

Wreath products and projective system of non Schurian association schemes

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Abstract

A wreath product is a method to construct an association scheme from two association schemes. We determine the automorphism group of a wreath product. We show a known result that a wreath product is Schurian if and only if both components are Schurian, which yields large families of non-Schurian association schemes and non-Schurian S -rings. We also study iterated wreath products. Kernel schemes by Martin and Stinson are shown to be iterated wreath products of class-one association schemes. The iterated wreath products give examples of projective systems of non-Schurian association schemes, with an explicit description of primitive idempotents.

Mathematics Subject Classifications: 05E30, 20D60, 20E18

1 Introduction

Association schemes are central objects in algebraic combinatorics, with many interactions with other areas of mathematics. The wreath product is a method to construct an association scheme from two association schemes. This notion appears in the monograph [17, P.45-47] by Weisfeiler, in a more general context of coherent configurations (in the terminology of cellular algebras). Song[16] gives a description in terms of association schemes and their adjacency matrices. Muzychuk[13] defines the wreath product of association schemes and its generalizations (he used the term “wedge product”). There is a construction of non-Schurian S -rings by using a generalized wreath product for S -rings, studied by Evdokimov and Ponomarenko[5].

In this paper, we construct projective systems of non-Schurian association schemes in Section 4. As preliminaries, we recall some basic facts on wreath products in Section 2 with a self-contained proof. In Section 3, we shall study the automorphism group of wreath products and their Schurian property. A large part of the results in Section 3.2

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is covered by more detailed discussions in the monograph by Chen and Ponomarenko [3, Sections 3.2 and 3.4], but we give proofs for self-containedness, e.g., on the known fact that the wreath product is Schurian if and only if both components are Schurian ([3, Corollary 3.4.7]). This yields a large class of non-Schurian association schemes and non-Schurian S -rings, from the known examples.

In Section 4, we consider iterated wreath products. The kernel scheme by Martin-Stinson[11] is an example. An infinite iterated wreath product gives a projective system of association schemes, namely, profinite association schemes in the sense of [12], with an explicit description of their primitive idempotents. In [12], only Schurian examples are given. One of our motivations is to give examples of projective systems of non-Schurian association schemes.

2 Wreath products

2.1 Category of association schemes

Let us recall the notion of association schemes briefly. See Bannai-Ito[1] and Delsarte[4] for details. We summarize basic terminologies.

Definition 2.1. Let X be a finite set. By $\#X$ we denote the cardinality of X . Let $C(X)$ denote the vector space of mappings from X to \mathbb{C} . By the multiplication of functions, $C(X)$ is a unital commutative ring. The set $C(X \times X)$ is naturally identified with the set of complex square matrices of size $\#(X)$, and the matrix product is given by $AB(x, z) = \sum_{y \in X} A(x, y)B(y, z)$. The Hadamard product \circ is given by the component-wise product, namely, $(A \circ B)(x, z) = A(x, z)B(x, z)$. (Note that \circ may denote the composition of mappings, but no confusion would occur.)

Definition 2.2. Let X, I be finite sets, and $R : X \times X \rightarrow I$ a surjection. We call (X, R, I) an association scheme, if the following properties (1), (2), and (3) are satisfied. For each $i \in I$, $R^{-1}(i)$ may be regarded as a relation on X , denoted by R_i . Let A_i be the corresponding adjacency matrix in $C(X \times X)$. The surjection R induces an injection $C(I) \rightarrow C(X \times X)$. Let $A_X \subset C(X \times X)$ be the image.

- (1) There is an $i_0 \in I$ such that A_{i_0} is an identity matrix.
- (2) $A_X \subset C(X \times X)$ is closed under the matrix product.
- (3) A_X is closed under the transpose of $C(X \times X)$.

The algebra A_X with the two multiplications (i.e. the Hadamard product and the matrix product) is called the Bose-Mesner algebra of (X, R, I) . The set of A_i ($i \in I$) is the set of primitive idempotents with respect to Hadamard products, and each is called an Hadamard primitive idempotent (or an adjacency matrix). We call X the underlying set of the association scheme, and I the set of the relations. The number $\#I - 1$ is called the number of classes. We may use the same notation i_0 for distinct association schemes. The number of 1 in a row of A_i is independent of the choice of the row, called the i -th valency

and denoted by k_i . It is also the number of 1 in each column of A_i . If A_X is commutative with respect to the matrix product, then (X, R, I) is said to be commutative.

In the following, we write simply an ‘‘association scheme X ’’ for an association scheme (X, R, I) by an abuse of language.

Definition 2.3. For a commutative association scheme X , it is known that A_X with matrix product is isomorphic to the direct product of $\#I$ copies of \mathbb{C} as a ring. Elements corresponding to $(0, \dots, 0, 1, 0, \dots, 0)$ are called the primitive idempotents. We fix an index set $J = J(X)$ of the set of primitive idempotents. Let E_j denote the primitive idempotent corresponding to $j \in J$. The notation j_0 is kept for $E_{j_0} = \frac{1}{\#X} \mathbf{J}$, where \mathbf{J} denotes the matrix whose components are all 1. We use the symbol j_0 for any commutative association schemes.

We follow MacLane [10] for the terminologies of the category theory, in particular, isomorphisms, functors, projective systems, and limits. The association schemes form a category, by the following (e.g. Hanaki[6] and Zieschang[19]).

Definition 2.4. Let (X, R, I) and (X', R', I') be association schemes. A morphism of association schemes from X to X' is a pair of functions $f : X \rightarrow X'$ and $\sigma : I \rightarrow I'$ such that the following diagram commutes:

$$\begin{array}{ccc}
 X \times X & \longrightarrow & I \\
 f \times f \downarrow & \circlearrowleft & \downarrow \sigma \\
 X' \times X' & \longrightarrow & I'
 \end{array} \tag{2.1}$$

By the commutativity, σ preserves i_0 . It is clear that σ is surjective if f is so. The morphism (f, σ) is said to be surjective if f is surjective.

2.2 Wreath products

Although originally the wreath product is introduced by Weisfeiler [17, P.45], this section follows Song[16, §4], since his description is due to association schemes and convenient for our purpose. The definition, a proof, and the eigenmatrix of wreath products are all given there. We recall these, partly because we use some different symbols and notations which make the definition and proof simpler, and partly because we use some part of the proof in the following arguments.

Definition 2.5. Let (X, R_X, I_X) and (Y, R_Y, I_Y) be association schemes. Then,

$$R_X \times R_Y : (X \times Y) \times (X \times Y) \rightarrow I_X \times I_Y$$

is an association scheme, called the direct product of X and Y [16, §3] (called the tensor product in [17]). In fact, this is the direct product in the category of association schemes.

It is easy to see the following.

Proposition 2.6. *The adjacency matrices of the direct product are*

$$\{A_{i_X} \otimes A_{i_Y} \mid i_X \in I_X, i_Y \in I_Y\}.$$

If both X and Y are commutative, then so is $X \times Y$, and the primitive idempotents are

$$\{E_{j_X} \otimes E_{j_Y} \mid j_X \in J_X, j_Y \in J_Y\}.$$

The wreath product is defined as follows. The non-standard symbol λ comes from the “L” of lexicographic.

