

Hurwitz Orbits on Reflection Factorizations of Parabolic Quasi-Coxeter Elements

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Abstract

We prove that two reflection factorizations of a parabolic quasi-Coxeter element in a finite Coxeter group belong to the same Hurwitz orbit if and only if they generate the same subgroup and have the same multiset of conjugacy classes. As a lemma, we classify the finite Coxeter groups for which every reflection generating set that is minimal under inclusion is also of minimum size.

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1 Introduction

For any group G , there is a natural action of the k -stranded braid group

$$\mathcal{B}_k = \langle \sigma_1, \dots, \sigma_{k-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \text{ and} \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \dots, k-1 \rangle$$

on the set of k -tuples of elements of G : the generator σ_i acts via the *(left) Hurwitz move*

$$\sigma_i(g_1, \dots, g_i, g_{i+1}, \dots, g_k) = (g_1, \dots, g_{i+1}, g_i^{-1} g_i g_{i+1}, \dots, g_k), \quad (1.1)$$

swapping two adjacent elements and conjugating one by the other so as to preserve the product of the tuple. We call this the *Hurwitz action* of \mathcal{B}_k on G^k . This action is of particular interest when G is a reflection group and the factors g_1, \dots, g_k are reflections: in this setting, it played an important role in Hurwitz's study of branched Riemann surfaces [Hur91] (for $G = \mathfrak{S}_n$ the symmetric group) and in Bessis's proof of the $K(\pi, 1)$ conjecture for complements of reflection arrangement [Bes15] (for G an arbitrary well generated complex reflection group).

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There has also been considerable interest in fully understanding the structure of the Hurwitz action, independent of its applications to geometric problems. For abstract examples, the transitivity of the action has been interpreted [MR20] in terms of connectivity properties of posets associated to the group G and a given generating set for it. For more explicit examples, the orbit structure for minimum-length factorizations in 3-cycles in the alternating group is completely understood [MN19]; the same is true for reflection factorizations in the symmetric group [Klu88, BIT03] and in dihedral groups [Sia09, Ber11]. For factorizations of the special *Coxeter elements*, much is known in Coxeter groups [Bes03, IS10, BDSW14, LR16, Weg20, WY20, WY23] and complex reflection groups [Rip10, Lew20, Pet18, GLRS21, MPRW21]. Of particular interest for us is the recent paper [BGRW17], which gives a beautiful classification of the elements in a finite Coxeter group W with the property that the Hurwitz action is transitive on their minimum-length reflection factorizations: they are precisely the *parabolic quasi-Coxeter elements* of W (defined below in Definition 5).

In addition to preserving the product $g_1 \cdots g_k$ of the tuple of group elements, it is easy to see that the Hurwitz action also preserves the subgroup $H := \langle g_1, \dots, g_k \rangle$ generated by the factors, as well as their multiset of H -conjugacy classes. In [Lew20, §5], it was conjectured that these invariants are sufficient to distinguish Hurwitz orbits.¹

Conjecture 1. Let W be a finite Coxeter group and $g \in W$ an arbitrary element. Then two reflection factorizations of g belong to the same Hurwitz orbit if and only if they generate the same subgroup $H \leq W$ and have the same multiset of H -conjugacy classes.

In the present paper, our main result (Theorem 18) is to show that Conjecture 1 is valid when the product of the factors belongs to the class of parabolic quasi-Coxeter elements. Along the way, we prove a lemma (Lemma 13) that seems interesting in its own right, classifying the real reflection groups in which every reflection generating set that is *minimal under inclusion* is also *of minimum size*. In particular, we show (Corollary 15) that all Weyl groups have this property.

The plan of the paper is as follows: in Section 2, we review the background on finite Coxeter groups necessary for the remainder of the paper, including the definition and properties of parabolic quasi-Coxeter elements. In Section 3, we show (Lemma 13) that every minimal generating set of reflections in a finite Coxeter group W is of minimum size, unless W contains as an irreducible factor a dihedral group of order $2m$ where m is divisible by three distinct primes. Finally, in Section 4, we prove our main result (Theorem 18), that Conjecture 1 is valid whenever the element g is a parabolic quasi-Coxeter element of W .

¹In fact, the question raised in [Lew20] applied to all *complex* reflection groups. However, it turns out that there are counterexamples among the minimum-length reflection factorizations in two of the exceptional complex reflection groups – see [LW22, Rem. 4.4].

2 Finite Coxeter groups and parabolic quasi-Coxeter elements

In this section, we provide background on finite Coxeter groups and their parabolic quasi-Coxeter elements, as necessary for the main results of the paper. For an in-depth treatment of finite Coxeter groups, the reader may consult the classic references [Hum90], [Kan01], [BB05], and [Bro10].

2.1 Finite Coxeter groups

A group W is called a *Coxeter group* if it is generated by a finite set $S := \{s_1, \dots, s_n\}$ with a presentation of the following form: for some numbers m_{ij} such that $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ for $1 \leq i < j \leq n$,

$$W = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

We call the elements $s_i \in S$ the *simple generators* of W and we say that the size $n := |S|$ is the *rank* of W ; the pair (W, S) will be called a *Coxeter system*. The numbers m_{ij} determine the *Coxeter diagram* associated with the system (W, S) ; this is the graph with vertex set $[n] := \{1, \dots, n\}$ and $m_{ij} - 2$ edges between the vertices i and j . If the Coxeter diagram of (W, S) is connected, we say that W is an *irreducible Coxeter group*; every Coxeter group is a direct product of irreducibles.

