Hamiltonian intervals in the lattice of binary paths

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Submitted: Jun 16, 2023; Accepted: Dec 5, 2023; Published: Feb 9, 2024

Abstract

Let $\mathcal{P}_n$ be the set of all binary paths (i.e., lattice paths with upsteps $u = (1, 1)$ and downsteps $d = (1, -1)$) of length n endowed with the pointwise partial ordering (i.e., $P \leqslant Q$ iff the lattice path P lies weakly below Q) and let G_n be its Hasse graph. For each path $P \in \mathcal{P}_n$, we denote by $I(P)$ the interval which contains the elements of $\mathcal{P}_n$ less than or equal to P, excluding the first two elements of $\mathcal{P}_n$, and by $G(P)$ the subgraph of G_n induced by $I(P)$. In this paper, it is shown that $G(P)$ is Hamiltonian iff P contains at least two peaks and $I(P)$ has equal number of elements with even and odd rank. The last condition is always true for paths ending with an upstep, whereas, for paths ending with a downstep, a simple characterization is given, based on the structure of the path.

Mathematics Subject Classifications: 05C45, 06A07

1 Introduction

Let $\mathcal{P}_n$, where n is a positive integer, be the set of all binary paths P of length $|P| = n$, i.e., lattice paths $P = p_1 p_2 \cdots p_n$ where each *step* p_i, $i \in [n]$, is either an *upstep* $u = (1, 1)$ or a *downstep* $d = (1, -1)$ and connects two consecutive points of the path P. The number of u's (reps. d's) in P is denoted by $|P|_u$ (resp. $|P|_d$). A maximal sequence of u's (resp. d's) in P is called *ascent* (resp. *descent*) of P. The last point of an ascent (resp. descent) is called *peak* (resp. *valley*) of the path. Clearly, every peak (resp. valley) corresponds to either an occurrence of ud (resp. du), or an occurrence of u (resp. d) at the end of the path. We note that this definition extends the usual definition of peaks and valleys appearing in the literature. It is convenient to consider that the starting point of a path is the origin of a pair of axes. The y-coordinate of a lattice point on P is called *height* of this point. We set $\mathcal{P} = \bigcup_{n \geqslant 0} \mathcal{P}_n$, where $\mathcal{P}_0$ consists of only the empty path ε (the path which has no steps). Obviously, the set $\mathcal{P}_n$ has cardinality 2^n.

A natural partial ordering on $\mathcal{P}_n$ is defined by the geometric representation of paths $P, Q \in \mathcal{P}_n$ where $P \leqslant Q$ whenever P lies (weakly) below Q. We note that Q covers

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P whenever Q is obtained from P by turning exactly one of P's valleys into a peak. It is well-known that the poset $(\mathcal{P}_n, \leqslant)$, or simply $\mathcal{P}_n$, is a finite, self-dual, distributive, graded lattice with minimum and maximum elements the paths $\mathbf{0}_n = d^n = \underbrace{dd\cdots d}_{n \text{ times}}$ and $\mathbf{1}_n = u^n = \underbrace{uu\cdots u}_{n \text{ times}}$ respectively and its rank function is

$$\rho(P) = \sum_{i=1}^{n} (n-i+1)[p_i = u], \tag{1}$$

where $[S]$ is the Iverson notation, i.e., for every proposition S, $[S] = 1$ if S is true, and 0 if S is false. We can easily check that the rank function of the concatenation of two paths $P, Q \in \mathcal{P}$ is given by the formula

$$\rho(PQ) = \rho(P) + \rho(Q) + |Q||P|_u. \tag{2}$$

In particular, we have that

$$\rho(uP) = \rho(P) + |P| + 1, \qquad \rho(dP) = \rho(P),$$
$$\rho(Pu) = \rho(P) + |P|_u + 1 \qquad \text{and} \qquad \rho(Pd) = \rho(P) + |P|_u.$$

A natural involution on $\mathcal{P}$ is defined by the operation of turning the last step of the path (i.e., u becomes d and vice versa). We call this the *switch involution* and we will use it several times in this work, because of its property of reversing the parity of the rank.

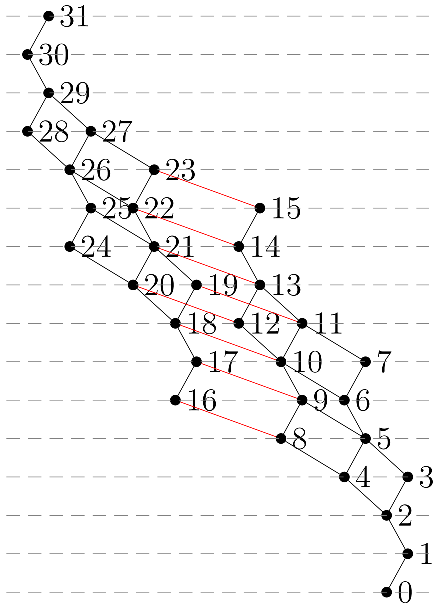
This lattice appears in the literature in various equivalent forms (e.g., sequences of integers [13], binary words [4, p. 92], subsets of $[n]$ [5], permutations of $[n]$ [15, p. 402], partitions of n into distinct parts [14], threshold graphs [7]). Ferrari and Pinzani [3] and Sapounakis et al. [9] have studied its sublattice of Dyck paths, Manes et al. have presented a bijection between comparable pairs of paths of this lattice and Dyck prefixes of odd length [6] and Tasoulas et al. have studied the chains with small intervals in this lattice [16].

In this paper we consider the Hasse graph G_n of $\mathcal{P}_n$, the edges of which are defined by the covering relation. For every $n \geqslant 2$, the graph G_n is decomposed into two copies of G_{n-1} consisting of the paths starting with u and d respectively. Two vertices not in the same copy are adjacent in the graph G_n iff they are of the form udP and duP respectively for some $P \in \mathcal{P}_{n-2}$. For example, for $n = 5$, see Fig. 1, where the vertices are encoded as integers and the edges with endpoints not in the same copy are colored red.

Using this recursive decomposition we can deduce some simple graph-theoretic properties and statistics of G_n. We can easily verify that the number $e(G_n)$ of edges and the number $c(G_n))$ of 4-cycles of G_n are equal to

$$e(G_n) = (n+1)2^{n-2},\ n \geqslant 1, \qquad c(G_n) = (n+1)(n-2)2^{n-5},\ n \geqslant 2.$$

For further information on these sequences, see seq. A001792 and A001793 in [11]. Ferrari and Munarini [2] have enumerated the edges in the Hasse graph of several related lattices of paths, including the subgraph of G_n consisting of Dyck paths.



num	path	num	path
15	duuuu	31	uuuuu
14	duuud	30	uuuud
13	duudu	29	uuudu
12	duudd	28	uuudd
11	duduu	27	uuduu
10	dudud	26	uudud
9	duddu	25	uuddu
8	duddd	24	uuddd
7	dduuu	23	uduuu
6	dduud	22	uduud
5	ddudu	21	ududu
4	ddudd	20	ududd
3	ddduu	19	udduu
2	dddud	18	uddud
1	ddddu	17	udddu
0	ddddd	16	udddd

Figure 1: The graph G_5. The graphs G_4 and G_3 are the subgraphs induced by the intervals $[0, 15]$ and $[0, 7]$ respectively.

The lattice $\mathcal{P}_n$ (resp. the graph G_n) is isomorphic to the lattice $M(n)$ (resp. the cover graph A_n of $M(n)$), introduced by Stanley [13] (resp. considered by Savage et al. [10]). Furthermore, in [10] (working on the isomorphic graph A_n) it is proved that for every $n \geqslant 3$ the subgraph of G_n on the set $\mathcal{P}_n^* = \mathcal{P}_n \setminus \{d^n, d^{n-1}u, u^n, u^{n-1}d\}$ is Hamiltonian. Obviously, this is the largest Hamiltonian subgraph of G_n, since the excluded vertices do not belong to any cycle (e.g., the vertices 0, 1, 30, 31 in Fig. 1, for $n = 5$). In a similar direction, Eades and Hickey [1] gave a sufficient and necessary condition for the subgraph of G_n on the interval $[d^{n-k}u^k, u^k d^{n-k}]$ to have a Hamiltonian path (iff $k \leqslant 1$ or $k \geqslant n-1$ or n is even and k is odd). The Hamiltonian cycles and paths obtained in the above two results correspond to combinatorial Gray codes for the poset elements serving as graph vertices. For a recent survey on combinatorial Gray codes see the paper by Mütze [8].

Working in the same direction, we consider the intervals of $\mathcal{P}_n$. For every $P \in \mathcal{P}_n^*$ we denote the ideal of P in $\mathcal{P}_n^*$ by $I(P)$, i.e., $I(P) = [d^{n-2}ud, P]$. Furthermore, for every subset A of $\mathcal{P}_n$ we denote by $G(A)$ the subgraph of G_n induced by A. In particular, for every $P \in \mathcal{P}_n^*$ we write $G(P)$ instead of $G(I(P))$ for simplicity. We note that, by the above, the graph $G(u^{n-2}du)$ is Hamiltonian. In this work, we investigate the Hamiltonicity of the graph $G(P)$, for an arbitrary path P. For $n \geqslant 3$ we set $\mathcal{H}_n = \{P \in \mathcal{P}_n^* : G(P) \text{ is Hamiltonian}\}$. We can exhaustively check (see Fig. 1) that

$$\mathcal{H}_3 = \{udu\} \qquad \text{and} \qquad \mathcal{H}_4 = \{dudu, ud^2u, udu^2, u^2du, udud\}.$$

It is easy to check that

- $P \in \mathcal{H}_n$ iff $d^iP \in \mathcal{H}_{n+i}$, for $i \geqslant 0$.

- If $P \in \mathcal{H}_n$ then P has at least two peaks.
- If $P \in \mathcal{H}_n$ then $|E(P)| = |O(P)|$, where $E(P)$ (resp. $O(P)$) is the set of paths $Q \leqslant P$ having even (resp. odd) rank. In particular, $|I(P)|$ is even.

According to the first property, for the Hamiltonicity of $G(P)$, it is enough to consider only paths P starting with u.

