Binding number, k-factor and spectral radius of graphs

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Submitted: Jun 30, 2023; Accepted: Nov 30, 2023; Published: Feb 9, 2024

Abstract

The binding number $b(G)$ of a graph G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. The association between the binding number and toughness is intricately interconnected, as both metrics function as pivotal indicators for quantifying the vulnerability of a graph. Conjectured by Brouwer and proved by Gu, a theorem asserts that for any d-regular connected graph G, the toughness $t(G)$ is always at least $\frac{d}{\lambda} - 1$, where λ denotes the second largest absolute eigenvalue of the adjacency matrix. Inspired by the work of Brouwer and Gu, in this paper, we investigate $b(G)$ from spectral perspectives, and provide tight sufficient conditions in terms of the spectral radius of a graph G to guarantee $b(G) \geqslant r$. The study of the existence of k-factors in graphs is a classic problem in graph theory. Katerinis and Woodall state that every graph with order $n \geqslant 4k-6$ satisfying $b(G) \geqslant 2$ contains a k-factor where $k \geqslant 2$. This leaves the following question: which 1-binding graphs have a k-factor? In this paper, we also provide the spectral radius conditions of 1-binding graphs to contain a perfect matching and a 2-factor, respectively.

Mathematics Subject Classifications: 05C50

1 Introduction

In 1973, Woodall[30] introduced the concept of binding number. For any $v \in V(G)$, let $N_G(v)$ ($N(v)$ for short) denote the neighborhood of v in G, and for $X \subseteq V(G)$, let $N_G(X) = \bigcup_{x \in X} N(x)$. The *binding number* $b(G)$ of G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. The binding number of a graph has many important applications in various fields, including graph theory, network science, quantum sensing and information processing.

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 https://doi.org/10.37236/12165

One issue concerning binding numbers involves characterizing its boundary by using other structural parameters of the graph, such as, degree sequence[3], minimum degree[4] and connectivity[30]. Similar to vertex-connectivity and edge-connectivity, toughness and binding number are measures of the vulnerability of a graph. A graph G is *t-tough* if $|S| \geqslant tc(G-S)$ for every subset $S \subseteq V(G)$ with $c(G-S) > 1$, where $c(G)$ is the number of components of a graph G. The *toughness* $\tau(G)$ of G is the maximum t for which G is t-tough. In 1973, Woodall[30] initially established the relationship between $\tau(G)$ and $b(G)$, and proved that $\tau(G) \geqslant b(G) - 1$. Later on, Goddard and Swart[16] observed that $b(G) \geqslant \tau(G)$ when $b(G) \leqslant 1$. In 2014, Bauer, Kahl, Schmeichel, Woodall and Yatauro[5] discovered a slightly better bound $\tau(G) \geqslant \min\{3(b(G)-1)/2, b(G)\}$ when $b(G) > 1$.

The study of toughness via eigenvalues was initiated by Alon [1] who showed that for any connected d-regular graph G, $\tau(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$, where λ is the second largest absolute eigenvalue of the adjacency matrix. Around the same time, Brouwer [8] independently discovered a slightly better bound $\tau(G) > \frac{d}{\lambda} - 2$, and he [8, 9] further conjectured that the lower bound can be improved to $\tau(G) \geqslant \frac{d}{\lambda} - 1$. Subsequently, Gu [17] strengthened the result of Brouwer and showed the lower bound can be improved to $\tau(G) > \frac{d}{\lambda} - \sqrt{2}$. Later on, he [18] completely confirmed the conjecture of Brouwer.

Let G be a graph with adjacency matrix $A(G)$. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called the *spectral radius* of G. For two graphs G_1 and G_2, let $G_1 \vee G_2$ be the graph obtained from the disjoint union $G_1 \cup G_2$ by adding all edges between G_1 and G_2. Very recently, Fan, Lin and Lu[15] provided a spectral radius condition for a graph to be t-tough, i.e. if G is a connected graph with $n \geqslant 4t^2 + 6t + 2$ vertices satisfying $\rho(G) \geqslant \rho(K_{2t-1} \vee (K_{n-2t} \cup K_1))$, then G is t-tough, unless $G \cong K_{2t-1} \vee (K_{n-2t} \cup K_1)$. Therefore, it is interesting to consider the spectral condition for a graph to be r-binding where r is a positive integer. In this paper, we study an extremal result for $b(G) \geqslant r$, as the following theorem.

Theorem 1. *Let r be a positive integer, and let G be a connected graph of order n. Then the following statements hold.*

(i) If $n \geqslant 14$ and $\rho(G) \geqslant \rho(K_1 \vee (K_{n-3} \cup 2K_1))$, then G is 1-binding, unless $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$.

(ii) If $n \geqslant \max\{\frac{r^3+5r^2+r}{2}, 5r^2+3\}$, $r \geqslant 2$ and $\rho(G) \geqslant \rho(K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1))$, then G is r-binding, unless $G \cong K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1)$.

Another important issue related to binding numbers is providing conditions for the existence of factors in a graph. An *$[a,b]$-factor* of a graph G is a spanning subgraph H such that $a \leqslant d_H(v) \leqslant b$ for each $v \in V(G)$. Particularly, a $[k,k]$-factor is called a k-factor. In 1971, Anderson[2] established an important threshold for the existence of perfect matchings in graphs by using binding number, and demonstrated that a graph G satisfying $b(G) \geqslant 4/3$ contains a perfect matching. Another seminal result due to Woodall[30] is that graphs with $b(G) \geqslant 3/2$ have a Hamiltonian cycle, and hence have a 2-factor. Later, Bauer and Schmeichel[4] further showed that a 2-connected graph on n vertices satisfying

$b(G) \geqslant 3/2$ and $\delta(G) > \frac{(2-b(G))n}{3-b(G)}$ is pancyclic. Soon afterwards, Katerinis and Woodall[22] extended them to a general factor, and discovered that a graph G on $n \geqslant 4k-6$ vertices satisfying $b(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$ contains a k-factor. Subsequently, Kano and Tokushige[23] proved that a connected graph G on n vertices satisfying $b(G) \geqslant \frac{(a+b-1)(n-1)}{an-(a+b)-3}$ and $n \geqslant \frac{bn-2}{a+b}$ has an f-factor, where $1 \leqslant a \leqslant b$ are integers and $f : V(G) \to \{a, a+1, \ldots, b\}$ is a function such that $\sum f(x) \equiv 0 (\bmod 2)$ for $x \in V(G)$.

There is a rich history of studying the existence of factors in graphs from the perspective of eigenvalues. The pioneering work of Brouwer and Haemers[7] established the relation between eigenvalues and matching number. They presented several sufficient conditions, in terms of the eigenvalues of the adjacency and Laplacian matrices, for a graph to contain a perfect matching, which were subsequently improved in[11, 12, 13]. Gu and Haemers[19] proved a Laplacian eigenvalue condition for a graph to contain an $[a, b]$-factor (and a k-factor, accordingly), by using their result on toughness and Laplacian eigenvalues. More recently, O [26] proved that if $n \geqslant 8$ or $n = 4$, and $\rho(G) > \rho(K_1 \vee (K_{n-3} \cup 2K_1))$, or if $n = 6$ and $\rho(G) > \rho(K_2 \vee 4K_1)$, then G has a perfect matching. Lately, Cho, Hyun, O and Park[10] posed a conjecture that a graph G of order $n \geqslant a+1$ satisfying $\rho(G) > \rho(K_{a-1} \vee (K_1 \cup K_{n-a}))$ contains an $[a, b]$-factor. Subsequently, Fan, Lin and Lu[14] proved this conjecture holds for $n \geqslant 3a + b - 1$. Very recently, Wei and Zhang[29] confirmed this conjecture completely. Recall that the result in [22] showed that a graph G with $n \geqslant 4k - 6$ vertices satisfying $b(G) \geqslant 2$ contains a k-factor. Denote by $\delta(G)$ the minimum degree of G. Note that $\delta(G) \geqslant k$ is a trivial necessary condition for a graph to contain a k-factor. Then we consider the following problem.

Problem 2. Which 1-binding graphs with $\delta(G) \geqslant k$ have a k-factor?

Concerning the Problem 2, in this paper, we provide preliminary work from the perspective of spectral radius, and determine spectral conditions to guarantee the existence of a perfect matching and a 2-factor in 1-binding graphs, respectively. That is, we solved Problem 2 for $k = 1, 2$. When $k \geqslant 3$, Problem 2 seems more complicated and this is left for possible future work.

Theorem 3. *Let G be a connected 1-binding graph of even order $n \geqslant 12$. If $\rho(G) \geqslant \rho(K_1 \vee (K_{n-5} \cup K_3 \cup K_1))$, then G contains a perfect matching, unless $G \cong K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$.*

Let H_n be the graph obtained from $K_2 \vee (K_{n-5} \cup 3K_1)$ by adding an edge between K_{n-5} and $3K_1$.

Theorem 4. *Suppose that G is a connected 1-binding graph of order $n \geqslant 21$ with minimum degree $\delta(G) \geqslant 2$. If $\rho(G) \geqslant \rho(H_n)$, then G contains a 2-factor, unless $G \cong H_n$.*

2 Proof of Theorem 1

For $X, Y \subseteq V(G)$, we denote by $e_G(X, Y)$ the number of edges with one endpoint in X and one endpoint in Y. For any vertex $v \in V(G)$ and any subset $S \subseteq V(G)$, let $d_S(v) = |N_G(v) \cap S|$. Recall that $N_G(X) = \bigcup_{x \in X} N_G(x)$ for $X \subseteq V(G)$. We first consider an edge condition to guarantee a connected graph to be r-binding.

Theorem 5. *Let r be a positive integer, and let G be a connected graph of order n. Then the following statements hold.*

(i) If $n \geqslant 14$ and $e(G) \geqslant \binom{n-2}{2}+2$, then G is 1-binding, unless $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$.

(ii) If $n \geqslant 3r+5$, $r \geqslant 2$ and $e(G) \geqslant \frac{n^2-n}{2} - \lceil \frac{n-r}{r} \rceil$, then G is r-binding, unless $G \cong K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)$.

