Nordhaus-Gaddum type inequalities for the kth largest Laplacian eigenvalues

Wen-Jun Li[a] Ji-Ming Guo[a]

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Abstract

Let G be a simple connected graph and $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ be the Laplacian eigenvalues of G. Let $\overline{G}$ be the complement of G. Einollahzadeh et al.[J. Combin. Theory Ser. B, 151(2021), 235–249] proved that $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant 1$. Grijó et al. [Discrete Appl. Math., 267(2019), 176–183] conjectured that $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ for any graph and proved it to be true for some graphs. In this paper, we prove $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ is true for some new graphs. Furthermore, we propose a more general conjecture that $\mu_k(G) + \mu_k(\overline{G}) \geqslant n - k$ holds for any graph G, with equality if and only if G or $\overline{G}$ is isomorphic to $K_{n-k} \vee H$, where H is a disconnected graph on k vertices and has at least $n - k + 1$ connected components. And we prove that it is true for $k \leqslant \frac{n+1}{2}$, for unicyclic graphs, bicyclic graphs, threshold graphs, bipartite graphs, regular graphs, complete multipartite graphs and c-cyclic graphs when $n \geqslant 2c + 8$.

Mathematics Subject Classifications: 05C50

1 Introduction

Let G be a simple graph of order $n(G)$ and size $m(G)$. If there's no ambiguity, we use n and m instead of $n(G)$ and $m(G)$. Let $\overline{G}$ be the complement of G. Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees of G. The matrix $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of G, and denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$, $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ and $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G)$, respectively. For two graphs G_1 and G_2, the union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with the vertex

[a]School of Mathematics, East China University of Science and Technology, Shanghai, P. R. China (leewj1375@163.com, Corresponding author. jimingguo@hotmail.com).

set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{st : s \in V(G_1), t \in V(G_2)\}$. We say a graph is H-free if it does not contain H as an induced subgraph.

A Nordhaus-Gaddum type inequality, or NG-inequality for simplicity, is defined as the bound of $p(G) + p(\overline{G})$, where $p(G)$ is a graph parameter. Such problems have been studied for a huge variety of graph parameters since it's first introduced by Nordhaus and Gaddum in [21] for the chromatic number of a graph G. See [3] for the comprehensive survey. Nikiforov [19] proposed the study of NG-inequality for all eigenvalues of a graph defining a function given by $\max\{|\lambda_k(G) + \lambda_k(\overline{G})| : |V(G)| = n\}$ for $k = 1, \ldots, n$. There are many results about it, for more details see [1, 6, 19, 20, 22, 23]. For the signless Laplacian eigenvalues, Ashraf and Tayfeh-Rezaie [4] showed that $q_1(G) + q_1(\overline{G}) \leqslant 3n - 4$. Huang and Lin [17] proved that $n - 2 \leqslant q_2(G) + q_2(\overline{G}) \leqslant 2n - 4$. As for the Laplacian eigenvalues, Zhai et al.[27] (see also You and Liu [26]) posed the conjecture that for any graph

$$\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant 1.$$

In 2014, Ashraf et al.[4] confirmed it for bipartite graphs and characterized the case when equality holds. Finally in 2021, Einollahzadeh and Karkhaneei [9] completely confirmed it.

Furthermore, Grijó et al.[12] studied NG-inequality for $\mu_{n-2}(G)$. They showed that $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ when G or $\overline{G}$ is disconnected, G is a bipartite graph, a regular graph, or when G and $\overline{G}$ have diameter not equal to 2. They also proposed a conjecture that for any graph

$$\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2,$$

and the equality holds if and only if G or $\overline{G}$ is isomorphic to $K_2 \vee H$, where H is a disconnected graph on $n - 2$ vertices and has at least 3 connected components.

In this paper, we prove that $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ is true for K_3-free graphs, the graph which has diameter 2 and whose complement has diameter not equal to 3. Furthermore, we propose the following more general conjecture.

Conjecture 1. Let G be a graph on n vertices and $\overline{G}$ be the complement of G. Then

$$\mu_k(G) + \mu_k(\overline{G}) \geqslant n - k,$$

for $k = 1, 2, \ldots, n-1$, with equality if and only if G or $\overline{G}$ is isomorphic to $K_{n-k} \vee H$, where H is a disconnected graph on k vertices and has at least $n - k + 1$ connected components.

We have checked that Conjecture 1 is true for all graphs with $n \leqslant 9$ vertices. We will prove that Conjecture 1 is true for unicyclic graphs, bicyclic graphs, threshold graphs, bipartite graphs, regular graphs, complete multipartite graphs and c-cyclic graphs when $n \geqslant 2c + 8$. And it is always true for $k \leqslant \frac{n}{2}$. In particular, if Brouwer's conjecture is true, then Conjecture 1 is true for $k = \frac{n+1}{2}$.

2 Preliminaries

In this section, we present some lemmas and terminologies which will be used in after sections. As usual, we denote the complete graph, path and cycle with n vertices by K_n, P_n

and C_n, respectively. The complete t-partite graph with the part sizes $n_1, n_2, \ldots, n_t (n = \sum_{i=1}^{n} n_i)$ $(t \geqslant 2)$ is denoted by $K_{n_1,n_2,\ldots,n_t}$. Let $\mu^{(s)}$ denote the Laplacian eigenvalue μ having the multiplicity s. The number of Laplacian eigenvalues of G in an interval I is denoted by $m_G I$. A vertex v is called an isolated vertex if $d(v) = 0$, and is called a pendant vertex if $d(v) = 1$. A quasipendant vertex of G is a vertex adjacent to at least one pendant vertex.

The following lemma illustrates the relationship of Laplacian eigenvalues between the graph and its complement graph.

Lemma 2 ([5],p.4). *For any graph G with n vertices, $\mu_n(\overline{G}) = 0$ and $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ $(i = 1, 2, \ldots, n-1)$.*

Note that $\overline{K_n} = nK_1$ and $\overline{K_{n_1,n_2,\ldots,n_t}} = \bigcup_{i=1}^{t} K_{n_i}$. By Lemma 2, the Laplacian eigenvalues of complete graphs and complete multipartite graphs are as follows.

Lemma 3. *Let n be a natural number.*
(1) *The Laplacian eigenvalues of K_n are $\{n^{(n-1)}, 0^{(1)}\}$.*
(2) *The Laplacian eigenvalues of $K_{n_1,n_2,\ldots,n_t}$ $(t \geqslant 2)$ are $\{n^{(t-1)}, (n-n_i)^{(n_i-1)}, 0^{(1)}\}$.*

By Lemma 2, studying the bound of $\mu_{n-1}(G) + \mu_{n-1}(\overline{G})$ can be translated into considering the bound of $\mu_1(G) - \mu_{n-1}(G)$, which is known as the Laplacian spread of graphs.