The main result of this work states that the latter two necessary conditions, when combined, give a sufficient condition for the Hamiltonicity of $G(P)$:

Theorem 1. *$P \in \mathcal{H}_n$ iff P has at least two peaks and $|E(P)| = |O(P)|$.*

We note that if the path P ends with u, i.e., $P = Qu$, for some path Q, then the interval $[d^n, P]$ is the disjoint union of the sets $\{Sd : S \leqslant Q\}$ and $\{Su : S \leqslant Q\}$, consisting of the paths ending with d and u respectively. Clearly, the switch involution defines a bijection between these two sets, thus giving $|E(P)| = |O(P)|$. Therefore, for paths ending with u, Theorem 1 reduces to: $P \in \mathcal{H}_n$ iff P has at least two peaks, which is proved in Section 2. In Section 3, we introduce the notion of a critical valley of a path P and, using this, we give a simple structural condition which is proved to be sufficient and necessary in order to have $|E(P)| = |O(P)|$. This condition is used in Section 4 in order to prove Theorem 1 also for paths ending with d.

We close this section with some notation and preliminary material that will be used in the next sections. For a path P, we denote by $f(P)$ (resp. $s(P)$) the path obtained by turning the first (resp. second) peak of P into a valley. More generally, we define inductively

$$f^i(P) = f(f^{i-1}(P)) \qquad \text{and} \qquad s^i(P) = s(s^{i-1}(P)),$$

where $i \geqslant 1$ and $f^0(P) = s^0(P) = P$.

Furthermore, we denote by $e(P)$ the edge $\{P, f(P)\}$. For two paths P_1 and P_2, we denote by P_1P_2 their concatenation and by $f(\{P_1, P_2\})$ the edge $\{f(P_1), f(P_2)\}$. If A is a set of paths and Q is a path, we set $QA = \{QP : P \in A\}$ and $AQ = \{PQ : P \in A\}$. In particular, for an edge $e = \{P_1, P_2\}$, we have $Qe = \{QP_1, QP_2\}$. Furthermore, if Q is a path and $\mathcal{F}$ is a subgraph of G_n, we define $Q\mathcal{F}$ as the graph with vertices and edges of the form QP and Qe respectively, where P is any vertex and e is any edge of $\mathcal{F}$. Two edges $e = \{P_1, P_2\}$ and $g = \{Q_1, Q_2\}$ with no common vertices are said to be parallel ($e \parallel g$) iff P_1, Q_1 and P_2, Q_2 are adjacent. For example, $e \parallel f(e)$, for every edge $e \neq e(P)$, $P \in \mathcal{P}$. More generally, two paths of vertices $(P_1, P_2, \ldots, P_r)$ and $(Q_1, Q_2, \ldots, Q_r)$ are said to be parallel iff $\{P_i, P_{i+1}\} \parallel \{Q_i, Q_{i+1}\}$, for every $i \in [r-1]$. Using parallel edges, we give a method for the construction of cycles.

Basic constructions

1. Let $(P_1, P_2, \ldots, P_r)$ and $(Q_1, Q_2, \ldots, Q_r)$ be two parallel paths of vertices of odd length and let C be a cycle containing the path $(Q_1, Q_2, \ldots, Q_r)$. Then, by replacing the edge $\{Q_i, Q_{i+1}\}$, for each odd $i \in [r-1]$, by the path of vertices $(Q_i, P_i, P_{i+1}, Q_{i+1})$, we expand C into a new cycle which also contains the vertices $P_1, P_2, \ldots, P_r$. (see Fig. 2).

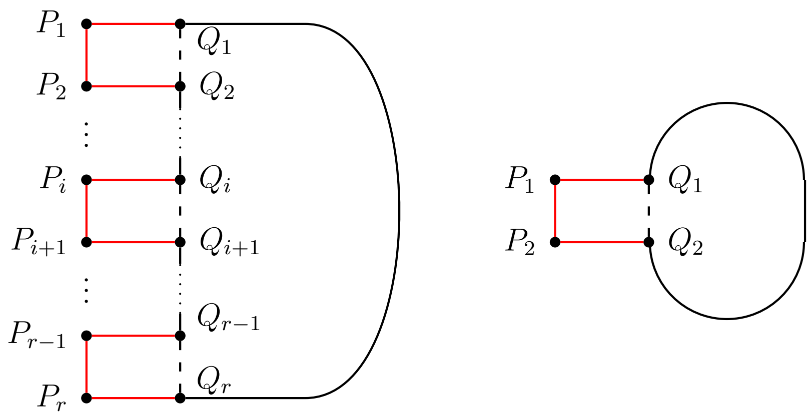


Figure 2: The join of a path of vertices (left) or an edge (right) and a cycle.

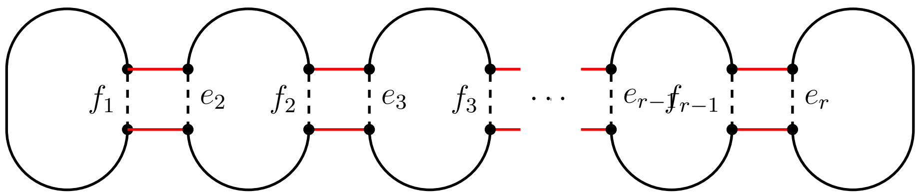


Figure 3: The join of the cycles C_i, $i \in [r]$.

2. Let $(C_i)_{i\in[r]}$, $r \geqslant 2$, be a finite sequence of disjoint cycles and let $(e_i)_{i\in[2,r]}$ and $(f_i)_{i\in[r-1]}$ be two finite sequences of edges such that e_i and f_i belong to C_i and $f_i \parallel e_{i+1}$ for every $i \in [r-1]$. Then, by deleting the edges of the two sequences and then connecting each vertex of f_i with the corresponding vertex of e_{i+1} for each $i \in [r-1]$, we can join the cycles C_i into a single cycle (see Fig. 3).

The above two constructions will be used repeatedly in the following sections, in most cases for $r = 2$.

Finally, throughout this paper, we denote by e_n^i, $i \in [3]$, the edges

$$e_n^1 = \{d^n, d^{n-1}u\}, \qquad e_n^2 = \{d^{n-2}ud, d^{n-3}ud^2\}, \qquad e_n^3 = \{d^{n-2}ud, d^{n-2}u^2\}.$$

For example, in Fig. 1, the edges e_n^1, e_n^2 and e_n^3 correspond to the edges $\{0,1\}, \{2,4\}$ and $\{2,3\}$ respectively. It is evident that e_n^1 does not belong to any cycle of G_n, whereas e_n^2 and e_n^3 belong to every cycle of G_n which contains the vertex $d^{n-2}ud$. The edges Qe_k^i, where $k \in [n]$, $i \in [3]$ and $Q \in \mathcal{P}_{n-k}$, will be used repeatedly in the following sections.

2 The Hamiltonicity for paths ending with u

In this section, we investigate the Hamiltonicity of the graph $G(P)$, for every path P ending with u.

Proposition 2. *If the path $P \in \mathcal{P}_n$ ends with u and has at least two peaks, then $P \in \mathcal{H}_n$.*

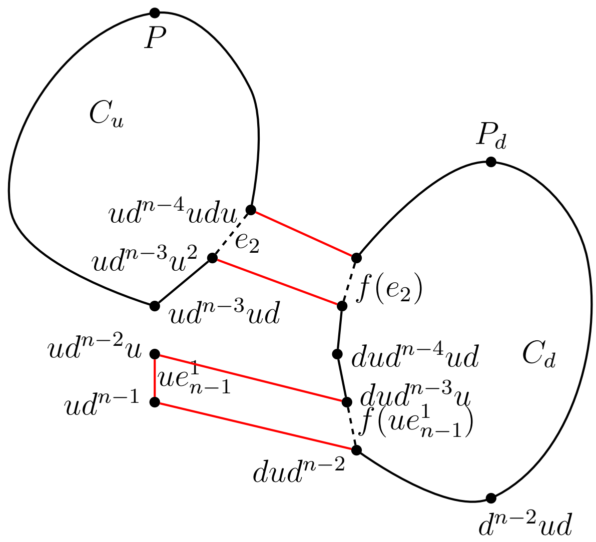


Figure 4: The construction of a Hamiltonian cycle of $G(P)$ when $P \neq u^k du^{n-1-k}$, $ud^k u^{n-1-k}$, $k \in [n-2]$.

Proof. We show the result by induction on the length n of P. Clearly, the result holds for $n = 3$ and $n = 4$. For $n \geqslant 5$, without loss of generality, we assume that P starts with u. We first show the result under the restriction $P \neq u^k du^{n-1-k}$, $ud^k u^{n-1-k}$, $k \in [n-2]$. Then, the path obtained by deleting the first u of P and the path P_d obtained by shifting the first d of P to the beginning of P, both end with u and have at least two peaks, so that by the induction hypothesis, we can find two Hamiltonian cycles C_u and C_d of $G([ud^{n-3}ud, P])$ and $G(P_d)$ respectively.

It is easy to see that $I(P)$ is partitioned as $I(P) = [ud^{n-3}ud, P] \cup I(P_d) \cup ue^1_{n-1}$, so that it is enough to join the cycles C_u, C_d and the edge ue^1_{n-1} into a single cycle.

Clearly, since the vertex $ud^{n-3}u^2$ (resp. $f(ud^{n-3}u^2) = dud^{n-4}u^2$) has degree 2 in $G([ud^{n-3}ud, P])$ (resp. 3 in $G(P_d)$), the cycle C_u (resp. C_d) contains both
$e_1 = \{ud^{n-3}u^2, ud^{n-3}ud\}$ and $e_2 = \{ud^{n-3}u^2, ud^{n-4}udu\}$ (resp. either one of $f(e_1)$ or $f(e_2)$). Then, for $e = e_1$ or $e = e_2$, we have that e belongs to C_u and $f(e)$ to C_d.

Furthermore, since the vertex $f(ud^{n-1}) = dud^{n-2}$ has degree 2 in $G(P_d)$, the edge $f(ue^1_{n-1}) = f(\{ud^{n-1}, ud^{n-2}u\}) = \{dud^{n-2}, dud^{n-3}u\}$ belongs to C_d.

Then, using basic constructions 1 and 2 for the pairs of parallel edges $(ue^1_{n-1}, f(ue^1_{n-1}))$ and $(e, f(e))$ respectively, we can join the cycles C_u, C_d and the edge ue^1_{n-1} into a single Hamiltonian cycle C of $G(P)$ (see Fig. 4, where $e = e_2$).