Proof. Suppose to the contrary that G is not r-binding where r is a positive integer, there exists some nonempty subset S of $V(G)$ with the maximum cardinality such that $\frac{|N_G(S)|}{|S|} < r$ and $N_G(S) \neq V(G)$. Take $S_1 = S \backslash (S \cap N_G(S))$ and $S_2 = S \cap N_G(S)$. It is clear that S_1 is an independent set and $e_G(S_1, S_2) = 0$. Let $N_1 = N_G(S) \backslash S_2$ and $N_2 = V(G) \backslash (S \cup N_G(S))$. Then $V(G) = S \cup N_1 \cup N_2$, $N_G(S_1) \subseteq N_1$ and $N_G(S) \cap N_2 = \varnothing$. Denote by $|N_i| = n_i$ and $|S_i| = s_i$ for $i = 1, 2$. Hence, $|N_G(S)| = s_2 + n_1$ and $|S| = s_1 + s_2$. Combining this with $\frac{|N_G(S)|}{|S|} < r$, we have

$$n_1 \leqslant rs_1 + (r-1)s_2 - 1. \tag{1}$$

We first assert that $N_2 = \varnothing$. Otherwise, $n_2 \geqslant 1$. In the case of $S_1 = \varnothing$, we obtain that $S = S_2 \neq \varnothing$. For $r = 1$, by $s_1 = 0$ and (1), we get $n_1 \leqslant -1$, a contradiction. Thus, we assume that $r \geqslant 2$. Putting (1) and $s_1 = 0$ into $n = s_1 + s_2 + n_1 + n_2$, we have $s_2 \geqslant \lceil \frac{n-n_2+1}{r} \rceil \geqslant 1$. This implies that $n > n_2$. Since $N_G(S_2) \cap N_2 = \varnothing$, it follows that $e_G(S_2, N_2) = 0$. Therefore,

$$\begin{aligned}
e(G) &\leqslant \frac{n^2-n}{2} - s_2 n_2 \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(s_2 n_2 - \left\lceil \frac{n-r}{r} \right\rceil \right) \\
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{(n-n_2+1)n_2}{r} - \frac{n}{r} \right) \text{ (since } s_2 \geqslant \tfrac{n-n_2+1}{r} \text{ and } \lceil \tfrac{n-r}{r} \rceil < \tfrac{n}{r}) \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \frac{(n_2-1)(n-n_2)}{r} \\
&\leqslant \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil \text{ (since } n_2 \geqslant 1 \text{ and } n > n_2),
\end{aligned}$$

a contradiction. In the case of $S_1 \neq \varnothing$, let $S' = S \cup N_2$. Observe that $N_G(S') \subseteq (N_G(S) \cup N_2) = V(G) \backslash S_1$. Thus, $|S'| = s_1 + s_2 + n_2$ and $|N_G(S')| \leqslant s_2 + n_1 + n_2$, and hence

$$\frac{|N_G(S')|}{|S'|} \leqslant \frac{s_2 + n_1 + n_2}{s_1 + s_2 + n_2} \leqslant \frac{r(s_1+s_2) + n_2 - 1}{s_1 + s_2 + n_2} < r$$

due to (1) and $r \geqslant 1$, which contradicts the maximality of S. This implies that $N_2 = \varnothing$ and $V(G) = S_1 \cup S_2 \cup N_1$. We next assert that $S_1 \neq \varnothing$. Otherwise, $N_G(S_2) = N_G(S) = S_2 \cup N_1 = V(G)$, contrary to the assumption that $N_G(S) \neq V(G)$. Hence, $s_1 \geqslant 1$. Next, we will divide the proof into the following two cases basing on the value of r.

Case 1. $r = 1$.

By (1), we have

$$n_1 \leqslant s_1 - 1. \tag{2}$$

Recall that $N_G(S_1) \subseteq N_1$ and $S_1 \neq \varnothing$. As G is a connected graph, we have $n_1 \geqslant 1$, and hence $s_1 \geqslant 2$ by (2). For $s_1 = 2$, we get $n_1 = 1$. One can verify that G is a spanning subgraph of $K_1 \vee (K_{n-3} \cup 2K_1)$. Therefore,

$$e(G) \leqslant e(K_1 \vee (K_{n-3} \cup 2K_1)) = \binom{n-2}{2} + 2,$$

with equality if and only if $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$, a contradiction. Thus, we consider $s_1 \geqslant 3$ in the following. If $S_2 = \varnothing$, then $n = s_1 + n_1$. Putting (2) into $n = n_1 + s_1$, we have $s_1 \geqslant (n+1)/2$. Since $N_G(S_1) \subseteq N_1$ and S_1 is an independent set, G is a spanning subgraph of $K_{n-s_1} \vee s_1 K_1$. Combining this with $n \geqslant 14$ and $s_1 \geqslant (n+1)/2$, we can deduce that

$$\begin{aligned}
e(G) &\leqslant e(K_{n-s_1} \vee s_1 K_1) \\
&= \frac{n^2 - n - s_1^2 + s_1}{2} \\
&= \binom{n-2}{2} + 2 - \Big(\frac{s_1^2 - s_1}{2} - 2n + 5\Big) \\
&\leqslant \binom{n-2}{2} + 2 - \frac{(n-3)(n-13)}{8} \text{ (since } s_1 \geqslant \tfrac{n+1}{2}) \\
&< \binom{n-2}{2} + 2 \text{ (since } n \geqslant 14),
\end{aligned}$$

a contradiction. If $S_2 \neq \varnothing$, since $S_2 = S \cap N_G(S)$, it follows that $d_{S_2}(v) \geqslant 1$ for each $v \in S_2$. This implies that $s_2 \geqslant 2$. Observe that G is a spanning subgraph of $K_{n_1} \vee (K_{n-s_1-n_1} \cup s_1 K_1)$. Combining this with $s_1 \geqslant 3$, $s_2 \geqslant 2$ and (2), we get

$$\begin{aligned}
e(G) &\leqslant e(K_{n_1} \vee (K_{n-s_1-n_1} \cup s_1 K_1)) \\
&= s_1 n_1 + \binom{n-s_1}{2} \\
&= \frac{n^2 - (2s_1+1)n + s_1^2 + s_1 + 2s_1 n_1}{2} \\
&= \binom{n-2}{2} + 2 - \frac{2(s_1-2)n - s_1^2 - (2n_1+1)s_1 + 10}{2} \\
&= \binom{n-2}{2} + 2 - \frac{s_1^2 + 2s_1 s_2 - 4n_1 - 5s_1 - 4s_2 + 10}{2} \text{ (since } n = n_1 + s_1 + s_2) \\
&\leqslant \binom{n-2}{2} + 2 - \frac{(s_1-2)(s_1+2s_2-7)}{2} \text{ (since } n_1 \leqslant s_1 - 1) \\
&\leqslant \binom{n-2}{2} + 2 \text{ (since } s_1 \geqslant 3 \text{ and } s_2 \geqslant 2),
\end{aligned}$$

where all above equalities hold if and only if $s_1 = 3$, $s_2 = 2$, $n_1 = 2$ and $n = s_1+s_2+n_1 = 7$. This is impossible because $n \geqslant 14$. It follows that $e(G) < \binom{n-2}{2} + 2$, which also deduces a contradiction.

Case 2. $r \geqslant 2$.

We first consider $S_2 = \varnothing$. Then $n_1 \leqslant rs_1 - 1$ by (1). Combining this with $n = n_1 + s_1$, we obtain that $s_1 \geqslant \lceil \frac{n+1}{r+1} \rceil$. Notice that G is a spanning subgraph of $K_{n-s_1} \vee s_1K_1$. Thus,

$$\begin{aligned}
e(G) &\leqslant e(K_{n-s_1} \vee s_1K_1) \\
&= s_1(n-s_1) + \binom{n-s_1}{2} \\
&= \frac{n^2 - n - s_1^2 + s_1}{2} \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{s_1^2 - s_1}{2} - \left\lceil \frac{n-r}{r} \right\rceil\right) \\
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{\left(\frac{n+1}{r+1}\right)^2 - \frac{n+1}{r+1}}{2} - \frac{n}{r}\right) \text{ (since } s_1 \geqslant \tfrac{n+1}{r+1} \text{ and } \lceil \tfrac{n-r}{r} \rceil < \tfrac{n}{r}) \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \frac{rn^2 - (3r^2+3r+2)n - r^2}{2r(r+1)^2} \\
&\leqslant \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \frac{5r^2+4r-10}{2r(r+1)^2} \text{ (since } n \geqslant 3r+5 \text{ and } r \geqslant 2) \\
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil \text{ (since } r \geqslant 2),
\end{aligned}$$

a contradiction. Thus, we consider $S_2 \neq \varnothing$ in the following. By using the same analysis as Case 1, we also deduce that $s_2 \geqslant 2$. Moreover, putting (1) into $n = s_1 + s_2 + n_1$, we have

$$s_2 \geqslant \left\lceil \frac{n-(r+1)s_1+1}{r} \right\rceil.$$

Therefore,

$$s_2 \geqslant \max\left\{2, \left\lceil \frac{n-(r+1)s_1+1}{r} \right\rceil\right\}. \tag{3}$$

If $\lceil \frac{n-(r+1)s_1+1}{r} \rceil < 2$, then $n \leqslant (r+1)s_1 + r - 1$, and hence $s_1 \geqslant \lceil \frac{n-r+1}{r+1} \rceil$. According to (3), we get $s_2 \geqslant 2$, and hence $n_1 = n - s_1 - s_2 \leqslant n - s_1 - 2$. One can verify that G is a spanning subgraph of $K_{n-s_1-2} \vee (K_2 \cup s_1K_1)$. Combining this with $s_1 \geqslant \lceil \frac{n-r+1}{r+1} \rceil$, $r \geqslant 2$ and $n \geqslant 3r+5$, we get

$$\begin{aligned}
e(G) &\leqslant e(K_{n-s_1-2} \vee (K_2 \cup s_1K_1)) \\
&= s_1(n-s_1-2) + \binom{n-s_1}{2} \\
&= \frac{n^2 - n - s_1^2 - 3s_1}{2} \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{s_1^2 + 3s_1}{2} - \left\lceil \frac{n-r}{r} \right\rceil\right)
\end{aligned}$$

$$
\begin{aligned}
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{\left(\frac{n-r+1}{r+1}\right)^2 + \frac{3(n-r+1)}{r+1}}{2} - \frac{n}{r} \right) \\
&\quad (\text{since } s_1 \geqslant \lceil \tfrac{n-r+1}{r+1} \rceil \geqslant \tfrac{n-r+1}{r+1} \text{ and } \lceil \tfrac{n-r}{r} \rceil < \tfrac{n}{r}) \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \frac{(nr+r^2+r-2)(n-2r)}{2r(r+1)^2} \\
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil \ (\text{since } r \geqslant 2 \text{ and } n \geqslant 3r+5),
\end{aligned}
$$

a contradiction.