Classification

In this paper we only work with *finite Coxeter groups*, namely Coxeter groups with finite *cardinality*. These turn out to be precisely the finite subgroups of $\mathrm{GL}(\mathbb{R}^n)$ generated by Euclidean reflections (the so-called *real reflection groups*) [Hum90, §6.4]. We will denote by $V \cong \mathbb{R}^n$ the ambient space on which they act. In [Cox34], Coxeter classified the irreducible real reflection groups into four infinite families – A_n (the symmetric groups), B_n (the hyperoctahedral groups of signed permutations), D_n (index-2 subgroups of the hyperoctahedral groups), and $I_2(m)$ (the dihedral groups) – and six exceptional types – H_3, H_4, F_4, E_6, E_7 , and E_8 – where the indices in all cases correspond to ranks.

We will also be interested (see Corollary 15) in a subclass of finite Coxeter groups known as *crystallographic* or *Weyl* groups. In terms of the numbers m_{ij} in the Coxeter presentation, Weyl groups are characterized by having $m_{ij} \in \{2, 3, 4, 6\}$ for all i, j . The irreducible Weyl groups are the infinite families A_n, B_n , and D_n , and the five exceptional types E_6, E_7, E_8, F_4 , and $G_2 := I_2(6)$.

Reflection factorizations

In a Coxeter group W , any element $t \in W$ that is conjugate to some simple generator will be called a *reflection*, and T will denote the set of reflections of W . Since $T \subset W$ is a generating set, we may consider the Cayley graph for W with respect to T . This determines a natural length function on the elements of the group: the length of an element $g \in W$ is the number of steps in the shortest path between the identity $1 \in W$

and g in the Cayley graph. Equivalently, this may be defined as the smallest number k for which there exist reflections t_1, \dots, t_k such that $g = t_1 \cdots t_k$. We call such factorizations *reduced reflection factorizations*, and the number $\ell_W^{\text{red}}(g) := k$ the *reflection length* of g . A more common construction in Coxeter groups involves the length function $\ell_S(g)$, where length is calculated in the Cayley graph with respect to the simple generators; we will *not* make use of it here.

The reflection length is subadditive over products: for any g, h one has $\ell_W^{\text{red}}(gh) \leq \ell_W^{\text{red}}(g) + \ell_W^{\text{red}}(h)$. Thus, it determines a partial order \leq_T (the *absolute order*) on the elements of W via

$$u \leq_T v \iff \ell_W^{\text{red}}(u) + \ell_W^{\text{red}}(u^{-1}v) = \ell_W^{\text{red}}(v).$$

In other words, $u \leq_T v$ if and only if the reduced reflection factorizations of u can be extended to give reduced reflection factorizations of v .

We will be particularly interested in a special type of reflection factorization that we introduce now. We say that a reflection factorization $g = t_1 \cdots t_N$ is *full* if the factors t_i generate the *full reflection group*, i.e., if $W = \langle t_1, \dots, t_N \rangle$. Notice that every reflection factorization is trivially full in the group generated by its factors. The *full reflection length* $\ell_W^{\text{full}}(g)$ of g is the minimum length of a full reflection factorization:

$$\ell_W^{\text{full}}(g) := \min \left\{ k : \exists t_1, \dots, t_k \in T \text{ such that } t_1 \cdots t_k = g \text{ and } \langle t_1, \dots, t_k \rangle = W \right\}.$$

Parabolic subgroups

For any subset $I \subset S$ of the simple generators S , the subgroup $W_I := \langle I \rangle$ they generate will be called a *standard parabolic subgroup*. Any subgroup $W' \leq W$ conjugate to some W_I will be called a *parabolic subgroup* of W . Parabolic subgroups are precisely the pointwise stabilizers W_U of arbitrary subsets U of the ambient space V [Kan01, §5.2], and by [Car72, Lem. 1] they are generated by the reflections whose fixed hyperplanes contain U . It is easy with this interpretation to see that intersections of parabolic subgroups are also parabolic (in fact the collection of parabolic subgroups forms a lattice). For any element $g \in W$ we write W_g for the *parabolic closure* of g in W , namely, the smallest parabolic subgroup that contains g . Again relying on the geometric interpretation, it is easy to see that the parabolic closure W_g must equal the pointwise stabilizer W_{V^g} of the fixed space V^g .

The following well known lemma gives a geometric characterization of the absolute order \leq_T ; we will rely on it in the proofs of our main theorems.

Lemma 2 (Carter's lemma). *Let W be a finite Coxeter group, g any of its elements, and W_g the parabolic closure of g . Then $\ell_W^{\text{red}}(g) = \text{codim}(V^g)$ and for a reflection $t \in W$, the following are equivalent:*

$$(1) \ t \leq_T g \qquad (2) \ V^t \supset V^g \qquad (3) \ t \in W_g.$$

Proof. The statement about the length $\ell_W^{\text{red}}(g)$ is what is usually known as Carter's lemma [Car72, Lem. 2]. Carter's proof is written for Weyl groups but applies verbatim for finite

Coxeter groups. The equivalence of parts (1) and (2) is implicit in the proof of [Car72, Lem. 2], see also [Bes03, Lem. 1.2.1]. Parts (2) and (3) are equivalent by Steinberg’s theorem [Bro10, Thm. 4.7] and since $W_g = W_{Vg}$. \square

Reflection generating sets

Let W be a Coxeter group of rank n . Every Coxeter group (finite or otherwise) has a natural geometric representation on \mathbb{R}^n in which each reflection fixes a hyperplane. In this representation, any collection of fewer than n reflections from W have nontrivial common fixed space, and therefore cannot generate the whole group W .