We now come to the remaining special cases.

First case: Let $P = u^k du^{n-1-k}$, $k \in [2, n-2]$. In this case, $I(P)$ is partitioned as

$$I(P) = [ud^{n-3}ud, P] \cup I(du^{n-3}du) \cup ue^1_{n-1} \cup \{du^{n-1}, du^{n-2}d\},$$

Thus, working as in the general case, where C_d is now a Hamiltonian cycle of $G(du^{n-3}du)$, we obtain a Hamiltonian cycle C of $G(I(P) \setminus \{du^{n-1}, du^{n-2}d\})$. Then, since the edge $\{udu^{n-2}, udu^{n-3}d\}$ belongs to C and is parallel to the edge $\{du^{n-1}, du^{n-2}d\}$, according

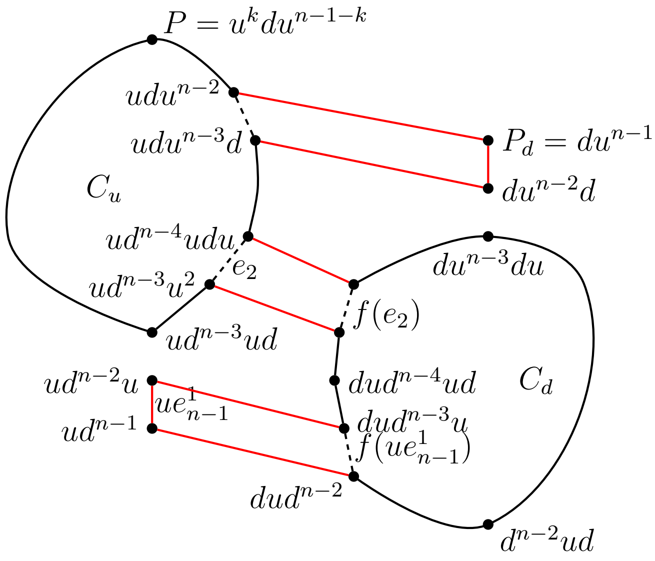


Figure 5: The construction of a Hamiltonian cycle of $G(P)$ when $P = u^k du^{n-1-k}$, $k \in [2, n-2]$.

to the basic construction 1, we can join $\{du^{n-1}, du^{n-2}d\}$ with C to obtain a Hamiltonian cycle of $G(P)$ (see Fig. 5).

Second case: Let $P = ud^k u^{n-1-k}$, $k \in [2, n-2]$. For $k \leqslant n-4$, $I(P)$ is partitioned as

$$I(P) = [ud^{n-3}ud, s^2(P)] \cup I(P_d) \cup ue^1_{n-1} \cup \{P, s(P)\},$$

where $s(P) = ud^k u^{n-2-k}d$ and $s^2(P) = ud^k u^{n-3-k}du$. Thus, working as in the general case, where C_u is now a Hamiltonian cycle of $G([ud^{n-2}ud, s^2(P)])$, we obtain a Hamiltonian cycle C of $G(I(P) \setminus \{P, s(P)\})$. Then, since the edge $f(\{P, s(P)\}) = \{dud^{k-1}u^{n-1-k}, dud^{k-1}u^{n-2-k}d\}$ belongs to C and is parallel to the edge $\{P, s(P)\}$, we can join $\{P, s(P)\}$ with C to obtain a Hamiltonian cycle of $G(P)$ (see Fig. 6).

For $k = n-3$ (resp. $k = n-2$), i.e., $P = ud^{n-3}u^2$ (resp. $P = ud^{n-2}u$), we have that $I(P) = I(P_d) \cup ue^1_{n-1} \cup \{P, s(P)\}$ (resp. $I(P_d) \cup ue^1_{n-1}$), and the required Hamiltonian cycle of $G(P)$ follows by joining the cycle C_d with the edges $ue^1_{n-1}, \{P, s(P)\}$ (resp. the edge $\{P, s(P)\} = ue^1_{n-1}$).

Third case: Let $P = udu^{n-2}$. In this case, $I(P)$ is partitioned as $I(P) = I(s^2(P)) \cup \{P, s(P)\} \cup \{f(P), f(s(P))\}$. Then, following the proof of the general case for the path $s^2(P) = udu^{n-4}du$, we can join the cycles C_u and C_d using the parallel edges $e = \{s^2(P), s^3(P)\}$ and $f(e)$, in order to obtain a Hamiltonian cycle $C_{s^2(P)}$ of $G(s^2(P))$, which contains the edge $\{s^2(P), f(s^2(P))\}$. (In Fig. 7, this cycle consists of all red and all non-dotted black edges.) Then, by replacing this edge in $C_{s^2(P)}$ by the path of vertices $(s^2(P), s(P), P, f(P), f(s(P)), f(s^2(P)))$, we obtain the required cycle of $G(P)$ (see Fig. 7). $\square$

We note that, for some paths, the desired Hamiltonian cycle is asked to satisfy certain conditions, as in the following result, which will be used in the proof of Proposition 9.

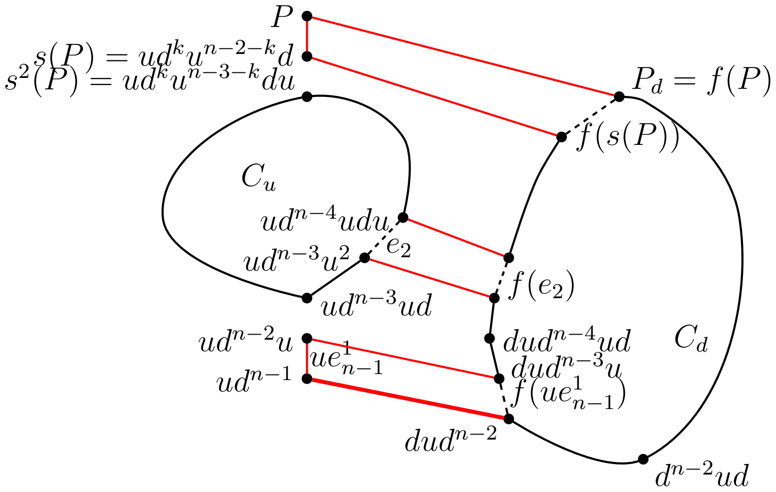


Figure 6: The construction of a Hamiltonian cycle of $G(P)$ when $P = ud^k u^{n-1-k}$, $k \in [2, n-4]$.

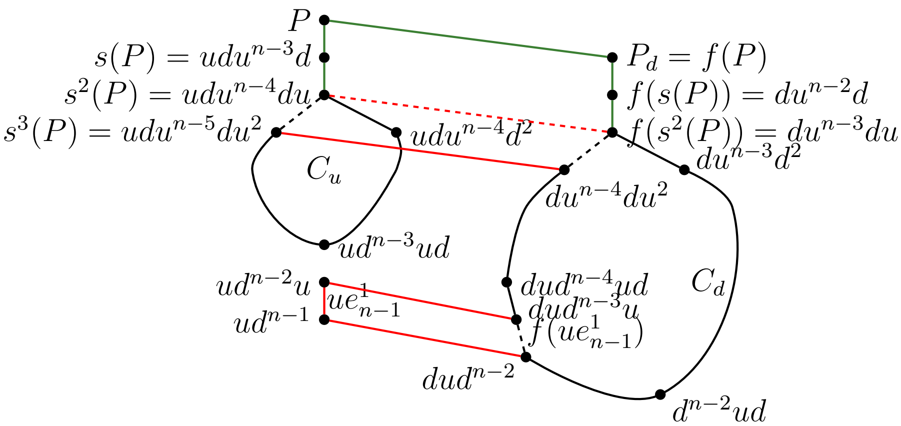


Figure 7: The construction of a Hamiltonian cycle of $G(P)$ when $P = udu^{n-2}$, $n \geqslant 5$.

Proposition 3. *For the path $P_\lambda = ud^\lambda u^r du \in \mathcal{P}_n$, where $\lambda, r \geqslant 0$, there exists a Hamiltonian cycle C_{P_λ} of $G(P_\lambda)$ which contains the path of vertices $(P_\lambda, f(P_\lambda), f^2(P_\lambda), \ldots, f^\lambda(P_\lambda))$ and the edges $ud^\lambda ue^1_{n-2-\lambda}$, for $r \geqslant 1$, $ud^\lambda ue^2_{n-2-\lambda}$, for $r \geqslant 2$ and $ud^\lambda ue^3_{n-2-\lambda}$, for $r \geqslant 3$.*

Proof. We use induction with respect to λ.

Let $\lambda = 0$, i.e., $P_0 = u^{r+1}du$. First, notice that since the vertex u^2d^{r+1} (resp. u^3d^2) has degree 2, for $r \geqslant 1$ (resp. $r = 2$), the edge $u^2e^1_{n-2} = \{u^2d^{r+1}, u^2d^r u\}$ (resp. $u^2e^2_{n-2} = \{u^2dud, u^3d^2\}$) belongs to any cycle of $G(P_0)$. For $r \geqslant 3$, starting from any Hamiltonian cycle C_1 of $G(u^{r-1}du)$ and applying the method used in the proof of Proposition 2 (first special case) for $C_u = uC_1$, we construct a Hamiltonian cycle C_2 of $G(u^r du)$ which contains all the edges of uC_1 except the deleted edges $\{udu^r, udu^{r-1}d\}$ and $\{ud^{r-2}udu, ud^{r-1}u^2\}$ (see Fig. 5 for $k = 1$ and $n = r + 2$). In particular, the edges ue^2_{n-1} and ue^3_{n-1} belong to C_2. Then, by applying once more the same method for $C_u = uC_2$, we obtain a cycle C_{P_0} of

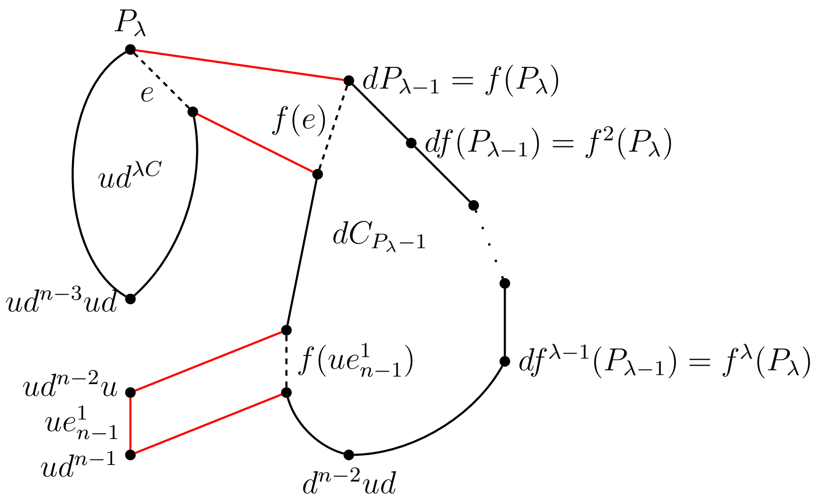


Figure 8: The construction of the cycle C_{P_λ} for $\lambda, r \geqslant 1$.