If $\lceil \frac{n-(r+1)s_1+1}{r} \rceil \geqslant 2$, then $n \geqslant (r+1)s_1 + r$, and hence $1 \leqslant s_1 \leqslant \lfloor \frac{n-r}{r+1} \rfloor$ and $s_2 \geqslant \lceil \frac{n-(r+1)s_1+1}{r} \rceil$ due to (3). We can find that G is a spanning subgraph of $K_{n-s_1-\lceil \frac{n-(r+1)s_1+1}{r} \rceil} \vee (K_{\lceil \frac{n-(r+1)s_1+1}{r} \rceil} \cup s_1K_1)$. For $s_1 = 1$, we get

$$
e(G) \leqslant e(K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)) = \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil,
$$

with equality if and only if $G \cong K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)$, a contradiction. For $2 \leqslant s_1 \leqslant \lfloor \frac{n-r}{r+1} \rfloor$, we have

$$
\begin{aligned}
e(G) &\leqslant e(K_{n-s_1-\lceil \frac{n-(r+1)s_1+1}{r} \rceil} \vee (K_{\lceil \frac{n-(r+1)s_1+1}{r} \rceil} \cup s_1K_1)) \\
&= s_1 \left(n - s_1 - \left\lceil \frac{n-(r+1)s_1+1}{r} \right\rceil \right) + \binom{n-s_1}{2} \\
&= \frac{n^2-n-s_1^2+s_1}{2} - s_1 \cdot \left\lceil \frac{n-(r+1)s_1+1}{r} \right\rceil \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{s_1^2-s_1}{2} + s_1 \cdot \left\lceil \frac{n-(r+1)s_1+1}{r} \right\rceil - \left\lceil \frac{n-r}{r} \right\rceil \right) \\
&< \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \left(\frac{s_1^2-s_1}{2} + \frac{s_1(n-(r+1)s_1+1)}{r} - \frac{n}{r} \right) \\
&= \frac{n^2-n}{2} - \left\lceil \frac{n-r}{r} \right\rceil - \frac{(-r-2)s_1^2 + (-r+2n+2)s_1 - 2n}{2r},
\end{aligned}
$$

where the penultimate inequality follows from the facts that $\lceil \frac{n-(r+1)s_1+1}{r} \rceil \geqslant \frac{n-(r+1)s_1+1}{r}$ and $\lceil \frac{n-r}{r} \rceil < \frac{n}{r}$. Let $f(s_1) = (-r-2)s_1^2 + (-r+2n+2)s_1 - 2n$. Then the symmetry axis of parabola $f(s_1)$ is $x = \frac{2n-r+2}{2r+4} > 2$ due to $n \geqslant 3r+5$ and $r \geqslant 2$. By a simple calculation, we have $f(2) = 2n - 6r - 4 > 0$. If $\lfloor \frac{n-r}{r+1} \rfloor > x$, then $f(s_1)$ is decreasing with respect to $s_1 \geqslant \lfloor \frac{n-r}{r+1} \rfloor$, and hence

$$
f\left(\left\lfloor \frac{n-r}{r+1} \right\rfloor \right) \geqslant f\left(\frac{n-r}{r+1} \right) = \frac{r(n+1)(n-3r-2)}{(r+1)^2} > 0
$$

due to $r \geqslant 2$ and $n \geqslant 3r+5$. Therefore, for $2 \leqslant s_1 \leqslant \lfloor \frac{n-r}{r+1} \rfloor$, we have

$$
f(s_1) \geqslant \min \left\{ f\left(\left\lfloor \frac{n-r}{r+1} \right\rfloor \right), f(2) \right\} > 0.
$$

This suggests that $e(G) < \frac{n^2-n}{2} - \lceil \frac{n-r}{r} \rceil$, a contradiction. Also, if $x \geqslant \lfloor \frac{n-r}{r+1} \rfloor$, then $f(s_1)$ is increasing with respect to $2 \leqslant s_1 \leqslant \lfloor \frac{n-r}{r+1} \rfloor$. Hence, $f(s_1) \geqslant f(2) > 0$, which also leads to a contradiction.

This completes the proof. $\square$

Let M be a real $n \times n$ matrix, and let $X = \{1, 2, \ldots, n\}$. Given a partition $\Pi = \{X_1, X_2, \ldots, X_k\}$ with $X = X_1 \cup X_2 \cup \cdots \cup X_k$, the matrix M can be partitioned as

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{pmatrix}.$$

The *quotient matrix* of M with respect to Π is defined as the $k \times k$ matrix $B_\Pi = (b_{i,j})_{i,j=1}^k$ where $b_{i,j}$ is the average value of all row sums of $M_{i,j}$. The partition Π is called *equitable* if each block $M_{i,j}$ of M has constant row sum $b_{i,j}$. Also, we say that the quotient matrix B_Π is *equitable* if Π is an equitable partition of M.

Lemma 6 (See [6]). *Let M be a real symmetric matrix, and let $\lambda_1(M)$ be the largest eigenvalue of M. If B_Π is an equitable quotient matrix of M, then the eigenvalues of B_Π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda_1(M) = \lambda_1(B_\Pi)$.*

Lemma 7 (See [20]). *Let G be a graph with n vertices and m edges. Then*

$$\rho(G) \leqslant \sqrt{2m - n + 1},$$

where the equality holds if and only if G is a star or a complete graph.

By Theorem 5, Lemmas 6 and 7, we give the proof of Theorem 1.

Proof of Theorem 1. Assume to the contrary that G is not r-binding where r is a positive integer, there exists some nonempty subset S of $V(G)$ with maximum cardinality such that $\frac{|N_G(S)|}{|S|} < r$ and $N_G(S) \neq V(G)$. Take $S_1 = S \backslash (S \cap N_G(S))$ and $S_2 = S \cap N_G(S)$. It is clear that S_1 is an independent set and $e_G(S_1, S_2) = 0$. Let $N_1 = N_G(S) \backslash S_2$ and $N_2 = V(G) \backslash (S \cup N_G(S))$. Then $V(G) = S \cup N_1 \cup N_2$, $N_G(S_1) \subseteq N_1$ and $N_G(S) \cap N_2 = \varnothing$. Denote by $|N_i| = n_i$ and $|S_i| = s_i$ for $i = 1, 2$. Therefore, $|N_G(S)| = s_2 + n_1$ and $|S| = s_1 + s_2$. Combining this with $\frac{|N_G(S)|}{|S|} < r$, we have

$$n_1 \leqslant rs_1 + (r-1)s_2 - 1. \tag{4}$$

We first assert that $N_2 = \varnothing$. Otherwise, $n_2 \geqslant 1$. In the case of $S_1 = \varnothing$, we obtain that $S = S_2 \neq \varnothing$ and $V(G) = S_2 \cup N_1 \cup N_2$. If $r = 1$, by $s_1 = 0$ and (4), we get $n_1 \leqslant -1$, a contradiction. Thus, we assume that $r \geqslant 2$. For $1 \leqslant n_2 \leqslant r+1$, let $w \in N_2$ and $S^* = S_2 \cup \{w\}$. Since $w \notin N_G(S_2)$, it follows that $N_G(S^*) \subseteq V(G) \backslash \{w\}$, and hence

$|N_G(S^*)| \leqslant n-1$. Combining this with $s_1 = 0$, $n_2 \leqslant r+1$, $n = s_2+n_1+n_2$ and (4), we deduce that

$$\frac{|N_G(S^*)|}{|S^*|} \leqslant \frac{n-1}{s_2+1} = \frac{s_2+n_1+n_2-1}{s_2+1} \leqslant \frac{n_2+rs_2-2}{s_2+1} \leqslant \frac{r(s_2+1)-1}{s_2+1} < r,$$

which contradicts the maximality of S. Thus, we consider $n_2 \geqslant r+2$ in the following. If $s_2 < n_2/2$, let $v \in S_2$ and $S' = N_2 \cup \{v\}$. As $N_G(v) \cap N_2 = \varnothing$, we deduce that $N_G(S') \subseteq V(G) \backslash \{v\}$. Notice that $|S'| = n_2+1$ and $|N_G(S')| \leqslant n-1$. Combining this with $s_1 = 0$, $s_2 < n_2/2$, $n_2 \geqslant r+2$, $r \geqslant 2$ and (4), we have

$$\frac{|N_G(S')|}{|S'|} \leqslant \frac{n-1}{n_2+1} = \frac{s_2+n_1+n_2-1}{n_2+1} \leqslant \frac{n_2+rs_2-2}{n_2+1} < \frac{r(n_2+1)-\frac{(r-2)n_2}{2}-r-2}{n_2+1} < r,$$

which also contradicts the maximality of S. Since $N_G(S_2) \cap N_2 = \varnothing$, it follows that $e_G(S_2, N_2) = 0$, and hence $e(G) \leqslant \frac{n(n-1)}{2} - n_2 s_2$. If $s_2 \geqslant n_2/2$, by $s_1 = 0$, (4) and Lemma 7, we get

$$\begin{aligned}
\rho(G) &\leqslant \sqrt{2e(G)-n+1} \\
&\leqslant \sqrt{n^2-2n+1-2s_2n_2} \text{ (since } e(G) \leqslant \tfrac{n(n-1)}{2} - n_2s_2) \\
&= \sqrt{(n-2)^2-(2s_2n_2-2n+3)} \\
&= \sqrt{(n-2)^2-(2s_2n_2-2(n_1+n_2+s_2)+3)} \text{ (since } n = s_2+n_1+n_2) \\
&\leqslant \sqrt{(n-2)^2-(2s_2n_2-2rs_2-2n_2+5)} \text{ (since } n_1 \leqslant (r-1)s_2-1) \\
&\leqslant \sqrt{(n-2)^2-(2s_2(n_2-r-2)+5)} \text{ (since } n_2 \leqslant 2s_2) \\
&< n-2 \text{ (since } n_2 \geqslant r+2).
\end{aligned}$$

Notice that $K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1)$ contains K_{n-1} as a proper subgraph. Thus,

$$\rho(G) < n-2 < \rho(K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1)),$$

a contradiction. In the case of $S_1 \neq \varnothing$, by using the same analysis as Theorem 5, we also deduce a contradiction. This implies that $N_2 = \varnothing$ and $V(G) = S_1 \cup S_2 \cup N_1$. We next assert that $S_1 \neq \varnothing$. Otherwise, $N_G(S_2) = N_G(S) = S_2 \cup N_1 = V(G)$, contrary to the assumption that $N_G(S) \neq V(G)$. Hence, $s_1 \geqslant 1$. Next, we will divide the proof into the following two cases basing on the value of r.

Case 1. $r = 1$.