Definition 2.7. Define

$$\text{lex} : I_X \times I_Y \rightarrow (I_X \setminus \{i_0\}) \amalg I_Y$$

by

$$\text{lex}(i_X, i_Y) = \begin{cases} i_X & \text{if } i_X \neq i_0, \\ i_Y & \text{if } i_X = i_0. \end{cases}$$

We denote

$$I_X \lambda I_Y := (I_X \setminus \{i_0\}) \amalg I_Y.$$

Definition 2.8. For a mapping $\sigma : I_X \rightarrow I_X$ preserving $i_0 \in I_X$ and a mapping $\tau : I_Y \rightarrow I_Y$ preserving $i_0 \in I_Y$, we define

$$\sigma \lambda \tau : I_X \lambda I_Y \rightarrow I_X \lambda I_Y$$

by σ on $I_X \setminus \{i_0\}$ and by τ on I_Y . (See Definition 2.4 for the preservation of i_0 .) We also define

$$\pi_{I_X} : I_X \lambda I_Y \rightarrow I_X, \quad i_X \mapsto i_X \text{ and } i_Y \mapsto i_0.$$

Remark 2.9. This remark is for category-oriented readers, and may be skipped since it is not essential in this paper. Let $\text{Sets}^!$ be the category of sets with a base point, namely, an object is a set with a base point and a morphism is a mapping preserving the base points. In other words, for a fixed singleton $\{!\}$, $\text{Sets}^!$ is the category of morphisms $\{!\} \rightarrow S$ with S being a set (a special case of comma categories). We remark that $I_X \lambda I_Y$ gives the coproduct functor $\text{Sets}^! \times \text{Sets}^! \rightarrow \text{Sets}^!$ (i.e. the pushout of $\{!\} \rightarrow I_X$ and $\{!\} \rightarrow I_Y$ in the category Sets of sets). The first definition in Definition 2.8 comes from this functoriality. Remark that lex is NOT a natural transformation

$$\text{lex} : \times \Rightarrow \lambda$$

where $\times : \text{Sets}^! \times \text{Sets}^! \rightarrow \text{Sets}^!$ is the direct product functor. On the other hand, there is a canonical natural transformation

$$\lambda \Rightarrow \times$$

given by

$$I_X \succcurlyeq I_Y \rightarrow I_X \times I_Y, \quad i_x \in I_X \mapsto (i_x, i_0), \text{ and } i_y \in I_Y \mapsto (i_0, i_y).$$

Consequently, if

$$\text{pr}_i : \text{Sets}^! \times \text{Sets}^! \rightarrow \text{Sets}^!$$

is the projection functor to the i -th component ($i = 1, 2$), then there is a natural transformation $\lambda \Rightarrow \text{pr}_i$ for each $i = 1, 2$, and π_{I_X} is the evaluation of $\lambda \Rightarrow \text{pr}_1$ at (I_X, I_Y) .

Definition 2.10. (Wreath products)

Let (X, R_X, I_X) and (Y, R_Y, I_Y) be association schemes. Then the composition

$$R_\lambda : (X \times Y) \times (X \times Y) \rightarrow I_X \times I_Y \xrightarrow{\text{lex}} I_X \succcurlyeq I_Y$$

is an association scheme, which is called the wreath product of X and Y , and denoted by

$$X \succcurlyeq Y.$$

Remark 2.11. The underlying set of $X \succcurlyeq Y$ is $X \times Y$, whereas the set of relations is $I_X \succcurlyeq I_Y$ with the cardinality $\#I_X + \#I_Y - 1$. The wreath product is a fusion of the direct product.

We first thought that the name ‘‘wreath product’’ seems misleading, since this seems to have no relation with the wreath product of groups (as Song [16] remarked in his introduction). However, there is a strong relation [17, P.47, §4.8] via transitive permutation groups. Still, we point out the similarity to the ‘‘lexicographic ordering,’’ since the above definition means that the relation of two elements in $X \times Y$ is determined by first looking at the X -components, and decided according to their relation when they are different, and otherwise by the relation at the Y -components. This procedure is similar to the lexicographic ordering. The next is a translation of [16, Theorem 4.1].

Theorem 2.12.

- (1) For association schemes X and Y , their wreath product $X \succcurlyeq Y$ is an association scheme.
- (2) Let A_{i_X} ($i_X \in I_X$) (and A_{i_Y} ($i_Y \in I_Y$), respectively) be the adjacency matrices of X (those of Y , respectively). Then, the adjacency matrices of $X \succcurlyeq Y$ are as follows.

$$\begin{aligned} \text{Type front:} & \quad A_{i_X} \otimes \mathbf{J} & \text{if } i_X \in I_X, i_X \neq i_0 \\ \text{Type rear:} & \quad \mathbf{I} \otimes A_{i_Y} & \text{if } i_Y \in I_Y, \end{aligned}$$

where \mathbf{I} denotes the identity matrix (of size $\#X$) and \mathbf{J} denotes the matrix whose components are all 1 (square of size $\#Y$).

Proof. Take an element $i_X \in I_X \setminus \{i_0\} \subset I_X \succcurlyeq I_Y$. Its preimage by lex is $\{(i_X, i_Y) \mid i_Y \in I_Y\}$. By definition,

$$\sum_{i_Y \in I_Y} A_{i_Y} = \mathbf{J}.$$

By Proposition 2.6, the inverse image of $i_X \in I_X \setminus \{i_0\} \subset I_X \wr I_Y$ by R_λ corresponds to

$$A_{i_X} \otimes \mathbf{J}.$$

Take an element $i_Y \in I_Y \subset I_X \wr I_Y$. Its inverse image by R_λ corresponds to

$$\mathbf{I} \otimes A_{i_Y}.$$

To check that these constitute an association scheme, we only need to show that this set of matrices is closed under the transpose and contains the unit, which are obvious, and that matrix products of these matrices are linear combinations of these. A product of the front-type matrices is a linear combination of those, since X is an association scheme. A product of the rear-type matrices is a linear combination of those, since Y is an association scheme. For the mixed case, by $\mathbf{J}A_{i_Y} = k_{i_Y}\mathbf{J}$, the product is a scalar multiple of a matrix of the front-type. \square

Here we start our own observations.

Corollary 2.13. *The wreath product $X \wr Y$ is not P -polynomial if $\#X > 1$ and $\#Y > 1$.*

Proof. The wreath product is P -polynomial, if there is an adjacency matrix that generates the Bose-Mesner algebra as a ring over \mathbb{C} . By $\#X > 1$, we have an $i_X \neq i_0$. By $\#Y > 1$, we have an $i_Y \neq i_0$. The linear span of the front-type matrices is identical with the set $A_X \otimes \mathbf{J}$ and is closed under the matrix multiplication. The linear span of the rear-type matrices is identical with the set $\mathbf{I} \otimes A_Y$ and is closed under the matrix multiplication. Thus, a front-type adjacency matrix does not generate a rear-type adjacency matrix $\mathbf{I} \otimes A_{i_Y}$, since $i_Y \neq i_0$ and hence $A_{i_Y} \neq \mathbf{J}$. A rear-type adjacency matrix does not generate a front-type adjacency matrix $A_{i_X} \otimes \mathbf{J}$ since $A_{i_X} \neq A_{i_0} = \mathbf{I}$. \square

Corollary 2.14. *The wreath product of association schemes is a commutative association scheme if and only if both components are commutative.*

Proof. This follows from the above description of the Hadamard primitive idempotents. \square

