $d_{V_i\backslash(L\cup W)}(v_0)\geqslant th$. Without loss of generality assume that $v_0\in V_1$. Since $v_0\notin L\cup W$, we have $d_G(v_0)>(1-\frac{1}{r}-6\xi)n$ and $d_{V_1}(v_0)<\frac{2r}{r-1}\xi n$. Recall $|V_1|\geqslant\frac{n}{r}-2\xi n$ and $\xi<\frac{1}{8r^3h}$. Then

$$|V_1\backslash(L\cup W)|\geqslant|V_1|-|L|-|W|\geqslant\frac{n}{r}-2\xi n-\xi n-\frac{r-1}{r}\xi n>\frac{n}{r}-4\xi n>\frac{2r}{r-1}\xi n>d_{V_1}(v_0).$$

So, $\overline{N}_{V_1\backslash(L\cup W)}(v_0)\neq\emptyset$. Let

$$G'=G+\sum_{v\in\overline{N}_{V_1\backslash(L\cup W)}(v_0)}vv_0.$$

Then we have $\rho(G')>\rho(G)$. By the maximality of $\rho(G)$, G' contains t disjoint $K_r^+(a_1,\ldots,a_r)$, say F, as a subgraph. By the construction of G', there exists a $K_r^+(a_1,\ldots,a_r)$ in F, say H, such that $v_0\in V(H)$. Let $F'=F-H\subseteq G$. Note that $d_{V_1\backslash(L\cup W)}(v_0)\geqslant th>(t-1)h=|V(F')|$. Then there exists a vertex $u_0\in N_{V_1\backslash(L\cup W)}(v_0)$ such that $u_0\notin V(F')$. Thus, v_0u_0 is an edge within $G\left[V_1\backslash(L\cup W\cup V(F'))\right]$. By Lemma 15, $G-(L\cup W\cup V(F'))$ contains a $K_r^+(a_1,\ldots,a_r)$, say H', such that $V(H')\bigcap V(F')=\emptyset$. Then $H'\cup F'$ is a copy of $tK_r^+(a_1,\ldots,a_r)$ in G, a contradiction. □

Lemma 17. *Let $\mu=\sum_{i=1}^{r}\mu_i$, where $\mu_i=\mu\left(G\left[V_i\backslash(L\cup W)\right]\right)$. Then $\mu\leqslant t-1$ and $G-(L\cup W)$ contains at least μ disjoint $K_r^+(a_1,\ldots,a_r)$.*

Proof. If $\mu=0$, then we are done. If $\mu\geqslant 1$, let $v_1v_2,\ldots,v_{2\mu-1}v_{2\mu}$ be μ independent edges in $\bigcup_{i=1}^{r}G[V_i\backslash(L\cup W)]$. Define $S_0=\{v_j\mid j=1,\ldots,2l\}$ where $l=\min\{\mu,t\}$ and let $S_1=S_0\backslash\{v_1,v_2\}$. Then $|S_1|=2l-2\leqslant(t-1)h$ and v_1v_2 is an edge within $G[V_i\backslash(L\cup W\cup S_1)]$ for some $i\in[r]$. By Lemma 15, $G-(L\cup W\cup S_1)$ contains a $K_r^+(a_1,\ldots,a_r)$ and we denote it by H_1. Let $S_2=(S_1\backslash\{v_3,v_4\})\cup V(H_1)$. Then $|S_2|=2(l-2)+h\leqslant(t-1)h$ and v_3v_4 is an edge within $V_i\backslash(L\cup W\cup S_2)$ for some $i\in[r]$. By Lemma 15, $G-(L\cup W\cup S_2)$ contains a $K_r^+(a_1,\ldots,a_r)$ and we denote it by H_2. Repeating the above steps, we obtain a sequence of subsets $S_1,\ldots S_l$. For $1\leqslant j\leqslant l$, $S_j=(S_{j-1}\backslash\{v_{2j-1},v_{2j}\})\cup V(H_{j-1})$ and $G-(L\cup W\cup S_j)$ contains a $K_r^+(a_1,\ldots,a_r)$ denoted by H_j. Thus, $H_1,\ldots,H_l$ are l disjoint $K_r^+(a_1,\ldots,a_r)$. Recall G is $tK_r^+(a_1,\ldots,a_r)$-free. Then $l\leqslant t-1$. By $l=\min\{\mu,t\}$, we have $l=\mu$. So, $G-(L\cup W)$ contains at least μ disjoint $K_r^+(a_1,\ldots,a_r)$. □