Since K_{n-2} is a proper subgraph of $K_1 \vee (K_{n-3} \cup 2K_1)$, it follows that $\rho(G) \geqslant \rho(K_1 \vee (K_{n-3} \cup 2K_1)) > n-3$. Combining this with Lemma 7, we have $e(G) \geqslant \binom{n-2}{2} + 2$. By Theorem 5, we can deduce that $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$ for $n \geqslant 14$, a contradiction.

Case 2. $r \geqslant 2$.

We first consider $S_2 = \varnothing$. Thus, $n_1 \leqslant rs_1 - 1$ by (4). Combining this with $n = s_1 + n_1$, we have $s_1 \geqslant \lceil \frac{n+1}{r+1} \rceil$. Let $p = \lceil \frac{n+1}{r+1} \rceil$. Then G is a spanning subgraph of $K_{n-p} \vee pK_1$, and hence

$$\rho(G) \leqslant \rho(K_{n-p} \vee pK_1), \tag{5}$$

with equality if and only if $G \cong K_{n-p} \vee pK_1$. Note that $A(K_{n-p} \vee pK_1)$ has the equitable quotient matrix

$$B_{\Pi_1} = \begin{bmatrix} 0 & n-p \\ p & n-p-1 \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of B_{Π_1} is

$$f(x) = x^2 + (-n+p+1)x - p(n-p).$$

Let $t = \lceil \frac{n-r}{r} \rceil$. Notice that $A(K_{n-t-1} \vee (K_t \cup K_1))$ has the equitable quotient matrix

$$C_{\Pi_2} = \begin{bmatrix} 0 & n-t-1 & 0 \\ 1 & n-t-2 & t \\ 0 & n-t-1 & t-1 \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of C_{Π_2} is

$$g(x) = x^3 - (n-3)x^2 - (2n-t-3)x + tn - t^2 - n + 1. \tag{6}$$

Since $n \geqslant 5r^2 + 3$ and $r \geqslant 2$, it follows that $p = \lceil \frac{n+1}{r+1} \rceil > 5$ and $t = \lceil \frac{n-r}{r} \rceil > 4$. Let

$$\tau(x) = xf(x) - g(x) = (p-2)x^2 + (-pn + p^2 + 2n - t - 3)x - tn + t^2 + n - 1.$$

Then the symmetry axis of parabola $\tau(x)$ is

$$\frac{(p-2)n - p^2 + t + 3}{2(p-2)} = \frac{n}{2} + \frac{-p^2 + t + 3}{2(p-2)} < \frac{n}{2} + \frac{t}{2(p-2)} < \frac{n}{2} + \frac{n}{2(p-2)r} < n - 2$$

due to $p > 5$, $r \geqslant 2$, $t < n/r$ and $n \geqslant 5r^2 + 3$. This implies that $\tau(x)$ is increasing with respect to $x \geqslant n - 2$. For $x \geqslant n - 2$,

$$\begin{aligned}
\tau(x) &\geqslant \tau(n-2) \\
&= (p^2 - 2p - 2t + 2)n - 2p^2 + t^2 + 4p + 2t - 3 \\
&= (p^2 - 2p - 2t + 2)(n-2) + (t-1)^2 \\
&> (p^2 - 2p - 2t + 2)(n-2) \text{ (since } t > 4) \\
&> \left(\left(\frac{n+1}{r+1} \right)^2 - \frac{2n+2}{r+1} - \frac{2n}{r} + 2 \right)(n-2) \text{ (since } p \geqslant \tfrac{n+1}{r+1} \text{ and } t < \tfrac{n}{r}) \\
&= \frac{(rn^2 - (4r^2 + 4r + 2)n + 2r^3 + 2r^2 + r)(n-2)}{r(r+1)^2} \\
&\geqslant \frac{(25r^5 - 20r^4 + 12r^3 - 20r^2 - 2r - 6)(n-2)}{r(r+1)^2} \text{ (since } n \geqslant 5r^2 + 3) \\
&> 0 \text{ (since } r \geqslant 2 \text{ and } n \geqslant 5r^2 + 3),
\end{aligned}$$

which leads to $xf(x) > g(x)$ for $x \geqslant n-2$. Since K_{n-1} is a proper subgraph of $K_{n-t-1} \vee (K_t \cup K_1)$, we have

$$\rho(K_{n-t-1} \vee (K_t \cup K_1)) > n-2. \tag{7}$$

It follows that $\lambda_1(B_{\Pi_1}) < \lambda_1(C_{\Pi_2})$. According to Lemma 6, we get

$$\rho\Big(K_{n-\lceil \frac{n+1}{r+1} \rceil} \vee \Big\lceil \frac{n+1}{r+1} \Big\rceil K_1\Big) < \rho(K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)).$$

Combining this with (5), we obtain that

$$\rho(G) < \rho(K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)),$$

a contradiction. Thus, we consider $S_2 \neq \varnothing$. By using a similar analysis as the Case 2 of Theorem 5, we can deduce that

$$s_2 \geqslant \max\Big\{2, \Big\lceil \frac{n-(r+1)s_1+1}{r} \Big\rceil\Big\}. \tag{8}$$

We have the following two subcases.

Subcase 2.1. $1 \leqslant s_1 \leqslant r$.

Since $1 \leqslant s_1 \leqslant r$, it follows that $\lceil \frac{n-(r+1)s_1+1}{r} \rceil > 2$ due to $n \geqslant 5r^2+3$. Thus, from (8), we have $s_2 \geqslant \lceil \frac{n-(r+1)s_1+1}{r} \rceil$. Assume that $a = s_1$ and $b = \lceil \frac{n-(r+1)s_1+1}{r} \rceil$. One can verify that G is a spanning subgraph of $K_{n-a-b} \vee (K_b \cup aK_1)$, and hence

$$\rho(G) \leqslant \rho(K_{n-a-b} \vee (K_b \cup aK_1)), \tag{9}$$

with equality if and only if $G \cong K_{n-a-b} \vee (K_b \cup aK_1)$. If $a = 1$, then $b = \lceil \frac{n-r}{r} \rceil$, and hence

$$\rho(G) \leqslant \rho(K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)),$$

with equality if and only if $G \cong K_{n-\lceil \frac{n-r}{r} \rceil -1} \vee (K_{\lceil \frac{n-r}{r} \rceil} \cup K_1)$, a contradiction. Thus, we consider $2 \leqslant a \leqslant r$. Note that $A(K_{n-a-b} \vee (K_b \cup aK_1))$ has the equitable quotient matrix

$$D_{\Pi_3} = \begin{bmatrix} 0 & n-a-b & 0 \\ a & n-a-b-1 & b \\ 0 & n-a-b & b-1 \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of D_{Π_3} is

$$h(x) = x^3 - (n-a-2)x^2 - ((a+1)n - a^2 - ab - a - 1)x - a(b-1)(a+b-n).$$

Combining this with (6), we have

$$h(x) - g(x) = (a-1)x^2 + ((1-a)n + a^2 + ab + a - t - 2)x - a(b-1)(a+b-n) - tn + t^2 + n - 1.$$

Define

$$\omega(x) = (a-1)x^2 + ((1-a)n + a^2 + ab + a - t - 2)x - a(b-1)(a+b-n) - tn + t^2 + n - 1.$$

The symmetry axis of parabola $\omega(x)$ is

$$\frac{(a-1)n-a^2-ab-a+t+2}{2(a-1)}<\frac{n}{2}+\frac{t}{2(a-1)}<\frac{n}{2}+\frac{n}{2(a-1)r}\leqslant\frac{3n}{4}<n-2$$

due to $a\geqslant 2$, $b>2$, $t<n/r$, $r\geqslant 2$ and $n\geqslant 5r^2+3$. This implies that $\omega(x)$ is increasing with respect to $x\geqslant n-2$. Hence,

$$\begin{aligned}
\omega(x)&\geqslant\omega(n-2)\\
&=(a^2+2ab-2a-2t+1)n-ab(a+b+1)-(a-1)^2+(t+1)^2-1\\
&>\Big(s_1^2+\frac{2s_1(n-(r+1)s_1+1)}{r}-2s_1-\frac{2n}{r}+1\Big)n-s_1\cdot\Big(\frac{n-(r+1)s_1+1}{r}+1\Big)\cdot\\
&\quad\Big(s_1+\Big(\frac{n-(r+1)s_1+1}{r}+1\Big)+1\Big)-(s_1-1)^2+\Big(\frac{n-r}{r}+1\Big)^2-1\\
&=\frac{1}{r^2}((2r-1)(s_1-1)n^2+((2-r^2-r)s_1^2-(2r^2+r+2)s_1+r^2)n+(-r-1)s_1^3+(r^2+\\
&\quad 4r+2)s_1^2+(-3r-1)s_1-2r^2),
\end{aligned}$$

where the penultimate inequality follows from the facts that $(n-r)/r\leqslant t<n/r$ and $(n-(r+1)s_1+1)/r\leqslant b<(n-(r+1)s_1+1)/r+1$. Let

$$\begin{aligned}
\varphi(n)=&(2r-1)(s_1-1)n^2+((2-r^2-r)s_1^2-(2r^2+r+2)s_1+r^2)n+(-r-1)s_1^3+(r^2+\\
&4r+2)s_1^2+(-3r-1)s_1-2r^2.
\end{aligned}$$

We take the derivative of $\varphi(n)$. Then

$$\begin{aligned}
\varphi'(n)&=2(2r-1)(s_1-1)n+(2-r^2-r)s_1^2-(2r^2+r+2)s_1+r^2\\
&\geqslant(2s_1-2)r^4+(9s_1-9)r^3+(4-s_1^2-5s_1)r^2+(1-s_1^2-2s_1)r+2s_1^2-2s_1\\
&\quad\text{(since } n\geqslant\tfrac{r^3+5r^2+r}{2},\ s_1\geqslant 2 \text{ and } r\geqslant 2)\\
&\geqslant(2s_1-2)r^4+(9s_1-9)r^3+(4-r^2-5r)r^2+(1-r^2-2r)r+2s_1^2-2s_1\ \text{(since } s_1\leqslant r)\\
&=(2s_1-3)r^4+(9s_1-15)r^3+2r^2+r+2s_1^2-2s_1\\
&>0\ \text{(since } s_1\geqslant 2 \text{ and } r\geqslant 2).
\end{aligned}$$

