Alon–Boppana-type bounds for weighted graphs

Alexander Polyanskii[a] Rinat Sadykov[b]

Submitted: Jul 13, 2023; Accepted: Jan 12, 2024; Published: Feb 9, 2024

Abstract

The *unraveled ball* of radius r centered at a vertex v in a weighted graph G is the ball of radius r centered at v in the universal cover of G. We present a general bound on the maximum spectral radius of unraveled balls of fixed radius in a weighted graph.

The weighted degree of a vertex in a weighted graph is the sum of weights of edges incident to the vertex. A weighted graph is called *regular* if the weighted degrees of its vertices are the same. Using the result on unraveled balls, we prove a variation of the Alon–Boppana theorem for regular weighted graphs.

Mathematics Subject Classifications: 05C22

1 Introduction

In 1993, Freidmen [1] refined the celebrated Alon–Boppana theorem [5]. He proved that for every d-regular graph G with diameter $2r$, the second largest eigenvalue of adjacency matrix of G, denoted by $\lambda_2(G)$, satisfies

$$\lambda_2(G) \geqslant 2\left(1 - \frac{\pi^2}{r^2} + O(r^4)\right)\sqrt{d-1}.$$

In 2005, Hoory [2, Theorem 1] studied the spectral radius of the universal cover of a non-regular graph. As a corollary he proved a variation of the Alon–Boppana theorem for graphs with r-robust average degree at least d, which was later improved by Jiang [3]. For a graph G, the *ball* of radius $r \geqslant 0$ centered at $v \in V(G)$, denoted by $G(v, r)$, is the induced subgraph of G on the vertices within distance r apart from v. We say that a graph has *r-robust average degree* at least d if for the induced subgraph obtained by deleting any ball or radius r, its average degree is at least d.

[a] Department of Mathematics, Emory University, 400 Dowman Drive, Atlanta, GA, 30322, U.S.A. (apolian@emory.edu).

[b] Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Russia (sadykov@phystech.edu).

Theorem 1. *[3, Theorem 8] Let $d \geqslant 1$ be a real number and let r be a positive integer. If a graph G has an r-robust average degree at least d, then*

$$\frac{\lambda_2(G)}{\lambda_1(P_r)} \geqslant \sqrt{d-1}.$$

Here $\lambda_1(P_r) = 2\cos\frac{\pi}{r+1}$ stands for the spectral radius of the path P_r with r vertices.

To prove Theorem 1, Jiang studied the maximum spectral radius of unraveled balls of a graph G, which are balls in the universal cover of G; see Definition 2. In 2022, Wang and Zhang applied the machinery developed in [3] and proved an analog of Theorem 1 for the normalized Laplacian of a graph [7, Theorem 1.6], improving the bounds from [8]; see also [6]. To show this result, they studied the maximum spectral radius of unraveled balls of a weighted graph.

Motivated by the works of Jiang and Wang–Zhang, we develop further their ideas and prove a generalization of Wang and Zhang's result on the spectral radius of unraveled balls for a weighted graph; see Theorem 6. This implies an analog of Theorem 1 for regular weighted graphs, as shown in Theorem 4.

A *weighted graph* is a graph without parallel edges and loops, in which every edge is assigned to a positive number. Formally, a weighted graph G is a triple $(V(G), E(G), w_G)$, where $V(G)$ and $E(G)$ are the vertex and edge sets of the graph G, respectively, and $w_G : E(G) \to \mathbb{R}_+$ is the weight function, with $\mathbb{R}_+$ being the set of positive real numbers. For sake of brevity, we write w_{ab} and w_{ba} for the weight $w_G(ab)$ of an edge $ab \in E(G)$. The *weighted degree* of a vertex v, denoted by w_v, is the sum of the weights of the edges incident to v, that is, $w_v = \sum_{vu\in E(G)} w_{vu}$. A weighted graph is called *w-regular* if the weighted degree of every vertex equals w. Throughout the paper, we regularly write "*a weighted graph with minimal degree at least 2*", which means that each vertex is incident to at least 2 edges (rather than the weighted degree of each vertex is at least 2).

A *non-backtracking walk* of length n in a weighted graph is a sequence of vertices $(v_0, \ldots, v_n)$ such that any two consecutive are adjacent and $v_i \neq v_{i+2}$ for all $i \in \{0, \ldots, n-2\}$. Denote by $W_i(G)$ the set of non-backtracking walks on a graph G of length i.