In the case of sets consisting of exactly n reflections, there are a variety of non-obvious restrictions on which sets of reflections can generate. For example, the following theorem of Wegener–Yahiatene shows that the conjugacy classes of the factors cannot be arbitrary.

Lemma 3 ([WY23, Lem. 6.4]). *In an arbitrary Coxeter group W of rank n , all size- n reflection generating sets of W determine the same multiset of conjugacy classes.*

2.2 The Hurwitz action

As illustrated in (1.1), the Hurwitz move σ_i applied to a tuple (g_1, \dots, g_k) moves the entry g_{i+1} one position to the left without changing its value. The inverse (*right Hurwitz move*) is given by

$$\sigma_i^{-1}(g_1, \dots, g_i, g_{i+1}, \dots, g_k) = (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_k) \quad (2.1)$$

and moves the entry g_i one position to the right, also without changing its value. By combining such moves, we can move any element of a factorization to any prescribed position (possibly conjugating other elements of the factorization), and more generally we can move any subsequence of entries to any prescribed set of positions unchanged as long as we do not alter their relative order in the factorization.

We will make use of the following structural lemma for the Hurwitz action on reflection factorizations throughout the paper. The proof given in [LR16] relies on the classification of finite Coxeter groups, but recently Wegener and Yahiatene gave a uniform proof [WY23].

Lemma 4 ([LR16, Cor. 1.4]). *Let W be a finite Coxeter group and $g \in W$ an element of reflection length $\ell_W^{\text{red}}(w) = k$. Then any factorization of g into N reflections (with $N \geq k$) lies in the Hurwitz orbit of some tuple (t_1, \dots, t_N) such that*

$$t_1 = t_2, \quad t_3 = t_4, \quad \dots \quad t_{N-k-1} = t_{N-k},$$

and (t_{N-k+1}, \dots, t_N) is a reduced factorization of g .

2.3 Parabolic quasi-Coxeter elements

The products $s_{i_1} \cdots s_{i_n}$ of the simple generators in any order are known as Coxeter elements and they are all conjugate to each other. We will call *any* element in this conjugacy class a *Coxeter element* of W . The Coxeter elements of parabolic subgroups are similarly called (generalized) *parabolic Coxeter elements* of W . Bessis showed [Bes03, Prop. 1.6.1] that the Hurwitz action on reduced reflection factorizations of parabolic Coxeter elements is transitive. Recently Baumeister et al. [BGRW17] studied the Hurwitz action on reduced factorizations of *arbitrary* elements in finite Coxeter groups, which led them to the following generalization of parabolic Coxeter elements.

Definition 5 (Parabolic quasi-Coxeter elements). In a finite Coxeter group W , we call an element $g \in W$ a *parabolic quasi-Coxeter element* if it admits a reduced reflection factorization $g = t_1 \cdots t_k$ whose factors t_i generate a parabolic subgroup of W . If the factors t_i generate the full group W , we say that g is a *quasi-Coxeter element*.

Notice that Definition 5 does not specify *which* parabolic subgroup should be generated by the factors t_i . However, we see in the next proposition that, in fact, the subgroup is completely determined by the parabolic quasi-Coxeter element g .

Proposition 6 ([Weg17, Prop. 2.4.11] + [BGRW17, Thm. 1.1]). *If W is a finite Coxeter group, $g \in W$ a parabolic quasi-Coxeter element, and $g = t_1 \cdots t_k$ a reduced reflection factorization, then $\langle t_1, \dots, t_k \rangle = W_g$.*

The following proposition may be seen as a characterization for *quasi-Coxeter* elements, analogous to the characterization of Coxeter elements as products of the reflections in a simple system.

Proposition 7. *Let W be a finite Coxeter group of rank n , and $\{t_1, \dots, t_n\} \subset W$ a generating set of reflections. Then the product $w = t_1 \cdots t_n$ (in any order) is a quasi-Coxeter element.*

Proof. The roots associated to a n -element generating set of reflections must be linearly independent; indeed, if they were not, the fixed spaces of the reflections would have nontrivial intersection. Then the result is an immediate corollary of [Car72, Lem. 3], which asserts that a product of reflections in a finite Coxeter group is reduced if and only if the associated roots are linearly independent. \square

Characterization of parabolic quasi-Coxeter elements

Our main result (Theorem 18) addresses Conjecture 1 for the class of parabolic quasi-Coxeter elements. We give below three different characterizations of parabolic quasi-Coxeter elements that we will rely on in Section 4.

Proposition 8. *In a finite Coxeter group W of rank n with an element $g \in W$, the following are equivalent.*

1. The element $g \in W$ is a parabolic quasi-Coxeter element.
2. The Hurwitz action on reduced reflection factorizations of g is transitive.
3. There exists a quasi-Coxeter element $w \in W$ such that $g \leq_T w$.
4. The full reflection length of g is given by $\ell_W^{\text{full}}(g) = 2n - \ell_W^{\text{red}}(g)$.