Lemma 18. *For $i\in[r]$, $G[V_i\backslash(L\cup W)]$ contains an independent set I_i with $|I_i|>|V_i\backslash(L\cup W)|-2(t-1)th$.*

Proof. Recall that $\mu_i=\mu\left(G\left[V_i\backslash(L\cup W)\right]\right)$. If $\mu_i=0$, then $V_i\backslash(L\cup W)$ is an independent set. If $\mu_i\geqslant 1$, let $v_1v_2,\ldots,v_{2\mu_i-1}v_{2\mu_i}$ be μ_i independent edges in $G[V_i\backslash(L\cup W)]$. Define

$$I_i=(V_i\backslash(L\cup W))\backslash\left(\bigcup_{j=1}^{2\mu_i}N_{V_i\backslash(L\bigcup W)}(v_j)\right).$$

We claim that I_i is an independent set. Otherwise, we have $E(G[I_i]) \neq \emptyset$. Then $\mu(G[V_i\backslash(L\cup W)]) \geqslant \mu_i+1 > \mu_i$, a contradiction. By Lemmas 16 and 17, we have

$$\begin{aligned}|I_i| &= |V_i\backslash(L\cup W)| - |\bigcup_{j=1}^{2\mu_i} N_{V_i\backslash(L\cup W)}(v_j)| \\ &\geqslant |V_i\backslash(L\cup W)| - 2\mu_i\Delta(V_i\backslash(L\cup W)) \\ &> |V_i\backslash(L\cup W)| - 2(t-1)th,\end{aligned}$$

as desired. □

In Lemmas 19-23, we assume that G is connected. By Perron-Frobenius theorem, there exists a positive eigenvector $\mathbf{x}$ corresponding to $\rho(G)$. For convenience, let x_v denote the coordinate of $\mathbf{x}$ such that x_v corresponds to the vertex $v \in V(G)$. Define $x_{v^*} = \max_{v\in V(G)} x_v$ and $x_{u^*} = \max_{v\in V(G)\backslash W} x_v$. Then we have

$$\rho(G)x_{v^*} \leqslant |W|x_{v^*} + (n-|W|)x_{u^*},\ \rho(G)x_{u^*} \leqslant |W|x_{v^*} + d_G(u^*)x_{u^*}.$$

By $\rho(G) > \frac{r-1}{r}n$ and $|W| \leqslant \frac{r-1}{r}\xi n$, we obtain

$$x_{u^*} \geqslant \frac{\rho(G)-|W|}{n-|W|}x_{v^*} > \frac{\rho(G)-|W|}{n}x_{v^*} \geqslant \frac{(r-1)(1-\xi)}{r}x_{v^*}, \tag{3.3}$$

and so

$$\begin{aligned}d_G(u^*) &\geqslant \rho(G) - |W|\frac{x_{v^*}}{x_{u^*}} \\ &> \frac{r-1}{r}n - \frac{r-1}{r}\xi n\frac{r}{(r-1)(1-\xi)} \\ &= \left(\frac{r-1}{r} - \frac{\xi}{1-\xi}\right)n \\ &> \left(\frac{r-1}{r} - 6\xi\right)n.\end{aligned}$$

Therefore, $u^* \notin L$. Recall $u^* \in V(G)\backslash W$. Then $u^* \in V(G)\backslash(L\cup W)$. Without loss of generality, assume that $u^* \in V_1$. We have

$$\begin{aligned}\rho(G)x_{u^*} &= \sum_{v\in N_{L\bigcup W}(u^*)} x_v + \sum_{v\in N_{V_1}(u^*)\backslash(L\cup W)} x_v + \sum_{\substack{v\in\bigcup_{i=2}^r N_{V_i}(u^*)\\ v\notin L\cup W}} x_v \\ &\leqslant \sum_{v\in N_{L\backslash W}(u^*)} x_v + \sum_{v\in N_W(u^*)} x_v + d_{V_1\backslash(L\cup W)}(u^*)x_{u^*} + \sum_{\substack{v\in\bigcup_{i=2}^r V_i\backslash I_i\\ v\notin L\cup W}} x_v + \sum_{v\in\bigcup_{i=2}^r I_i} x_v \\ &< |L|x_{u^*} + |W|x_{v^*} + thx_{u^*} + 2(r-1)thx_{u^*} + \sum_{v\in I_2\cup\cdots\cup I_r} x_v,\end{aligned}$$

where I_i $(2 \leqslant i \leqslant r)$ is an independent set of $G[V_i \backslash (L \cup W)]$ such that $|I_i| > |V_i \backslash (L \cup W)| - 2(t-1)th$. Thus,

$$\sum_{v \in I_2 \cup \cdots \cup I_r} x_v > (\rho(G) - |L| - (2r-1)th) x_{u^*} - |W| x_{v^*}. \tag{3.4}$$

Lemma 19. $L = \emptyset$.