$G(P_0)$ which contains all the edges of uC_2 except $\{udu^{r+1}, udu^r d\}$ and $\{ud^{r-1}udu, ud^r u^2\}$. It follows that the edges $u^2e^2_{n-2}$ and $u^2e^3_{n-2}$ belong to C_{P_0}.

For $\lambda \geqslant 1$, using the induction hypothesis, there exists a Hamiltonian cycle $C_{P_{\lambda-1}}$ of $G(P_{\lambda-1})$ which contains the path of vertices $(P_{\lambda-1}, f(P_{\lambda-1}), \ldots, f^{\lambda-1}(P_{\lambda-1}))$. For $r = 0$, i.e., $P_\lambda = ud^{\lambda+1}u$, the cycle C_{P_λ} is obtained by joining the cycle $dC_{P_{\lambda-1}}$ with the edge ue^1_{n-1}. For $r \geqslant 1$, consider any Hamiltonian cycle C of $G(u^r du)$. As in the case where $\lambda = 0$, C contains the edge $ue^1_{n-2-\lambda}$ and, for $r \geqslant 2$, the edge $ue^2_{n-2-\lambda}$. Furthermore, for $r \geqslant 3$, by taking $C = C_2$ (of case $\lambda = 0$), we may also assume that C contains the edges $ue^2_{n-2-\lambda}$ and $ue^3_{n-2-\lambda}$. Since $I(P_\lambda)$ is partitioned as

$$I(P_\lambda) = [ud^{n-3}ud, P_\lambda] \cup I(dP_\lambda) \cup ue^1_{n-1},$$

it is enough to join the cycles $ud^\lambda C$, $dC_{P_{\lambda-1}}$ and the edge ue^1_{n-1} to a Hamiltonian cycle C_{P_λ} of $G(P_\lambda)$. This is done in the same way as in the proof of the general case in Proposition 2. The only difference is that here we use P_λ instead of $ud^{n-3}u^2$ and e is either $\{P_\lambda, ud^\lambda u^{r-1}du^2\}$ or $\{P_\lambda, ud^\lambda u^r d^2\}$ (see Fig. 8).

It is easy to check that the path of vertices

$$(P_\lambda, f(P_\lambda), f^2(P_\lambda), \ldots, f^\lambda(P_\lambda)) = (P_\lambda, dP_{\lambda-1}, df(P_{\lambda-1}), \ldots, df^{\lambda-1}(P_{\lambda-1}))$$

belongs to the cycle C_{P_λ}. Furthermore, since the cycle C_{P_λ} contains all the edges of the cycle $ud^\lambda C$, except the deleted edge e, it will contain the edges $ud^\lambda ue^1_{n-2-\lambda}$, for $r \geqslant 1$, $ud^\lambda ue^2_{n-2-\lambda}$, for $r \geqslant 2$ and $ud^\lambda ue^3_{n-2-\lambda}$, for $r \geqslant 3$. □

3 Counting the difference of even and odd ranks

In this section, we give a sufficient and necessary condition for a path P, in order to have $|E(P)| = |O(P)|$, which will be used in the next section, for the proof of Theorem 1 for paths ending with d. For this, we consider the difference mapping Δ on $\mathcal{P}$ defined by

$$\Delta(P) = |E(P)| - |O(P)| = \sum_{S \leqslant P} (-1)^{[\rho(S) \text{ is odd}]},$$

and we study its properties. We note that $\Delta(Pu) = 0$, $\Delta(dP) = \Delta(P)$ and $\Delta(d^n) = 1$, $n \geqslant 0$.

We first give some definitions and notation. For any set of paths A, we define

$$E(A) = \{Q \in A : \rho(Q) \text{ is even}\},\ O(A) = \{Q \in A : \rho(Q) \text{ is odd}\}$$

$$\text{and } \Delta(A) = |E(A)| - |O(A)|.$$

In particular, we have

$$E(P) = E([d^{|P|}, P]), \quad O(P) = O([d^{|P|}, P]) \quad \text{and} \quad \Delta(P) = \Delta(I(P)).$$

Moreover, we define

$$I_0(P) = \{Q \leqslant P : |Q|_u = |P|_u\} \quad \text{and} \quad \Delta_0(P) = \Delta(I_0(P)).$$

Finally, for a path $P \neq d^n$, we set $\theta(P)$ to be the path obtained by deleting the rightmost u of P. We note that $\theta(Pd) = \theta(P)d$ and $I_0(\theta(P)u)$ is the set of paths in $I_0(P)$ ending with u.

In the following result we summarize the properties of Δ_0 and Δ, as well as the connection between them.

Lemma 4. *For any path $P \in \mathcal{P}$, we have that:*

i) $\Delta(P) = \Delta_0(P) - \Delta_0(\theta(P)u)$, *for* $P \neq d^{|P|}$.

ii) $\Delta_0(Pu) = (-1)^{|P|_u+1}\Delta_0(P)$.

iii) $\Delta_0(Pd) = (-1)^{|P|_u} (\Delta_0(P) + \Delta_0(\theta(P)d))$, *for* $P \neq d^{|P|}$.

iv) $\Delta(Pd) = (-1)^{|P|_u}\Delta_0(P)$.

v) $\Delta(Pdd) = (-1)^{|P|_u}(\Delta(Pd) - \Delta(\theta(P)dd))$, *for* $P \neq d^{|P|}$.

vi) $\Delta(Pud^i) = (-1)^{|P|_u+1} (\Delta(Pud^{i-1}) - \Delta(Pd^i))$, *for* $i \geqslant 1$.

vii) $\Delta(Pud^i) = \Delta(Pud^{i-2}) - \Delta(\theta(P)d^i)$, *for* $i \geqslant 2$ *and* $P \neq d^{|P|}$.

viii) $(-1)^{\binom{|P|_u}{2}}\Delta(P) \geqslant 0$.

Proof. *i*) The restriction of the switch involution on the set $[d^{|P|}, P] \setminus \{S \in I_0(P) : S \text{ ends with } d\}$ is still an involution, so that

$$\begin{aligned}\Delta(P) &= \Delta(\{S \in I_0(P) : S \text{ ends with } d\}) \\ &= \Delta(I_0(P)) - \Delta(\{S \in I_0(P) : S \text{ ends with } u\}) \\ &= \Delta_0(P) - \Delta_0(\theta(P)u).\end{aligned}$$

ii) For any path $Q \in I_0(P)$ we have that $\rho(Qu) = \rho(Q) + |Q|_u + 1 = \rho(Q) + |P|_u + 1$, and since $I_0(Pu) = I_0(P)u$, it follows that

$$\Delta_0(Pu) = \Delta(I_0(P)u) = \begin{cases} \Delta_0(P), & |P|_u + 1 \text{ is even} \\ -\Delta_0(P), & |P|_u + 1 \text{ is odd} \end{cases} = (-1)^{|P|_u+1}\Delta_0(P).$$

iii) Note that the set $I_0(Pd)$ is the disjoint union of the sets $I_0(P)d$ and $I_0(\theta(Pd)u)$. Using relation *ii*) and the fact that the switch involution defines a bijection between $I_0(P)d$ and $I_0(P)u$ that reverses the parity of the rank, we have that

$$\begin{aligned}\Delta_0(Pd) &= \Delta(I_0(P)d) + \Delta(I_0(\theta(Pd)u)) = -\Delta(I_0(P)u) + \Delta(I_0(\theta(Pd)u)) \\ &= -\Delta_0(Pu) + (-1)^{|\theta(Pd)|_u+1}\Delta_0(\theta(Pd)) \\ &= (-1)^{|P|_u}\Delta_0(P) + (-1)^{|P|_u}\Delta_0(\theta(P)d)\end{aligned}$$

iv) When $P = d^{|P|}$, the result holds trivially. Otherwise, using *i*), *ii*) and *iii*), we have that

$$\begin{aligned}\Delta(Pd) &= \Delta_0(Pd) - \Delta_0(\theta(Pd)u) \\ &= (-1)^{|P|_u}(\Delta_0(P) + \Delta_0(\theta(P)d)) - (-1)^{|P|_u}\Delta_0(\theta(Pd)),\end{aligned}$$

and the result follows, since $\theta(P)d = \theta(Pd)$.

v) Using relations *iii*) and *iv*), we have that

$$\begin{aligned}\Delta(Pdd) &= (-1)^{|P|_u}\Delta_0(Pd) = \Delta_0(P) + \Delta_0(\theta(P)d) \\ &= (-1)^{|P|_u}(\Delta(Pd) - \Delta(\theta(P)dd))\end{aligned}$$

vi) Usings relations *ii*) and *iv*), we obtain that

$$\Delta(Pud) = (-1)^{|P|_u+1}\Delta_0(Pu) = \Delta_0(P) = (-1)^{|P|_u}\Delta(Pd),$$

so that the required equality holds for $i = 1$, since $\Delta(Pu) = 0$. Moreover, for $i \geqslant 2$, the result follows immediately by applying *v*) for the path Pud^{i-2}.

vii) By applying twice relation *vi*) and then relation *v*) for the path Pd^{i-2}, we obtain that

$$\begin{aligned}\Delta(Pud^i) &= (-1)^{|P|_u+1}\left(\Delta(Pud^{i-1}) - \Delta(Pd^i)\right) \\ &= \Delta(Pud^{i-2}) - \Delta(Pd^{i-1}) + \Delta(Pd^{i-1}) - \Delta(\theta(Pd^{i-2})dd) \\ &= \Delta(Pud^{i-2}) - \Delta(\theta(P)d^i).\end{aligned}$$

viii) The result follows easily by induction on the length and using *vi*) and *vii*). ☐

In the sequel, we give some conditions in order to have $\Delta(P) = 0$, for a path P. Clearly, since $\Delta(Pu) = 0$ and $\Delta(dP) = \Delta(P)$, it is enough to restrict ourselves to paths starting with u and ending with d. We start with some necessary conditions in the next result.