Proof. This is a combination of known results. The equivalence (1) \leftrightarrow (2) is [BGRW17, Thm. 1.1], while (1) \leftrightarrow (3) is given as [BGRW17, Cor. 6.11], and finally (1) \leftrightarrow (4) is [DLM24, Thm. 5.8]. \square

We record here a corollary of the characterization that is implicit in [BGRW17] and [Gob17], and of which we will also make use later on.

Corollary 9. *In a finite Coxeter group W with a parabolic quasi-Coxeter element $g \in W$, if $W' \leq W$ is any reflection subgroup that contains g , then $W_g \leq W'$ and g is parabolic quasi-Coxeter in W' .*

Proof. Let's start with a reduced reflection factorization $g = t_1 \cdots t_k$ in W' . By Carter's lemma (Lemma 2), we have that $\ell_{W'}^{\text{red}}(g) = \text{codim}(V^g) = \ell_W^{\text{red}}(g)$, so that this is also a reduced W -reflection factorization of g . By Proposition 6, this forces the equality $\langle t_1, \dots, t_k \rangle = W_g$, and therefore $W_g \leq W'$.

Similarly, *all* reduced W' -reflection factorizations of g are also reduced in W . Since g is parabolic quasi-Coxeter in W , Proposition 8 [(1) \leftrightarrow (2)] for the pair (W, g) implies that they must all be Hurwitz equivalent. Applying Proposition 8 [(1) \leftrightarrow (2)] again but now in the opposite direction and for the pair (W', g) completes the proof. \square

3 Generating sets of reflections: minimum versus minimal

Let W be a finite Coxeter group and X a set of reflections that generates W . We say that X is a *minimal generating set* if no proper subset of X generates W . For example, since every Coxeter group of rank n may be generated by n reflections but not fewer, every generating set of n reflections is minimal. However, there may be minimal generating sets of larger size.

Example 10 ([BB05, Ch. 1, Exer. 6]). Consider the dihedral group $I_2(30)$, of order 2×30 , acting on \mathbb{R}^2 . Let $R := \{r_0, r_2, r_{-3}\}$ be the set containing the following three reflections (illustrated in Figure 1): r_0 is the reflection across the x -axis, r_2 is the reflection across the line $y = \tan(2\pi/30)x$, and r_{-3} is the reflection across the line $y = \tan(-3\pi/30)x$.

It is easy to see that R generates $I_2(30)$: for example, the pair of reflections r_{-3} and $r_0 r_2 r_0$ have fixed lines that make an angle of $\pi/30$ and hence they generate $I_2(30)$. It is also easy to see that the subset $\{r_0, r_2\}$ generates a subgroup of order 2×15 , the subset $\{r_0, r_{-3}\}$ generates a subgroup of order 2×10 , and the subset $\{r_2, r_{-3}\}$ generates a subgroup of order 2×6 . Thus R is a minimal generating set of reflections, even though it is not (and does not contain) a *minimum-size* generating set of reflections.

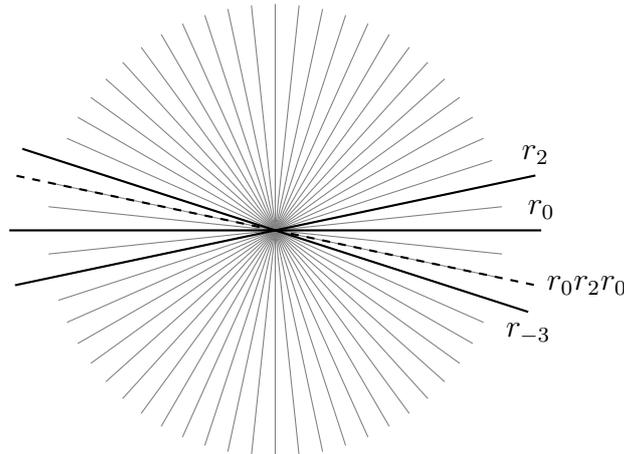


Figure 1: The reflecting lines associated with the group $I_2(30)$, as in Example 10.

This example motivates the following definition.

Definition 11. Say that a finite Coxeter group W of rank n satisfies the *minimum-equals-minimal property* if any set X of reflections of W that generates W contains a subset $\{t_1, \dots, t_n\} \subset X$ of exactly n reflections that generates W . Equivalently, the property is that every minimal generating set of reflections is actually a generating set of minimum size.

Since the reflections of $W_1 \times W_2$ are the union of the reflections of W_1 and the reflections of W_2 , a finite Coxeter group has the minimum-equals-minimal property if and only if each of its irreducible factors also has the property. Thus, it suffices to consider the case of irreducible groups. In the next two results, we classify the irreducible finite Coxeter groups with the minimum-equals-minimal property, beginning with a generalization of Example 10.

Proposition 12. *If W is the dihedral group $I_2(m)$ and m has at least three distinct prime factors, then W does not satisfy the minimum-equals-minimal property.*

Proof. As discussed in [LR16, §3.2],² for any three distinct reflections r_1, r_2, r_3 in $I_2(m)$, we can choose roots $\alpha_1, \alpha_2, \alpha_3$ orthogonal to their reflecting lines such that the angles between the α_i have measures $\frac{\pi}{m}A_{12}$, $\frac{\pi}{m}A_{13}$, and $\frac{\pi}{m}A_{23}$ where A_{12}, A_{13}, A_{23} are integers in $\{1, \dots, m-1\}$ with sum $2m$; moreover, the three reflections generate the whole group if and only if $\gcd(A_{12}, A_{13}, A_{23}, m) = 1$. Likewise, it's easy to see that reflections r_i and r_j generate the group if and only if $\gcd(A_{ij}, m) = 1$.