Corollary 2.15. *Suppose that both X and Y are commutative. Let E_{j_X} ($j_X \in J_X$) (and E_{j_Y} ($j_Y \in J_Y$), respectively) be the primitive idempotents of X (of Y , respectively). Then, the primitive idempotents of $X \wr Y$ are as follows.*

$$\begin{array}{ll} \text{Type front:} & E_{j_X} \otimes \frac{1}{\#Y}\mathbf{J} \quad \text{if } j_X \in J_X \\ \text{Type rear:} & \mathbf{I} \otimes E_{j_Y} \quad \text{if } j_Y \in J_Y, j_Y \neq j_0. \end{array}$$

Thus,

$$J(X \wr Y) = J_X \coprod (J_Y \setminus \{j_0\}).$$

Proof. One sees that these elements are linear combinations of the adjacency matrices in Theorem 2.12 because X and Y are commutative association schemes, and hence in the Bose-Mesner algebra of $X \wr Y$. It is easy to check that these are idempotents and the product of any two distinct elements is zero, and hence are linearly independent. Since the number of these matrices is $\#X + \#Y - 1$, which is the cardinality of $I_X \wr I_Y$, these idempotents span the Bose-Mesner algebra. \square

Corollary 2.16. *Let X and Y be commutative association schemes. Then, $X \wr Y$ is not Q -polynomial, if $\#X > 1$ and $\#Y > 1$.*

Proof. A similar proof to that of Corollary 2.13 applies to. \square

Corollary 2.17. *We use $E_{j_0} = \frac{1}{\#Y} \mathbf{J}$. The first eigenmatrix of $X \wr Y$ is given by*

$$\begin{aligned} (A_{i_X} \otimes \mathbf{J})(E_{j_X} \otimes E_{j_0}) &= P_{i_X}(j_X) \#Y (E_{j_X} \otimes E_{j_0}), \\ (\mathbf{I} \otimes A_{i_Y})(E_{j_X} \otimes E_{j_0}) &= k_{i_Y}(E_{j_X} \otimes E_{j_0}), \\ (A_{i_X} \otimes \mathbf{J})(\mathbf{I} \otimes E_{j_Y}) &= 0(\mathbf{I} \otimes E_{j_Y}), \\ (\mathbf{I} \otimes A_{i_Y})(\mathbf{I} \otimes E_{j_Y}) &= P_{i_Y}(j_Y)(\mathbf{I} \otimes E_{j_Y}), \end{aligned}$$

where $P_{i_X}(j_X)$ is the eigenvalue of A_{i_X} for eigenvector E_{j_X} , and $P_{i_Y}(j_Y)$ is the eigenvalue of A_{i_Y} for eigenvector E_{j_Y} .

Proposition 2.18. *Let X and Y be association schemes. The projection*

$$p : X \times Y \rightarrow X$$

of the underlying sets and the mapping

$$\pi_{I_X} : I_X \wr I_Y \rightarrow I_X$$

in Definition 2.8 give a surjective morphism of association schemes

$$\pi : X \wr Y \rightarrow X.$$

Proof. This follows by a diagram chasing. \square

3 Automorphisms and Schurian property

3.1 Automorphism group of a wreath product

Definition 3.1. For an association scheme X , we denote by $\text{Aut}(X)$ the group of automorphisms in the category of association schemes (Definition 2.4). On the other hand, for any set X equipped with any structure (such as an association scheme or a group),

$$\text{Aut}_{\text{Sets}}(X)$$

denote the group of bijections from X to X , neglecting the structure of X .

As stated before, the wreath product of association schemes is studied and generalized by Muzychuk[13], where in Proposition 3.2 the automorphism group of generalized wreath products of thin-association schemes is determined. Chen and Ponomarenko [3, Theorem 3.4.6] determined the automorphism group of the wreath product, but their definition of automorphism groups is the normal subgroup $\text{Aut}(X|I) < \text{Aut}(X)$ in Lemma 3.5 below, and thus different from ours.¹ We give a description of the automorphism group of wreath products of association schemes.

Theorem 3.2. *Let X and Y be association schemes, and $p : X \times Y \rightarrow X$ be the projection. Then, for any $(f, \sigma) \in \text{Aut}(X \wr Y)$, f maps each fiber $Y_x := p^{-1}(x)$ to another fiber. Thus, f induces an element $\pi(f) \in \text{Aut}_{\text{Sets}}(X)$. This gives a surjective morphism of groups*

$$\text{Aut}(X \wr Y) \xrightarrow{P} \text{Aut}(X)$$

with a natural splitting morphism. Thus, we have a group isomorphism

$$\text{Aut}(X \wr Y) \cong K \rtimes \text{Aut}(X),$$

where $K := \ker P$, whose structure is described in Proposition 3.3 below. We have a natural embedding of $\text{Aut}(Y)$ into K that acts trivially on X and diagonally on each fiber Y_x . The image of the embedding commutes with the image of the splitting. Thus, $\text{Aut}(X \wr Y)$ has a subgroup isomorphic to $\text{Aut}(Y) \times \text{Aut}(X)$.

Proof. We construct P . An automorphism in $\text{Aut}(X \wr Y)$ is a pair of bijections

$$f : X \times Y \rightarrow X \times Y,$$

$$\tau : I_X \wr I_Y \rightarrow I_X \wr I_Y$$

which makes the diagram

$$\begin{array}{ccc} (X \times Y) \times (X \times Y) & \xrightarrow{f \times f} & (X \times Y) \times (X \times Y) \\ R_{\wr} \downarrow & \circlearrowleft & \downarrow R_{\wr} \\ I_X \wr I_Y & \xrightarrow{\tau} & I_X \wr I_Y \end{array} \quad (3.1)$$

commute. We claim that τ permutes $I_X \setminus \{i_0\}$ and I_Y separately. Take an element $i_X \in I_X \setminus \{i_0\}$. Then the corresponding valency is $k_{i_X} \#Y \geq \#Y$, by Theorem 2.12. For $i_Y \in I_Y$, the valency is $k_{i_Y} \leq \#Y$, with equality holds only if $I_Y = \{i_0\}$. Thus, there

¹Our definition of Aut coincides with Iso in [3, Definition 2.2.1], which Chen and Ponomarenko say “definitely more natural than Aut , but here we follow a long tradition.”

is no automorphism that maps i_X to i_Y . Thus we have $\pi(\tau)$ which makes the following commute:

$$\begin{array}{ccc}
 I_X \wr I_Y & \xrightarrow{\tau} & I_X \wr I_Y \\
 \pi_I \downarrow & \circlearrowleft & \downarrow \pi_I \\
 I_X & \xrightarrow{\pi(\tau)} & I_X
 \end{array} . \tag{3.2}$$

For (x, y) and (x, y') , since

$$R_\times((x, y), (x, y')) \in I_Y \subset I_X \wr I_Y,$$

we have

$$R_\times(f(x, y), f(x, y')) \in I_Y \subset I_X \wr I_Y.$$

This means that the X -component of $f(x, y)$ is the same with that of $f(x, y')$, i.e., f maps a fiber of p to another fiber, which shows the unique existence of $\pi(f)$ that makes a commutative diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{f} & X \times Y \\
 p \downarrow & \circlearrowleft & \downarrow p \\
 X & \xrightarrow{\pi(f)} & X.
 \end{array} \tag{3.3}$$