Definition 2. Given a weighted graph G, we define the weighted tree $\tilde{G}(v, r)$ as follows. Its vertex set is the set of all non-backtracking walks of length at most r that start at v, where two vertices are adjacent if one is a simple extension of the other. Specifically, vertices $(v_0, \ldots, v_n)$ and $(u_0, \ldots, u_m)$ with $n < m$ are adjacent if and only if $m = n + 1$ and $v_i = u_i$ for all $i \in \{0, \ldots, n\}$. We say this edge of $\tilde{G}(v, r)$ is extended by the edge $u_{m-1}u_m$ in the graph G. Two vertices of the same length are never adjacent. We assign a weight to each edge in $\tilde{G}(v, r)$ equal to the weight of its extending edge in G.

In other words, the graph $\tilde{G}(v, r)$, which we call an *unraveled ball*, is isomorphic to a ball of radius r in the universal cover $\tilde{G}$ of G. Slightly abusing notation, in the current paper, we say $\tilde{G}(v, r)$ is an induced subgraph of $\tilde{G}$

It is worth mentioning that we may look at the set $W_1(G)$ as the set of directed edges of a graph G, that is, for any edge $xy \in E(G)$, there are two corresponding non-backtracking edges (x, y) and (y, x) in $W_1(G)$. So, we write $w_{(x,y)}$ for w_{xy} if $xy \in E(G)$.

The *weighted adjacency matrix* A_G of a weighted graph G is a matrix whose rows and columns correspond to vertices of the graph G. The elements of the matrix A_G are defined in the following way

$$A_{G x,y} = \begin{cases} w_{xy} & \text{if } xy \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the matrix A_G is symmetric and has zeros on its main diagonal.

Let $\lambda_i(G)$ be the i-th largest eigenvalue of G. The central result of the paper is presented below.

Theorem 3. *Let G be a w-regular weighted graph with minimum degree at least 2. Then for any positive integer r, there is a vertex v such that*

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \frac{\sum_{e \in W_1(G)} w_e^{3/2}(w - w_e)^{1/2}}{w|V(G)|}.$$

This theorem generalizes the main result from the paper of Jiang [3, Theorem 1]; see also the proof of Lemma 19 in [4].

We define the *average combinatorial degree* of a weighted graph G as $\frac{2|E(G)|}{|V(G)|}$. As a corollary of Theorem 3, we obtain a variation of the Alon–Boppana theorem.

Theorem 4. *Let G be a w-regular weighted graph with combinatorial degree $d \geqslant 39$ and let r be any positive integer. Assume that for every vertex v there is an induced subgraph of $G \setminus G(v, r+1)$ with minimum weighted degree at least $2w\sqrt{d-1}/d$. Then we have*

$$\frac{\lambda_2(G)}{\lambda_1(P_{r+1})} \geqslant \frac{w\sqrt{d-1}}{d}.$$

A slightly more involved argument allows to show that Theorem 4 holds for $d \geqslant 7.1980\ldots$; see Remark 13.

The rest of the paper is organized as follows. Section 2 presents the proof of Theorem 6, a generalization of Theorem 3 for weighted graphs that are not necessary w-regular. In Section 3, we utilize Theorem 6 to derive Corollary 8, a general form of Theorem 3 for w-regular graphs. Within the same section, we apply Corollary 8 to show Corollary 9, a slightly weaker form of Theorem 3. The latter corollary serves as a key ingredient in the proof of Theorem 4 presented in Section 4.

2 Bounding the maximum spectral radius of an unraveled ball

Definition 5. Given a weighted graph G with minimum degree at least 2, a stationary Markov chain $(E_i)_{i=1}^{\infty}$ on $W_1(G)$ is *assigned to* G if its transition matrix $P = (p_{e_1,e_2})_{e_1,e_2 \in W_1(G)}$ satisfies

$$p_{e_1,e_2} = \Pr\big(E_i = e_2 | E_{i-1} = e_1\big) = 0 \text{ if } e_2 \text{ does not prolong } e_1.$$

We say that $e_2 \in W_1(G)$ *prolongs* $e_1 \in W_1(G)$ if there are three distinct vertices $x, y, z \in V(G)$ such that $e_1 = (z, y)$ and $e_2 = (y, x)$. For the sake of brevity, we write $e_1 \to e_2$ if e_2 prolongs e_1.