²However, the reader should note a small error in the discussion there: [LR16] omit the final “ m ” in the GCD, writing the condition as $\gcd(A_{12}, A_{13}, A_{23}) = 1$. One can see the failure of this version already in the case $m = 3$, when three equally spaced vectors give $A_{12} = A_{13} = A_{23} = 2$ but the associated reflections generate the group.

Suppose that m is divisible by at least three primes; then we can write $m = p \cdot q \cdot r$ where $p, q, r > 1$ are pairwise relatively prime (though not necessarily prime themselves). By the Chinese Remainder Theorem, there are (unique) integers $a, b \in \{1, \dots, m - 1\}$ that satisfy the systems of congruences

$$\begin{cases} a \equiv 0 \pmod{p} \\ a \equiv 1 \pmod{q} \\ a \equiv 1 \pmod{r} \end{cases} \quad \text{and} \quad \begin{cases} b \equiv 1 \pmod{p} \\ b \equiv 0 \pmod{q} \\ b \equiv -1 \pmod{r} \end{cases}.$$

If $a + b > m$, define $A_{12} := a$, $A_{13} := b$, $A_{23} := 2m - a - b$; otherwise, define $A_{12} := m - a$, $A_{13} := m - b$, $A_{23} := a + b$. Now, pick an arbitrary root α_1 of $I_2(m)$ and let α_2, α_3 be the uniquely determined roots so that the triple $(\alpha_1, \alpha_2, \alpha_3)$ corresponds to the triple (A_{12}, A_{13}, A_{23}) , and let r_1, r_2, r_3 be the three associated reflections. In either case, by construction, $\gcd(A_{12}, m) = p$, $\gcd(A_{13}, m) = q$, and $\gcd(A_{23}, m) = r$, so no pair of r_1, r_2, r_3 generates the group. However, the three reflections together do generate $I_2(m)$, as

$$\gcd(A_{12}, A_{13}, A_{23}, m) = \gcd(a, b, m) = \gcd(\gcd(a, m), \gcd(b, m)) = \gcd(p, q) = 1. \quad \square$$

Lemma 13. *Let W be an irreducible finite Coxeter group. Then W satisfies the minimum-equals-minimal property if and only if it belongs to one of the following two categories:*

1. *it is not of dihedral type, or*
2. *it is of dihedral type $I_2(m)$ with m having at most two distinct prime factors.*

Proof. The “only if” direction is covered by Proposition 12. Thus, it remains to show that if a finite Coxeter group belongs to categories (1) or (2), then it has the minimum-equals-minimal property. Our proof has two parts: **first**, we show that if a finite Coxeter group W has a minimal but non-minimum generating set of reflections, then there is some (proper or not) reflection subgroup $W' \leq W$ with a minimal generating set of reflections of size exactly $\text{rk}(W') + 1$; **second**, we show that in every irreducible group W in the categories (1) and (2), every reflection generating set of size $\text{rk}(W) + 1$ is non-minimal (i.e., contains a proper subset that also generates W).

We now explain why these two claims suffice to prove the lemma. Suppose that W contains a minimal but non-minimum generating set of reflections. By the first claim, W contains a reflection subgroup W' that has a minimal generating set consisting of $\text{rk}(W') + 1$ reflections. Consider the decomposition of W' into irreducible components. Each of the reflections in the generating set belongs to exactly one of these components, each component contains at least as many reflections as its rank, and the sum of the ranks of the components is $\text{rk}(W')$; therefore one of the components (call it W'') contains exactly $\text{rk}(W'') + 1$ of the reflections, and these form a minimal generating set of reflections for $W'' \leq W$. By the second claim, this means that W'' cannot belong to either of categories (1) or (2). On the other hand, for a group belonging to either category (1) or (2), all of its irreducible reflection subgroups also belong to (1) or (2) (since each dihedral subgroup

belongs to a parabolic dihedral subgroup, whose index m must equal a label in the Coxeter diagram of W). Therefore, supposing W contains a minimal but non-minimum generating set of reflections, we have shown (assuming the two claims) that W is not in categories (1) or (2). Therefore, to prove the proposition, it suffices to prove the two claims.

For the first claim, we proceed by induction on the cardinality of W . The base case $W = A_1$ is trivially true since A_1 does not have any minimal but non-minimum generating set. Now pick a finite Coxeter group W with a minimal but non-minimum generating set X . If $|X| = n + 1$ we are done. If, on the other hand, $|X| \geq n + 2$, choose a reflection $t \in X$, and let $W' = \langle X \setminus t \rangle$. Since X is a minimal generating set of W , $W' \subsetneq W$. Moreover, the set $X \setminus t$ must be a *minimal* generating set for W' (or else a proper subset of $X \setminus t$ together with t would generate W). Finally, since $|X \setminus t| \geq n + 1 > n \geq \text{rank}(W')$ and we are done by the inductive hypothesis for W' .

For the second claim, we consider separately the infinite families $A_{n-1} = \mathfrak{S}_n$, B_n , D_n , $I_2(m)$, and the exceptional types.