Proof. Suppose to the contrary that $L \neq \emptyset$. Then there exists a vertex $v_0 \in L$. Without loss of generality, assume that $v_0 \in V_1$. Let

$$G' = G - \sum_{v \in N_G(v_0)} vv_0 + \sum_{v \in I_2 \cup \cdots \cup I_r} vv_0.$$

By Lemma 14 and (3.3), (3.4), we have

$$\begin{aligned}
\rho(G') - \rho(G) &\geqslant \mathbf{x}^T (A(G') - A(G)) \mathbf{x} \\
&= 2x_{v_0} \left(\sum_{v \in \bigcup_{i=2}^r I_i} x_v - \sum_{v \in N_G(v_0)} x_v \right) \\
&= 2x_{v_0} \left(\sum_{v \in \bigcup_{i=2}^r I_i} x_v - \sum_{v \in N_W(v_0)} x_v - \sum_{v \in N_{V(G) \backslash W}(v_0)} x_v \right) \\
&> 2x_{v_0} \left((\rho(G) - |L| - (2r-1)th - d_G(v_0)) x_{u^*} - 2|W| x_{v^*} \right) \\
&> 2x_{v_0} x_{v^*} \left(\left(\left(1 - \frac{1}{r}\right) n - \xi n - \xi n - \left(1 - \frac{1}{r} - 6\xi\right) n \right) \right. \\
&\quad \left. \times \frac{(r-1)(1-\xi)}{r} - 2 \cdot \frac{r-1}{r} \xi n \right) \\
&= \frac{4}{r}(r-1)\xi(1-2\xi) n x_{v_0} x_{v^*} \\
&> 0.
\end{aligned}$$

To get a contradiction, we just need to show that G' is $tK_r^+(a_1, \ldots, a_r)$-free. Otherwise, G' contains a copy of $tK_r^+(a_1, \ldots, a_r)$ and we denote it by F. By the construction of G', there exists a $K_r^+(a_1, \ldots, a_r)$, say H, in F such that $v_0 \in V(H)$. Let $N_H(v_0) = \{v_1, \ldots, v_{d_H(v_0)}\}$. Note that $N_H(v_0) \subseteq \bigcup_{i=2}^r I_i \subseteq \bigcup_{i=2}^r V_i \backslash (L \cup W)$. Then for any vertex $v \in N_H(v_0) \bigcap I_i$ $(2 \leqslant i \leqslant r)$, we have $d_G(v) > \left(1 - \frac{1}{r} - 6\xi\right) n$ and $d_{V_i}(v) < \frac{2r}{r-1} \xi n \leqslant 2r\xi n$. Moreover, for $j \in [r]$ and $j \neq i$, we get

$$\begin{aligned}
d_{V_j}(v) &\geqslant d_G(v) - d_{V_i}(v) - (r-2)\left(\frac{1}{r} + 2\xi\right) n \\
&> \left(1 - \frac{1}{r} - 6\xi\right) n - 2r\xi n - (r-2)\left(\frac{1}{r} + 2\xi\right) n \\
&= \frac{n}{r} - 2(2r+1)\xi n.
\end{aligned}$$

By $\xi < \frac{1}{8r^3h}$ and $d_H(v_0) < h$, we have

$$\begin{aligned}\left|\bigcap_{k=1}^{d_H(v_0)} N_{V_1}(v_k)\right| &\geqslant \sum_{k=1}^{d_H(v_0)} d_{V_1}(v_k) - (d_H(v_0)-1)|V_1| \\ &> d_H(v_0)\left(\frac{n}{r} - 2(2r+1)\xi n\right) - (d_H(v_0)-1)\left(\frac{n}{r}+2\xi n\right) \\ &= \left(\frac{1}{r}+2\xi\right)n - 4d_H(v_0)(r+1)\xi \\ &> \frac{n}{r} - 8rh\xi n \\ &> h = |V(H)|.\end{aligned}$$