This implies that $\varphi(n)$ is increasing with respect to $n\geqslant\frac{r^3+5r^2+r}{2}$. Therefore,

$$\begin{aligned}
\varphi(n)&\geqslant\varphi\Big(\frac{r^3+5r^2+r}{2}\Big)\\
&=-(r+1)s_1^3+\Big(2-\frac{r^5}{2}-3r^4-2r^3+\frac{11r^2}{2}+5r\Big)s_1^2+\Big(-1+\frac{r^7}{2}+\frac{19r^6}{4}+10r^5\\
&\quad-\frac{29r^4}{4}-\frac{13r^3}{2}-\frac{23r^2}{4}-4r\Big)s_1-\frac{2r^7+19r^6+42r^5-17r^4-10r^3+7r^2}{4}\\
&=\psi(s_1).
\end{aligned}$$

Also, we take derivative of $\psi(s_1)$. Then

$$\psi'(s_1) = -3(r+1)s_1^2+2\Big(2-\frac{r^5}{2}-3r^4-2r^3+\frac{11r^2}{2}+5r\Big)s_1-1+\frac{r^7}{2}+\frac{19r^6}{4}+10r^5-\frac{29r^4}{4}-\frac{13r^3}{2}-\frac{23r^2}{4}-4r,$$

and the symmetry axis of parabola $\psi'(s_1)$ is $\frac{2-\frac{r^5}{2}-3r^4-2r^3+\frac{11r^2}{2}+5r}{3(r+1)} < 0$ due to $r \geqslant 2$. Since

$$\psi'(r) = \frac{2r^7+15r^6+16r^5-45r^4+6r^3+5r^2-4}{4} > 0$$

due to $r \geqslant 2$, it follows that $\psi(s_1)$ is increasing with respect to $2 \leqslant s_1 \leqslant r$. Therefore,

$$\psi(s_1) \geqslant \psi(2) = \frac{2r^7+19r^6+30r^5-89r^4-74r^3+35r^2+16r-8}{4} > 0$$

due to $r \geqslant 2$, which leads to $\varphi(n) > 0$, and hence $\omega(x) > 0$ for $x \geqslant n-2$. This implies that $h(x) > g(x)$ for $x \geqslant n-2$. According to (7), we obtain that $\lambda_1(D_{\Pi_3}) < \lambda_1(C_{\Pi_2})$. Furthermore, by Lemma 6, we can deduce that

$$\rho(K_{n-s_1-\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \vee (K_{\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \cup s_1K_1)) < \rho(K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1))$$

for $2 \leqslant s_1 \leqslant r$. Combining this with (9), we have

$$\rho(G) < \rho(K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1)),$$

a contradiction.

Subcase 2.2. $s_1 \geqslant r+1$.

If $\lceil\frac{n-(r+1)s_1+1}{r}\rceil \geqslant 2$, then $n \geqslant (r+1)s_1+r$, and hence $r+1 \leqslant s_1 \leqslant \lfloor\frac{n-r}{r+1}\rfloor$ and $s_2 \geqslant \lceil\frac{n-(r+1)s_1+1}{r}\rceil$ due to (8). Therefore, G is a spanning subgraph of $K_{n-s_1-\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \vee (K_{\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \cup s_1K_1)$. It follows that

$$\begin{aligned} e(G) &\leqslant e(K_{n-s_1-\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \vee (K_{\lceil\frac{n-(r+1)s_1+1}{r}\rceil} \cup s_1K_1)) \\ &= s_1\Big(n-\Big\lceil\frac{n-(r+1)s_1+1}{r}\Big\rceil-s_1\Big)+\binom{n-s_1}{2} \\ &= \frac{n^2-n-s_1^2+s_1}{2}-s_1\cdot\Big\lceil\frac{n-(r+1)s_1+1}{r}\Big\rceil \\ &\leqslant \frac{rn^2-(r+2s_1)n+(r+2)s_1^2+(r-2)s_1}{2r}, \end{aligned}$$

where the last inequality follows from the fact that $\lceil\frac{n-(r+1)s_1+1}{r}\rceil \geqslant \frac{n-(r+1)s_1+1}{r}$. Combining this with Lemma 7, we have

$$\rho(G) \leqslant \sqrt{2e(G)-n+1} \leqslant \sqrt{\frac{(r+2)s_1^2+(r-2n-2)s_1+r(n-1)^2}{r}}.$$

Let $\phi(s_1)=(r+2)s_1^2+(r-2n-2)s_1+r(n-1)^2$. Then the symmetry axis of parabola $\phi(s_1)$ is $y=\frac{n+1-\frac{r}{2}}{r+2}$. If $y<\lfloor\frac{n-r}{r+1}\rfloor$, then $\phi(s_1)$ is increasing with respect to $s_1\geqslant\lfloor\frac{n-r}{r+1}\rfloor$. Notice that $n\geqslant 5r^2+3$, $\lfloor\frac{n-r}{r+1}\rfloor\leqslant\frac{n-r}{r+1}$ and $r\geqslant 2$. Thus,

$$\begin{aligned}
&\phi(r+1)-\phi\Big(\Big\lfloor\frac{n-r}{r+1}\Big\rfloor\Big)\\
&\geqslant (r+2)(r+1)^2+(r-2n-2)(r+1)-(r+2)\Big(\frac{n-r}{r+1}\Big)^2-(r-2n-2)\Big(\frac{n-r}{r+1}\Big)\\
&=\frac{r(n-r^2-3r-1)(n-r^2-4r-2)}{(r+1)^2}\\
&>0.
\end{aligned}$$

This implies that, for $r+1\leqslant s_1\leqslant\lfloor\frac{n-r}{r+1}\rfloor$, the maximum value of $\phi(s_1)$ is attained at $s_1=r+1$, and hence

$$\rho(G)\leqslant\sqrt{\frac{\phi(r+1)}{r}}=\sqrt{(n-2)^2-\frac{2n-r^3-5r^2-r}{r}}\leqslant n-2$$

due to $n\geqslant\frac{r^3+5r^2+r}{2}$. Combining this with (7), we can deduce that

$$\rho(G)\leqslant n-2<\rho(K_{n-\lceil\frac{n-r}{r}\rceil-1}\vee(K_{\lceil\frac{n-r}{r}\rceil}\cup K_1)),$$

a contradiction. If $y\geqslant\lfloor\frac{n-r}{r+1}\rfloor$, then $\phi(s_1)$ is decreasing with respect to $r+1\leqslant s_1\leqslant\lfloor\frac{n-r}{r+1}\rfloor$. By using a similar analysis as above, we also deduce a contradiction.

If $\lceil\frac{n-(r+1)s_1+1}{r}\rceil<2$, then $n\leqslant(r+1)s_1+r-1$, and hence $s_1\geqslant\lceil\frac{n-r+1}{r+1}\rceil$. By (8), we get $s_2\geqslant 2$. Thus, G is a spanning subgraph of $K_{n-s_1-2}\vee(K_2\cup s_1K_1)$, and hence

$$e(G)\leqslant s_1(n-s_1-2)+\binom{n-s_1}{2}=\frac{n^2-n-s_1^2-3s_1}{2}.$$

Combining this with Lemma 7, we have

$$\begin{aligned}
\rho(G)&\leqslant\sqrt{2e(G)-n+1}\\
&\leqslant\sqrt{n^2-2n-s_1^2-3s_1+1}\\
&\leqslant\sqrt{n^2-2n-\Big(\frac{n-r+1}{r+1}\Big)^2-3\Big(\frac{n-r+1}{r+1}\Big)+1}\ (\text{since } s_1\geqslant\tfrac{n-r+1}{r+1})\\
&=\sqrt{(n-2)^2-\frac{n^2+(-2r^2-3r+3)n+r^2+4r+7}{(r+1)^2}}\\
&\leqslant\sqrt{(n-2)^2-\frac{15r^4-15r^3+40r^2-5r+25}{(r+1)^2}}\ (\text{since } n\geqslant 5r^2+3)\\
&<n-2\ (\text{since } r\geqslant 2).
\end{aligned}$$

Again by (7), we get

$$\rho(G) < n-2 < \rho(K_{n-\lceil\frac{n-r}{r}\rceil-1} \vee (K_{\lceil\frac{n-r}{r}\rceil} \cup K_1)),$$

which also leads to a contradiction.

This completes the proof. $\square$

3 Proof of Theorem 3

Lemma 8 (See [27]). *A graph G has a perfect matching if and only if for every subset $S \subseteq V(G)$,*

$$o(G-S) \leqslant |S|$$

where $o(H)$ is the number of odd components in a graph H.

Lemma 9. *Let $n = \sum_{i=1}^{t} n_i + s$ where $s \geqslant 1$. If $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$, $n_2 \geqslant 3$ and $n_1 < n-s-t-1$, then*

$$\rho(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})) < \rho(K_s \vee (K_{n-s-t-1} \cup K_3 \cup (t-2)K_1)).$$

Proof. Let $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})$, and let x denote the Perron vector of $A(G)$. By symmetry, we can suppose that $x(v) = x_i$ for all $v \in V(K_{n_i})$, where $1 \leqslant i \leqslant t$, and $x(u) = y_1$ for all $u \in V(K_s)$. Since G contains K_{n_1+s} as a proper subgraph, it follows that $\rho(G) > \rho(K_{n_1+s}) = n_1 + s - 1 > n_1 - 1$. Note that $n_1 \geqslant n_i$ for $2 \leqslant i \leqslant t$. Then by $A(G)x = \rho(G)x$, we get

$$(\rho(G) - (n_i - 1))(x_1 - x_i) = (n_1 - n_i)x_1 \geqslant 0,$$

where $2 \leqslant i \leqslant t$. This implies that $x_1 \geqslant x_i$ for $2 \leqslant i \leqslant t$. Let $G' = K_s \vee (K_{n-s-t-1} \cup K_3 \cup (t-2)K_1)$. Then

$$\begin{aligned}
\rho(G') - \rho(G) &\geqslant x^T(A(G') - A(G))x \\
&= 2(n_2-3)x_2\Big(n_1x_1 + \sum_{j=3}^{t}(n_j-1)x_j - 3x_2\Big) + 2\sum_{j=3}^{t}(n_j-1)x_j(n_1x_1 - x_j) \\
&\quad + 2\sum_{i=3}^{t-1}\sum_{j=i+1}^{t}(n_i-1)(n_j-1)x_ix_j \\
&> 0
\end{aligned}$$

due to $3 \leqslant n_2 \leqslant n_1 < n-s-t-1$, $n_j \geqslant 1$ for $3 \leqslant j \leqslant t$ and $x_1 \geqslant x_i$ for $2 \leqslant i \leqslant t$. Thus, the result follows. $\square$

Now, we shall give the proof of Theorem 3.