In the group $A_{n-1} = \mathfrak{S}_n$, any set X of reflections corresponds to a graph on $[n]$ by identifying the transposition (ij) with an edge joining vertices i and j , and generating sets correspond to connected graphs (see, e.g., [DLM22, Prop. 2.1]). Every connected graph contains a spanning tree. The subset of X corresponding to the spanning tree consists of $n - 1$ reflections and generates \mathfrak{S}_n . Therefore any generating set of reflections for \mathfrak{S}_n contains a generating subset of cardinality $n - 1 = \text{rank}(\mathfrak{S}_n)$.

In the group B_n , any set of reflections corresponds to a signed graph on $[n]$ (with diagonal reflections corresponding to loops). In order for a set of reflections to generate the whole group B_n , the graph must be connected (otherwise it generates a subgroup of a conjugate of $B_k \times B_{n-k}$ for some k) and must contain at least one loop (otherwise it generates a subgroup of D_n). Any connected graph with a loop contains a spanning tree with a loop; the subset of reflections corresponding to this subgraph is a minimum generating set for B_n .

In the group D_n , any set of reflections corresponds to a loopless signed graph on $[n]$. In order for a set of reflections to generate the whole group D_n , the graph must be connected (otherwise it generates a subgroup of a conjugate of $D_k \times D_{n-k}$ for some k) and must contain at least one negative cycle (i.e., a cycle with an odd number of negative edges) (otherwise, it generates a subgroup conjugate to \mathfrak{S}_n , since every signed spanning tree generates a conjugate of \mathfrak{S}_n [Shi05, Lem. 2.7] that includes all the signed transpositions that do *not* create a negative cycle with the tree). Given a loopless connected signed graph with at least one negative cycle, fix a negative cycle, and delete edges from the graph one by one provided that the deleted edges do not belong to the cycle and removing them does not disconnect the graph. The result is a signed *unicycle* (a connected graph with exactly one cycle) whose unique cycle is the chosen negative cycle. All unicycles on n vertices have precisely n edges, and all signed unicycles whose cycle is negative correspond to a generating set of reflections in D_n , so this subgraph corresponds to a minimum generating set for D_n .

In the group $I_2(m)$ with m having at most two (distinct) prime factors p and q , consider any three reflections r_1, r_2, r_3 that generate the whole group $I_2(m)$ and

the associated integers A_{12}, A_{13}, A_{23} described in the proof of Proposition 12. The integers A_{ij} are constructed so that $A_{12} + A_{13} + A_{23} = 2m$ and the assumption that the r_i 's generate the full group implies (again, see [LR16, §3.2]) that we must have $\gcd(A_{12}, A_{13}, A_{23}, m) = 1$. From the first equation, if p divides two of the A_{ij} then it divides all three, which contradicts the second equation; likewise for q . But then one of the A_{ij} is divisible by neither p nor q and so is relatively prime to m ; in this case the reflections r_i and r_j are sufficient to generate the whole group W .

In the exceptional types, we used an exhaustive computer calculation, checking (for each group of rank n) all $(n + 1)$ -element generating subsets of reflections and confirming that they each contain an n -element generating set. (For the larger groups, this computation is made tractable by considering sets of reflections up to conjugation by W , as in [LR16, §3.6].) \square

Remark 14. For the simply laced types there is a simpler argument that combines two existing results: it was shown in [BGRW17, Lem. 5.12] that if W is simply laced, then a set of reflections in W generates W if and only if the roots orthogonal to their reflecting hyperplanes form a \mathbb{Z} -spanning set for the root lattice of W . By [BH19, Thm. 4.1], if a set of roots generates the root lattice, then it contains a \mathbb{Z} -basis for the root lattice. And finally again by [BGRW17, Lem. 5.12], the reflections corresponding to this \mathbb{Z} -basis form a minimum generating set for W .

All reflection subgroups of Weyl groups are themselves Weyl groups and, moreover, the only irreducible dihedral groups that are crystallographic are $I_2(3) = \mathfrak{S}_3$, $I_2(4) = B_2$, and $I_2(6) = G_2$. The following is then immediate.

Corollary 15. *All Weyl groups W satisfy the minimum-equals-minimal property of Definition 11.*

Remark 16. For real reflection groups, Proposition 12 establishes that minimal-but-not-minimum generating sets of reflections are reasonably well behaved: restricted to dihedral types and controlled by some elementary number theory. In complex reflection groups, generating sets with this property are much more complicated; perhaps this is a reason why Conjecture 1 is truly a conjecture for the real types and fails in some complex cases.

4 Main result

In this section, we prove our main result on the Hurwitz orbits of factorizations of parabolic quasi-Coxeter elements. We begin with the case of minimum-length full reflection factorizations.

Proposition 17. *If W is a finite Coxeter group and $g \in W$ a parabolic quasi-Coxeter element, then the Hurwitz action is transitive on minimum-length full reflection factorizations of g .*

Proof. Denote the reflection length of g by k (i.e., $k = \ell_W^{\text{red}}(g)$). By Proposition 8 [(1) \leftrightarrow (4)], the minimum length of full factorizations of g is $2n - k$, where n is the rank of W .