If the minimum degree of a weighted (connected) graph G is at least 2, then a stationary Markov chain assigned to G has no absorbing states and its stationary distribution $\pi = (\pi_e)_{e\in W_1(G)}$ is well defined. Since the Markov chain is stationary, we have

$$\Pr\big(E_i = e\big) = \pi_e \text{ for any positive integer } i.$$

Since the ending vertex of E_i and the starting vertex of E_{i+1} are the same, we can concatenate $E_1, \dots, E_i$ to form a random non-backtracking walk of length i, denoted by the random variables $Y_i = (X_0, \dots, X_i)$.

With these definitions, we can state and prove the following general theorem.

Theorem 6. *Let G be a weighted graph with minimum degree at least 2, let $(E_i)_{i=1}^{+\infty}$ be a stationary Markov chain assigned to G with transition matrix $P = (p_{e_1,e_2})_{e_1,e_2\in W_1(G)}$ and stationary distribution $\pi = (\pi_e)_{e\in W_1(G)}$. For any function $g : W_1(G) \to \mathbb{R}$ and a positive integer r, there is a vertex v of G such that*

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \Big(\sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} w_{e_2} g(e_1) g(e_2) \pi_{e_1} \sqrt{p_{e_1,e_2}} \Big) \Big(\sum_{e\in W_1(G)} g^2(e)\pi_e \Big)^{-1}.$$

Proof. Let us begin by defining the weighted forest F_G as the disjoint union of all graphs $\tilde{G}(v,r)$, where $v \in V(G)$, that is,

$$F_G = \bigcup_{v\in V(G)} \tilde{G}(v,r).$$

Thus, the vertex set of F_G is $\bigcup_{i=0}^{r+1} W_i(G)$. Since F_G is a union of disjoint trees, we have

$$\lambda_1(F_G) = \max\big\{\lambda_1(\tilde{G}(v,r)) : v \in V(G)\big\}.$$

Therefore, it is sufficient to show that

$$\frac{\lambda_1(F_G)}{\lambda_1(P_{r+1})} \geqslant \Big(\sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} w_{e_2} g(e_1) g(e_2) \pi_{e_1} \sqrt{p_{e_1,e_2}} \Big) \Big(\sum_{e\in W_1(G)} g^2(e)\pi_e \Big)^{-1}.$$

Denote by $(x_1, \dots, x_{r+1}) \in \mathbb{R}^{r+1}$ the eigenvector of the spectral radius $\lambda_1(P_{r+1})$ of the path P_{r+1} of length r. Then the Rayleigh principle yields

$$\sum_{i=2}^{r+1} 2x_{i-1}x_i = \lambda_1(P_{r+1}) \sum_{i=1}^{r+1} x_i^2. \tag{1}$$

Define a vector $f \in \mathbb{R}^{V(F)}$ by setting, for $\omega = (v_0, v_1, \dots, v_i) \in W_i(G)$ and $\omega' = (v_{i-1}, v_i) \in W_1(G)$,

$$f(\omega) := \begin{cases} 0, & \text{if } i = 0,\\ x_i g\big(\omega'\big)\sqrt{\Pr(Y_i = \omega)}, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned}\langle f,f\rangle &= \sum_{i=1}^{r+1}\sum_{\omega\in W_i(G)} f(\omega)^2\\ &= \sum_{i=1}^{r+1} x_i^2 \sum_{\omega\in W_i(G)} g^2(\omega')\Pr(Y_i=\omega)\\ &= \sum_{i=1}^{r+1} x_i^2 \sum_{\omega'\in W_1(G)} g^2(\omega')\Pr(E_i=\omega')\\ &= \sum_{i=1}^{r+1} x_i^2 \sum_{e\in W_1(G)} g^2(e)\pi_e. \qquad (2)\end{aligned}$$

Denoting $\omega^- = (v_0,\ldots,v_{i-1})$ and $\omega'' = (v_{i-2},v_{i-1})$, we obtain

$$\begin{aligned}\langle f, A_{F_G} f\rangle &= \sum_{i=2}^{r+1}\sum_{\omega\in W_i(G)} 2x_{i-1}x_i w_{\omega^-} f(\omega)f(\omega^-)\\ &= \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{\omega\in W_i(G)} w_{\omega'} g(\omega'')g(\omega')\sqrt{\Pr(Y_{i-1}=\omega^-)\Pr(Y_i=\omega)}.\end{aligned}$$