Proposition 5. *Let P be a path in $\mathcal{P}$ that starts with u, ends with d and satisfies $\Delta(P) = 0$. Then,*

i) *$|P|_d$ is even.*

ii) *If $|P|$ is even, then P ends with ud.*

Proof. *i)* Assume that the result is false and let P be a path with minimum length such that P starts with u, ends with d, $\Delta(P) = 0$ and $|P|_d$ is odd. We consider the following two cases:

First case: P ends with ud, i.e., $P = Qud$, $Q \in \mathcal{P}$. Clearly, since $\Delta(ud) = 1$, we have that $Q \neq \varepsilon$, so that Q starts with u. Then, from *vi)* of Lemma 4, we have that

$$0 = \Delta(Qud) = (-1)^{|Q|_u+1}(\Delta(Qu) - \Delta(Qd)) = (-1)^{|Q|_u}\Delta(Qd),$$

which is a contradiction, since $|Qd| < |P|$ and $|Qd|_d$ is odd.

Second case: P ends with dd, i.e., $P = Qud^i$, $Q \in \mathcal{P}$, $i \geqslant 2$. Clearly, since $|ud^i|_d = i$ and $\Delta(ud^i) = [i \text{ is odd}]$, we obtain that $Q \neq \varepsilon$, so that Q starts with u. Then, from relation *vii)* of Lemma 4, we have that

$$0 = \Delta(Qud^i) = \Delta(Qud^{i-2}) - \Delta(\theta(Q)d^i).$$

Using the above equality and since $|Qud^{i-2}|_u = |P|_u$ and $|\theta(Q)d^i|_u = |P|_u - 2$, from *viii)* of Lemma 4, we obtain that $(-1)^{\binom{|P|_u}{2}+\binom{|P|_u-2}{2}}\Delta^2(\theta(Q)d^i) \geqslant 0$. Since the exponent of -1 is odd, we deduce that $\Delta(\theta(Q)d^i) = 0$, which also implies that $|\theta(Q)|_u > 0$. Thus, $\theta(Q)d^i$ is a path of length less than $|P|$ that starts with u, ends with d^2 and satisfies $\Delta(\theta(Q)d^i) = 0$ and $|\theta(Q)d^i)|_d$ is odd, which is a contradiction.

ii) The proof is similar to the above second case and it is omitted. □

Our aim is to give a sufficient and necessary condition in order to have $\Delta(P) = 0$ for a path P. As we will see, this condition is based on the valleys of the path. For this, we need some more definitions.

A path P is decomposed with respect to a certain valley as $P = LduR$, or $P = Ld$ if this valley is the last step of P, where $L, R \in \mathcal{P}$. This valley is called *odd* whenever $|L|$ is even. Moreover, if both $|L|_u$ and $|L|_d$ are even, then it is called *critical*. For example, the path $P = uududdduud$ has three valleys (colored red), the first (leftmost) is critical, the second is not odd and the third is odd but not critical, whereas the path $Q = uduudduudd$ has no critical valleys.

Similarly, if $P = LudR$ or $P = Lu$, the peak following L is called *odd* whenever $|L|$ is even. In the following, we will refer to every occurrence of either a peak or a valley in a path P as a peak/valley of P. We note that a path has no odd peak/valley iff it has no ascent or descent of odd length. This implies that if the leftmost odd peak/valley of a path is a valley, then it is critical. On the other hand, if the leftmost odd peak/valley is a peak, it is possible that the path has no critical valleys. (An example is the path $P = uuddudud$, where the leftmost odd peak/valley is colored red.)

Next, we introduce an involution ϕ on $\mathcal{P}$, which is similar to the one presented by Ruskey (see [12], p.133). (Ruskey defines ϕ on the interval $I_0(u^k d^{n-k})$ in order to prove a sufficient and necessary condition for a Hamiltonian path to exist in the subgraph induced by this interval.)

For every path P we define $\phi(P)$ to be the path obtained from P by turning the leftmost odd peak/valley (a peak becomes valley and vice versa). It is straightforward to see that ϕ is an involution on $\mathcal{P}$ that changes the rank of paths with an odd peak/valley by one. On the other hand, it fixes the paths with no odd peak/valley, i.e., paths with no ascent or descent of odd length. Obviously, these paths always have even length. Moreover, the parity of their rank depends on the number of u's. In particular,

$$\phi(P) = P \Rightarrow \rho(P) \equiv \rho(d^{|P|_d} u^{|P|_u}) = \frac{|P|_u(|P|_u + 1)}{2} \pmod 2. \tag{3}$$

This can be easily verified using the fact that each u in P has an even number of d's on its right.

The use of ϕ simplifies the evaluation of $\Delta(P)$. Indeed, every pair of distinct paths S, T in $[d^{|P|}, P]$, such that $\phi(S) = T$ have ranks of different parity, thus giving zero contribution to the value of $\Delta(P)$. Hence,

$$\Delta(P) = \Delta(U_P), \qquad \text{where } U_P = \{S \leqslant P : \phi(S) = S \text{ or } \phi(S) \nleqslant P\}. \tag{4}$$

In the next result, we give the necessary conditions in order to have $\Delta(P) = 0$, for a path P with no critical valleys.

Proposition 6. *Let $P \in \mathcal{P}$ be a path starting with u, ending with d and having no critical valleys. Then, $\Delta(P) = 0$, if either $|P|$ is odd, or $|P|, |P|_d$ are even and P ends with ud.*

Proof. Firstly, assume $|P|$ is odd. Then, there exists no path $S \leqslant P$ such that $\phi(S) = S$ (any such path must have even length). Moreover, since P has no critical valleys, there exists no path $S \leqslant P$ such that $\phi(S) \nleqslant P$. Indeed, any such path must have a common valley with P, which is the leftmost odd peak/valley of S, hence a critical valley of S. Then, it follows easily that this valley must also be a critical valley of P, which is a contradiction. From the above, it follows that U_p is empty, so that $\Delta(P) = \Delta(U_P) = 0$.

Secondly, assume $|P|$, $|P|_d$ are even and P ends with ud, i.e., $P = Qud$. Then $|Q|_u$ is odd and Lemma 4vi) gives $\Delta(Qud) = (-1)^{|Q|_u}\Delta(Qd) = 0$, since Qd has odd length and no critical valleys. ☐

We now come to the main result of this section.

Theorem 7. *A path $P \in \mathcal{P}$ starting with u and ending with d has $|E(P)| = |O(P)|$ (i.e., $\Delta(P) = 0$) iff P has no critical valleys and i) $|P|$ is odd, or ii) $|P|, |P|_d$ are even and P ends with ud.*

Proof. In view of Propositions 5 and 6, it is enough to show that if P has a critical valley, then $\Delta(P) \neq 0$. For the rest of the proof, we assume that $|P|_d$ is even and that if $|P|$ is even then P ends with ud, otherwise the result follows from Proposition 5. It follows that the last step of P is not a critical valley. Let $P = LduR$, where $L, R \in \mathcal{P}$, be the decomposition of P with respect to its leftmost critical valley and let $Q = LudR$.

We first show that

$$\Delta(P) = \Delta(Q) + (-1)^{|P|+1+|L|_u/2}|\Phi_0(L)|\Delta(R), \tag{5}$$

where $\Phi_0(L) = \{T \leqslant L : \phi(T) = T, |T|_u = |L|_u\}$. Note that $|\Phi_0(L)| > 0$, since $d^{|L|_d}u^{|L|_u} \in \Phi_0(L)$. In order to prove equality (5), we first note that $U_Q \subseteq U_P$ and every path $S \in U_P \setminus U_Q$ satisfies $S \leqslant P$, $\phi(S) \leqslant Q$ and $\phi(S) \nleqslant P$, therefore the leftmost odd peak/valley of S coincides with the leftmost critical valley of P, i.e., $S = TduZ$, where $T \in \Phi_0(L)$, $Z \leqslant R$. Using relations (2) and (3), it follows that

$$\begin{aligned}
\rho(S) &= \rho(T) + \rho(duZ) + |duZ||T|_u = \rho(T) + \rho(Z) + |Z| + 1 + (|Z|+2)|T|_u \\
&\equiv \frac{|T|_u(|T|_u+1)}{2} + \rho(Z) + |Z| + 1 + (|Z|+2)|T|_u \equiv \rho(Z) + |Z| + 1 + |T|_u/2 \\
&\equiv \rho(Z) + |P| + 1 + |L|_u/2 \pmod 2.
\end{aligned}$$

Then, using equation (4), we have that

$$\begin{aligned}
\Delta(P) - \Delta(Q) = \Delta(U_P \setminus U_Q) &= \sum_{S \in U_P \setminus U_Q} (-1)^{[\rho(S) \text{ is odd}]} \\
&= \sum_{T \in \Phi_0(L)} \sum_{Z \leqslant R} (-1)^{[\rho(Z)+|P|+1+|L|_u/2 \text{ is odd}]} \\
&= (-1)^{|P|+1+|L|_u/2}|\Phi_0(L)| \sum_{Z \leqslant R} (-1)^{[\rho(Z) \text{ is odd}]} \\
&= (-1)^{|P|+1+|L|_u/2}|\Phi_0(L)|\Delta(R),
\end{aligned}$$

so that equality (5) holds.

Since $|P|_d$ is even, it is easy to see that the two summands of the right-hand side of equation (5) never have opposite signs. Indeed, using Lemma 4 viii), we have that

$$\begin{aligned}
&(-1)^{|P|+1+|L|_u/2}\Delta(Q)\Delta(R) \geqslant 0 \\
&\Leftrightarrow \frac{(|R|_u + |L|_u + 1)(|R|_u + |L|_u) + |R|_u(|R|_u - 1) + |L|_u}{2} + |P| + 1 \equiv 0 \pmod 2 \\
&\Leftrightarrow |R|_u + |P| + 1 \equiv 0 \pmod 2 \\
&\Leftrightarrow |P|_d \equiv 0 \pmod 2.
\end{aligned}$$

Therefore, it is enough to prove that always one of the summands is non-zero.