Now we have

$$\begin{array}{ccccc}
 & & (X \times Y) \times (X \times Y) & \xrightarrow{f \times f} & (X \times Y) \times (X \times Y) & & \\
 & \swarrow & \downarrow & \circlearrowleft & \downarrow & \searrow & \\
 X \times X & \circlearrowleft & I_X \wr I_Y & \xrightarrow{\tau} & I_X \wr I_Y & \circlearrowleft & X \times X \\
 & \swarrow & \downarrow & \circlearrowleft & \downarrow & \searrow & \\
 & & I_X & \xrightarrow{\pi(\tau)} & I_X, & &
 \end{array} \tag{3.4}$$

where the commutativity of the two triangles follows from Proposition 2.18. This commutativity, (3.3), and the surjectivity of the projection $X \times Y \rightarrow X$ conclude the com-

mutativity of

$$\begin{array}{ccc}
 X \times X & \xrightarrow{\pi(f) \times \pi(f)} & X \times X \\
 \downarrow & \circlearrowleft & \downarrow \\
 I_X & \xrightarrow{\pi(\tau)} & I_X
 \end{array}, \tag{3.5}$$

and hence gives an automorphism of the association scheme X , which gives the group homomorphism P . Conversely, an automorphism (g, τ) of X , namely, a pair of bijections $g : X \rightarrow X$, $\tau : I_X \rightarrow I_X$ with an additional commutativity condition gives a pair of bijections $g \times \text{id}_Y : X \times Y \rightarrow X \times Y$ and $\tau \lambda \text{id}_{I_Y} : I_X \lambda I_Y \rightarrow I_X \lambda I_Y$ (see Definition 2.8), which is an automorphism of the association scheme $X \lambda Y$, splitting the homomorphism P . On the other hand, it is easy to show that any automorphism (g, σ) of Y induces an automorphism (f, τ) of $X \lambda Y$, by putting $f = \text{id}_X \times g$ and $\tau = \text{id}_{I_X} \lambda \sigma$ (see Definition 2.8), which lies in the kernel of P and commutes with the image of the splitting of P . \square

On the kernel of P , we show the following.

Proposition 3.3. *The kernel K of $P : \text{Aut}(X \lambda Y) \rightarrow \text{Aut}(X)$ is a subgroup of $\prod_{x \in X} \text{Aut}(Y_x)$. More precisely, we have*

$$K = \prod_{\tau \in T} \left(\prod_{x \in X} \text{Aut}(Y_x)_\tau \right), \tag{3.6}$$

where T is the image in $\text{Aut}_{\text{Sets}}(I_Y)$ of $\text{Aut}(Y)$, and $\text{Aut}(Y_x)_\tau$ denotes the set of elements in $\text{Aut}(Y_x)$ whose $\text{Aut}_{\text{Sets}}(I_Y)$ -component is τ . (Note that for $(f, \tau) \in \text{Aut}(Y)$, f uniquely determines τ , hence the above union is disjoint.)

Thus, there is an injective group homomorphism

$$\text{Aut}(X \lambda Y) \hookrightarrow \text{Aut}(Y_x) \wr_X \text{Aut}(X),$$

where the right-hand side is the wreath product of groups with respect to the action of $\text{Aut}(X)$ on X .

Proof. We compute the kernel K of P . It is clear that

$$K \subset \prod_{x \in X} \text{Aut}_{\text{Sets}}(Y_x).$$

Take any $(k, \sigma) \in K$ and $x \in X$, and look at the action on Y_x . Clearly $\sigma = (\text{id}_{I_X} \lambda (\sigma|_{I_Y}))$. There exists $k_x \in \text{Aut}_{\text{Sets}}(Y)$ such that $k(x, y) = (x, k_x(y))$. For any $y_1, y_2 \in Y$,

$$\begin{aligned}
 R_Y(k_x(y_1), k_x(y_2)) &= R_\lambda((x, k_x(y_1)), (x, k_x(y_2))) \\
 &= R_\lambda(k(x, y_1), k(x, y_2)) \\
 &= \sigma \circ R_\lambda((x, y_1), (x, y_2)) \\
 &= \sigma|_{I_Y} \circ R_Y(y_1, y_2)
 \end{aligned}$$

implies that $(k_x, \sigma|_{I_Y}) \in \text{Aut}(Y)$. This holds for any $x \in X$, and thus

$$(k, \text{id}_{I_X} \wr (\sigma|_{I_Y})) \in \prod_{x \in X} \text{Aut}(Y_x)_{\sigma|_{I_Y}}.$$

It follows that

$$K \subset \prod_{\tau \in T} \prod_{x \in X} \text{Aut}(Y_x)_\tau.$$

Conversely, take an element from the right-hand side of (3.6):

$$k = \prod_{x \in X} k_x \in \prod_{x \in X} \text{Aut}_{\text{Sets}}(Y_x) \subset \text{Aut}_{\text{Sets}}(X \times Y).$$

Thus, k_x shares $\tau \in \text{Aut}_{\text{Sets}}(I_Y)$ for any x such that $(k_x, \tau) \in \text{Aut}(Y)$. We claim that $(k, \text{id}_{I_X} \wr \tau) \in K$. In fact, for (x_1, y_1) and (x_2, y_2) , if $x_1 \neq x_2$, then

$$\begin{aligned} R_\wr(k(x_1, y_1), k(x_2, y_2)) &= R_\wr((x_1, k_{x_1}(y_1)), (x_2, k_{x_2}(y_2))) \\ &= R_X(x_1, x_2) \\ &= (\text{id}_{I_X} \wr \tau) \circ R_\wr((x_1, y_1), (x_2, y_2)), \end{aligned}$$

and if $x_1 = x_2$, then

$$\begin{aligned} R_\wr(k(x_1, y_1), k(x_2, y_2)) &= R_\wr(x_1, k_{x_1}(y_1), x_2, k_{x_2}(y_2)) \\ &= R_Y(k_{x_1}(y_1), k_{x_2}(y_2)) \\ &= R_Y(k_{x_1}(y_1), k_{x_1}(y_2)) \\ &= \tau \circ R_Y(y_1, y_2) \\ &= (\text{id}_{I_X} \wr \tau) \circ R_\wr((x_1, y_1), (x_2, y_2)), \end{aligned}$$

which imply $(k, \text{id}_{I_X} \wr \tau) \in K$. Thus

$$K = \prod_{\tau \in T} \prod_{x \in X} \text{Aut}(Y_x)_\tau$$

follows. The rest of the claims, i.e., the relation to the wreath product, holds at the level of permutation groups. That is, the set of permutations of $X \times Y$ that permute Y_x block-wise is isomorphic to

$$\text{Aut}_{\text{Sets}}(Y_x) \wr_X \text{Aut}_{\text{Sets}}(X).$$

The detail is omitted. □

3.2 Construction of non-Schurian schemes

The results of this subsection are known to the researchers in this area, and proofs are given in Chen and Ponomarenko [3, Sections 3.2 and 3.4] with more systematic and detailed studies. However, we felt that it might not be easy to follow the details, and decided to include our proofs, for the reader's convenience.

Definition 3.4. (Schurian association schemes)

Let G be a group and X a set with left transitive action of G . Then the quotient by the diagonal action

$$R : X \times X \rightarrow G \backslash (X \times X) =: I$$

is known to be a (possibly non-commutative) association scheme. An association scheme isomorphic to this type is called a Schurian association scheme.

We may replace G with its image in $\text{Aut}_{\text{Sets}}(X)$. Then, a pair (g, id_I) for $g \in G$ is an automorphism of the Schurian scheme, and G transitively acts on $R^{-1}(i)$ for each $i \in I$. The following is immediate from this observation.