For the Markov chain, we have

$$\frac{\Pr(Y_i=\omega)}{\Pr(Y_{i-1}=\omega^-)} = \Pr(E_i=\omega'|E_{i-1}=\omega'') = p_{\omega'',\omega'}.$$

Substituting this in the equation for $\langle f, A_{F_G} f\rangle$, we get

$$\begin{aligned}\langle f, A_{F_G} f\rangle &= \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{\omega\in W_i(G)} \frac{w_{\omega'} g(\omega'')g(\omega')}{\sqrt{p_{\omega'',\omega'}}}\Pr(Y_i=\omega)\\ &= \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} \frac{w_{e_2} g(e_1)g(e_2)}{\sqrt{p_{e_1,e_2}}}\Pr\big(E_i=e_2, E_{i-1}=e_1\big).\end{aligned}$$

Since $\Pr\big(E_i=e_2,E_{i-1}=e_1\big) = \Pr\big(E_i=e_2|E_{i-1}=e_1\big)\Pr\big(E_{i-1}=e_1\big) = p_{e_1,e_2}\pi_{e_1}$, we easily conclude that

$$\langle f, A_{F_G} f\rangle = \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} w_{e_2} g(e_1)g(e_2)\pi_{e_1}\sqrt{p_{e_1,e_2}}.$$

Combining this equality with (1), (2), and the Rayleigh principle
$\lambda_1(F_G) \geqslant \langle f, A_{F_G} f\rangle / \langle f,f\rangle$, we finish the proof. □

Remark 7. Recall that $\tilde{G}(v,r)$ is a ball of radius r centered at v in the universal cover $\tilde{G}$ of G. So by the monotonicity of spectral radius, $\lambda_1(\tilde{G}) \geqslant \lambda_1(\tilde{G}(v,r))$. Since $\lambda_1(P_{r+1}) \to 2$ as $r \to +\infty$, under the assumptions of Theorem 6, we obtain

$$\lambda_1(\tilde{G}) \geqslant 2\Big(\sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} w_{e_2} g(e_1)g(e_2)\pi_{e_1}\sqrt{p_{e_1,e_2}}\Big)\Big(\sum_{e\in W_1(G)} g^2(e)\pi_e\Big)^{-1}.$$

3 Proofs of corollaries on regular weighted graphs

To prove special cases of Theorem 6, the authors of [3, Theorem 1] and [7, Theorem 1.5] consider the following stationary Markov chain on $W_1(G)$ such that its stationary distribution can be easily found. Namely, they assumed that given the stage $E_i = (v_{i-1}, v_i)$, the stage E_{i+1} is chosen among $\{(v_i, u) \in W_1(G) : u \neq v_{i+2}\}$ uniformly at random. Hence the transition matrix $P = (p_{e_1,e_2})_{e_1,e_2\in W_1(G)}$ of this Markov chain is defined by

$$p_{(x,y),(z,t)} = \begin{cases} \dfrac{1}{\deg y - 1} & \text{if } y = z \text{ and } t \neq x; \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that the distribution $\pi = (\pi_e)_{e\in W_1(G)}$ with $\pi_e = \frac{1}{|W_1(G)|}$ is stationary.

In the next general corollary of Theorem 6, we use another stationary Markov chain assigned to a regular weighted graph such that its stationary distribution be explicitly found.

Corollary 8. *Let G be a w-regular weighted graph with minimum degree at least 2. For any function $g : W_1(G) \to \mathbb{R}$ and any positive integer r, there is a vertex v of G such that*

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \Big(\sum_{\substack{e_1,e_2\in W_1(G)\\ e_1\to e_2}} g(e_1)g(e_2)w_{e_1}w_{e_2}^{3/2}(w-w_{e_1})^{1/2}\Big)\Big(\sum_{e\in W_1(G)} g^2(e)w_e(w-w_e)\Big)^{-1}.$$

Particularly, choosing $g(e) = (w - w_e)^{-1/2}$, we have

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \frac{\sum_{e\in W_1(G)} w_e^{3/2}(w-w_e)^{1/2}}{w|V(G)|}.$$