Lemma 4 ([9], Theorem 1). *Let G be a graph on n vertices and $\overline{G}$ be the complement of G. Then $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant 1$, with equality if and only if G or $\overline{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n-1$.*

The following lemma is known as the interlacing theorem on Laplacian eigenvalues.

Lemma 5 ([11], Theorem 13.6.2). *Let G be a graph with n vertices and let G' be a graph obtained from G by inserting a new edge into G. Then the Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

Let $N(v)$ denote the set of vertices adjacent to the vertex v. The following upper bound for $\mu_1(G)$ is always less than or equal to n.

Lemma 6 ([8], Theorem 2.1). *If $G = (V, E)$ is a graph, then $\mu_1(G) \leqslant \max\{d(u) + d(v) - |N(u) \cap N(v)| : uv \in E\}$.*

Let $\lambda_i(M) (1 \leqslant i \leqslant n)$ denote the ith largest eigenvalue of a matrix M with order n. The following lemma is well-known as Weyl's inequality.

Lemma 7 ([16], Theorem 4.3.1). *Let B and C be Hermitian matrices of order n and let $1 \leqslant i, j \leqslant n$. Then*
(1) $\lambda_i(B) + \lambda_j(C) \leqslant \lambda_{i+j-n}(B + C)$, *if* $i + j \geqslant n + 1$
(2) $\lambda_i(B) + \lambda_j(C) \geqslant \lambda_{i+j-1}(B + C)$, *if* $i + j \leqslant n + 1$.

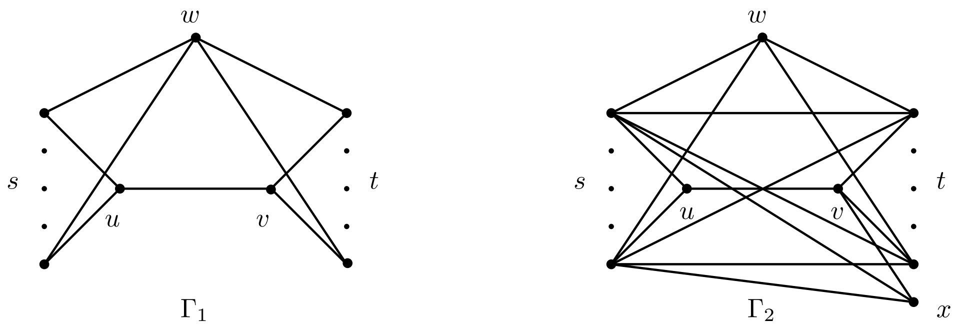

Figure 1: Γ_i, $i = 1, 2$.

Denote by $\Phi(B) = \Phi(B; x) = det(xI - B)$ the characteristic polynomial of the matrix B. If $v \in G$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to the vertex v.

Lemma 8 ([13], Lemma 8)**.** *Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge. Then*

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

Let G_1, u and G_2, v be two disjoint rooted graphs with roots u and v, respectively. The coalescence of two rooted graphs G_1, u and G_2, v, denoted by $G_1 \cdot G_2$, is the graph formed by identifying the two roots u and v.

Lemma 9 ([15], Corollary 2.3)**.** *If G_1 and G_2 are two rooted graphs with roots u and v, respectively, then the Laplacian characteristic polynomial of the coalescence $G_1 \cdot G_2$ is*

$$\Phi(L(G_1 \cdot G_2)) = \Phi(L(G_1))\Phi(L_v(G_2)) + \Phi(L(G_2))\Phi(L_u(G_1)) - x\Phi(L_u(G_1))\Phi(L_v(G_2)).$$

3 NG-inequality for $\mu_{n-2}(G)$

We first introduce some graphs. Let $T_{uv}(s,t)(s,t \geqslant 1)$ denote the tree of diameter 3 having exactly two quasipendant vertices u and v, where u is adjacent to s pendant vertices and v is adjacent to t pendant vertices. Let Γ_1 be a graph obtained from a $T_{uv}(s,t)$ and a vertex w by joining w to all vertices in $T_{uv}(s,t)$ except $\{u, v\}$(See Figure 1). In $T_{uv}(s,t)$, let x be some vertex in $N(v) \setminus \{u\}$. Let Γ_2 be a graph obtained from a $T_{uv}(s,t)$ and a vertex w by joining w to all vertices in $T_{uv}(s,t)$ except $\{u, v, x\}$ and joining each vertex in $N(u) \setminus \{v\}$ to all vertices in $N(v) \setminus \{u\}$(See Figure 1).

Lemma 10. *Let $G = \Gamma_i(i = 1, 2)$ be the graph as defined above. Then $m_G(n-2, n] \leqslant 1$, with the only exception that $G = C_5$, in which case $m_{C_5}(n-2, n] = 2$.*

Proof. First suppose $G = \Gamma_1$. It is easy to check that $m_{C_5}(3,5] = 2$. Next suppose $G \neq C_5$. Without loss of generality, suppose $s \geqslant t$. Then $s \geqslant 2$. Let R_n be the graph obtained by merging an edge between K_{n-1} and K_3. By Lemma 2, it follows that $\mu_{n-1}(R_n) = n - \mu_1(\overline{R_n}) = n - \mu_1(2K_1 \cup K_{1,n-3}) = 2$. It is easy to see that $\overline{G}$ has K_{s+t} as a subgraph. Note that in $\overline{G}$, v is adjacent to all vertices in K_s and w is adjacent to u. Then $\overline{G}$ has $K_2 \cup R_{n-2}$ as a spanning subgraph. Hence by Lemma 5, we have $\mu_{n-2}(\overline{G}) \geqslant \mu_{n-2}(K_2 \cup R_{n-2}) = 2$. And then by Lemma 2, we have $\mu_2(G) \leqslant n-2$. Therefore, $m_G(n-2,n] \leqslant 1$ holds.

Next suppose $G = \Gamma_2$. It is easy to see that $\overline{G}$ has $K_{s+1} \cup K_{t+1}$ as a subgraph. Note that in $\overline{G}$, w is adjacent to both u and x. Then $\overline{G}$ has $K_{s+1} \cup R_{t+2}$ as a spanning subgraph. By Lemma 5, it follows that $\mu_{n-2}(\overline{G}) \geqslant \mu_{n-2}(K_{s+1} \cup R_{t+2}) = 2$. And then by Lemma 2, we have $\mu_2(G) \leqslant n-2$. Therefore, $m_G(n-2,n] \leqslant 1$ holds. This completes the proof. □

Ahanjideh et al. [2](Theorem 5.4) showed that if G is K_3-free, then $m_G(n-1,n] \leqslant 1$. We improve this result as follows.