Thus, there exists one vertex $u_0 \in \bigcap_{k=1}^{d_H(v_0)} N_{V_1}(v_k)\backslash V(H)$. Assume that $F' = F - V(H)$. By replacing $\{v_0v_1, \ldots, v_0v_{d_H(v_0)}\}$ with $\{u_0v_1, \ldots, u_0v_{d_H(v_0)}\}$, we obtain a $K_r^+(a_1, \ldots, a_r)$, say H', in $G - V(F')$. So, $F' \cup H'$ is a copy of $tK_r^+(a_1, \ldots, a_r)$ in G, a contradiction. $\square$

Lemma 20. *For any vertex set $S \subseteq V(G)$ with $|S| \leqslant th$, if there exists a vertex $v_0 \in W$, then $G - (W \cup S\backslash\{v_0\})$ contains a $K_r^+(a_1, \ldots, a_r)$.*

Proof. By Lemma 19 and the definition of L, we have $d_G(v) > \left(1 - \frac{1}{r} - 6\xi\right)n$ for all vertex $v \in V(G)$. Assume that $v_0 \in V_1$. Recall $V(G) = V_1 \cup \cdots \cup V_r$ is the vertex partition such that $\sum\limits_{1\leqslant i<j\leqslant r} e(V_i, V_j)$ attains the maximum. So, $d_{V_1}(v_0) \leqslant \frac{1}{r}d_G(v_0)$. Then we have

$$\begin{aligned}d_{V_2}(v_0) &\geqslant d_G(v_0) - d_{V_1}(v_0) - (r-2)\left(\frac{n}{r}+2\xi n\right) \\ &> \left(1-\frac{1}{r}\right)\left(1-\frac{1}{r}-6\xi\right)n - (r-2)\left(\frac{n}{r}+2\xi n\right) \\ &= \frac{n}{r^2} - 2\left(1+r-\frac{3}{r}\right)\xi n \\ &> \frac{n}{r^2} - 2(r+1)\xi n.\end{aligned}$$

Recall $v_0 \in W$. Then $d_{V_1}(v_0) \geqslant \frac{2r}{r-1}\xi n$ and

$$d_{V_1\backslash(W\cup S)}(v_0) \geqslant d_{V_1}(v_0) - |W| - |S| \geqslant \left(\frac{2r}{r-1} - \frac{r-1}{r}\right)\xi n - th > 0.$$

Therefore, there exists a vertex $u_0 \in N_{V_1\backslash(W\cup S)}(v_0)$.

Note that $|V_1\backslash(W\cup S)| \geqslant |V_1| - |W| - |S| \geqslant \frac{n}{r} - 2\xi n - \xi n - th > a_1$. Choose $v_{1,1}, \ldots, v_{1,a_1-1} \in V_1\backslash(W \cup S \cup \{u_0\})$ and let $u_0 = v_{1,a_1}$. Then for $i \in [a_1]$, we have $d_{V_1}(v_{1,i}) < \frac{2r}{r-1}\xi n$ and

$$d_{V_2}(v_{1,i}) \geqslant d_G(v_{1,i}) - d_{V_1}(v_{1,i}) - (r-2)\left(\frac{n}{r}+2\xi n\right)$$

$$
\begin{aligned}
&> \left(1-\frac{1}{r}-6\xi\right)n-\frac{2r}{r-1}\xi n-\frac{r-2}{r}n-(r-2)2\xi n\\
&= \frac{n}{r}-\frac{2(r^2+r-1)}{r-1}\xi n\\
&\geqslant \frac{n}{r}-2(1+2r)\xi n.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left|\left(N_{V_2}(v_0)\bigcap(\bigcap_{i=1}^{a_1}N_{V_2}(v_{1,i}))\right)\backslash(W\cup S)\right| \geqslant& d_{V_2}(v_0)+\sum_{i=1}^{a_1}d_{V_2}(u_{1,i})-a_1|V_2|-|W|-|S|\\
\geqslant&\frac{n}{r^2}-2(r+1)\xi n+a_1(\frac{n}{r}-2(1+2r)\xi n\\
&-\frac{n}{r}-2\xi n)-\xi n-th\\
>&\frac{n}{r^2}-2(2a_1+1)(r+2)\xi n\\
>&a_2.
\end{aligned}
$$