Lemma 3.5. *Let (X, R, I) be an association scheme. Let $\text{Aut}(X|I) \subset \text{Aut}(X)$ be the subgroup consisting of elements that act on I trivially. Then, X is Schurian if and only if $\text{Aut}(X|I)$ acts transitively on $R^{-1}(i)$ for each $i \in I$.*

The next is [3, Corollary 3.2.22].

Proposition 3.6. *Let X and Y be association schemes. The following are equivalent.*

- (1) *Both X and Y are Schurian.*
- (2) *The direct product $X \times Y$ is Schurian.*

Proof. Assume (1). Then, it is easy to check that we have a natural group homomorphism

$$\text{Aut}(X|I_X) \times \text{Aut}(Y|I_Y) \rightarrow \text{Aut}(X \times Y|I_X \times I_Y).$$

For any $(i_X, i_Y) \in I_X \times I_Y$, the subset

$$R_{X \times Y}^{-1}(i_X, i_Y) \subset (X \times Y) \times (X \times Y)$$

is identified with

$$R_X^{-1}(i_X) \times R_Y^{-1}(i_Y)$$

under the identification $(X \times Y) \times (X \times Y) = (X \times X) \times (Y \times Y)$, on which $\text{Aut}(X|I_X) \times \text{Aut}(Y|I_Y)$ acts transitively by the assumption, and by Lemma 3.5, we conclude (2).

Conversely, assume (2). By symmetry, it suffices to show that X is Schurian. We claim that there is a natural group morphism

$$\text{Aut}(X \times Y|I_X \times I_Y) \rightarrow \text{Aut}(X|I_X).$$

Let $p : X \times Y \rightarrow X$ be the projection. Take $(f, \text{id}) \in \text{Aut}(X \times Y|I_X \times I_Y)$. Take any $x \in X$ and $y, y' \in Y$. Then

$$R_{X \times Y}(f(x, y), f(x, y')) = R_{X \times Y}((x, y), (x, y')) = (i_0, R_Y(y, y'))$$

holds. Thus $p(f(x, y)) = p(f(x, y'))$, hence we have the existence of $\pi(f)$ that makes the diagram (3.3) commute. Take any $y \in Y$. If we denote $I_X \times I_Y \rightarrow I_X$ by the same symbol p , we have

$$\begin{aligned} R_X(\pi(f)(x), \pi(f)(x')) &= R_X(p \circ f(x, y), p \circ f(x', y)) \\ &= p \circ R_{X \times Y}(f(x, y), f(x', y)) \\ &= p \circ R_{X \times Y}((x, y), (x', y)) \\ &= R_{X \times Y}(p(x, y), p(x', y)) \\ &= R_X(x, x'), \end{aligned}$$

which implies $(\pi(f), \text{id}) \in \text{Aut}(X|I_X)$. Suppose that

$$R_X(x_1, x'_1) = R_X(x_2, x'_2) = i_X.$$

Take any $y \in Y$. Then

$$R_{X \times Y}((x_1, y), (x'_1, y)) = R_{X \times Y}((x_2, y), (x'_2, y)) = (i_X, i_0).$$

Thus by the assumption (2), we have $(f, \text{id}) \in \text{Aut}(X \times Y|I_X \times I_Y)$ such that

$$f(x_1, y) = (x_2, y), \quad f(x'_1, y) = (x'_2, y).$$

Then, by taking the image of p ,

$$\pi(f)(x_1) = x_2, \quad \pi(f)(x'_1) = x'_2.$$

Thus, $\text{Aut}(X|I_X)$ transitively acts on $R_X^{-1}(i_X)$, which shows (1). □

The next is [3, Corollary 3.4.7].

Theorem 3.7. *Let X and Y be association schemes. The following are equivalent.*

- (1) *Both X and Y are Schurian.*
- (2) *The wreath product $X \wr Y$ is Schurian.*

Proof. Assume (2). Put $G := \text{Aut}(X \wr Y|I_X \wr I_Y)$ as in Lemma 3.5. Take any (x_1, x'_1) , (x_2, x'_2) and $i_X \in I_X$ with

$$R_X(x_1, x'_1) = R_X(x_2, x'_2) = i_X.$$

We take an arbitrary $y \in Y$, then

$$R_{\wr}((x_1, y), (x'_1, y)) = R_{\wr}((x_2, y), (x'_2, y))$$

holds by the case division for $x_1 = x'_1$ or not. We consider the image $P(G) \subset \text{Aut}(X)$ in Theorem 3.2. Since $X \wr Y$ is Schurian, there is an automorphism $(f, \tau) \in G$ such that $\tau = \text{id}_{I_X \wr I_Y}$, $f(x_1, y) = (x_2, y)$ and $f(x'_1, y) = (x'_2, y)$. By Theorem 3.2, there is an

automorphism $P((f, \tau)) = (\pi(f), \pi(\text{id})) \in P(G)$ of X , which maps x_1 to x_2 and x'_1 to x'_2 , hence X is Schurian. For Y , take arbitrary (y_1, y'_1) , (y_2, y'_2) and $i_Y \in I_Y$ with

$$R_Y(y_1, y'_1) = R_Y(y_2, y'_2) = i_Y.$$

Fix an arbitrary $x \in X$, identify $Y = \{x\} \times Y \subset X \times Y$, and let G_x be the stabilizer of $\{x\} \times Y$ in G (i.e. the set of elements in G that preserve $\{x\} \times Y$ as a set). Hence G_x acts on Y , and by definition of G , G_x trivially acts on I_Y and consequently $G_x \subset \text{Aut}(Y|I_Y)$ follows by the commutativity of

$$\begin{array}{ccc} (\{x\} \times Y) \times (\{x\} \times Y) & \xrightarrow{R_X|_{\{x\}} \times R_Y} & \{i_0\} \times I_Y \\ \downarrow & & \downarrow \\ (X \times Y) \times (X \times Y) & \xrightarrow{R_\lambda} & I_X \times I_Y \end{array}.$$

Since

$$R_\lambda((x, y_1), (x, y'_1)) = R_\lambda((x, y_2), (x, y'_2)) = i_Y$$

and $X \times Y$ is Schurian, there is an automorphism $(f, \text{id}_{I_X \times I_Y}) \in G$ with both $f(x, y_1) = (x, y_2)$ and $f(x, y'_1) = (x, y'_2)$ holding. By (3.3), this means $\pi(f)(x) = x$ and $(f, \text{id}) \in G_x$. Thus, $G_x \subset \text{Aut}(Y|I_Y)$ acts transitively on $(R_X|_{\{x\}} \times R_Y)^{-1}(i_0, i_Y)$, and hence Y is Schurian. These imply (1).

Assume (1). Take four points in $X \times Y$ such that

$$R_\lambda((x_1, y_1), (x'_1, y'_1)) = R_\lambda((x_2, y_2), (x'_2, y'_2)). \quad (3.7)$$

Assume that $x_1 = x'_1$. Then, by the definition of R_λ , $x_2 = x'_2$, and $R_Y(y_1, y'_1) = R_Y(y_2, y'_2)$. Thus, there is an $(f, \text{id}_X) \in \text{Aut}(X|I_X)$ with

$$f \times f : (x_1, x'_1) \mapsto (x_2, x'_2)$$

(since both lie in $R_X^{-1}(i_0)$). There is a $(g, \text{id}_Y) \in \text{Aut}(Y|I_Y)$ with

$$g \times g : (y_1, y'_1) = (y_2, y'_2).$$

By the last statement of Theorem 3.2, we have

$$(f, \text{id}_X) \circ (g, \text{id}_Y) \in \text{Aut}(X \times Y|I_X \times I_Y)$$

that maps $(x_1, y_1) \mapsto (x_2, y_2)$ and $(x'_1, y'_1) \mapsto (x'_2, y'_2)$, which implies (2).