Theorem 11. *If G is a K_3-free graph of order n, then $m_G(n-2,n] \leqslant 1$, with the only exception that $G = C_5$, in which case $m_{C_5}(n-2,n] = 2$.*

Proof. If $\mu_1(G) \leqslant n-2$, then $m_G(n-2,n] = 0 < 2$. Next suppose that $\mu_1(G) > n-2$. Then by Lemma 6, there are two adjacent vertices, say u and v, such that $d(u)+d(v) = n$ or $d(u)+d(v) = n-1$. Note that both $N(u)$ and $N(v)$ are independent sets. If $d(u)+d(v) = n$, then G is bipartite with two parts $N(u)$ and $N(v)$. By Lemmas 3 and 5, one can see that $m_G(n-2,n] \leqslant m_{K_{d(u),d(v)}}(n-2,n] \leqslant 1$. If $d(u)+d(v) = n-1$, suppose $w \notin N(u) \cup N(v)$. First suppose that w is adjacent to all vertices in $N(u) \cup N(v) \setminus \{u, v\}$. Note that $N(w)$ is an independent sets. Then G is isomorphic to Γ_1. And by Lemma 10, it follows that $m_G(n-2,n] \leqslant 1$, with the only exception that $G = C_5$, in which case $m_{C_5}(n-2,n] = 2$. Otherwise, without loss of generality, suppose there exists a vertex $x \in N(v)$ and $x \neq u$ such that x is not adjacent to w. Then G is a subgraph of Γ_2. Hence by Lemmas 5 and 10, $m_G(n-2,n] \leqslant m_{\Gamma_2}(n-2,n] \leqslant 1$. This completes the proof. □

Remark: Note that for $G = K_{2,n-2}$, we have $m_G[n-2,n] = 2 > 1$. It follows that $m_G[n-2,n] \leqslant 1$ is not true for all K_3-free graphs.

Note that $\overline{C_5} = C_5$ and $\mu_{n-2}(C_5) \approx 1.3 > 1$. Then $\mu_{n-2}(C_5) + \mu_{n-2}(\overline{C_5}) > 2$. By Theorem 11, if G is K_3-free and $G \neq C_5$, then $\mu_2(G) \leqslant n-2$. By Lemma 6, we have $\mu_2(G) \leqslant \mu_1(G) \leqslant n$ for any graph G. Since $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ is equivalent to $\mu_2(G) + \mu_2(\overline{G}) \leqslant 2n-2$ by Lemma 2, we obtain the following corollary immediately.

Corollary 12. *If G is K_3-free, then $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$.*

We denote the diameter of a graph G by $D(G)$. Einollahzadeh and Karkhaneei [9] proved that the algebraic connectivity of the graph with diameter less than 3 is no less than 1.

Lemma 13 ([9], Lemma 5)**.** *If $D(G) \leqslant 2$, then $\mu_{n-1}(G) \geqslant 1$.*

The class of graphs $G_D(n_1, n_2, \ldots, n_D, n_{D+1})$ is composed by $D+1$ cliques K_{n_i} of sizes $n_i (n_i \geqslant 1,\ 1 \leqslant i \leqslant D+1)$ in such a way that each clique K_{n_i} is connected to its neighboring cliques $K_{n_{i-1}}$ and $K_{n_{i+1}}$ by a join operation for each $2 \leqslant i \leqslant D$. For examples, $G_3(2,1,3,2)$ and $G_3(1,2,4,1)$ are shown in Figure 2. Obviously, the class of graphs $G_D(n_1, n_2, \ldots, n_D, n_{D+1})$ has diameter D. Let $\mu_{i_{max}}(n, D)$ be the maximum of the ith largest Laplacian eigenvalue among all graphs $G(n, D)$ with order n and diameter D for each $i = 1, \ldots, n$.

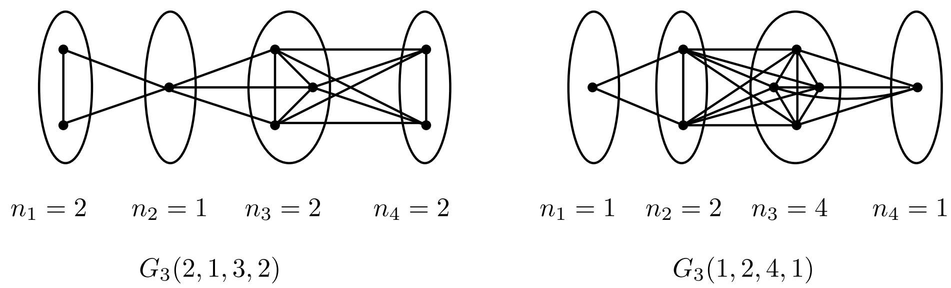

Figure 2: The graphs $G_3(2,1,3,2)$ and $G_3(1,2,4,1)$.

The following two lemmas indicate the relationship between the diameter and ith largest Laplacian eigenvalue.

Lemma 14 ([25], Theorem 4)**.** *Any graph $G(n, D)$ is a subgraph of at least one graph in the class $G_D(n_1 = 1, n_2, \ldots, n_D, n_{D+1} = 1)$ with $n = \sum_{i=1}^{D+1} n_i$.*

Lemma 15 ([25], Theorem 16)**.** *Given the order n, the maximum of $\mu_{i_{max}}(n, D)$, $i = 1, \ldots, n$ is non-increasing as the diameter D increases, i.e. $\mu_{i_{max}}(n, D+1) \leqslant \mu_{i_{max}}(n, D)$.*

Let $H(s, t)$ be the graph obtained from $K_{2,s}$ and $K_{2,t}$ by identifying u and v, where u (resp. v) is a vertex of degree s (resp. t) in $K_{2,s}$ (resp. $K_{2,t}$). Let $K^e_{2,n-2}$ be the graph of order n obtained by inserting an edge to the partite set of order 2 in $K_{2,n-2}$. Note that $\overline{K^e_{2,n-2}} = K_{n-2} \cup 2K_1$. By Lemmas 2 and 3, we have $\mu_{n-2}(K^e_{2,n-2}) \geqslant 2$.

Theorem 16. *Let G be a graph of order n. If $D(G) = 2$ and $D(\overline{G}) \neq 3$, then $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$.*

Proof. Suppose first that $D(G) = D(\overline{G}) = 2$. Then by Lemma 13, we have $\mu_{n-2}(G) \geqslant \mu_{n-1}(G) \geqslant 1$ and $\mu_{n-2}(\overline{G}) \geqslant \mu_{n-1}(\overline{G}) \geqslant 1$. Hence the result holds.