Assume the opposite, i.e, $\Delta(P) = \Delta(Q) = \Delta(R) = 0$, and take P to have the minimum positive number of critical valleys. Clearly, P has exactly one critical valley, otherwise

Q would contradict the minimality of P. Since, $\Delta(d^{|R|}) = 1$, the path R has at least one u, therefore it is decomposed as $R = d^i R'$, $i \geqslant 0$, where R' starts with u. According to Proposition 5i), relation $\Delta(R') = \Delta(R) = 0$ implies that $|R'|_d$ is even. Furthermore, since $|P|_d$ and $|L|_d$ are even, it follows that i is odd. Then, we can easily check that the path uR' has no critical valleys, so that, by Proposition 6, we have that $\Delta(uR') = 0$. By partitioning the interval $[d^{|R'|+1}, uR']$ into $[ud^{|R'|}, uR']$ and $[d^{|R'|+1}, dT]$, where T is the path obtained from uR' by deleting its first d, we have that

$$0 = \Delta(uR') = \Delta([ud^{|R'|}, uR']) + \Delta(dT) = (-1)^{|R'|+1}\Delta(R') + \Delta(T) = \Delta(T).$$

This contradicts Proposition 5i), since $|T|_d$ is odd. □

The above result will be used repeatedly in the following section.

4 The Hamiltonicity for paths ending with d

In this section, we show Theorem 1 for paths ending with d. It is enough to consider paths starting with u. For this, we denote by $\mathcal{K}_n$, $n \geqslant 4$, the set of paths $P \in \mathcal{P}_n$ that start with u and end with d, have at least two peaks and satisfy $|E(P)| = |O(P)|$, and we set $\mathcal{K} = \bigcup_{n \geqslant 4} \mathcal{K}_n$. In Proposition 9, we show that the graph $G(P)$ is Hamiltonian for every path $P \in \mathcal{K}$, completing the proof of Theorem 1 for paths ending with d. Any path $P \in \mathcal{K}$ is decomposed as

$$P = u^{k_1} d^{\lambda_1} u^{k_2} d^{\lambda_2} \cdots u^{k_\rho} d^{\lambda_\rho}, \qquad \text{where } \rho \geqslant 2 \text{ and } k_i, \lambda_i \geqslant 1, \text{ for all } i \in [\rho].$$

We first give the following Lemma, which is an immediate consequence of Theorem 7 and will be used repeatedly and without further reference in the proof of Proposition 9.

Lemma 8. *Let $P = u^{k_1} d^{\lambda_1} u^{k_2} d^{\lambda_2} \cdots u^{k_\rho} d^{\lambda_\rho} \in \mathcal{P}$ and define $P_3 = u^{k_3} d^{\lambda_3} \cdots u^{k_\rho} d^{\lambda_\rho}$ whenever $\rho \geqslant 3$. If $P \in \mathcal{K}$, then the following paths also belong to $\mathcal{K}$:*

1. *$f(P)$, whenever k_1 is even.*
2. *$s(P)$, whenever k_1, k_2 are odd and $\rho \geqslant 3$.*
3. *$f^2(P)$, whenever k_1 is odd and $k_1 \geqslant 3$.*
4. *$s^2(P)$, whenever k_1 is even and $k_2 \geqslant \rho = 2$, or k_1 odd, k_2 even and either $\rho \geqslant 3$ or $\lambda_2 \geqslant \rho = 2$.*
5. *$u^{k_2} d^{\lambda_2} P_3$, whenever k_1 is even and $\rho \geqslant 3$.*
6. *$u d^{\lambda_1} u^{k_2} d^{\lambda_2} \cdots u^{k_\rho} d^{\lambda_\rho}$, whenever k_1 is odd and $k_1 \geqslant 3$.*
7. *$u^{k_1} d^{\lambda_2} P_3$, whenever k_2, λ_1 are even and $\rho \geqslant 3$.*
8. *$u^{k_1} d^{\lambda_2 - 1} P_3$, whenever k_1, λ_1 are odd, k_2 is even and either $\rho \geqslant 3$ and $\lambda_2 \geqslant 2$, or $\rho \geqslant 4$ and $\lambda_2 = 1$.*

9. P_3, *whenever* k_1, k_2 *are odd and* $\rho \geqslant 4$.

10. $u^{k_1}d^{\lambda_1}u^{k_2}d^{\lambda_2}f^2(P_3)$, *whenever* k_1, k_2 *are odd,* $\rho = 3$ *and* $k_3 \geqslant 2$.

Proposition 9. *For every path* $P = u^{k_1}d^{\lambda_1}u^{k_2}d^{\lambda_2}\cdots u^{k_\rho}d^{\lambda_\rho} \in \mathcal{K}_n$, *there exists a Hamiltonian cycle* C_P *of* $G(P)$. *Moreover, if* k_1 *is odd, then* C_P *satisfies the following properties:*

1. $u^{k_1}d^{\lambda_1}ue^1_{n-k_1-\lambda_1-1}$ *belongs to* C_P, *whenever* $k_2 \geqslant 2$ *or* $k_2 = 1$, λ_1 *is odd and* $|P|_u \geqslant k_1 + 2$.

2. *Either* $u^{k_1}d^{\lambda_1}ue^2_{n-k_1-\lambda_1-1}$ *or* $u^{k_1}d^{\lambda_1}ue^3_{n-k_1-\lambda_1-1}$ *belong to* C_P, *whenever* $|P|_u \geqslant k_1+2$ *and* $n \geqslant k_1 + \lambda_1 + 4$.

3. $e(P)$ *belongs to* C_P.

4. $e(f(P))$ *belongs to* C_P, *whenever* $k_1 = 1$, k_2 *and* λ_1 *are even and* $\rho \geqslant 3$.

5. *The path of vertices* $< P > = u^{k_1}d^{\lambda_1}u^{k_2}d^{\lambda_2}\cdots u^{k_{\rho-1}}d^{\lambda_{\rho-1}}u^{k_\rho-1}(ud^{\lambda_\rho}, f(ud^{\lambda_\rho}), \ldots, f^{\lambda_\rho+1}(ud^{\lambda_\rho}) = d^{\lambda_\rho+1})$ *belongs to* C_P, *whenever* $\rho = 3$, $\lambda_1 = 1$ *and* k_2 *is odd.*

6. $e(u^{k_1}d^{\lambda_1}ud^{n-k_1-\lambda_1-1})$ *belongs to* C_P, *whenever* λ_1 *is even.*

7. $e(u^{k_1}d^{\lambda_1}u^{k_2}dud^{n-k_1-\lambda_1-k_2-2})$ *belongs to* C_P, *whenever* λ_1 *is odd,* k_2 *is even and* $\rho \geqslant 3$.

Proof. We show the result by induction on the length and the ordering of the paths. This is done by considering several cases.

Case A: k_1 even In this case, in view of Theorem 7, λ_1 is also even and $f(P) = u^{k_1-1}dud^{\lambda_1-1}u^{k_2}d^{\lambda_2}\cdots u^{k_\rho}d^{\lambda_\rho} \in \mathcal{K}_n$. Then, by the induction hypothesis, there exists a Hamiltonian cycle $C_{f(P)}$ of $G(f(P))$ satisfying the associated properties. Clearly,

$$I(P) \setminus I(f(P)) = [u^{k_1}d^{n-k_1}, P].$$

We consider three subcases:

(*i*) $\rho \geqslant 3$. Then, the path $S = u^{k_2}d^{\lambda_2}\cdots u^{k_\rho}d^{\lambda_\rho}$ also belongs to $\mathcal{K}$, so that, by the induction hypothesis, there exists a Hamiltonian cycle C_S of $G(S)$. It follows that the cycle $C = u^{k_1}d^{\lambda_1}C_S$ is a Hamiltonian cycle of $G([u^{k_1}d^{n-k_1-2}ud, P])$. The vertex $u^{k_1}d^{n-k_1-2}ud$ has two valleys, therefore it has two neighbors in this graph, so that the edges $u^{k_1}e^2_{n-k_1}$ and $u^{k_1}e^3_{n-k_1}$ belong to C. Moreover, according to properties 1 and 2 for the path $f(P)$, the edge $f(u^{k_1}e^1_{n-k_1}) = u^{k_1-1}due^1_{n-k_1-1}$ and either one of the edges $f(u^{k_1}e^2_{n-k_1}) = u^{k_1-1}due^2_{n-k_1-1}$ or $f(u^{k_1}e^3_{n-k_1}) = u^{k_1-1}due^3_{n-k_1-1}$ belong to $C_{f(P)}$. Then, using the basic constructions 1 and 2 for the pairs of parallel edges $(u^{k_1}e^1_{n-k_1}, f(u^{k_1}e^1_{n-k_1}))$ and $(e, f(e))$, where $e = u^{k_1}e^2_{n-k_1}$ or $e = u^{k_1}e^3_{n-k_1}$, we join the cycles C and $C_{f(P)}$ and the edge $u^{k_1}e^1_{n-k_1}$ into a Hamiltonian cycle C_P of $G(P)$ (see Fig. 9(*i*)).