Proof of Theorem 3. Suppose that G is a connected 1-binding graph of even order $n \geqslant 12$ and contains no perfect matchings. By Lemma 8, there exists some subset S of $V(G)$ such that $o(G-S) \geqslant |S|+2$. Assume that $|S| = s$ and $o(G-S) = q$. Thus, $q \geqslant s+2$. We first assert that $S \neq \varnothing$. Otherwise, $S = \varnothing$. Since G is a connected graph of even order, it follows that $0 = o(G) = o(G-S) \geqslant 2$, a contradiction. This implies that $s \geqslant 1$. Let $O_1, O_2, \ldots, O_q$ be the odd components of $G-S$ and $|O_i| = n_i$ for $1 \leqslant i \leqslant q$. Without loss of generality, we assume that $n_q \geqslant n_{q-1} \geqslant \cdots \geqslant n_1$. We next assert that $n_{s+1} \geqslant 3$. Otherwise, $n_i = 1$ for $1 \leqslant i \leqslant s+1$. Let $S' = V(O_1 \cup O_2 \cup \cdots \cup O_{s+1})$. Then $N_G(S') \subseteq S$. Notice that $|S'| = s+1$ and $|N_G(S')| \leqslant |S| = s$. Thus,

$$\frac{|N_G(S')|}{|S'|} \leqslant \frac{s}{s+1} < 1,$$

which is impossible because G is 1-binding. This implies that $n_i \geqslant 3$ for $i \geqslant s+1$. One can verify that G is a spanning subgraph of $G_s^1 = K_s \vee (K_{n_1} \cup \cdots \cup K_{n_{s+1}} \cup K_{n-s-\sum_{i=1}^{s+1} n_i})$. Hence

$$\rho(G) \leqslant \rho(G_s^1), \tag{10}$$

where the equality holds if and only if $G \cong G_s^1$. Define $G_s^2 = K_s \vee (K_{n-2s-3} \cup K_3 \cup sK_1)$. Note that $n - s - \sum_{i=1}^{s+1} n_i \geqslant \sum_{i=s+2}^{q} n_i \geqslant n_{s+1} \geqslant 3$, $s \geqslant 1$ and $n_j \geqslant 1$ for $1 \leqslant j \leqslant s$. Then by Lemma 9, we can deduce that

$$\rho(G_s^1) \leqslant \rho(G_s^2), \tag{11}$$

where the equality holds if and only if $G_s^1 \cong G_s^2$. If $s = 1$, then $G_s^2 \cong K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$. From (10) and (11), we have

$$\rho(G) \leqslant \rho(K_1 \vee (K_{n-5} \cup K_3 \cup K_1)),$$

where the equality holds if and only if $G \cong K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$. Next, we consider $s \geqslant 2$ in the following. Observe that $A(K_s \vee (K_{n-2s-3} \cup K_3 \cup sK_1))$ has the equitable quotient matrix

$$A_\Pi^s = \begin{bmatrix} 0 & 0 & s & 0 \\ 0 & 2 & s & 0 \\ s & 3 & s-1 & n-2s-3 \\ 0 & 0 & s & n-2s-4 \end{bmatrix}.$$

By a simple computation, the characteristic polynomial of A_Π^s is

$$\begin{aligned} \varphi(A_\Pi^s, x) =& x^4 + (s-n+3)x^3 + (n-s^2-4s-6)x^2 + (s+1)(ns-2s^2+2n-6s-8)x - \\ & 2ns^2 + 4s^3 + 8s^2. \end{aligned}$$

Notice that $A(K_1 \vee (K_{n-5} \cup K_3 \cup K_1))$ has the equitable quotient matrix A_Π^1, which is obtained by replacing s with 1 in A_Π^s. Thus,

$$\begin{aligned} \varphi(A_\Pi^s, x) - \varphi(A_\Pi^1, x) =& (s-1)(x^3 - (s+5)x^2 + ((s+4)n - 2s^2 - 10s - 24)x - 2ns \\ & + 4s^2 - 2n + 12s + 12). \end{aligned} \tag{12}$$

Let $\gamma(x) = x^3-(s+5)x^2+((s+4)n-2s^2-10s-24)x-2ns+4s^2-2n+12s+12$. We take the derivative of $\gamma(x)$. Note that $n \geqslant 2s+6$. For $x \geqslant n-5$, we obtain that

$$\begin{aligned}\gamma'(x) &= 3x^2 - 2(s+5)x + (s+4)n - 2s^2 - 10s - 24 \\ &\geqslant 3n^2 - (s+36)n - 2s^2 + 101 \text{ (since } x \geqslant n-5) \\ &\geqslant 8s^2 - 6s - 7 \text{ (since } n \geqslant 2s+6) \\ &> 0 \text{ (since } s \geqslant 2).\end{aligned}$$

It follows that $\gamma(x)$ is increasing with respect to $x \geqslant n-5$. Thus,

$$\gamma(x) \geqslant \gamma(n-5) = n^3 - 16n^2 + (-2s^2 - 7s + 79)n + 14s^2 + 37s - 118. \tag{13}$$

Let $p(n) = n^3 - 16n^2 + (-2s^2 - 7s + 79)n + 14s^2 + 37s - 118$. By using a similar analysis as above, we also deduce that $p(n)$ is increasing with respect to $n \geqslant 2s+6$. For $s \geqslant 3$ and $n \geqslant 2s+6$, we have

$$p(n) \geqslant p(2s+6) = 4s^3 - 4s^2 - 15s - 4 > 0.$$

For $s = 2$ and $n \geqslant 12$, we also obtain that $p(n) \geqslant p(12) = 120$. Combining this with (12), (13) and $s \geqslant 2$, we have

$$\varphi(A^s_\Pi, x) > \varphi(A^1_\Pi, x)$$

for $x \geqslant n-5$. Since $K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$ contains K_{n-4} as a proper subgraph, we have $\rho(K_1 \vee (K_{n-5} \cup K_3 \cup K_1)) > \rho(K_{n-4}) = n-5$, and hence $\lambda_1(A^s_\Pi, x) < \lambda_1(A^1_\Pi, x)$ for $s \geqslant 2$. Combining this with Lemma 6, (10) and (11), we obtain that

$$\rho(G) \leqslant \rho(G^1_s) \leqslant \rho(K_s \vee (K_{n-2s-3} \cup K_3 \cup sK_1)) < \rho(K_1 \vee (K_{n-5} \cup K_3 \cup K_1))$$

where $s \geqslant 2$.

Concluding the above results, we have

$$\rho(G) \leqslant \rho(K_1 \vee (K_{n-5} \cup K_3 \cup K_1)),$$

where the equality holds if and only if $G \cong K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$. Note that $K_1 \vee (K_{n-5} \cup K_3 \cup K_1)$ is a 1-binding graph and contains no perfect matchings. Then the result follows. □

Remark 10. When $n = 10$, we can obtain that the extremal graph is not the same as this in Theorem 3. Notice that $n \geqslant 2s+6$. Thus, $1 \leqslant s \leqslant (n-6)/2$. For $n = 10$, we get $1 \leqslant s \leqslant 2$, and hence $G^2_s \cong K_1 \vee (K_5 \cup K_3 \cup K_1)$ or $G^2_s \cong K_2 \vee (2K_3 \cup 2K_1)$. By a simple computation, we get $\rho(K_1 \vee (K_5 \cup K_3 \cup K_1)) = 5.22034$ and $\rho(K_2 \vee (2K_3 \cup 2K_1)) = 5.34085$. It follows that

$$\rho(K_2 \vee (2K_3 \cup 2K_1)) > \rho(K_1 \vee (K_5 \cup K_3 \cup K_1)).$$

Combining this with (10) and (11), we have

$$\rho(G) \leqslant \rho(K_2 \vee (2K_3 \cup 2K_1)),$$

where the equality holds if and only if $G \cong K_2 \vee (2K_3 \cup 2K_1)$. It is easy to see that $K_2 \vee (2K_3 \cup 2K_1)$ is a 1-binding graph and contains no perfect matchings. This suggests that a graph G of order $n = 10$ satisfying $\rho(G) \geqslant \rho(K_2 \vee (2K_3 \cup 2K_1))$ either contains a perfect matching or is isomorphic to $K_2 \vee (2K_3 \cup 2K_1)$.

4 Proof of Theorem 4

Lemma 11 (See [28]). *Let k be a positive integer, and let G be a graph. Then G contains a k-factor if and only if*

$$\delta_G(S,T) = k|S| + \sum_{v\in T} d_G(v) - k|T| - e_G(S,T) - q_G(S,T) \geqslant 0$$

for all disjoint subsets $S, T \subseteq V(G)$, where $q_G(S,T)$ is the number of the components C of $G-(S\cup T)$ such that $e_G(V(C),T) + k|V(C)| \equiv 1 (\mathrm{mod}\ 2)$. Moreover, $\delta_G(S,T) \equiv k|V(G)| (\mathrm{mod}\ 2)$.

Lemma 12 (See [21, 25]). *Let G be a graph on n vertices and m edges with minimum degree $\delta \geqslant 1$. Then*

$$\rho(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}},$$

with equality if and only if G is either a δ-regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Lemma 13 (See [25]). *For nonnegative integers p and q with $2q \leqslant p(p-1)$ and $0 \leqslant x \leqslant p-1$, the function $f(x) = (x-1)/2 + \sqrt{2q - px + (1+x)^2/4}$ is decreasing with respect to x.*

Lemma 14 (See [24]). *Let G be a connected graph, and let u, v be two vertices of G. Suppose that $v_1, v_2, \ldots, v_s \in N_G(v)\backslash N_G(u)$ with $s \geqslant 1$, and G^* is the graph obtained from G by deleting the edges vv_i and adding the edges uv_i for $1 \leqslant i \leqslant s$. Let x be the Perron vector of $A(G)$. If $x(u) \geqslant x(v)$, then $\rho(G) < \rho(G^*)$.*

Denote by $G - v$ and $G - uv$ the graphs obtained from G by deleting the vertex $v \in V(G)$ and the edge $uv \in E(G)$, respectively. Similarly, $G + uv$ is obtained from G by adding the edge $uv \notin E(G)$. Recall that H_n is the graph obtained from $K_2 \vee (K_{n-5} \cup 3K_1)$ by adding an edge between K_{n-5} and $3K_1$.