We claim that $\mu_2(G) \leqslant n-2$ for any G with $D(G) \geqslant 4$. For $D(G) = 4$, by Lemmas 5 and 14, we only need to prove that $\mu_2(G) \leqslant n-2$ for any G of the class $G_4(1, n_2, n_3, n_4, 1)$. Without loss of generality, we may assume $n_4 \geqslant n_2 \geqslant 1$. If $n_4 = n_2 = 1$, then $\overline{G}$ has $K^e_{2,n_3} \cup K_2$ as a spanning subgraph. By Lemma 5, we have $\mu_{n-2}(\overline{G}) \geqslant \mu_{n-2}(K^e_{2,n_3} \cup K_2) = 2$. If $n_2 = 1$ and $n_4 \geqslant 2$, then $\overline{G}$ has $H(n_3, n_4)$ as a spanning subgraph. By Lemma 9, we have the characteristic polynomial of $L(H(n_3, n_4))$ is

$$\begin{aligned}\Phi(L(H(n_3,n_4))) =&\Phi(L(K_{2,n_3}))\Phi(L_v(K_{2,n_4})) + \Phi(L(K_{2,n_4}))\Phi(L_u(K_{2,n_3}))\\ &- x\Phi(L_u(K_{2,n_3}))\Phi(L_v(K_{2,n_4}))\\ =&x(x-(n_3+2))(x-n_3)(x-2)^{n_3-1}((x-2)(x-n_4)-n_4)(x-2)^{n_4-1}\\ &+x(x-(n_4+2))(x-n_4)(x-2)^{n_4-1}((x-2)(x-n_3)-n_3)(x-2)^{n_3-1}\\ &-x((x-2)(x-n_4)-n_4)(x-2)^{n_4-1}((x-2)(x-n_3)-n_3)(x-2)^{n_3-1}\\ =&x(x-2)^{n_3+n_4-2}(x^4-(2n_3+2n_4+4)x^3-((n_4+2)n_3+2n_4)(n_3+n_4+2)x\\ &+(n_3^2+(3n_4+6)n_3+(n_4+2)^2+2n_4)x^2+n_3n_4(n_3+n_4+3))\\ \triangleq&x(x-2)^{n_3+n_4-2}f(x).\end{aligned}$$

Note that $n_3 + n_4 \geqslant 3$, then we have

$$\begin{aligned}&f(0) = n_3n_4(n_3+n_4+3) > 0, f(1) = -n_3^2 - n_4^2 + 1 < 0,\\ &f(2) = -n_3n_4(n_3+n_4-3) \leqslant 0, f(\frac{n_3+n_4}{2}+1) = \frac{((n_3+n_4)^2-8)(n_3-n_4)^2+16}{16} > 0,\\ &f(n_3+n_4+1) = -n_3^2 - n_4^2 + 1 < 0, f(n_3+n_4+2) = n_3n_4(n_3+n_4+3) > 0.\end{aligned}$$

Therefore, we have $\mu_{n-2}(H(n_3, n_4)) = 2$. Then by Lemma 5, we have $\mu_{n-2}(\overline{G}) \geqslant 2$. If $n_4 \geqslant n_2 \geqslant 2$, then $\overline{G}$ has $K_{n_2,n_4} \cup K^e_{2,n_3}$ as a spanning subgraph. Then by Lemmas 3 and 5, we have $\mu_{n-2}(\overline{G}) \geqslant \mu_{n-2}(K_{n_2,n_4} \cup K^e_{2,n_3}) \geqslant 2$. Thus, we have $\mu_{n-2}(\overline{G}) \geqslant 2$ for any G of the class $G_4(1, n_2, n_3, n_4, 1)$. By Lemma 2, we have $\mu_2(G) \leqslant n - 2$ for any G of the class $G_4(1, n_2, n_3, n_4, 1)$. For $D(G) \geqslant 5$, by Lemma 15, we have $\mu_2(G) \leqslant \mu_{2_{max}}(n, D) \leqslant \mu_{2_{max}}(n, 4) \leqslant n - 2$.

Assume now that $D(G) = 2$ and $D(\overline{G}) \geqslant 4$. by Lemma 2, we have $\mu_{n-2}(G)+\mu_{n-2}(\overline{G}) \geqslant 2$. This completes the proof. □

In [12](see Theorem 7), the authors proved that $\mu_{n-2}(G) + \mu_{n-2}(\overline{G}) \geqslant 2$ is true for the graph G when $D(G) \neq 2$ and $D(\overline{G}) \neq 2$. Combining with this result, $\mu_{n-2}(G)+\mu_{n-2}(\overline{G}) \geqslant 2$ is proved to be true for all graphs except the graph G which satisfys $D(G) = 2$, $D(\overline{G}) = 3$, and both G and $\overline{G}$ have a K_3 as a subgraph.

4 NG-inequality for $\mu_k(G)$

In this section, we consider the kth largest Laplacian eigenvalue. Firstly, we consider the equality case.

Lemma 17. *Let $G = K_{n-k} \vee H$ be the graph of order n for some integer $k \in [\frac{n+1}{2}, n-1]$, where H is a disconnected graph on k vertices and has at least $n - k + 1$ connected components. Then $\mu_k(G) + \mu_k(\overline{G}) = n - k$.*

Proof. We denote the Laplacian spectrum of H as $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{2k-n-1}$ and $0^{(n-k+1)}$. Then by Lemma 2, the Laplacian spectrum of $\overline{H}$ are $k^{(n-k)}$ and $k-\mu_{2k-n-1} \geqslant \cdots \geqslant k-\mu_1$

and 0. Since $\overline{G} = \overline{K_{n-k}} \cup \overline{H}$, it follows that the Laplacian spectrum of $\overline{G}$ are $k^{(n-k)}$, $k - \mu_{2k-n-1} \geqslant \cdots \geqslant k - \mu_1$ and $0^{(n-k+1)}$. By Lemma 2, the Laplacian spectrum of G are $n^{(n-k)}$, $n - k + \mu_1 \geqslant \cdots \geqslant n - k + \mu_{2k-n-1}$, $(n-k)^{(n-k)}$ and 0. It is easy to see that $\mu_k(G) + \mu_k(\overline{G}) = n - k + 0 = n - k$ for $k \in [\frac{n+1}{2}, n-1]$. This completes the proof. □

Let G be a graph with n vertices and $S_k(G) = \sum_{i=1}^{k} \mu_i(G)$ be the sum of the first k largest Laplacian eigenvalues of G. Brouwer [5] conjectured that for any graph G with n vertices and for any $k \in \{1, 2, \ldots, n\}$, $S_k(G) \leqslant m + \binom{k+1}{2}$. Recently, Vladimir et al.[24] claimed that if Brouwer conjecture is true for $n \leqslant 10^{27}$, then Brouwer conjecture is true. Now we have the following result if $S_k(G) \leqslant m + \binom{k+1}{2}$ for $k \in \{1, 2, \ldots, n\}$.