Proof. Consider a stationary Markov chain assigned to G defined by its transition matrix $P = (p_{e_1,e_2})_{e_1,e_2\in W_1(G)}$ as follows

$$p_{e_1,e_2} = \begin{cases} \dfrac{w_{e_2}}{w - w_{e_1}} & \text{if } e_1 \to e_2; \\ 0 & \text{otherwise.} \end{cases}$$

Using $\pi = \pi P$ for the stationary distribution $\pi = (\pi_e)_{e\in W_1(G)}$, one can easily verify that

$$\pi_e = \frac{w_e(w-w_e)}{S}, \quad \text{where } S = \sum_{e\in W_1(G)} w_e(w-w_e).$$

Applying Theorem 6 to this Markov chain, we easily obtain the first desired inequality.

Assuming that $g(e) = (w - w_e)^{-1/2}$, we have

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \Big(\sum_{\substack{e_1,e_2 \in W_1(G) \\ e_1 \to e_2}} w_{e_1} w_{e_2}^{3/2} (w - w_{e_2})^{-1/2} \Big) \Big(\sum_{e \in W_1(G)} w_e \Big)^{-1}. \tag{3}$$

The multipliers of the right-hand side of (3) can be easily found

$$\sum_{e \in W_1(G)} w_e = w|V(G)|.$$

$$\begin{aligned} \sum_{\substack{e_1,e_2 \in W_1(G) \\ e_1 \to e_2}} w_{e_1} w_{e_2}^{3/2} (w - w_{e_2})^{-1/2} &= \sum_{e_2 \in W_1(G)} w_{e_2}^{3/2} (w - w_{e_2})^{-1/2} \sum_{e_1 : e_1 \to e_2} w_{e_1} \\ &= \sum_{e \in W_1(G)} w_e^{3/2} (w - w_e)^{1/2}. \end{aligned}$$

Substituting these equalities in (3), we obtain the second desired inequality. □

Corollary 9. *Let G be a w-regular weighted graph with minimum degree at least 2. Assume that its average combinatorial degree d satisfies $2\sqrt{d-1}/d \leqslant \mu$, where $\mu = \frac{3-\sqrt{3}}{4}$ (that is, $d \geqslant 38.7620\ldots$). Then, for any positive integer r, there exists a vertex v of G such that*

$$\frac{\lambda_1(\tilde{G}(v,r))}{\lambda_1(P_{r+1})} \geqslant \frac{w\sqrt{d-1}}{d}.$$

Proof. There are two possible cases.

Case 1. There is $e = (v,u) \in W_1(G)$ such that $w_e \geqslant \mu w$. Define a vector $f' \in \mathbb{R}^{V(\tilde{G}(v,r))}$ by setting

$$f'(x) = \begin{cases} 1 & \text{if } x \in \{(v), (v,u)\}; \\ 0 & \text{otherwise.} \end{cases}$$

Using the Rayleigh principle, we have

$$\lambda_1(\tilde{G}(v,r)) \geqslant \frac{\langle f', A_{\tilde{G}(v,r)} f' \rangle}{\langle f', f' \rangle} = w_e \geqslant \mu w,$$

which finishes the proof in this case as $2\sqrt{d-1}/d \leqslant \mu$ and $\lambda_1(P_{r+1}) < 2$.

Case 2. For any vertex $e \in W_1(G)$, we have $w_e \leqslant \mu w$. By a straightforward computation, we obtain that the function $x \mapsto x^{3/2}(w-x)^{1/2}$ is convex on the interval $[0, \mu w]$ and concave on $[\mu w, w]$, where $\mu = \frac{3-\sqrt{3}}{4}$. By Jensen's inequality, we obtain

$$\sum_{e \in W_1(G)} w_e^{3/2} (w - w_e)^{1/2} \geqslant |W_1(G)| \Big(\frac{w}{d}\Big)^{3/2} \Big(w - \frac{w}{d}\Big)^{1/2} = |V(G)| \frac{w^2\sqrt{d-1}}{d}.$$

Substituting this inequality in the second inequality of Corollary 8, we finish the proof. □