Assume that $x_1 \neq x'_1$. In this case, $R_X(x_1, x'_1) = R_X(x_2, x'_2)$ and $x_2 \neq x'_2$, hence there is an $f \in \text{Aut}(X|I_X)$ with

$$f \times f : (x_1, x'_1) \mapsto (x_2, x'_2).$$

Then $(f \times \text{id}_Y | \text{id}) \in \text{Aut}(X \wr Y | I_X \wr I_Y)$ maps

$$(x_1, y_1) \mapsto (x_2, y_1), \quad (x'_1, y'_1) \mapsto (x'_2, y'_1). \quad (3.8)$$

Recall that $\text{Aut}(Y_{x_2} | I_{Y_{x_2}}) = \text{Aut}(Y_{x_2})_{\text{id}_{I_Y}}$ in Proposition 3.3 lies in K , and since $x_2 \neq x'_2$,

$$\text{Aut}(Y_{x_2})_{\text{id}_{I_Y}} \times \text{Aut}(Y_{x'_2})_{\text{id}_{I_Y}} \in K.$$

Thus, since Y is Schurian, there are a $g \in \text{Aut}(Y_{x_2} | I_{Y_{x_2}})$ mapping $y_1 \mapsto y_2$ and a $g' \in \text{Aut}(Y_{x'_2} | I_{Y_{x'_2}})$ mapping $y'_1 \mapsto y'_2$. Thus, $g \circ g' \in \text{Aut}(X \wr Y | I_X \wr I_Y)$ maps

$$(x_2, y_1) \mapsto (x_2, y_2), \quad (x'_2, y'_1) \mapsto (x'_2, y'_2). \quad (3.9)$$

Together with (3.8), we conclude (1). \square

Corollary 3.8. *The wreath product of a non-Schurian association scheme and an association scheme (in both order of product) is non-Schurian. The same statement holds for the direct product.*

Thus, there exists a large family of non-Schurian schemes. We remark that there are substantial studies on construction of non-Schurian schemes, e.g., Evdokimov-Ponomarenko[5], Hanaki-Hirai-Ponomarenko[7], and Hirasaka-Kim[8]. Non-Schurian Schur rings are of particular interest since historically Wielandt[18, Theorem 26.4] found such an example, answering a question by Schur. To avoid confusion, we use the term S -rings for Schur rings (see Definition 3.10 below).

The results for S -rings stated in the rest of this section are closely related with the results by Evdokimov-Ponomarenko[5]. They used generalized wreath products to construct non-Schurian S -rings in a cyclic group, using delicate arguments. We deal with only direct products of groups and the usual wreath products, but still give a construction of non-Schurian S -rings.

We start with a definition of Cayley association schemes, which is equivalent to the notion of S -rings. We denote by e the unit of a group. We use the terminologies such as S -rings and Schurian S -rings according to a survey by Muzychuk and Ponomarenko [14]. The following definition of Cayley association schemes is given in [1, II.6] (without naming), as well as the equivalence to the notion of S -rings. We changed $g_2 g_1^{-1}$ in the definition there to $g_1^{-1} g_2$ because we consider the left action. A detailed study on Cayley association schemes and S -rings is found in [3, Section 2.4].

Definition 3.9. (Cayley association schemes)

Let G be a finite group. If there is a surjective mapping $r : G \rightarrow I$ such that the composition

$$G \times G \rightarrow G \rightarrow I, \quad (g_1, g_2) \mapsto r(g_1^{-1} g_2)$$

is an association scheme, then it is called a Cayley association scheme.

This notion is equivalent to the following notion of S -rings. The conditions on $r : G \rightarrow I$ are equivalent, and the Bose-Mesner algebra of a Cayley association scheme is naturally isomorphic to the corresponding S -ring.

Definition 3.10. (*S*-rings)

Let G be a finite group, and $r : G \rightarrow I$ a surjective mapping. Let $\mathbb{C}[G]$ be the group ring. For a subset $S \subset G$, define

$$\underline{S} := \sum_{s \in S} s \in \mathbb{C}[G].$$

Let

$$A_r = \text{the } \mathbb{C}\text{-linear span of } \underline{r^{-1}(i)} \subset \mathbb{C}[G] \text{ for } i \in I .$$

Then A_r is called an *S*-ring, if the following conditions are satisfied.

- (1) $\{e\} = r^{-1}(i_0)$ for some $i_0 \in I$.
- (2) A_r is closed under the product in $\mathbb{C}[G]$.
- (3) For any $i \in I$, there is $i' \in I$ with $\{g^{-1} \mid g \in r^{-1}(i)\} = r^{-1}(i')$.

We want to discuss on the Schurian *S*-rings which are defined below.

Definition 3.11. (Schurian *S*-rings)

Let X be a finite set. Let Γ be a group transitively acting on X , with a subgroup G acting transitively and faithfully on X . Fix $x \in X$. Then we have a bijection

$$G \rightarrow X, \quad g \mapsto gx,$$

and through this bijection Γ acts on G . Let Γ_e be the stabilizer of $e \in G$. Consider

$$r : G \rightarrow \Gamma_e \backslash G =: I.$$

Then $r : G \rightarrow I$ gives an *S*-ring, which is called a Schurian *S*-ring.

The next proposition is well-known, and the proof is omitted.

Proposition 3.12. *Let G be a finite group, and assume that $r : G \rightarrow I$ gives a Cayley association scheme, and equivalently, an *S*-ring. Then, the following are equivalent.*

- (1) *The Cayley association scheme is Schurian in the sense of Definition 3.4.*
- (2) *The *S*-ring is Schurian in the sense of Definition 3.11.*

The following proposition is a direct consequence of the definitions and the equivalence between Cayley association schemes and *S*-rings.

Proposition 3.13. *The wreath product of two *S*-rings is an *S*-ring. The direct product of two *S*-rings is an *S*-ring.*

Proof. Let G_1 and G_2 be finite groups. Let $r_1 : G_1 \rightarrow I_1$ and $r_2 : G_2 \rightarrow I_2$ be the corresponding mappings. Then, their wreath product is given by

$$\begin{array}{ccc}
(G_1 \times G_2) \times (G_1 \times G_2) & \longrightarrow & G_1 \times G_2 \\
\downarrow & & \downarrow \\
(G_1 \times G_1) \times (G_2 \times G_2) & \longrightarrow & I_1 \times I_2 \longrightarrow I_1 \wr I_2
\end{array}$$

where the top arrow is $((g_1, g_2), (g_3, g_4)) \mapsto (g_1, g_2)^{-1}(g_3, g_4) = (g_1^{-1}g_3, g_2^{-1}g_4)$. The definition of the wreath product is via the left bottom corner. By the commutativity of the diagram, it is an S -ring. The claim for the direct product follows in a similar manner, by merely removing $I_1 \wr I_2$ from the above diagram. \square

Proposition 3.12, Proposition 3.13 and Theorem 3.7 imply the following proposition.

Proposition 3.14. *The wreath product of a non-Schurian S -ring and an S -ring (in both order) is non-Schurian. The same statement holds for the direct product.*

The existence of a large number of non-Schurian S -rings follows. Using generalized wreath products, Evdokimov-Ponomarenko[5] proved the following theorem.