Theorem 18. *Let G be a graph on n vertices and $\overline{G}$ be the complement of G. Then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n$ for $k \leqslant \frac{n}{2}$. In particular, if Brouwer's conjecture holds, then $\mu_{\frac{n+1}{2}}(G) + \mu_{\frac{n+1}{2}}(\overline{G}) \geqslant n - \frac{n+1}{2}$.*

Proof. For $k \leqslant \frac{n}{2}$, by Lemma 7, we have $\mu_k(G) + \mu_k(\overline{G}) \geqslant \mu_{2k-1}(K_n) = n$. Next we consider the case $k = \frac{n+1}{2}$, which implies that n is odd. By Lemma 2, we have

$$\mu_{\frac{n-1}{2}}(G) + \mu_{\frac{n+1}{2}}(\overline{G}) = n. \tag{1}$$

Now we show that

$$\mu_{\frac{n-1}{2}}(G) \leqslant \mu_{\frac{n+1}{2}}(G) + \frac{n+1}{2}. \tag{2}$$

We prove (2) by contradiction. We may assume that $\mu_{\frac{n-1}{2}}(G) > \mu_{\frac{n+1}{2}}(G) + \frac{n+1}{2}$ for G. Let $T_{\frac{n-1}{2}}(G) = \sum_{i=\frac{n+1}{2}}^{n-1} \mu_i(G)$. Then $S_{\frac{n-1}{2}}(G) - T_{\frac{n-1}{2}}(G) > \frac{n+1}{2} \cdot \frac{n-1}{2} = \frac{n^2-1}{4}$. By Lemma 2,

$$\begin{aligned} S_{\frac{n-1}{2}}(G) - T_{\frac{n-1}{2}}(G) &= S_{\frac{n-1}{2}}(G) - \left(n \cdot \frac{n-1}{2} - S_{\frac{n-1}{2}}(\overline{G})\right) \\ &= S_{\frac{n-1}{2}}(G) + S_{\frac{n-1}{2}}(\overline{G}) - \frac{n(n-1)}{2}. \end{aligned}$$

Now we can deduce that

$$S_{\frac{n-1}{2}}(G) + S_{\frac{n-1}{2}}(\overline{G}) > \frac{n(n-1)}{2} + \frac{n^2-1}{4}.$$

On the other hand, if Brouwer's conjecture holds, we have

$$S_{\frac{n-1}{2}}(G) + S_{\frac{n-1}{2}}(\overline{G}) \leqslant m + \overline{m} + 2 \cdot \binom{\frac{n+1}{2}}{2} = \frac{n(n-1)}{2} + \frac{n^2-1}{4},$$

a contradiction. In consequence, by (1) and (2), if Brouwer's conjecture holds, then $\mu_{\frac{n+1}{2}}(G) + \mu_{\frac{n+1}{2}}(\overline{G}) \geqslant n - \frac{n+1}{2}$. This completes the proof. □

The graphs having no induced subgraphs $2K_2$, P_4, C_4 are called threshold graphs. Threshold graphs have many known characterizations(see [10] for example), one of which is that the vertices can be partitioned into a clique K and an independent set I so that the sets of neighbors of the vertices of I (and therefore also of K) are nested, i.e., totally ordered by inclusion. Using this property, we give a vertex partition to the connected threshold graph.

For a connected threshold graph G, suppose K_{t+1} is a maximal clique of G. Let $V(K_{t+1}) = \{v_1, v_2, \ldots, v_t, v_{t+1}\}$ such that $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_{t+1})$. Let $V_i = \{v_i\}$ and $U_j = \{v : N(v) = \{v_1, v_2, \ldots, v_j\}\}$ for $i, j = 1, 2, \ldots, t$. Then $V(G) = \bigcup_{i=1}^{t} V_i \cup \bigcup_{i=1}^{t} U_i$. Clearly, if $u' \in U_i$ and $u'' \in U_{i+1}$, then $N(u') \subset N(u'')$, and this explains the nested property. Let $s_j = |U_j|$. Then $s_t \geqslant 1$ and $s_j \geqslant 0$ for $j = 1, 2, \ldots, t-1$. For example, if $G = K_n$, let $V(K_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, then $V_i = \{v_i\}$ for $i = 1, 2, \ldots, n-1$, $U_i = \emptyset$ for $i = 1, 2, \ldots, n-2$ and $U_{n-1} = \{v_n\}$.

We remark that $\sum_{i=s}^{t} a_i = 0$ if $s > t$.

Theorem 19. *Let G be a threshold graph of order n. Then*

$$\mu_k(G) + \mu_k(\overline{G}) \geqslant n - k,$$

for $k = 1, 2, \ldots, n-1$, with equality if and only if G or $\overline{G}$ is isomorphic to $K_{n-k} \vee H$, where H is a disconnected graph on k vertices and has at least $n-k+1$ connected components.

Proof. Since the complement of a threshold graph is also a threshold graph, we now consider the connected threshold graph G. Let K_{t+1} be the maximal clique in G and $V(K_{t+1}) = \{v_1, v_2, \ldots, v_t, v_{t+1}\}$. Using the above vertex partition and notations, we have $n = t + \sum_{i=1}^{t} s_i$, $d(u) = j$ for $u \in U_j$ and $d(v_j) = t - 1 + \sum_{i=j}^{t} s_i$, where $j = 1, 2, \ldots, t$. For the better understanding, the degree sequence of G are as follows in non-decreasing order,

$$\{\underbrace{1, \ldots, 1}_{s_1}, \underbrace{2, \ldots, 2}_{s_2}, \ldots, \underbrace{t, \ldots, t}_{s_t}, t-1+s_t, t-1+s_t+s_{t-1}, \ldots, t-1+\sum_{i=1}^{t} s_i\}.$$

By Theorem 2 in [18], for a threshold graph G, $\mu_i(G) = d_i^*(G)$, where $d_i^*(G) = |\{v \in V(G) : d(v) \geqslant i\}|$ for $i = 1, 2, \ldots, n$. Then from the degree sequence of G, it is easy to obtain that the Laplacian eigenvalues of G are as follows in non-increasing order,

$$\{n^{(1)}, (n-s_1)^{(1)}, (n-s_1-s_2)^{(1)}, \ldots, \left(n - \sum_{i=1}^{t-1} s_i\right)^{(1)}, t^{(s_t-1)}, (t-1)^{(s_{t-1})}, (t-2)^{(s_{t-2})}, \ldots, 1^{(s_1)}, 0^{(1)}\}.$$

By Lemma 2, the Laplacian eigenvalues of $\overline{G}$ are as follows in non-increasing order,

$$\{(n-1)^{(s_1)}, (n-2)^{(s_2)}, \ldots, (n-t+1)^{(s_{t-1})}, (n-t)^{(s_t-1)}, \left(\sum_{i=1}^{t-1} s_i\right)^{(1)}, \left(\sum_{i=1}^{t-2} s_i\right)^{(1)}, \ldots, s_1^{(1)}, 0^{(2)}\}.$$

First suppose $\sum_{i=1}^{j_0-1} s_i + 1 \leqslant k \leqslant \sum_{i=1}^{j_0} s_i$, for $j_0 \in \{1, 2, \ldots, t-1\}$. Then $\mu_k(\overline{G}) = n - j_0$. If $\mu_k(G) = n - \sum_{i=1}^{k-1} s_i$, then we have $\mu_k(G) + \mu_k(\overline{G}) \geqslant n - j_0 > n - k$ when $k > j_0$. When $k \leqslant j_0$, we have $\sum_{i=1}^{k-1} s_i \leqslant \sum_{i=1}^{j_0-1} s_i \leqslant k-1$. Then we have $\mu_k(G) + \mu_k(\overline{G}) \geqslant n - k + 1 + n - j_0 > n - k$. If $\mu_k(G) = t$, then $\mu_k(G) + \mu_k(\overline{G}) = n - j_0 + t > n > n - k$ since $j_0 < t$. If $\mu_k(G) = t - j'$ for $j' \in \{1, 2, \ldots, t-1\}$, then $k \geqslant t$. Since $j' < t$ and $j < t$, we have $\mu_k(G) + \mu_k(\overline{G}) = t - j' + n - j_0 > n - t \geqslant n - k$.