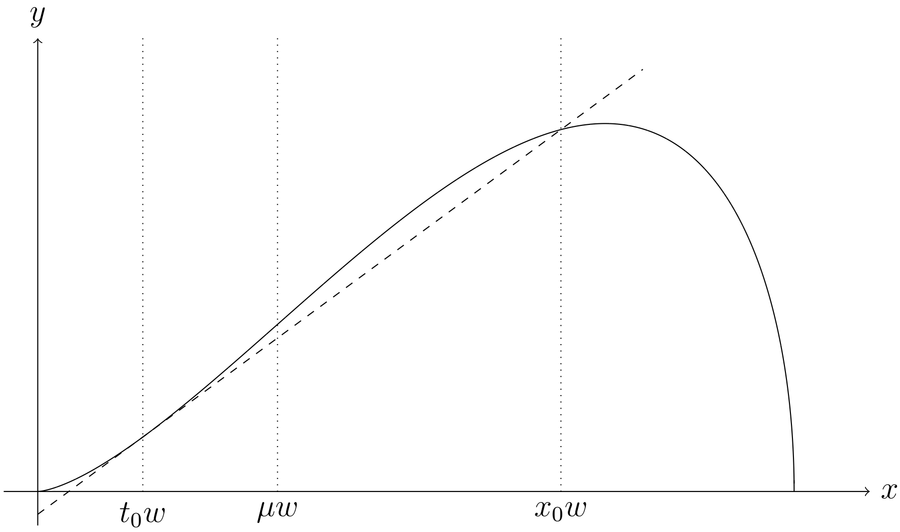

Figure 1: Graphs of g and ℓ_t (dashed).

Remark 10. Slightly modifying the argument, we can prove Corollary 9 for $d \geqslant 1/t_0 = 7.1980\ldots$, where t_0 is defined below.

Let $g : [0, w] \to \mathbb{R}$ be a function given by $g(y) = y^{3/2}(w - y)^{1/2}$. For any $t \in (0, 1)$, let $\ell_t : \mathbb{R} \to \mathbb{R}$ be an affine function defining the tangent line of g at the point $tw \in (0, w)$; see Figure 1. Consider the following system of equations and inequalities

$$\begin{cases} \ell_t(xw) = g(xw); \\ x = \frac{2\sqrt{1/t-1}}{1/t}; \\ 0 < t < \mu = \frac{3-\sqrt{3}}{4} < x, \end{cases}$$

which has only one solution: $x = x_0 = 0.6917\ldots, t = t_0 = 0.1389\ldots$; see Figure 1.

First, we may assume that

$$w_e \leqslant wx_0 = \frac{2w\sqrt{1/t_0 - 1}}{1/t_0}$$

for all $e \in W_1(G)$; otherwise, we use the proof of the first case in Corollary 9.

Next, consider the function $h : [0, x_0w] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} g(x) & \text{if } 0 \leqslant x \leqslant t_0w; \\ \ell_{t_0}(x) & \text{if } t_0w < x \leqslant x_0w. \end{cases}$$

Recall that the function g is convex of $[0, \mu w]$. Since $0 < t_0 < \mu = \frac{3-\sqrt{3}}{4}$ and ℓ_{t_0} defines the tangent line for the graph of g at the point t_0w, we conclude that the function h is convex on $[0, x_0w]$ and $g(y) \geqslant h(y)$ for any $0 \leqslant y \leqslant x_0w$. Therefore, we can apply

Jensen's inequality for h and obtain the desired inequality

$$\begin{aligned}\sum_{e\in W_1(G)} w_e^{3/2}(w-w_e)^{1/2} =& \sum_{e\in W_1(G)} g(w_e)\\ \geqslant& \sum_{e\in W_1(G)} h(w_e)\\ \geqslant& |W_1(G)|h\big(\frac{w}{d}\big)\\ =& |W_1(G)|g\big(\frac{w}{d}\big) = |V(G)|\frac{w^2\sqrt{d-1}}{d}.\end{aligned}$$

(Here we use that $d \geqslant 1/t_0$, and thus $h(\frac{w}{d}) = g(\frac{w}{d})$ by the definition of h.) Substituting this inequality in the second inequality of Corollary 8, we finish the proof.

Remark 11. Using Corollaries 8 and 9, we can prove lower bounds for the spectral radius for the universal cover $\tilde{G}$ of a w-regular weighted graph G as it is shown in Remark 7.

4 Proof of Theorem 4

The following lemma connects the spectral radii of balls $G(v,r)$ and $\tilde{G}(v,r)$.