By Lemma 4 (followed by some left Hurwitz moves to bring the reduced factorization in the beginning), every reflection factorization of g as given in the statement is Hurwitz-equivalent to one of the form

$$g = t_1 \cdots t_k \cdot t_{k+1} \cdot t_{k+1} \cdots t_n \cdot t_n, \quad (4.1)$$

where $t_1 \cdots t_k = g$ is a reduced reflection factorization of g . It is sufficient to show that the factorization (4.1) is Hurwitz-equivalent to any other of the form

$$g = t'_1 \cdots t'_k \cdot t'_{k+1} \cdot t'_{k+1} \cdots t'_n \cdot t'_n \quad (4.2)$$

with $\langle t'_i \rangle = W$. We proceed by induction on the common length $2n - k$ of the two factorizations. In the base case, the length takes the smallest possible value $2n - k = n$. This means that the factorization $g = t_1 \cdots t_n$ is simultaneously reduced and full, so g is a quasi-Coxeter element and Proposition 8 [(1) \leftrightarrow (2)] applies.

Now assume that $2n - k > n$. We aim to find a factorization in the same Hurwitz orbit as (4.1) whose last factor is t'_n . We first turn (4.1) to a factorization of the form

$$g = t_1 \cdots t_n \cdot t''_{n+1} \cdots t''_{2n-k}, \quad (4.3)$$

by sliding some terms t_i with left Hurwitz moves (1.1). Since the factorization (4.1) is full, we must further have that $\langle t_i \rangle = W$, and then by Proposition 7 that $w := t_1 \cdots t_n$ is a quasi-Coxeter element of W . Since $w = t_1 \cdots t_n$ is quasi-Coxeter, its parabolic closure is the full group W and hence (by Lemma 2) all reflections lie below w in absolute order \leq_T . In particular, w has a reduced reflection factorization whose last factor is t'_n . Since w is quasi-Coxeter, we can reach that factorization from $t_1 \cdots t_n$ via Hurwitz moves; then we can slide t'_n to the last position of (4.3) via right Hurwitz moves (2.1). This all means that the first factorization (4.1) of g is Hurwitz-equivalent to one of the following form:

$$g = \widehat{t}_1 \cdots \widehat{t}_{2n-k-1} \cdot t'_n. \quad (4.4)$$

We would now like to show that the factorizations (4.2) and (4.4) are Hurwitz-equivalent. It is sufficient to prove the equivalence of the following two factorizations:

$$gt'_n = \widehat{t}_1 \cdots \widehat{t}_{2n-k-1} \quad \text{and} \quad gt'_n = t'_1 \cdots t'_k \cdot t'_{k+1} \cdot t'_{k+1} \cdots t'_{n-1} \cdot t'_{n-1} \cdot t'_n. \quad (4.5)$$

Define

$$w' := t'_1 \cdots t'_k \cdot t'_{k+1} \cdots t'_n = t'_1 \cdots t'_k \cdot t'_n \cdot (t'_n t'_{k+1} t'_n) \cdots (t'_n t'_{n-1} t'_n). \quad (4.6)$$

The element w' is quasi-Coxeter for the same reasons that w is, and the two factorizations in (4.6) are reduced factorizations of w' . Therefore, from the second factorization in (4.6), we have that $t'_1 \cdots t'_k t'_n = gt'_n$ lies below w' in the absolute order \leq_T . Then by Proposition 8 [(1) \leftrightarrow (3)], gt'_n is a parabolic quasi-Coxeter element. It follows that the two factorizations in (4.5) are length- $(2n - k - 1)$ reflection factorizations of a parabolic quasi-Coxeter element whose reflection length is $k + 1$; the second is manifestly full (it shares the same set of factors as the full factorization (4.2)), and so to conclude using the inductive hypothesis it suffices to show that the first factorization in (4.5) is also full.

Let $W' := \langle \widehat{t}_1, \dots, \widehat{t}_{2n-k-1} \rangle$. By construction we have that $\langle W', t'_n \rangle = W$. Since gt'_n is parabolic quasi-Coxeter in W and also belongs to W' , we have by Corollary 9 that W' contains the parabolic closure $W_{gt'_n} = \langle t'_1, \dots, t'_k, t'_n \rangle$. In particular, W' contains t'_n . Since $\langle W', t'_n \rangle = W$, this means $W' = W$; that is, the first factorization of (4.5) is indeed full. Now the inductive assumption guarantees that the two factorizations in (4.5) are Hurwitz-equivalent, and so (by the preceding arguments) that the factorizations (4.1) and (4.2) are equivalent. The result follows by induction. \square

We are now prepared for the proof of our main result.

Theorem 18. *Conjecture 1 is true whenever g is a parabolic quasi-Coxeter element in the finite Coxeter group W .*

Proof. It is sufficient to prove the theorem for irreducible groups. The case of the dihedral groups has already been covered in [Ber11]. By Lemma 13, all the non-dihedral types satisfy the minimum-equals-minimal property of Definition 11, and so it is enough to prove it for this class of groups. Therefore, let W be a finite Coxeter group with the minimum-equals-minimal property and let g be a parabolic quasi-Coxeter element of W , and consider two reflection factorizations of g that generate the same subgroup $W' \leq W$ and have the same multiset of W' -conjugacy classes.

By Corollary 9, g is parabolic quasi-Coxeter in W' . Further, it follows from the proof of Lemma 13 that if W has the minimum-equals-minimal property, then the same is true of all of its reflection subgroups; in particular, it is true of W' . So without loss of generality we may as well relabel W' as W and consider the case that the two factorizations of g are full. Let $k = \ell_W^{\text{red}}(g)$ be the reflection length of g and let $n = \text{rank}(W)$ be the rank of W , so that the length of the two factorizations is at least $\ell_W^{\text{full}}(g) = 2n - k$ (by Proposition 8 [(1) \leftrightarrow (4)]). We now proceed by induction on the length of the factorizations.