Next suppose $\sum_{i=1}^{t-1} s_i + 1 \leqslant k \leqslant n - t - 1$. Then $\mu_k(\overline{G}) = n - t$. If $\mu_k(G) = n - \sum_{i=1}^{k-1} s_i$, then $1 \leqslant k \leqslant t$. Hence, $\sum_{i=1}^{k-1} s_i \leqslant \sum_{i=1}^{t-1} s_i \leqslant k-1$. Then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n - t + n - \sum_{i=1}^{k-1} s_i \geqslant n - t + n - k + 1 > n - k$. If $\mu_k(G) = t$, then $\mu_k(G) + \mu_k(\overline{G}) = n > n - k$. If $\mu_k(G) = t - j_0$ for $j_0 \in \{1, 2, \ldots, t-1\}$, then $k \geqslant t$. Since $j_0 < t$, we have $\mu_k(G) + \mu_k(\overline{G}) = n - j_0 > n - t \geqslant n - k$.

Finally, suppose $n - t \leqslant k \leqslant n - 1$. Then $\mu_k(\overline{G}) = \sum_{i=1}^{n-1-k} s_i$. If $\mu_k(G) = n - \sum_{i=1}^{k-1} s_i$, then $1 \leqslant k \leqslant t$. Recall that $n = t + \sum_{i=1}^{t} s_i$ and $s_t \geqslant 1$. Then $\sum_{i=1}^{k-1} s_i \leqslant \sum_{i=1}^{t-1} s_i < n - t$. So $\mu_k(G) + \mu_k(\overline{G}) \geqslant \mu_k(G) > t \geqslant n - k$. If $\mu_k(G) = t$, then $t + 1 \leqslant k \leqslant t + s_t - 1$. We have $\mu_k(G) + \mu_k(\overline{G}) = t + \sum_{i=1}^{n-k-1} s_i \geqslant t \geqslant n - k$. Equality holds if and only if $t = n - k$, $\sum_{i=1}^{n-k-1} s_i = 0$, and $n - t \geqslant t + 1$. It follows that $k = n - t$ and $s_1 = s_2 = \cdots = s_{t-1} = 0$. Then $G = K_t \vee (n-t)K_1$, the result holds. If $\mu_k(G) = t - j_0$ for $j_0 \in \{1, \ldots, t-1\}$, then $k \geqslant n - \sum_{i=1}^{t-j_0} s_i$. When $t - j_0 \geqslant n - k$, we have $\mu_k(G) + \mu_k(\overline{G}) = \sum_{i=1}^{n-k-1} s_i + t - j_0 \geqslant n - k$. Equality holds if and only if $t - j_0 = n - k$ and $\sum_{i=1}^{n-k-1} s_i = 0$. It follows that $k = n - t + j_0$ and $s_1 = s_2 = \cdots = s_{t-j_0-1} = 0$. Then $G = K_{t-j_0} \vee H$, where $V(K_{t-j_0}) = \{v_1, \ldots, v_{t-j_0}\}$ and $H = G[U_{t-j_0} \cup U_{t-j_0+1} \cup \cdots \cup U_t \cup \{v_{t-j_0+1}, \ldots, v_t\}]$. Recall that $\sum_{i=1}^{t-j_0} s_i \geqslant n - k$, we have $s_{t-j_0} \geqslant n - k$. Then H is a disconnected graph on $n - t + j_0$ vertices and has at least $t - j_0 + 1$ connected components. When $1 \leqslant t - j_0 \leqslant n - k - 1$, we have $\sum_{i=1}^{n-k-1} s_i \geqslant \sum_{i=1}^{t-j_0} s_i \geqslant n - k$. Then $\mu_k(G) + \mu_k(\overline{G}) = \sum_{i=1}^{n-k-1} s_i + t - j_0 > n - k$. This completes the proof. □

The barbell graph $B^e_{s,t}(s \geqslant t)$ is constructed by connecting two complete graphs $K_s(s \geqslant 1)$ and $K_t(t \geqslant 1)$ by a bridge e.

Lemma 20. *Let G be the barbell graph $B^e_{s,t}$ with $s > t > 1$. Then $\mu_s(G) > t$.*

Proof. Let $e = uv$, $u \in K_s$ and $v \in K_t$. By Lemma 8, the characteristic polynomial of $L(G)$ is as follows:

$$\begin{aligned}\Phi(L(G)) =&\Phi(L(K_s))\Phi(L(K_t)) - \Phi(L(K_s))\Phi(L_v(K_t)) - \Phi(L(K_t))\Phi(L_u(K_s))\\ =&x(x-s)^{s-1}x(x-t)^{t-1} - x(x-s)^{s-1}(x-1)(x-t)^{t-2}\\ &-x(x-t)^{t-1}(x-1)(x-s)^{s-2}\\ =&x(x-s)^{s-2}(x-t)^{t-2}(x^3-(s+t+2)x^2+(st+s+t+2)x-(s+t))\\ \triangleq&x(x-s)^{s-2}(x-t)^{t-2}g(x)\end{aligned}$$

Let $x_1 \geqslant x_2 \geqslant x_3$ be the roots of the equation $g(x) = 0$. Then the Laplacian spectrum of G are $\{x_1^{(1)}, s^{(s-2)}, x_2^{(1)}, t^{(t-2)}, x_3^{(1)}, 0^{(1)}\}$. Since

$$\begin{aligned}&g(0) = -s-t < 0, g(1) = st-(s+t)+1 > 0,\\ &g(t) = (t-1)(s-t) > 0, g(t+1) = -s+1 < 0,\\ &g(s) = (s-1)(t-s) < 0, g(s+2) = s^2+(3-t)s-3t+4 > 0,\end{aligned}$$

we have $0 < x_3 < 1$, $t < x_2 < t+1$, and $s < x_1 < s+2$. Consequently, $\mu_s(G) = x_2 > t$, as required. □

Let $\lceil x \rceil$ denote the least integer not less than x. In [4], Ashraf et al. proved $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant 1$ for bipartite graphs. We improve this result as follows.