Lemma 12. *[7, Lemma 4.2] For any vertex v of a graph H and any positive integer r, we have*

$$\lambda_1(H(v,r)) \geqslant \lambda_1(\tilde{H}(v,r)).$$

Suppose that G has a vertex of degree 1, that is, incident to one edge. Then there is a connected component of G that is a path of length 1 with weight of its only edge equal to w. Clearly, this component has eigenvalue w, which is large enough to finish the proof in this case. Therefore, we can assume that there are no vertices of degree 1 in G.

Lemma 12 and Corollary 9 yield that there exists a vertex $v \in G$ such that

$$\lambda_1(G(v,r)) \geqslant \lambda_1(\tilde{G}(v,r)) \geqslant \lambda_1(P_{r+1})\frac{w\sqrt{d-1}}{d}.$$

Denote by $f_1 \in \mathbb{R}^{V(G)}$ the vector that coincides on $V(G(v,r))$ with the eigenvector of the spectral radius of $G(v,r)$ and is zero on $V(G) \setminus V(G(v,r))$. By the Rayleigh principle, we have

$$\lambda_1(G(v,r)) = \frac{\langle f_1, A_G f_1\rangle}{\langle f_1, f_1\rangle}.$$

Let G' be an induced subgraph of $G \setminus G(v,r+1)$ with minimum weighted degree at least $2w\sqrt{d-1}/d$. Define a vector $f_2 \in \mathbb{R}^{V(G)}$ by setting

$$f_2(x) = \begin{cases} 1 & \text{if } x \in V(G');\\ 0 & \text{otherwise.}\end{cases}$$

Hence by the Rayleigh principle, we obtain

$$\lambda_1(G') \geqslant \frac{\langle f_2, A_G f_2\rangle}{\langle f_2, f_2\rangle} \geqslant 2w\frac{\sqrt{d-1}}{d}.$$

One can choose scalars c_1 and c_2 such that the vector $f = c_1 f_1 + c_2 f_2 \neq 0$ is perpendicular to the eigenvector $(1, \dots, 1)$ of the spectral radius $\lambda_1(G) = w$. Therefore, by the Rayleigh principle, we obtain

$$\lambda_2(G) \geqslant \frac{\langle f, A_G f\rangle}{\langle f, f\rangle} = \frac{c_1^2\langle f_1, A_G f_1\rangle + c_2^2\langle f_2, A_G f_2\rangle}{c_1^2\langle f_1, f_1\rangle + c_2^2\langle f_2, f_2\rangle} \geqslant \lambda_1(P_{r+1})\frac{w\sqrt{d-1}}{d},$$

which finishes the proof. □

Remark 13. As shown in Remark 10, Corollary 9 holds for $d \geqslant 7.1980\dots$. So using the same argument we can prove Theorem 4 for $d \geqslant 7.1980\dots$.

Acknowledgments.

Part of the research was carried out when A.P. was a Senior Research Fellow at the Moscow Institute of Physics and Technology and was supported by the Young Russian Mathematics award.

The authors thank Alexander Golovanov for fruitful discussions and careful proofreading of the paper.

References

[1] J. Friedman. Some geometric aspects of graphs and their eigenfunctions. *Duke Math. J.*, 69(3):487–525, 1993.

[2] S. Hoory. A lower bound on the spectral radius of the universal cover of a graph. *Journal of Combinatorial Theory, Series B*, 93(1):33–43, 2005.

[3] Z. Jiang. On spectral radii of unraveled balls. *Journal of Combinatorial Theory, Series B*, 136:72–80, 2019.

[4] Z. Jiang and A. Polyanskii. Forbidden subgraphs for graphs of bounded spectral radius, with applications to equiangular lines. *Israel Journal of Mathematics*, 236(1):393–421, 2020.

[5] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.

[6] N. Srivastava and L. Trevisan. An Alon–Boppana type bound for weighted graphs and lower bounds for spectral sparsification. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1306–1315. SIAM, 2018.

[7] Y. Wang and X.-D. Zhang. On weighted spectral radius of unraveled balls and normalized Laplacian eigenvalues. *Discrete Mathematics*, 345(12):113173, 2022.

[8] S. J. Young. The weighted spectrum of the universal cover and an Alon–Boppana result for the normalized Laplacian. *Journal of Combinatorics*, 13(1):23–40, 2022.