Lemma 15. *Let G be a graph with $n \geqslant 21$ vertices obtained from $K_1 \vee (K_2 \cup K_1 \cup K_{n-4})$ by adding an edge between the pendent vertex and K_{n-4}. Then $\rho(G) < \rho(H_n)$.*

Proof. Observe that K_{n-3} is a proper subgraph of G. Then $\rho(G) > n-4$. The labels of the vertices in G are shown in Fig.1. Let x be the Perron vector of $A(G)$. By symmetry, we see that $x(v_1) = x(v_2)$ and $x(w_1) = x(w_i)$ where $3 \leqslant i \leqslant n-4$. Then, from $A(G)x = \rho(G)x$, we obtain

$$\begin{cases} \rho(G)x(v_1) = x(v_1) + x(u), \\ \rho(G)x(w_1) = (n-6)x(w_1) + x(w_2) + x(u), \end{cases}$$

which gives that

$$(\rho(G) - (n-6))(x(w_1) - x(v_1)) = x(w_2) + (n-7)x(v_1) > 0$$

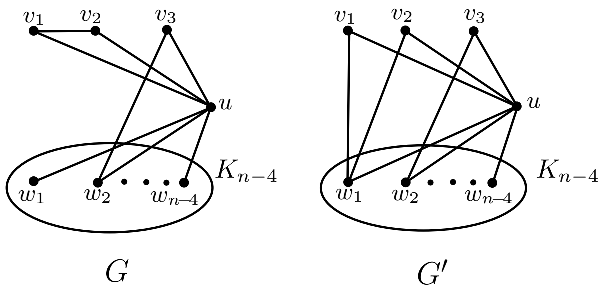

Figure 1: Graphs G and G'

due to $n \geqslant 21$ and $\rho(G) > n-4$. Hence $x(w_1) > x(v_1)$. Let $G' = G - v_1v_2 + v_1w_1 + v_2w_1$. Then $\rho(G') > \rho(G)$ by Lemma 14. Furthermore, G' is a proper spanning subgraph of H_n. Thus,

$$\rho(G) < \rho(G') < \rho(H_n),$$

as required. □

Lemma 16. *Let a and b be two positive integers. If $a \geqslant b \geqslant 3$, then*

$$\binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}.$$

Proof. Note that $a \geqslant b \geqslant 3$. Then

$$\binom{a+1}{2} + \binom{b-1}{2} - \binom{a}{2} - \binom{b}{2} = a - b + 1 > 0.$$

Thus the result follows. □

For any $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S and $e(S)$ be the number of edges in $G[S]$. Now, we shall give the proof of Theorem 4.

Proof of Theorem 4. Suppose to the contrary that G contains no 2-factors. By Lemma 11, there exist two disjoint subsets $S, T \subseteq V(G)$ satisfying $|S \cup T|$ as large as possible such that

$$\delta_G(S,T) = 2|S| + \sum_{t \in T} d_G(t) - 2|T| - e_G(S,T) - q_G(S,T) \leqslant -2, \tag{14}$$

where $q_G(S,T)$ is the number of the components C of $G-(S \cup T)$ such that $e_G(V(C),T) \equiv 1 (\mathrm{mod}\ 2)$. Assume that $|S| = s$, $|T| = t$ and $q_G(S,T) = q$. Let $C_1, C_2, \ldots, C_q$ be the components of $G - (S \cup T)$ such that $e_G(V(C_i),T) \equiv 1 (\mathrm{mod}\ 2)$ where $1 \leqslant i \leqslant q$. Now, we divide the proof into the following five claims.

Claim 1. If $q \geqslant 1$, then $|V(C_i)| \geqslant 2$ for $1 \leqslant i \leqslant q$.

Otherwise, there exists some C_j such that $|V(C_j)| = 1$ where $1 \leqslant j \leqslant q$. Let $C_j = \{v\}$. Since $e_G(v,T) \equiv 1 \pmod 2$, it follows that $e_G(v,T) \geqslant 2l-1$ where l is a positive integer. If $e_G(v,T) = 1$, let $T' = T \cup \{v\}$, we have

$$\begin{aligned}
\delta_G(S,T') &= 2|S| + \sum_{t\in T'} d_G(t) - 2|T'| - e_G(S,T') - q_G(S,T') \\
&= 2s + \Big(\sum_{t\in T} d_G(t) + d_G(v)\Big) - 2(t+1) - (e_G(S,T) + d_S(v)) - (q-1) \\
&= 2s + \sum_{t\in T} d_G(t) - 2t - e_G(S,T) - q + e_G(v,T) - 1 \\
&= \delta_G(S,T) \quad (\text{since } e_G(v,T) = 1) \\
&\leqslant -2 \quad (\text{by } (14)),
\end{aligned}$$

which contradicts the maximality of $|S \cup T|$. If $e_G(v,T) \geqslant 3$, let $S' = S \cup \{v\}$, we have

$$\begin{aligned}
\delta_G(S',T) &= 2|S'| + \sum_{t\in T} d_G(t) - 2|T| - e_G(S',T) - q_G(S',T) \\
&= 2(s+1) + \sum_{t\in T} d_G(t) - 2t - (e_G(S,T) + e_G(v,T)) - (q-1) \\
&= 2s + \sum_{t\in T} d_G(t) - 2t - e_G(S,T) - q - e_G(v,T) + 3 \\
&\leqslant \delta_G(S,T) \quad (\text{since } e_G(v,T) \geqslant 3) \\
&\leqslant -2 \quad (\text{by } (14)),
\end{aligned}$$

which also leads to a contradiction. $\square$

Claim 2. $t \geqslant s+1$.

Otherwise, $s \geqslant t$. If $q = 0$, then $\delta_G(S,T) = 2(s-t) + \sum_{t\in T} d_{G-S}(t) \geqslant 0$, which contradicts (14). If $q \geqslant 1$, since $e_G(V(C_i),T) \geqslant 1$ for $1 \leqslant i \leqslant q$, we have

$$\sum_{t\in T} d_{G-S}(t) \geqslant \sum_{i=1}^{q} e_G(V(C_i),T) \geqslant q, \tag{15}$$

and hence

$$\delta_G(S,T) = 2(s-t) + \sum_{t\in T} d_{G-S}(t) - q \geqslant 0,$$

which also contradicts (14). This implies that $t \geqslant s+1$, as required. $\square$

By Lemmas 12, 13, and the fact $\delta(G) \geqslant 2$, we obtain

$$\rho(G) \leqslant \frac{1}{2} + \sqrt{2e(G) - 2n + \frac{9}{4}}. \tag{16}$$

Note that $\rho(G) \geqslant \rho(H_n) > \rho(K_{n-3}) = n-4$. Combining this with (16), we have

$$e(G) \geqslant \binom{n-3}{2} + 4. \tag{17}$$

Without loss of generality, we assume that $|V(C_q)| \geqslant |V(C_{q-1})| \geqslant \cdots \geqslant |V(C_1)| \geqslant 2$.

Claim 3. $q \leqslant 1$.

Otherwise, $q \geqslant 2$. Claim 2 implies that $t \geqslant 1$. In the case of $t = 1$, by Claim 2, we can deduce that $s = 0$. Let $T = \{z\}$. Then

$$q \leqslant \sum_{i=1}^{q} e_G(z, V(C_i)) \leqslant d_G(z) \leqslant q + 2t - 2s + e_G(S, z) - 2 = q$$

due to (14) and (15). It follows that $d_G(z) = q$. Combining this with $e_G(z, V(C_i)) \geqslant 1$, we have $e_G(z, V(C_i)) = 1$ where $1 \leqslant i \leqslant q$. We assert that $|V(C_i)| \geqslant 3$ for $1 \leqslant i \leqslant q$. If not, there exists some C_j such that $|V(C_j)| = 2$ for $1 \leqslant j \leqslant q$. Since $e_G(z, V(C_j)) = 1$ and $S = \varnothing$, there must exist some vertex in C_j with degree exactly one in G. This is impossible because $\delta(G) \geqslant 2$. Therefore, $|V(C_i)| \geqslant 3$ for $1 \leqslant i \leqslant q$, and hence $n \geqslant 3q+1$. Combining this with Lemma 16, we get

$$\begin{aligned}
e(G) &\leqslant d_G(z) + \sum_{i=1}^{q-1} \binom{|V(C_i)|}{2} + \binom{n-1-\sum_{i=1}^{q-1}|V(C_i)|}{2} \\
&\leqslant q + (q-1)\binom{3}{2} + \binom{n-1-3(q-1)}{2} \text{ (since } d_G(z) = q \text{ and } q \geqslant 2) \\
&= \binom{n-3}{2} + 4 - \left((3q-5)n + 12 - \frac{9q^2}{2} + \frac{q}{2}\right) \\
&\leqslant \binom{n-3}{2} + 4 - \frac{(q-1)(9q-14)}{2} \text{ (since } n \geqslant 3q+1 \text{ and } q \geqslant 2) \\
&< \binom{n-3}{2} + 4 \text{ (since } q \geqslant 2),
\end{aligned}$$

which contradicts (17). For $t = 2$, by Claim 2, we have $0 \leqslant s \leqslant 1$, and hence $e(S) = 0$. Putting $t = 2$ into (14) yields that $\sum_{t \in T} d_G(t) \leqslant q+2$. Note that $|V(C_i)| \geqslant 2$ for $1 \leqslant i \leqslant q$ and $t = 2$. Then $n \geqslant 2q + s + 2$. Combining this with $\sum_{t \in T} d_G(t) \leqslant q + 2$, $|V(C_i)| \geqslant 2$ for $1 \leqslant i \leqslant q$ and Lemma 16, we obtain that

$$\begin{aligned}
e(G) &\leqslant \sum_{t \in T} d_G(t) + \sum_{i=1}^{q-1} \binom{|V(C_i)|}{2} + \binom{n-t-s-\sum_{i=1}^{q-1}|V(C_i)|}{2} + s(n-s-t) + e(S) \\
&\leqslant q+2+(q-1)\binom{2}{2} + \binom{n-s-2q}{2} + s(n-s-2) \text{ (since } t=2,\ e(S)=0 \text{ and } q \geqslant 2) \\
&= \binom{n-3}{2} + 4 - \left((2q-3)n + 9 - 3q - 2q^2 - 2qs + \frac{s^2}{2} + \frac{3s}{2}\right) \\
&\leqslant \binom{n-3}{2} + 4 - \left(2q^2 - 5q + \frac{s^2-3s}{2} + 3\right) \text{ (since } n \geqslant 2q+s+2 \text{ and } q \geqslant 2) \\
&\leqslant \binom{n-3}{2} + 4 - \frac{(s-1)(s-2)}{2} \text{ (since } q \geqslant 2)
\end{aligned}$$

$$\leqslant \binom{n-3}{2} + 4 \text{ (since } s \leqslant 1),$$

where all equalities hold if and only if $s = 1$, $q = 2$ and $n = 2q+s+2 = 7$. This is impossible because $n \geqslant 21$. Furthermore, $e(G) < \binom{n-3}{2} + 4$, which contradicts (17). We consider $t \geqslant 3$ in the following. It is not hard to see that at least $\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} |V(C_i)||V(C_j)|$ edges here are not in G. Note that $|V(C_q)| \geqslant \cdots \geqslant |V(C_1)| \geqslant 2$. Then

$$\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} |V(C_i)||V(C_j)| \geqslant |V(C_1)|\Big(\sum_{i=1}^{q} |V(C_i)| - |V(C_1)|\Big) \geqslant 2\sum_{i=1}^{q} |V(C_i)| - 4 \geqslant 4(q-1).$$