Theorem 21. *Let $G =$ be a bipartite graph of order $n(n = r+s)$ with partition (V_1, V_2) and $|V_1| = r \geqslant |V_2| = s$. Then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n-k$ for $k = 1, 2, \ldots, n-1$, with equality if and only if $k = n-1$ and G or $\overline{G}$ is a star with at most 1 isolated vertex.*

Proof. If $s = 1$, then G is a star with some isolated vertices. Suppose $G = K_{1,t-1} \cup (n-t)K_1 (2 \leqslant t \leqslant n)$. Then the Laplacian spectrum of G are $t^{(1)}$, $1^{(t-2)}$ and $0^{(n-t+1)}$. By Lemma 2, the Laplacian spectrum of $\overline{G}$ are $n^{(n-t)}$, $(n-1)^{(t-2)}$, $(n-t)^{(1)}$ and $0^{(1)}$. If $1 \leqslant k \leqslant n-t$, then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n > n-k$. If $n-t+1 \leqslant k \leqslant n-2$, then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n-1 \geqslant n-k$. Equality is possible if $k = 1$ and $t = n$. And then $\mu_1(G)+\mu_1(\overline{G}) = n+n-1 > n-1$. Next suppose $k = n-1$. If $t = n$, then $G = K_{1,n-1}$. Then $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) = 1$. If $t \leqslant n-1$, then $\mu_{n-1}(G) + \mu_{n-1}(\overline{G}) \geqslant n-t \geqslant 1$. With equality if and only if $t = n-1$. Then $G = K_{1,n-2} \cup K_1$. It follows that $\overline{G} = K_1 \vee (K_1 \cup K_{n-2})$. The result holds.

Next suppose $s \geqslant 2$. Since $r \geqslant s$, we have $r \geqslant \lceil \frac{n}{2} \rceil \geqslant \frac{n}{2}$ and then $n - r \leqslant \frac{n}{2}$. Note that $\overline{G}$ has $H = K_r \cup K_s$ as a spanning subgraph. Then by Lemma 5, it follows that $\mu_k(\overline{G}) \geqslant \mu_k(H)$. By Lemma 2, the Laplacian spectrum of H are as follows.

$$\mu_k(H) = \begin{cases} r, & 1 \leqslant k \leqslant r-1\\ s, & r \leqslant k \leqslant n-2\\ 0, & k = n-1, n\end{cases}$$

For $1 \leqslant k \leqslant n-r$, by Theorem 18, we have $\mu_k(G)+\mu_k(\overline{G}) > n-k$. For $n-r+1 \leqslant k \leqslant r-1$, we have $\mu_k(G)+\mu_k(\overline{G}) \geqslant \mu_k(H) = r \geqslant n-k+1 > n-k$. For $r \leqslant k \leqslant n-2$, we have $\mu_k(G)+\mu_k(\overline{G}) \geqslant \mu_k(H) = s = n-r \geqslant n-k$. Equality holds if and only if $k=r$, $\mu_k(G) = 0$ and $\mu_k(\overline{G}) = s$. Now we prove that no such bipartite graph exists. Suppose first that $r > s$. By Lemma 20, we have $\mu_r(B^e_{r,s}) > s$. Then $\mu_r(\overline{G}) = s$ implies that $\overline{G} = H$. It follows that $\mu_r(G) = n - \mu_s(\overline{G}) = n-r = s$, which is contradict to $\mu_r(G) = 0$. Next suppose that $r = s$. Since $\mu_r(G) = 0$, G has at least $r+1$ connected components. It follows that G has at least 1 isolated vertex. Therefore, $\overline{G}$ has $K_1 \vee (K_r \cup K_{r-1})$ as a spanning subgraph. By Lemmas 2 and 3, we have $\mu_r(K_1 \vee (K_r \cup K_{r-1})) = s+1$. Then by Lemma 5, $\mu_k(\overline{G}) \geqslant s+1$, a contradiction. It is true for $k = n-1$ due to Lemma 4. This completes the proof. □

Theorem 22. *Let G be a complete t-partite graph $K_{n_1,n_2,\ldots,n_t}$ with $t \geqslant 3$ and $\sum\limits_{i=1}^{t} n_i = n$. Then $\mu_k(G)+\mu_k(\overline{G}) \geqslant n-k$ for $k = 1,2,\ldots,n-1$, with equality if and only if $k = n-t+1$ and $G = K_{n-t+1,1,\ldots,1}$.*

Proof. Without loss of generality, we assume that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t$. By Lemmas 2 and 3, the Laplacian eigenvalues of G are $\{n^{(t-1)}, (n-n_t)^{(n_t-1)}, \ldots, (n-n_1)^{(n_1-1)}, 0^{(1)}\}$ and the Laplacian eigenvalues of $\overline{G}$ are $\{n_1^{(n_1-1)}, n_2^{(n_2-1)}, \ldots, n_t^{(n_t-1)}, 0^{(t)}\}$. Sort the eigenvalues by non-increasing order. Then in G, for every $1 \leqslant r \leqslant t$, the first $n-n_r$ is $\sum\limits_{i=r+1}^{t} n_i + r$ largest eigenvalues. In $\overline{G}$, for every $1 \leqslant s \leqslant t$, the first n_s is $\sum\limits_{i=1}^{s-1} n_i - s + 2$ largest eigenvalues. For $1 \leqslant k \leqslant t-1$, we have $\mu_k(G)+\mu_k(\overline{G}) \geqslant n > n-k$. For $t \leqslant k \leqslant n-t$, $\mu_k(G)+\mu_k(\overline{G})$ has the form $n - n_r + n_s$ for some $r, s (1 \leqslant r, s \leqslant t)$. If $r \geqslant s$, which implies that $n_r \leqslant n_s$, then $n - n_r + n_s \geqslant n > n-k$. Next we suppose $r \leqslant s-1$. Since $n_i \geqslant 1$ for $i = 1, \ldots, t$, it follows that

$$\sum_{i=1}^{s-1} n_i - s + 2 \geqslant n_r > n_r - n_s.$$

Hence $k \geqslant \max\{\sum\limits_{i=1}^{s-1} n_i - s + 2, \sum\limits_{i=r+1}^{t} n_i + r\} > n_r - n_s$. Therefore $n - n_r + n_s > n-k$. Consequently, $\mu_k(G)+\mu_k(\overline{G}) > n-k$ for $t \leqslant k \leqslant n-t$. For $n-t+1 \leqslant k \leqslant n-1$, $\mu_k(G)+\mu_k(\overline{G}) = \mu_k(G) = n - n_r$ for some $r (1 \leqslant r \leqslant t)$. Since $k \geqslant n-t+1 = \sum\limits_{i=1}^{t} n_i - t + 1 \geqslant n_r + t - 1 - t + 1 = n_r$ for some $r (1 \leqslant r \leqslant t)$, we have $\mu_k(G)+\mu_k(\overline{G}) \geqslant n-k$. Equality holds if and only if $n_1 = n-t+1$, $n_2 = \cdots = n_t = 1$ and $k = n-t+1$. Then $G = K_{n-t+1,1,\ldots,1} = K_{t-1} \vee (n-t+1)K_1$. □