Theorem 3.15. *Let $n = p_1 p_2 p_3 p_4 n'$ be an integer where p_1, p_2, p_3, p_4 are prime numbers with the condition $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$ and n' is a positive integer. Put $d := \text{lcm}(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1)$. If $d > 2$, then the cyclic group of order n has a non-Schurian S -ring.*

As another example, Hanaki-Hirai-Ponomarenko[7] proved a generalization of Wielandt's construction:

Theorem 3.16. *Let p be a prime. Let G be an elementary abelian p -group of even rank except for the orders 2^2 , 3^2 , and 2^4 . Then G has a non-Schurian S -ring.*

Starting from these examples, by taking the wreath product with any S -rings or association schemes, we have a large family of non-Schurian S -rings and non-Schurian association schemes.

4 Iterated product and profinite association schemes

A special case of an iterated wreath product is implicitly used in the construction of the kernel scheme by Martin-Stinson[11] (the notation here follows [12]).

Definition 4.1. Let n be a positive integer, and V a finite set of alphabet with cardinality $v \geq 2$. Let X_n be V^n , and $I_n := \{1, 2, \dots, n\} \cup \{\infty\}$. (We use ∞ instead of a natural notation $n+1$, since this is i_0 and to be distinguished when considering a projective system in the next section.) Define $R_n : X_n \times X_n \rightarrow I_n$ as follows. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be elements of X_n . Let $R(x, y)$ be the smallest index i for which $x_i \neq y_i$. If $x = y$, then $R(x, y) = \infty$. This is a symmetric (and hence commutative) association scheme, with $R^{-1}(\infty)$ being the identity relation. This is called a kernel scheme, and denoted by $\overrightarrow{k(n, v)}$.

Definition 4.2. Let X be an association scheme. We define inductively its wreath power for $n \in \mathbb{N}$ by:

- $X^{\wedge 1} := X$.
- $X^{\wedge n} := (X^{\wedge n-1}) \wr X$ for $n \geq 2$.

Definition 4.3. Let $v \geq 2$ be an integer. Let $H(1, v)$ be the class-one association scheme of size v , namely, the unique association scheme with $I = \{i_0, i_1\}$ with $\#X = v$ (the notation follows Delsarte[4, §4.1.1]).

Proposition 4.4.

$$\overrightarrow{k(n, v)} = H(1, v)^{\wedge n}.$$

Proof. This can be proved in a straightforward manner by induction on n . The definition of $X \wr Y$ is “to determine the relation of (x, y) and (x', y') , first look at the X -component; if $x \neq x'$ then the relation is decided by them in X . If not, then the relation is decided by those of y and y' .” This is compatible with the definition of the kernel schemes. \square

Remark that in Martin-Stinson [11], the kernel schemes are shown to be association schemes by computing the intersection numbers. The above proposition gives another proof.

4.1 Profinite association schemes

One of the motivations of this study is to construct a projective system of non-Schurian association schemes. This section follows Matsumoto-Ogawa-Okuda[12]. Proofs of the statements are given there.

Proposition 4.5. *Let X be an association scheme. Let A_X be its Bose-Mesner algebra. We define a convolution product \bullet on A_X as a normalization of the matrix product*

$$A \bullet B := \frac{1}{\#X} AB.$$

- (1) *Let $p : X \rightarrow X'$ be a surjective morphism of association schemes. Through the identification of $A_X = C(I_X)$, we have a canonical injection $\Psi : A_{X'} \hookrightarrow A_X$ by $C(I_{X'}) \hookrightarrow C(I_X)$. Then, Ψ preserves Hadamard product, Hadamard unit, and the convolution product \bullet (does not preserve the convolution unit if $\#X > \#X'$).*
- (2) *Suppose that X and X' are commutative. Then, the set of primitive idempotents of A_X with respect to the convolution product is naturally identified with that to the matrix product (and hence with $J(X)$). The former set is obtained by multiplying each element of $J(X)$ by $\#X$. From now on, $J(X)$ means the set of primitive idempotents with respect to \bullet .*

(3) An element of $J(X')$ is mapped by Ψ to a non-zero idempotent in A_X , and thus a non-empty sum of elements of $J(X)$. For distinct elements of $J(X')$, the corresponding non-empty subsets of $J(X)$ have no intersection. This gives a one-to-many (and non-empty) correspondence $J(X') \rightarrow J(X)$, in other words, a partial surjection from $J(X)$ to $J(X')$, denoted by $J(X) \dashrightarrow J(X')$.

Definition 4.6. Let Λ be a directed ordered set, namely, a partial ordered set where any two elements have an upper bound. A profinite association scheme $(X_\lambda)_{\lambda \in \Lambda}$ is a projective system of association schemes with surjective morphisms, namely:

- (1) A family of association schemes $(R_\lambda, X_\lambda, I_\lambda)$ for $\lambda \in \Lambda$.
- (2) For any $\lambda \geq \mu \in \Lambda$, a surjective morphism $p_{\lambda,\mu} : X_\lambda \rightarrow X_\mu$ is specified.
- (3) For any λ , $p_{\lambda,\lambda} = \text{id}_{X_\lambda}$.
- (4) For any $\lambda \geq \mu \geq \nu$,

$$p_{\lambda,\nu} = p_{\mu,\nu} \circ p_{\lambda,\mu}.$$

We define its underlying set by

$$X^\wedge = \varprojlim X_\lambda,$$

and the set of relations by

$$I^\wedge = \varprojlim I_\lambda,$$

in the category of sets. Then, X^\wedge and I^\wedge have natural (profinite) topologies, where X_λ and I_λ are finite sets with the discrete topology. If every X_λ is commutative, we have a projective system of partial surjections of $J_\lambda := J(X_\lambda)$. We define

$$J^\wedge = \varprojlim J_\lambda,$$

which is proved to have a discrete topology. We define the Bose-Mesner algebra of $(X_\lambda)_{\lambda \in \Lambda}$ as the inductive limit

$$A_{X^\wedge} := \varinjlim A_{X_\lambda},$$

which has Hadamard product with unit, and the convolution product (without unit if J^\wedge is infinite). It has a linear basis J^\wedge consisting of all the primitive idempotents, and is isomorphic, as a ring with Hadamard product, to the space of locally constant functions on I^\wedge . For subsets $J_D \subset J^\wedge$, $I_C \subset I^\wedge$, and a finite multi-subset $Y \subset X^\wedge$, a property “ I_C -free code and J_D -design” of Y is defined.

As described in [12], the concept of profinite association schemes has an application to Quasi Monte Carlo methods. More precisely, as a projective system of ordered Hamming schemes, one can obtain a profinite association scheme which is called the pro-ordered Hamming scheme. Then (t, s) -sequences which have been studied deeply as Quasi Monte Carlo methods (see [15]) can be explained by design theory on the pro-ordered Hamming scheme.

4.2 Iterated wreath products

Iterated wreath products give examples of profinite association schemes. We begin with preparation.