Let $v_{2,1},\dots,v_{2,a_2}$ be the common neighbor of $v_0,v_{1,1},\dots,v_{1,a_1}$ in $V_2\backslash(W\cup S)$. For any integer $2\leqslant l\leqslant r-1$, assume that $v_{l,1},\dots,v_{l,a_l}$ are the common neighbors of $\{v_0\}\bigcup\{v_{i,1},\dots,v_{i,a_i}|1\leqslant i\leqslant l-1\}$ in $V_l\backslash(W\cup S)$. By a similar discussion as those of $d_{V_2}(v_0)$ and $d_{V_2}(v_{1,k})$ $(1\leqslant k\leqslant a_1)$, we have $d_{V_{l+1}}(v_0)>\frac{n}{r^2}-2\left(r+1\right)\xi n$ and $d_{V_{l+1}}(u_{i,j})>\frac{n}{r}-2(2r+1)\xi n$ for $i\in[l]$ and $j\in[a_i]$. Note that $\sum\limits_{i=1}^{l}a_i<\sum\limits_{i=1}^{r}a_i=h$. Then

$$
\begin{aligned}
&\left|N_{V_{l+1}}(v_0)\bigcap(\bigcap_{i\in[l],\ j\in[a_i]}N_{V_{l+1}}(v_{i,j}))\backslash(W\cup S)\right|\\
\geqslant& d_{v_{l+1}}(v_0)+\sum_{i=1}^{l}\sum_{j=1}^{a_i}d_{V_{l+1}}(v_{i,j})-\sum_{i=1}^{l}a_i|V_{l+1}|-|W|-|S|\\
>&\left(\frac{n}{r^2}-2(r+1)\xi n\right)+\sum_{i=1}^{l}a_i\left(\frac{n}{r}-2(2r+1)\xi n-\frac{n}{r}-2\xi n\right)-\xi n-th\\
>&\frac{n}{r^2}-8rh\xi n\\
>&a_{l+1}.
\end{aligned}
$$

Let $v_{l+1,1}\dots,v_{l+1,a_{l+1}}$ be the common neighbors of $\{u_0\}\cup\{v_{i1},\dots,v_{i,a_i}|1\leqslant i\leqslant l\}$ in $V_{l+1}\backslash(W\cup S)$. Then the subgraph induced by $\{v_0\}\cup\{v_{i1},\dots,v_{i,a_i}|1\leqslant i\leqslant r\}$ contains a copy of $K_r^+(a_1,\dots,a_r)$, i.e., $G-(W\cup S\backslash\{v_0\})$ contains a $K_r^+(a_1,\dots,a_r)$. □

Lemma 21. *For any vertex $v\in V(G)$, we have $x_v>\frac{r-1}{r}\xi x_{v^*}$.*

Proof. Suppose there exists one vertex $v_0 \in V(G)$ such that $x_{v_0} \leqslant \frac{r-1}{r}\xi x_{v^*}$. Let

$$G' = G - \sum_{v \in N_G(v_0)} vv_0 + \sum_{v \in \bigcup_{i=2}^r I_i} vv_0.$$

By a similar discussion as the proof of Lemma 19, we have G' is $tK_r^+(a_1, \dots, a_r)$-free. Recall $\rho(G) > \left(1 - \frac{1}{r}\right) n$ and $|W| \leqslant \left(1 - \frac{1}{r}\right) \xi n$. Then we have $|W| < \xi\rho(G)$. Together with $\xi < \frac{1}{8r^3 h}$, we have

$$\begin{aligned}
\rho(G') - \rho(G) \geqslant & \sum_{v \in \bigcup_{i=2}^r I_i} 2x_v x_{v_0} - \sum_{v \in N_G(v_0)} 2x_v x_{v_0} \\
=& 2x_{v_0} \left(\sum_{v \in \bigcup_{i=2}^r I_i} x_v - \sum_{v \in N_G(v_0)} x_v \right) \\
>& 2x_{v_0} \left((\rho(G) - (2t-1)th) \frac{(r-1)(1-\xi)}{r} x_{v^*} \right. \\
& \left. - |W| x_{v^*} - \rho(G) \left(\frac{r-1}{r} \right) \xi x_{v^*} \right) \\
>& 2\rho(G) x_{v_0} x_{v^*} \left(\frac{(r-1)(1-\xi)}{r} - \frac{3r+2}{r}\xi - \xi - \left(\frac{r-1}{r} \right) \xi \right) \\
=& 2\rho(G) x_{v_0} x_{v^*} \left(1 - \frac{1}{r} - 6\xi \right) \\
>& 0,
\end{aligned}$$

a contradiction. □

Lemma 22. $|W| = t - 1$ *and* $\mu = 0$.