Combining this with (14), $n \geqslant 2q + s + t$ and Claim 2, we have

$$\begin{aligned}
e(G) &\leqslant \sum_{t\in T} d_G(t) + \binom{n-t}{2} - \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} |V(C_i)||V(C_j)| \\
&\leqslant 2t - 2s + st + q - 2 + \binom{n-t}{2} - 4(q-1) \\
&= \binom{n-3}{2} + 4 - \Big((t-3)n + 8 - \frac{5t+t^2}{2} + 2s - st + 3q\Big) \\
&\leqslant \binom{n-3}{2} + 4 - \Big((2t-3)q + \frac{t^2-11t}{2} - s + 8\Big) \text{ (since } n \geqslant 2q+s+t \text{ and } t \geqslant 3) \\
&\leqslant \binom{n-3}{2} + 4 - \Big(\frac{t^2-3t}{2} + 2 - s\Big) \text{ (since } q \geqslant 2 \text{ and } t \geqslant 3) \\
&\leqslant \binom{n-3}{2} + 4 - \frac{(t-2)(t-3)}{2} \text{ (since } s \leqslant t-1) \\
&\leqslant \binom{n-3}{2} + 4 \text{ (since } t \geqslant 3),
\end{aligned}$$

where all equalities hold if and only if $t = 3$, $s = 2$, $q = 2$ and $n = 2q + s + t = 9$. This is impossible because $n \geqslant 21$. Thus, $e(G) < \binom{n-3}{2} + 4$, which also leads to a contradiction. This implies that $q \leqslant 1$, as required. □

By Claim 3, $\delta(G) \geqslant 2$ and (14), we can deduce that

$$2t \leqslant \sum_{t\in T} d_G(t) \leqslant 2t - 2s + e_G(S,T) + q - 2 \leqslant 2t - 2s + st - 1. \tag{18}$$

Claim 4. $n \geqslant s + t + 3$.

Otherwise, $n = s + t + a$ where $0 \leqslant a \leqslant 2$. From Claim 2 and $n \geqslant 21$, we obtain that $21 \leqslant n \leqslant 2t - 1 + a$, and hence $t \geqslant 11 - a/2$. Combining this with (18), we have

$$e(G) \leqslant \sum_{t\in T} d_G(t) + \binom{n-t}{2}$$

$$
\begin{aligned}
&\leqslant 2t-2s+st-1+\frac{(n-t)(n-t-1)}{2}\\
&=\binom{n-3}{2}+4-\Big((t-3)n+11-\frac{5t+t^2}{2}+2s-ts\Big)\\
&=\binom{n-3}{2}+4-\Big(\frac{t^2}{2}-\frac{(11-2a)t}{2}-s-3a+11\Big)\ (\text{since } n=s+t+a)\\
&\leqslant\binom{n-3}{2}+4-\Big(\frac{t^2}{2}+\frac{(2a-13)t}{2}-3a+12\Big)\ (\text{since } s\leqslant t-1)\\
&\leqslant\binom{n-3}{2}+4-\Big(1+\frac{23a}{4}-\frac{3a^2}{8}\Big)\ (\text{since } 0\leqslant a\leqslant 2 \text{ and } t\geqslant 11-a/2)\\
&<\binom{n-3}{2}+4\ (\text{since } 0\leqslant a\leqslant 2),
\end{aligned}
$$

which contradicts (17). This implies that $n \geqslant s+t+3$, as required. □

For $s=0$, we get $2t \leqslant \sum_{t\in T} d_G(t) \leqslant 2t-1$ by (18), a contradiction. Thus, we consider $s \geqslant 1$. Again by (18), we have

$$2t \leqslant \sum_{t\in T} d_G(t) \leqslant 2t-2s+st-1,$$

and hence $t \geqslant 2+1/s$, which implies that $t \geqslant 3$ because t is a positive integer.

Claim 5. $t=3$.

Otherwise, $t \geqslant 4$. For $4 \leqslant t \leqslant 7$, by (18), we get

$$
\begin{aligned}
e(G) &\leqslant \sum_{t\in T} d_G(t)+\binom{n-t}{2}\\
&\leqslant 2t-2s+st-1+\frac{(n-t)(n-t-1)}{2}\\
&=\binom{n-3}{2}+4-\Big((t-3)n+11-\frac{5t+t^2}{2}+(2-t)s\Big)\\
&\leqslant\binom{n-3}{2}+4-\Big((t-3)n+9+\frac{t-3t^2}{2}\Big)\ (\text{since } s\leqslant t-1)\\
&\leqslant\binom{n-3}{2}+4-\Big(\frac{43t-3t^2}{2}-54\Big)\ (\text{since } n\geqslant 21 \text{ and } t\geqslant 4)\\
&<\binom{n-3}{2}+4\ (\text{since } 4\leqslant t\leqslant 7),
\end{aligned}
$$

which contradicts (17). For $t \geqslant 8$, by Claims 2 and 4, we obtain that

$$e(G) \leqslant \sum_{t\in T} d_G(t)+\binom{n-t}{2}$$

$$
\begin{aligned}
&\leqslant 2t-2s+st-1+\frac{(n-t)(n-t-1)}{2}\\
&=\binom{n-3}{2}+4-\Big((t-3)n+11-\frac{5t+t^2}{2}+(2-t)s\Big)\\
&\leqslant \binom{n-3}{2}+4-\Big(\frac{t^2-5t}{2}-s+2\Big) \text{ (since } n\geqslant s+t+3 \text{ and } t\geqslant 8)\\
&\leqslant \binom{n-3}{2}+4-\Big(\frac{t^2-7t}{2}+3\Big) \text{ (since } s\leqslant t-1)\\
&< \binom{n-3}{2}+4 \text{ (since } t\geqslant 8),
\end{aligned}
$$

which also contradicts (17). This implies that $t = 3$, as required. □

By Claims 2 and 5, we obtain that $0 \leqslant s \leqslant 2$. Note that $t = 3$. For $s = 0$, by (18), we obtain that

$$6 \leqslant \sum_{t\in T} d_G(t) \leqslant 2t-2s+st-1 = 5,$$

a contradiction. For $s = 1$, let $S = \{u\}$. According to $\delta(G) \geqslant 2$, $t = 3$ and (14), we have

$$6 \leqslant \sum_{t\in T} d_G(t) \leqslant 2t-2s+e_G(u,T)+q-2 = 2+e_G(u,T)+q,$$

which gives that $e_G(u,T)+q \geqslant 4$. Observe that $e_G(u,T) \leqslant t = 3$ and $q \leqslant 1$ due to Claim 3. It follows that $q = 1$ and $e_G(u,T) = 3$, and hence $\sum_{t\in T} d_G(t) = 6$. Considering that

$$\sum_{t\in T} d_{G-u}(t) = \sum_{t\in T} d_G(t) - e_G(u,T) = 3,$$

we can deduce that $e(T) \leqslant 1$. Suppose that $T = \{v_1, v_2, v_3\}$. If $e(T) = 1$, without loss of generality, we may assume that $v_1v_2 \in G[T]$. Since $\delta(G) \geqslant 2$, there must exist some vertex, say w_2, in $V(G)\backslash(\{u\}\cup T)$ such that $v_3w_2 \in E(G)$. One can verify that G is a spanning subgraph of $K_1 \vee (K_2 \cup K_1 \cup K_{n-4}) + e$, where e is an edge between the pendent vertex and K_{n-4}. Combining this with Lemma 15, we get

$$\rho(G) \leqslant \rho(K_1 \vee (K_2 \cup K_1 \cup K_{n-4}) + e) < \rho(H_n),$$

a contradiction. Thus, we assume that $e(T) = 0$. Since $\sum_{t\in T} d_{G-u}(t) = 3$, we have $1 \leqslant |N_{G-u}(T)| \leqslant 3$. We assert that $|N_{G-u}(T)| \geqslant 2$. Otherwise, $|N_{G-u}(T)| = 1$. Combining this with $u \in N_G(T)$, we have $|N_G(T)| = 2$, and hence $\frac{|N_G(T)|}{|T|} = \frac{2}{3} < 1$. This is impossible because G is 1-binding. It follows that $2 \leqslant |N_{G-u}(T)| \leqslant 3$. Recall that $\delta(G) \geqslant 2$ and $e(u,T) = t = 3$. If $|N_{G-u}(T)| = 3$, let $N_{G-u}(T) = \{w_1, w_2, w_3\}$ and $v_iw_i \in E(G)$ for $1 \leqslant i \leqslant 3$. Let x be the Perron vector of $A(G)$. Without loss of generality, we may assume that $x(w_1) \geqslant x(w_2)$. Suppose that $G^* = G - v_2w_2 + v_2w_1$. Clearly, G^* is a proper spanning subgraph of H_n. Combining this with Lemma 14, we obtain that

$$\rho(G) < \rho(G^*) < \rho(H_n),$$

a contradiction. If $|N_{G-u}(T)| = 2$, then G is a proper spanning subgraph of H_n. By using a similar analysis as above, we can also deduce a contradiction. For $s = 2$, we have $\sum_{t\in T} d_{G-S}(t) \leqslant 2t - 2s + q - 2 = q \leqslant 1$ due to (14), Claims 3 and 5. It follows that $e(T) = 0$. Note that G is a spanning subgraph of H_n. Then

$$\rho(G) \leqslant \rho(H_n),$$

where the equality holds if and only if $G \cong H_n$, which also leads to a contradiction.

This completes the proof. □

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and helpful suggestions.

Dandan Fan was supported by the National Natural Science Foundation of China (Grant Nos. 12301454) and Natural Science Foundation of Xinjiang Uygur Autonomous Region (Grant Nos. 2022D01B103). Huiqiu Lin was supported by the National Natural Science Foundation of China (Grant Nos. 12271162), Natural Science Foundation of Shanghai (Nos. 22ZR1416300 and 23JC1401500) and the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning (No. TP2022031).

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