Lemma 23 ([28], Theorem 3.4)**.** *Let G be a d-regular graph of order n. Then for $k = 1,2,\ldots,n-1$, $\mu_k(G) \leqslant \frac{1}{n-1}(nd + \sqrt{\frac{n-k-1}{k} nd(n-d-1)})$.*

Theorem 24. *Let G be a d-regular graph of order n. Then $\mu_k(G)+\mu_k(\overline{G}) > n-k$ for $k = 1,2,\ldots,n-1$.*

Proof. By Lemma 2, we have $\mu_k(G) = n - \mu_{n-k}(\overline{G})$ for $k = 1, 2, \ldots, n-1$. Then $\mu_k(G) + \mu_k(\overline{G}) > n-k$ is equivalent to $\mu_{n-k}(G) + \mu_{n-k}(\overline{G}) < n+k$ for $k = 1, 2, \ldots, n-1$. That is $\mu_k(G) + \mu_k(\overline{G}) < 2n-k$ for $k = 1, 2, \ldots, n-1$. It is clear that $\overline{G}$ is a $(n-1-d)$-regular graph. Then by Lemma 23,

$$\begin{aligned}\mu_k(G) + \mu_k(\overline{G}) &\leqslant \frac{1}{n-1}(nd + \sqrt{\frac{n-k-1}{k}nd(n-d-1)}) \\ &+ \frac{1}{n-1}(n(n-1-d) + \sqrt{\frac{n-k-1}{k}n(n-d-1)d}) \\ &= \frac{1}{n-1}(n(n-1) + 2\sqrt{\frac{n-k-1}{k}nd(n-d-1)}) \\ &= n + \frac{2}{n-1}\sqrt{\frac{n-k-1}{k}nd(n-d-1)}.\end{aligned}$$

We next to show that $\frac{2}{n-1}\sqrt{\frac{n-k-1}{k}nd(n-d-1)} < n-k$. That is to show $4\frac{n-k-1}{k}nd(n-1-d) < (n-k)^2(n-1)^2$. Since $k \leqslant n$, we have $n\frac{n-k-1}{k} = \frac{(n-k)^2+(n-k)(k-1)-k}{k} < \frac{k(n-k)^2}{k} = (n-k)^2$. By AM-GM inequality, we have $d(n-d-1) \leqslant (\frac{d+n-d-1}{2})^2 = \frac{(n-1)^2}{4}$. Hence $4\frac{n-k-1}{k}nd(n-1-d) < (n-k)^2(n-1)^2$ holds. This completes the proof. ☐

Lemma 25 ([14], Theorem 4)**.** *Let T be a tree with n vertices. Then $\mu_k(T) \leqslant \lceil \frac{n}{k} \rceil$ for $1 \leqslant k \leqslant n$.*

We say that a connected graph G is c-cyclic, or it has c cycles, if it has $n-1+c$ edges.

Theorem 26. *Let G be a c-cyclic graph of order n. If $n \geqslant 2c+8$, then $\mu_k(G) + \mu_k(\overline{G}) > n-k$ for $k = 2, 3, \ldots, n-1$.*

Proof. By Lemma 2, it's equivalent to proving that $\mu_k(G) + \mu_k(\overline{G}) < 2n-k$ for $k = 1, 2, \ldots, n$. Let T be a spanning tree of G. Let $F_1, F_2, \ldots, F_t$ be the connected components of $G-T$. Suppose $\mu_1(F_1) \geqslant \mu_1(F_i)$, $i = 2, \ldots, t$. Note that $m(F_1) \geqslant n(F_1)-1$. By Lemma 6, we have $\mu_1(F_1) \leqslant n(F_1) \leqslant m(F_1)+1$. By Lemmas 7 and 25,

$$\begin{aligned}\mu_k(G) &\leqslant \mu_k(T) + \mu_1(G-T) \leqslant \mu_k(T) + \mu_1(F_1) \\ &\leqslant \mu_k(T) + m(F_1) + 1 \leqslant \lceil \frac{n}{k} \rceil + c + 1 \\ &< \frac{n}{k} + c + 2.\end{aligned}$$

For $2 \leqslant k \leqslant n-c-4$, we next to show that $\frac{n}{k} + c + 2 \leqslant n-k$ when $n \geqslant 2c+8$. That is to show $\frac{k^2+(c-n+2)k+n}{k} \leqslant 0$. Since $k > 0$, we only need to show that $k^2+(c-n+2)k+n \leqslant 0$ for $2 \leqslant k \leqslant n-c-4$. Let $g(k) = k^2 + (c-n+2)k + n$. It's not difficult to know that the roots of $g(k) = 0$ are $k_1, k_2 = \frac{n-c-2}{2} \mp \frac{\sqrt{c^2-2cn+n^2+4c-8n+4}}{2}$. Since $n \geqslant 2c+8$, we have $c^2 - 2cn + n^2 + 4c - 8n + 4 \geqslant (n-(c+6))^2$. Hence $k_1 \leqslant \frac{n-c-2}{2} - \frac{n-(c+6)}{2} = 2$, $k_2 \geqslant \frac{n-c-2}{2} + \frac{n-(c+6)}{2} = n-c-4$. Then $g(k) \leqslant 0$ for $2 \leqslant k \leqslant n-c-4$. Therefore, we have $\mu_k(G) < n-k$ for $2 \leqslant k \leqslant n-c-4$. Since $\mu_k(\overline{G}) \leqslant n$ by Lemma 6, we have $\mu_k(G) + \mu_k(\overline{G}) < 2n-k$ for $2 \leqslant k \leqslant n-c-4$.

For $n-c-3 \leqslant k \leqslant n-1$, we have $k \geqslant n-c-3 \geqslant \frac{n}{2}+1 > \frac{n+1}{2}$ since $n \geqslant 2c+8$. By Lemma 7, we have $\mu_k(G) + \mu_k(\overline{G}) \leqslant \mu_{2k-n}(K_n) = n < 2n-k$. This completes the proof. □

Using a computer, we check that Conjecture 1 is true for all graphs with at most 9 vertices and for bicyclic graphs with at most 11 vertices. Then by Lemma 4 and Theorem 26, we immediately obtain the following corollaries.

Corollary 27. *Let G be a unicyclic graph of order n. Then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n-k$ for $k = 1, 2, \ldots, n-1$.*

Corollary 28. *Let G be a bicyclic graph of order n. Then $\mu_k(G) + \mu_k(\overline{G}) \geqslant n-k$ for $k = 1, 2, \ldots, n-1$.*

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