Proof. By Lemma 17, we have $\mu = \mu\left(\bigcup_{i=1}^r G[V_i \backslash W]\right) \leqslant t - 1$ and there exist μ disjoint $K_r^+(a_1, \dots, a_r)$, say $H_1, \dots, H_\mu$, in $G - W$.

We claim that $|W| \leqslant t-1-\mu$. Otherwise, we have $|W| \geqslant t-\mu$. Let $S_0 = \{v_1, \dots, v_{t-\mu}\}$ be a subset of W. Define $S_1 = S_0 \backslash \{v_1\} \bigcup \left(\bigcup_{i=1}^\mu V(H_i)\right)$. Then $|S_1| = (t-\mu-1)+\mu h \leqslant th$. Note that $v_1 \in W$. By Lemma 20, $G-(W \cup S_1 \backslash \{v_1\})$ contains a $K_r^+(a_1, \dots, a_r)$, say $H_{\mu+1}$, such that $V(H_{\mu+1}) \bigcap S_0 \subseteq \{v_1\}$. If $t-\mu \geqslant 2$, let $S_2 = (S_1 \backslash \{v_2\}) \cup V(H_{\mu+1})$. Then $|S_2| = (t-\mu-2)+(\mu+1)h \leqslant th$. By Lemma 20, $G-(W \cup S_2 \backslash \{v_2\})$ contains a $K_r^+(a_1, \dots, a_r)$, say $H_{\mu+2}$, such that $V(H_{\mu+2}) \bigcap S_0 \subseteq \{v_2\}$. Repeating the above steps, we obtain a sequence of subsets $S_1, \dots, S_{t-\mu}$. For $1 \leqslant j \leqslant t-\mu$, we have $S_j = (S_{j-1} \backslash \{v_j\}) \bigcup V(H_{\mu+j-1})$ and $G - (W \cup S_j \backslash \{v_j\})$ contains a $K_r^+(a_1, \dots, a_r)$, say $H_{\mu+j}$, such that $V(H_{\mu+j}) \bigcap S_0 \subseteq \{v_j\}$. Thus, $H_1, \dots, H_\mu, H_{\mu+1}, \dots, H_t$ are t disjoint $K_r^+(a_1, \dots, a_r)$ in G, a contradiction.

Clearly, if $|W| = t-1$, then $\mu = 0$. Hence, we just need to show that $|W| = t-1$. Suppose to the contrary that $|W| \leqslant t-2$. Recall $|V_1| \geqslant \frac{n}{r} - 2\xi n$. Let S be a subset of $V_1 \backslash W$ such that $|S| = t-1-|W|$. Define $F = \bigcup_{i=1}^r G[V_i \backslash W]$. By Lemmas 16 and

17, we have $\mu(F) = \mu \leqslant t-1$ and $\Delta(F) < th$. Furthermore, by Lemma 8, $e(F) \leqslant f(\mu(F), \Delta(F)) \leqslant (t-1)(th+1)$. Let

$$G' = G - \sum_{uv \in E(F)} uv + \sum_{\substack{u \in S \\ v \in V_1 \setminus (W \cup S)}} uv.$$

Then G' is a spanning subgraph of $K_{|W \cup S|} \vee K_{|V_1 \setminus (W \cup S)|, |V_2 \setminus W|, \ldots, |V_r \setminus W|}$. Note that $|W \cup S| = t-1$. Then G' is $tK_r^+(a_1, \ldots, a_r)$-free. Moreover, by Lemma 21, we have

$$\begin{aligned}
\rho(G') - \rho(G) &\geqslant \sum_{\substack{u \in S \\ v \in V_1 \setminus (W \cup S)}} 2x_u x_v - \sum_{uv \in E(F)} 2x_u x_v \\
&\geqslant |S|(|V_1| - |W \cup S|) \left(\frac{r-1}{r}\right)^2 \xi^2 x_{v^*}^2 - 2e(F) x_{v^*}^2 \\
&\geqslant |S| \left(\frac{n}{r} - 2\xi n - (t-1)\right) \left(\frac{r-1}{r}\right)^2 \xi^2 x_{v^*}^2 - 2(t-1)(th+1) x_{v^*}^2 \\
&> 0,
\end{aligned}$$

which contradicts the maximality of $\rho(G)$. Thus, $|W| = t-1$ and we get $\mu = 0$. □