The base case is Proposition 17. Now suppose the two full factorizations of g have length $2n - k + 2s$ for $s > 0$. By Lemma 4, we can assume that the two factorizations have the form

$$g = t_1 \cdots t_k \cdot t_{k+1} \cdot t_{k+1} \cdots t_{n+s} \cdot t_{n+s} \quad \text{and} \quad g = t'_1 \cdots t'_k \cdot t'_{k+1} \cdot t'_{k+1} \cdots t'_{n+s} \cdot t'_{n+s}. \quad (4.7)$$

Our strategy is to apply some Hurwitz moves to make the two factorizations agree in their last two factors while remaining full factorizations of g . Toward that end, observe that $\{t_1, \dots, t_{n+s}\}$ and $\{t'_1, \dots, t'_{n+s}\}$ are generating sets of W (because both factorizations are full) that are not minimum (because $s > 0$). Therefore, since W is assumed to have the minimum-equals-minimal property, in each set there is a reflection that can be removed to leave a generating set of reflections. Moreover, since the two factorizations have the same multiset of conjugacy classes and (by Lemma 3) all minimal generating sets have the same multiset of conjugacy classes, we can even arrange to choose the “unnecessary” reflections t_i and t'_j to be conjugate in W . We now explain how to produce factorizations in the same Hurwitz orbits as those in (4.7) that end in a pair of equal factors $\cdots t' \cdot t'$ that are conjugate to t_i .

Suppose first that $i > k$. In that case, we can use Hurwitz moves to slide the two copies of t_i to the end of the factorization, producing a new factorization

$$g = t_1 \cdots t_k \cdot t_{k+1} \cdot t_{k+1} \cdots t_{i-1} \cdot t_{i-1} \cdot t_{i+1} \cdot t_{i+1} \cdots t_{n+s} \cdot t_{n+s} \cdot t_i \cdot t_i$$

in which the prefix is a full reflection factorization of g .

On the other hand, suppose that $i \leq k$. Apply the Hurwitz moves $\sigma_{i-1}^{-1}, \sigma_{i-2}^{-1}, \dots, \sigma_1^{-1}$ in that order to produce a new factorization $g = t' \cdot t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_k \cdot t_{k+1} \cdot t_{k+1} \cdots t_{n+s} \cdot t_{n+s}$ for some reflection t' that belongs to the same conjugacy class as t_i and still lies below g is absolute order. Since the remaining factors have not changed, they still include an n -element generating set for W . Now apply the Hurwitz moves $\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_{2n-k+2s-1}^{-1}$ in that order to produce a new factorization in which t' is at the end (instead of the beginning) and all other factors have been conjugated by t' . Conjugating a generating set gives another generating set, so the remaining factors (omitting t') contain an n -element generating set; we can use Hurwitz moves to bring these n factors to consecutive positions. As in the proof of Proposition 17, the product w of these n factors is a quasi-Coxeter element (Proposition 7) and $t' \leq_T w$ (Lemma 2), so by Proposition 17 we can apply some Hurwitz moves to these n factors and produce a factorization with a second copy of t' . We may again use Hurwitz moves to slide this second copy of t' to the end, and then apply Lemma 4 to the prefix of $2n - k + 2s - 2$ factors to produce a new factorization of the form

$$g = t''_1 \cdots t''_k \cdot t''_{k+1} \cdot t''_{k+1} \cdots t''_{n+s-1} \cdot t''_{n+s-1} \cdot t' \cdot t'. \quad (4.8)$$

This is in the desired form, but we must explain why the prefix (omitting the final two t' factors) is full; this will finally use the fact that $t' \leq_T g$. Let $W' = \langle t''_1, \dots, t''_{n+s-1} \rangle$. By Corollary 9, $W_g \leq W'$, and by Lemma 2, $t' \in W_g$. Therefore $t' \in W'$. Thus $W' = \langle t''_1, \dots, t''_{n+s-1}, t' \rangle = W$ (where the last equality comes because the group generated by the factors is invariant under the Hurwitz action). Thus the prefix (dropping the two copies of t') is full, as claimed.

The last two paragraphs show that the first factorization in (4.7) is Hurwitz-equivalent to a factorization in which the last two factors are equal and come from a prescribed conjugacy class, and for which the prefix (omitting those factors) is a full factorization of g . The same argument applies to the second factorization in (4.7); and, moreover, because the two factorizations in (4.7) have the same multiset of conjugacy classes, we can do this in such a way that the final two factors in the two new factorizations belong to the same conjugacy class. To finish, we use the same technique as in [LR16, Thm. 1.1], using Hurwitz moves to bring factors from the prefix to the third position from the end, applying the Hurwitz moves (with $M = 2n - k + 2s$)

$$\begin{aligned} (\dots, t, t', t') &\xrightarrow{\sigma_{M-2}} (\dots, t', t'tt', t') \xrightarrow{\sigma_{M-1}} (\dots, t', t', t) \xrightarrow{\sigma_{M-1}} \\ &(\dots, t', t, tt't) \xrightarrow{\sigma_{M-2}} (\dots, t, tt't, tt't), \end{aligned}$$

and then restoring the prefix to its original form. Since the prefix generates W , we can by repeated application of this strategy replace the final two factors with any conjugate

pair of reflections; in particular, we may arrange so that the two factorizations in question agree on their final two factors. Since the prefixes are full and have shorter length, we are done by induction. \square

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