Lemma 4.7. *The wreath product is associative, i.e., there is a canonical isomorphism*

$$(X \wr Y) \wr Z \rightarrow X \wr (Y \wr Z)$$

for association schemes X, Y and Z . Hence we may write

$$X \wr Y \wr Z.$$

Proof. This follows from the identification

$$\begin{aligned} (I_X \wr I_Y) \wr I_Z &= ((I_X \setminus \{i_0\}) \amalg I_Y \setminus \{i_0\}) \amalg I_Z \\ &= (I_X \setminus \{i_0\}) \amalg (I_Y \setminus \{i_0\}) \amalg I_Z \\ &= I_X \wr (I_Y \wr I_Z) \end{aligned}$$

and the commutativity of

$$\begin{array}{ccc} I_X \times I_Y \times I_Z & \xrightarrow{\text{id} \times \text{lex}} & I_X \times (I_Y \wr I_Z) \\ \text{lex} \times \text{id} \downarrow & & \downarrow \text{lex} \\ (I_X \wr I_Y) \times I_Z & \xrightarrow{\text{lex}} & (I_X \wr I_Y) \wr I_Z = I_X \wr (I_Y \wr I_Z). \end{array}$$

□

Lemma 4.8. *Let X and Y be association schemes. The projection $\pi : X \wr Y \rightarrow X$ given in Proposition 2.18 induces an injection*

$$A_X \rightarrow A_{X \wr Y}, \quad A \mapsto A \otimes \mathbf{J}_Y.$$

Proof. Because of the definition of

$$\pi_{I_X} : I_X \wr I_Y \rightarrow I_X$$

in Definition 2.8, the preimage of $i_X \in I_X$ in $I_X \times I_Y$ is

$$\{(i_X, i_Y) \mid i_Y \in I_Y\},$$

and hence the image of $A_{i_X} \in A_X$ in $A_{X \wr Y}$ is

$$\sum_{i_Y \in I_Y} A_{i_X} \otimes A_{i_Y} = A_{i_X} \otimes \mathbf{J}_Y$$

by Proposition 2.6. Since A_X is the linear span of A_{i_X} , the statement follows. □

Proposition 4.9. Let X_1, X_2, \dots, X_n be any sequence of association schemes. Let I_1, I_2, \dots, I_n be the set of their relations. Then, their wreath product $X_1 \wr X_2 \wr X_3 \wr \dots \wr X_n$ has underlying set

$$X_1 \times \dots \times X_n$$

and the set of relations

$$I_1 \wr I_2 \wr I_3 \wr \dots \wr I_n = (I_1 \setminus \{i_0\}) \amalg (I_2 \setminus \{i_0\}) \amalg \dots \amalg (I_{n-1} \setminus \{i_0\}) \amalg I_n.$$

Proof. The structure of the underlying set follows by definition. The structure of the set of relations follows by induction from Definition 2.7. \square

Proposition 4.10. Let $X_1, X_2, \dots,$ be an infinite series of association schemes. Then, the series of the wreath products

$$(X_1 \wr \dots \wr X_n)_{n \in \mathbb{N}_{>0}}$$

form a projective system of association schemes. The mappings of the underlying sets are given by projections

$$X_1 \times \dots \times X_n \rightarrow X_1 \times \dots \times X_{n-1}.$$

The mappings of the sets of relations

$$(I_1 \setminus \{i_0\}) \amalg \dots \amalg (I_{n-1} \setminus \{i_0\}) \amalg I_n \rightarrow (I_1 \setminus \{i_0\}) \amalg \dots \amalg (I_{n-2} \setminus \{i_0\}) \amalg I_{n-1}$$

are given by mapping the elements in I_n to the i_0 in I_{n-1} .

The projective limit of the underlying set is the direct product (with direct product topology, hence compact and Hausdorff)

$$X^\wedge = \prod_{i=1}^{\infty} X_i.$$

The projective limit I^\wedge of $I_1 \wr \dots \wr I_n$ is the one-point compactification of the discrete topological set

$$\prod_{i=1}^{\infty} (I_i \setminus \{i_0\}).$$

Proof. The mapping between the underlying set is the projection by Proposition 2.18. It is a general fact that the projective limit of finite direct products is the infinite direct product.

The projective system $(I_1 \wr \dots \wr I_n)_{n \in \mathbb{N}_{>0}}$ is given by mapping the last I_n to the i_0 of I_{n-1} , by Proposition 2.18. We consider its projective limit. Except for i_0 , every element in the coproduct is a clopen point in the projective limit, and the set of open neighborhoods of the limit of i_0 is the set of the union of $\{i_0\}$ and the complement of a finite set of $\prod_{i=1}^{\infty} (I_i \setminus \{i_0\})$. \square

Proposition 4.11. *Suppose that every X_n is commutative in Proposition 4.10. Then, the primitive idempotents of $X_1 \wr X_2 \wr X_3 \cdots \wr X_n$ is*

$$J_{X_1} \prod (J_{X_2} \setminus \{j_0\}) \prod (J_{X_3} \setminus \{j_0\}) \prod \cdots \prod (J_{X_n} \setminus \{j_0\}). \quad (4.1)$$

Its inductive limit is

$$J^\wedge := J_{X_1} \prod_{i=2}^{\infty} (J_{X_i} \setminus \{j_0\}). \quad (4.2)$$

Proof. By induction using Corollary 2.15, (4.1) is equal to $J(X_1 \wr X_2 \wr X_3 \cdots \wr X_n)$. By Lemma 4.8 the one-to-many correspondence (3) in Proposition 4.5

$$J(X_1 \wr X_2 \wr X_3 \wr \cdots \wr X_{n-1}) \rightarrow J(X_1 \wr X_2 \wr X_3 \wr \cdots \wr X_n) \quad (4.3)$$

is given by

$$E \mapsto E \otimes \mathbf{J}_{X_n},$$

where the right-hand side is a primitive idempotent (w.r.t. \bullet) in $A_{X_1 \wr X_2 \wr X_3 \cdots \wr X_n}$ by Theorem 2.12. This is a natural inclusion of (4.1) for $n - 1$ to that for n . Thus, the partial surjection (3) in Proposition 4.5, namely,

$$J(X_1 \wr X_2 \wr X_3 \wr \cdots \wr X_n) \rightarrow J(X_1 \wr X_2 \wr X_3 \wr \cdots \wr X_{n-1})$$

is induced by the natural inclusion (4.3). The projective limit of the partial surjections is in this case equal to the inductive limit of injections, hence is a union (4.2). \square

By Proposition 4.4, the kernel schemes in Definition 4.1 is a special case of Proposition 4.10, where each I_i (and consequently J_i) has the cardinality two. They form a projective system, where $I_n \rightarrow I_{n-1}$ is mapping $i \mapsto i$ for $i < n$, $n \mapsto \infty$, and $\infty \mapsto \infty$, as proved in Proposition 4.10. $J_{n-1} \rightarrow J_n$ is a canonical inclusion, as proved in Proposition 4.11. The above iterated wreath products give examples of profinite association schemes whose X^\wedge , I^\wedge , and J^\wedge are explicitly described. There is a closely related earlier research by Barg and Skrikanov [2, Section 8], where they treat similar objects coming from a profinite abelian group, and obtain the duality theorems and the structural constants.

Our final remark is about a relation with Kurihara-Okuda[9]. There, for any compact Hausdorff group G and its closed subgroup H , the notion of Bose-Mesner algebra for the homogeneous space G/H is given (which may be seen as an analogue to a Schurian scheme).

Any profinite group G is compact and Hausdorff, and for any closed subgroup H , G/H can be viewed as both a homogeneous space (as in [9]) and a profinite association scheme as in [12]. Both methods yield the same Bose-Mesner algebra. In this case, G/H yields a projective system of Schurian association schemes. Theorem 3.7 and the iterated wreath products imply that there is a large class of projective systems of finite non-Schurian association schemes.

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