Lemma 23. *For any vertex $u \in W$, we have $d_G(u) = n-1$.*

Proof. Suppose to the contrary that there exists a vertex $v_0 \in W$ such that $d_G(v_0) < n-1$. Then there exists a vertex $u_0 \in V(G)$ such that $v_0 u_0 \notin E(G)$. Let $G' = G + v_0 u_0$. Clearly, $\rho(G') > \rho(G)$. By the maximality of $\rho(G)$, G' contains a copy of $tK_r^+(a_1, \ldots, a_r)$ and we denote it by F. Furthermore, there exists a $K_r^+(a_1, \ldots, a_r)$ in F, say H, such that $u_0 v_0 \in E(H)$. Let $F' = F - V(H)$. We have $|F'| = (t-1)h$. Recall $v_0 \in W$. By Lemma 20, $G - ((W \cup V(F')) \setminus \{v_0\})$ contains a $K_r^+(a_1, \ldots, a_r)$, say H', such that $V(H') \cap V(F') = \emptyset$. Thus, $H' \cup F'$ is a copy of $tK_r^+(a_1, \ldots, a_r)$ in G, a contradiction. □

Now we are ready to prove Theorem 3.

The proof of Theorem 3. We prove Theorem 3 according to the following two cases.

Case 1. G is connected. Let $n_i = |V_i \setminus W|$ for $i \in [r]$. By Lemmas 22 and 23, we have $G \subseteq K_{t-1} \vee K_r(n_1, \ldots, n_r)$. Note that $\rho(G)$ attains the maximum. Then $G \cong K_{t-1} \vee K_r(n_1, \ldots, n_t)$. Without loss of generality, assume that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$. By symmetry, let $x_u = x_i$ for each vertex $u \in V_i \setminus W$ $(1 \leqslant i \leqslant r)$ and let $x_v = x_0$ for each vertex $v \in W$. Then we have

$$\rho(G) x_0 = (t-2) x_0 + \sum_{j=1}^{r} n_j x_j$$

and

$$\rho(G) x_i = (t-1) x_0 + \sum_{j=1}^{r} n_j x_j - n_i x_i$$

for $1 \leqslant i \leqslant r$. By some calculations, we get $x_i = \frac{\rho(G)+1}{\rho(G)+n_i}x_0$, $i = 1, \ldots, r$. To get $G \cong K_{t-1} \vee T_{n-t+1,r}$, it suffices to show that $n_i - n_j \leqslant 1$ for every $1 \leqslant i < j \leqslant r$. Suppose to the contrary that there exist i_0, j_0 with $1 \leqslant i_0 < j_0 \leqslant r$ such that $n_{i_0} - n_{j_0} \geqslant 2$. Choose $v_{i_0} \in V_{i_0} \backslash W$ and let

$$G' = G - \sum_{v \in V_{j_0} \backslash W} vv_{i_0} + \sum_{v \in V_{i_0} \backslash (W \cup \{v_{i_0}\})} vv_{i_0}.$$

Then $G' \cong K_{t-1} \vee K_r(n_1, \ldots, n_{i_0} - 1, \ldots, n_{j_0} + 1, \ldots, n_r)$ and so G' is $tK_r^+(a_1, \ldots, a_r)$-free. Furthermore, by $n_{i_0} - n_{j_0} > 2$, we have

$$\begin{aligned}\rho(G') - \rho(G) &\geqslant 2x_0((n_{i_0} - 1)x_{i_0} - n_{j_0}x_{j_0}) \\ &= 2x_0^2 \frac{(\rho(G) + 1)((n_{i_0} - n_{j_0} - 1)\rho(G) - n_{j_0})}{(\rho(G) + n_{i_0})(\rho(G) + n_{j_0})} \\ &\geqslant 2x_0^2 \frac{(\rho(G) + 1)(\rho(G) - n_{j_0})}{(\rho(G) + n_{i_0})(\rho(G) + n_{j_0})}.\end{aligned}$$

If $r = 2$, by $n_1 \geqslant n_2 + 2$, we have $\rho(G) > \frac{n}{2} > n_2 = n_{j_0}$. If $r \geqslant 3$, by $n_{j_0} = |V_{j_0} \backslash W| \leqslant \frac{n}{r} + 2\xi n - (t-1)$, we have $\rho(G) \geqslant \left(1 - \frac{1}{r}\right) n + \frac{2t-2}{r} - \frac{(2t+r-2)^2}{4nr} > n_{j_0}$. So, $\rho(G) - n_{j_0} > 0$ and we get $\rho(G') - \rho(G) > 0$, a contradiction.

Case 2. G is not connected. Let $G_1, G_2, \ldots, G_m$ be connected components of G. Then $\rho(G) = \max_{1 \leqslant i \leqslant m} \rho(G_i)$. Without loss of generality, assume that $\rho(G) = \rho(G_1)$. Then G_1 is a connected n_1-vertex $t'K_r^+(a_1, a_2, \ldots, a_r)$-free graph where $n_1 < n$ and $t' \leqslant t$. We claim that n_1 is large enough. Otherwise, $\rho(G_1) \leqslant \rho(K_{n_1}) = n_1 - 1 < \frac{(r-1)n}{r} < \rho(K_{t-1} \vee K_{n-t+1,r})$. By Case 1, we have $G_1 \cong K_{t'-1} \vee K_{n_1-t'+1,r}$. By $n_1 < n$ and $t' \leqslant t$, we have G_1 is a proper subgraph of $K_{t-1} \vee T_{n-t+1,r}$. So, $\rho(G) = \rho(G_1) < \rho(K_{t-1} \vee T_{n-t+1,r})$. Note that $K_{t-1} \vee T_{n-t+1,r}$ is $tK_r^+(a_1, a_2, \ldots, a_r)$-free. We get a contradiction.

This completes the proof. □

At last we give the proof of Theorem 5 (based on Theorems 1 and 3).

The proof of Theorem 5. Note that F_i is a color-critical graph with $\chi(F_i) = r + 1$, $1 \leqslant i \leqslant t$. Hence, for $1 \leqslant i \leqslant r$, there exists a graph $K_r^+(a_{i1}, a_{i2}, \ldots, a_{ir})$ containing a copy of F_i as a subgraph. Let $a_j = \max_{1 \leqslant j \leqslant r} a_{ij}$ for $1 \leqslant j \leqslant r$. Then one sees that $\bigcup_{i=1}^t F_i$ is a subgraph of $tK_r^+(a_1, a_2, \ldots, a_r)$. So, $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^t F_i) \leqslant \mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \ldots, a_r))$.

By Theorem 3, $\mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \ldots, a_r)) = \rho(K_{t-1} \vee T_{n-t+1,r})$ and $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \ldots, a_r))$. Together with Theorem 1, $K_{t-1} \vee T_{n-t+1,r}$ is a $\bigcup_{i=1}^t F_i$-free graph with order n. Hence, $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^t F_i)$ for sufficiently large n. □

4 Concluding remark

In this paper, we characterize the extremal graph of sufficiently large order n with respect to $\mathrm{ex}_{sp}(n, tK_r^+(a_1, a_2, \ldots, a_r))$ (see Theorem 3), where $t \geqslant 1$, $r \geqslant 2$ are two positive integers. Let $K_r^+(a_{11}, a_{12}, \ldots, a_{1r}), \ldots, K_r^+(a_{t1}, a_{t2}, \ldots, a_{tr})$ be t disjoint graphs.

We also obtain the unique extremal graph of sufficiently large order n with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} K_r^+(a_{i1}, a_{i2}, \ldots, a_{ir}))$ (see Corollary 4). Note that for any color-critical graph F with $\chi(F) = r + 1$, there exist some integers $a_1, \ldots, a_r$ such that F is a subgraph of $K_r^+(a_1, a_2, \ldots, a_r)$. For t disjoint color-critical graphs $F_1, \ldots, F_t$ with $\chi(F_i) = r + 1$ $(1 \leqslant i \leqslant t)$, we determine the unique extremal graph of sufficiently large order n with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} F_i)$ (see Theorem 5).

Note that complete graph K_{r+1} and odd cycle C_{2k+1} are both color-critical graphs. Then the following two results are direct consequences of our main result (Theorem 5).

Corollary 24 ([30])**.** *For positive integers $t \geqslant 2, r \geqslant 2$ and sufficiently large n, $K_{t-1} \vee T_{n-t+1,r}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, tK_{r+1})$.*

Corollary 25. *Given some positive integers $t, r_1, \ldots, r_t$ and let $C_{2r_1+1}, \ldots, C_{2r_t+1}$ be t disjoint odd cycles. Then $K_{t-1} \vee T_{n-t+1,2}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, \bigcup_{i=1}^{t} C_{2r_i+1})$ for sufficiently large n.*

Especially, if $r_1 = r_2 = \cdots = r_t = r$, we get the next result.

Corollary 26 ([15])**.** *For positive integers t, r and sufficiently large $n, K_{t-1} \vee T_{n-t+1,2}$ is the unique extremal graph with respect to $\mathrm{ex}_{sp}(n, tC_{2r+1})$.*

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