(*ii*) $\rho = 2$ and $k_2 = 1$, i.e., $P = u^{k_1}d^{\lambda_1}ud^{\lambda_2}$. In this case, the set $I(P) \setminus I(f(P))$ consists entirely of the vertices of

$$<P> = u^{k_1}d^{\lambda_1}(ud^{\lambda_2}, dud^{\lambda_2-1}, \ldots, d^{\lambda_2}u, d^{\lambda_2+1}).$$

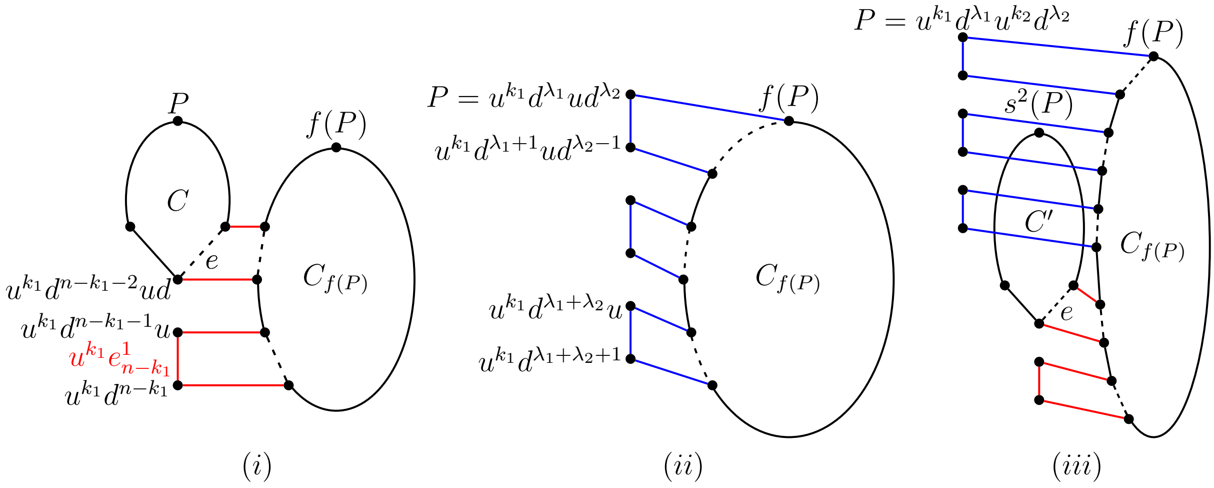


Figure 9: The construction of C_p for k_1 even, in each one of the three subcases $i)$ $\rho \geqslant 3$, $ii)$ $\rho = 2$ and $k_2 = 1$, $iii)$ $\rho = 2$ and $k_2 \geqslant 2$.

On the other hand, according to property 5 for the path $f(P)$, the path of vertices

$$< f(P) > = u^{k_1-1}dud^{\lambda_1-1}(ud^{\lambda_2}, dud^{\lambda_2-1}, \dots, d^{\lambda_2}u, d^{\lambda_2+1}).$$

belongs to $C_{f(P)}$. The paths of vertices $< P >$ and $< f(P) >$ are parallel and they both have length $\lambda_2 + 1$, which is odd, since λ_2 must be even. Then, using the basic construction 1, we obtain the required cycle C_P, which consists of the vertices of $C_{f(P)}$ and the vertices of $< P >$ (see Fig. 9(ii)).

(iii) $\rho = 2$ and $k_2 \geqslant 2$, i.e., $P = u^{k_1}d^{\lambda_1}u^{k_2}d^{\lambda_2}$. Clearly, since by Proposition 5 the number $|P|_d = \lambda_1 + \lambda_2$ is even, we have that λ_2 is also even. It follows from Theorem 7 that n is odd, so that k_2 is also odd.

The construction of C_p in this case is a combination of the constructions in the previous two cases. Firstly, by applying the method used in the first case for the path $s^2(P) = u^{k_1}d^{\lambda_1}u^{k_2-2}du^2d^{\lambda_2-1}$, we obtain a Hamiltonian cycle C' of $G([u^{k_1}d^{n-k_1-2}ud, s^2(P)])$ which can be joined with the cycle $C_{f(P)}$ into a Hamiltonian cycle C'' of $G([u^{k_1}d^{n-k_1-2}ud, s^2(P)] \cup I(f(P)))$. Clearly, by property 5 of the cycle $C_{f(P)}$, we deduce that C'' contains the path of vertices $< f(P) >$. Next, since the set $I(P) \setminus ([u^{k_1}d^{n-k_1-2}ud, s^2(P)] \cup I(f(P)))$ consists entirely of the vertices of $< P >$, as in the proof of case (ii), the cycle C'' and the path of vertices $< f(P) >$ can be joined to give the required cycle C_p (see Fig. 9(iii)).

Case B: k_1 odd, $k_1 \geqslant 3$ In this case, $I(P)$ is decomposed as

$$I(P) = u^{k_1-1}I(R) \cup I(f^2(P)) \cup u^{k_1-1}e^1_{n-k_1+1},$$

where $R = ud^{\lambda_1}u^{k_2}d^{\lambda_2} \cdots u^{k_\rho}d^{\lambda_\rho}$ and $f^2(P) = u^{k_1-2}du^2d^{\lambda_1-1}u^{k_2}d^{\lambda_2} \cdots u^{k_\rho}d^{\lambda_\rho}$. Clearly, since $R \in \mathcal{K}_{n-k_1+1}$ and $f^2(P) \in \mathcal{K}_n$, by the induction hypothesis, there exist Hamiltonian cycles C_R of $G(R)$ and $C_{f^2(P)}$ of $G(f^2(P))$ which satisfy the associated properties.

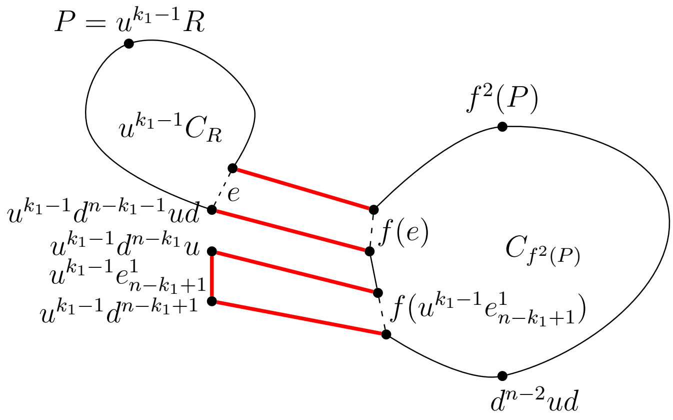


Figure 10: The construction of C_P for k_1 odd, $k_1 \geqslant 3$.

First, note that $f(u^{k_1-1}e^i_{n-k_1+1}) = u^{k_1-2}due^i_{n-k_1}$, for every $i \in [3]$. Then, by property 2 for the path $f^2(P)$, we obtain that either one of the edges $f(u^{k_1-1}e^2_{n-k_1+1})$ or $f(u^{k_1-1}e^3_{n-k_1+1})$ belongs to the cycle $C_{f^2(P)}$. On the other hand, since both $u^{k_1-1}e^2_{n-k_1+1}$ and $u^{k_1-1}e^3_{n-k_1+1}$ belong to the cycle $u^{k_1-1}C_R$, we obtain that, for $e = u^{k_1-1}e^2_{n-k_1+1}$ or $e = u^{k_1-1}e^3_{n-k_1+1}$, the edge e belongs to $u^{k_1-1}C_R$ and $f(e)$ belongs to $C_{f^2(P)}$. Furthermore, by property 1 for the path $f^2(P)$, we obtain that also the edge $f(u^{k_1-1}e^1_{n-k_1+1})$ belongs to the cycle $C_{f^2(P)}$. Then, using the basic constructions 1 and 2, for the pairs $(u^{k_1-1}e^1_{n-k_1+1}, f(u^{k_1-1}e^1_{n-k_1+1}))$ and $(e, f(e))$ respectively, we can join the cycles $u^{k_1-1}C_R$, $C_{f^2(P)}$ and the edge $u^{k_1-1}e^1_{n-k_1+1}$ into a Hamiltonian cycle C_P of $G(P)$ (see Fig. 10). Finally, notice that the cycle C_P contains all the edges of the cycle $u^{k_1-1}C_R$, except the edge e, therefore, from the properties of C_R, we deduce automatically the required properties of C_P.

Case C: $\boldsymbol{k_1 = 1, \rho = 2, \lambda_2 = 1}$ In this case, $P = ud^{\lambda_1}u^{k_2}d$ and we have that

$$I(P) \setminus I(s(P)) = \{f^i(P) : 0 \leqslant i \leqslant \lambda_1\}, \qquad \text{where } s(P) = ud^{\lambda_1}u^{k_2-1}du.$$

From Proposition 3, it follows that there exists a Hamiltonian cycle $C_{s(P)}$ of $G(s(P))$ containing the path of vertices $(s(P), f(s(P)), \ldots, f^{\lambda_1}(s(P)))$, and the edges $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ for $k_2 \geqslant 2$, $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ for $k_2 \geqslant 3$, $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$ for $k_2 \geqslant 4$.

Clearly, since $P \in \mathcal{K}_n$, it follows that $|P|_d$ is even, so that λ_1 is odd. Thus, the parallel paths of vertices $(P, f(P), \ldots, f^{\lambda_1}(P))$ and $(s(P), f(s(P)), \ldots, f^{\lambda_1}(s(P)))$ have odd length. It follows, using the basic construction 1 (see Fig. 2), that the cycle $C_{s(P)}$ can be expanded into a cycle C_P, which contains also the edges $f^i(P)$, $0 \leqslant i \leqslant \lambda_1$. Clearly, the cycle C_P is a Hamiltonian cycle of $G(P)$ satisfying the required properties (i.e., 1, 2 and 3).

Case D: $\boldsymbol{k_1 = 1, k_2}$ even and if $\boldsymbol{\rho = 2}$ then $\boldsymbol{\lambda_2 \geqslant 2}$ In this case, we set $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}Q$, where $Q = \varepsilon$ and $\lambda_2 \geqslant 2$, or $Q = u^{k_3}d^{\lambda_3} \cdots u^{k_\rho}d^{\lambda_\rho}$. Then, since the path

$$s^2(P) = \begin{cases} ud^{\lambda_1+1}u^2d^{\lambda_2-1}Q, & \text{for } k_2 = 2, \\ ud^{\lambda_1}u^{k_2-2}du^2d^{\lambda_2-1}Q, & \text{for } k_2 > 2, \end{cases}$$

belongs to $\mathcal{K}_n$, by the induction hypothesis, there exists a Hamiltonian cycle $C_{s^2(P)}$ of $G(s^2(P))$ which satisfies the associated properties. Furthermore, using the equality

$$I(P) \setminus I(s^2(P)) = \bigcup_{i=0}^{\lambda_1} f^i(ud^{\lambda_1}u^{k_2-1})[d^{|T|}, T],$$

where $T = ud^{\lambda_2}Q$, we will construct a Hamiltonian cycle C of $G(I(P) \setminus I(s^2(P)))$ which is joined suitably with the cycle $C_{s^2(P)}$ in order to construct the desired cycle C_P. This is done by considering several subcases:

D1. $\boldsymbol{P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}}$, $\boldsymbol{k_2}$ even and $\boldsymbol{\lambda_2 \geqslant 2}$ In this case $T = ud^{\lambda_2}$, so that $I(P) \setminus I(s^2(P))$ is a grid of paths with $\lambda_2 + 2$ rows and $\lambda_1 + 1$ columns. More precisely, we have that

$$I(P) \setminus I(s^2(P)) = \{P_{j,i} : i \in [0, \lambda_1], j \in [0, \lambda_2 + 1]\},$$

where

$$P_{j,i} = f^i(ud^{\lambda_1}u^{k_2-1})f^j(ud^{\lambda_2}) = \begin{cases} d^i ud^{\lambda_1-i}u^{k_2-1}d^j ud^{\lambda_2-j}, & j \leqslant \lambda_2, \\ d^i ud^{\lambda_1-i}u^{k_2-1}d^{\lambda_2+1}, & j = \lambda_2 + 1. \end{cases}.$$

Clearly, since $|P|_d$ is even, in this case, we have that λ_1, λ_2 have the same parity, which leads to two further subcases:

i) If λ_1, λ_2 are even, we construct first the $\dfrac{\lambda_2}{2} + 1$ (horizontal) cycles $(P_{j,0}, P_{j,1}, \ldots, P_{j,\lambda_1}, P_{j+1,\lambda_1}, \ldots, P_{j+1,1}, P_{j+1,0}, P_{j,0})$, $j \in [0, \lambda_2 + 1]$, j even and join them into a single cycle C, according to the basic construction 2, using the pairs of parallel edges $(\{P_{j-1,\lambda_1-1}, P_{j-1,\lambda_1}\}, \{P_{j,\lambda_1-1}, P_{j,\lambda_1}\})$, $j \in [2, \lambda_2]$, j even (see Fig. 11 for $\lambda_1 = 4$ and $\lambda_2 = 6$). Next, using the parallel edges $e(s(P)) = \{P_{1,0}, P_{1,1}\}$ of C and $e(s^2(P))$ of $C_{s^2(P)}$, we join the cycles C and $C_{s^2(P)}$ into a Hamiltonian cycle C_P of $G(P)$

Next, we verify that the cycle C_P satisfies the required properties (i.e., 1, 2, 3 and 6). Indeed, the edge $e(P) = \{P_{0,0}, P_{0,1}\}$ belongs to C_P. Furthermore, for $k_2 = 2$, the edges

$$ud^{\lambda_1}ue^1_{n-\lambda_1-2} = \{P_{\lambda_2,0}, P_{\lambda_2+1,0}\}, \qquad ud^{\lambda_1}ue^2_{n-\lambda_1-2} = \{P_{\lambda_2-2,0}, P_{\lambda_2-1,0}\},$$

$$e(ud^{\lambda_1}ud^{n-\lambda_1-2}) = \{P_{\lambda_2+1,0}, P_{\lambda_2+1,1}\}$$

belong to C_P. On the other hand, for $k_2 > 2$, by properties 1, 2 and 6 for $s^2(P)$, it follows that the edges $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$, $e(ud^{\lambda_1}ud^{n-\lambda_1-2})$ and either one of $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ and $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$ belong to $C_{s^2(P)}$ and thus also to C_P.

ii) If λ_1, λ_2 are odd and $\lambda_2 \geqslant 3$, first we construct the $(\lambda_1 + 1)/2$ (vertical) cycles

$$(P_{0,i}, P_{1,i}, \ldots, P_{\lambda_2+1,i}, P_{\lambda_2+1,i+1}, P_{\lambda_2,i+1}, \ldots, P_{0,i+1}, P_{0,i}), \quad i \in [0, \lambda_1],\ i \text{ is even},$$

and, for $\lambda_1 \geqslant 3$, we join them into a cycle C, according to the basic construction 2, using the pairs of parallel edges $(\{P_{0,i-1}, P_{1,i-1}\}, \{P_{0,i}, P_{1,i}\})$, $i \in [2, \lambda_1 - 1]$, i even, whereas, for $\lambda_1 = 1$, $C = (P_{0,0}, P_{1,0}, \ldots, P_{\lambda_2+1,0}, P_{\lambda_2+1,1}, \ldots, P_{0,1}, P_{0,0})$ (see Fig. 12, for $\lambda_2 = 5$ and either $\lambda_1 = 1$ or $\lambda_1 = 5$).

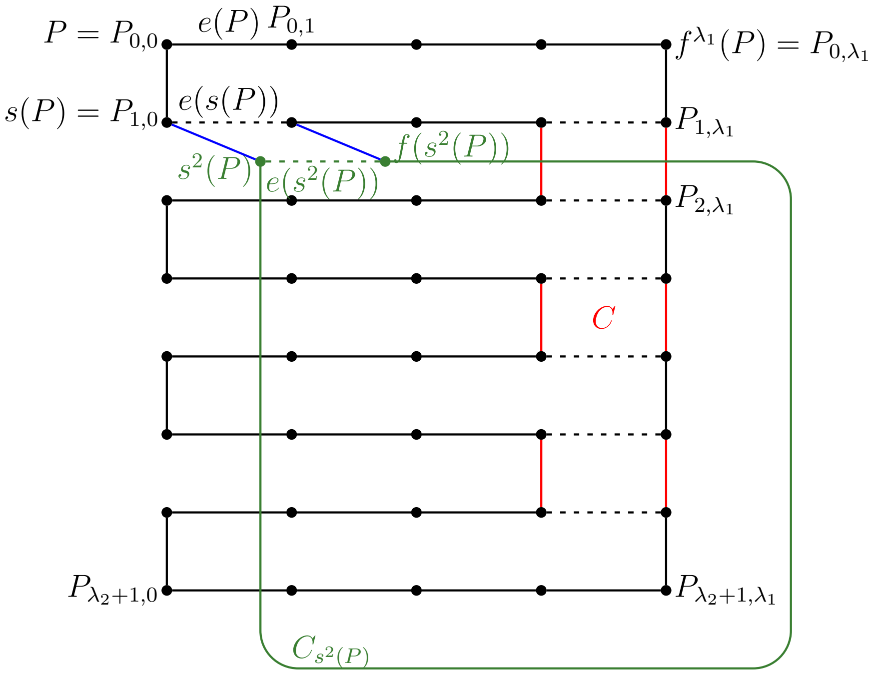


Figure 11: The construction of C_p for $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}$, for k_2, λ_1, λ_2 even.

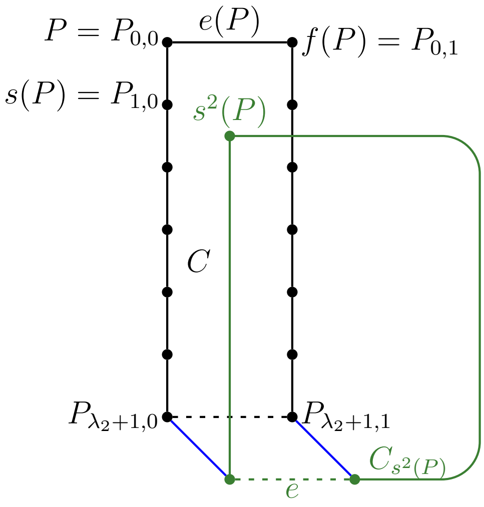
Clearly, by applying property 6 (resp. 7) for the path $s^2(P)$, when $k_2 = 2$ (resp. $k_2 > 2$), we deduce that the edge $e = e(ud^{\lambda_1}u^{k_2-2}dud^{\lambda_2})$ belongs to the cycle $C_{s^2(P)}$. Thus, using this edge and its parallel edge $e(ud^{\lambda_1}u^{k_2-1}d^{\lambda_2+1}) = \{P_{\lambda_2+1,0}, P_{\lambda_2+1,1}\}$ of C, we can join the cycles C and $C_{s^2(P)}$, according to the basic construction 2, into a Hamiltonian cycle C_P of $G(P)$ (see Fig. 12). Clearly, as in the previous subcase, the cycle C_P satisfies the required properties (i.e., 1, 2 and 3).

D2. $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}Q$, where $Q = u^{k_3}d^{\lambda_3}\cdots u^{k_\rho}d^{\lambda_\rho}$, $\rho \geqslant 3$ and k_2 even We consider two subcases:

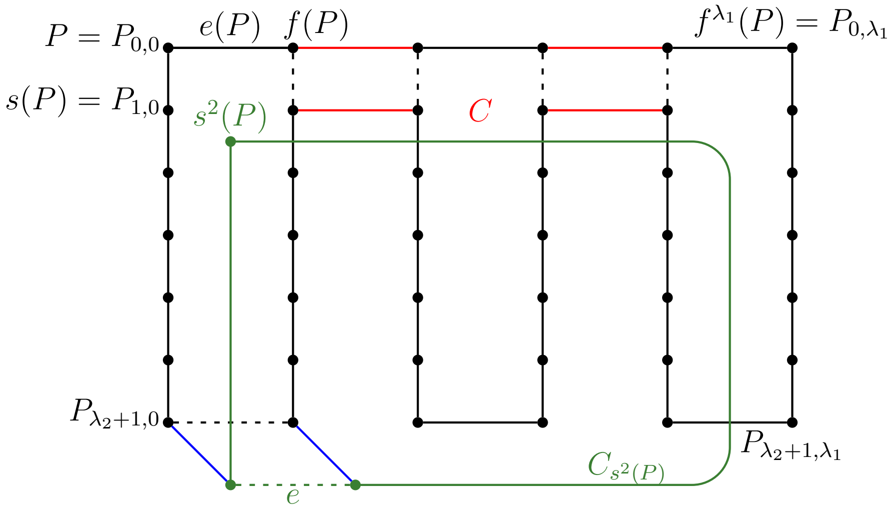
i) λ_1 even. In this case the path $T = ud^{\lambda_2}Q$ belongs to $\mathcal{K}$, so that by the induction hypothesis there exists a Hamiltonian cycle C_T of $G(T)$ satisfying the associated properties. We will construct the Hamiltonian cycle C of $G(I(P) \setminus I(s^2P))$ into two steps.

First, using the pairs (e_i, e_{i+1}), $0 \leqslant i \leqslant \lambda_1 - 2$ and i even, and (g_i, g_{i+1}), $1 \leqslant i \leqslant \lambda_1 - 1$ and i odd, where $e_i = f^i(ud^{\lambda_1}u^{k_2-1})\{T, f(T)\}$ and $g_i = f^i(ud^{\lambda_1}u^{k_2-1})\{T, S\}$, where S is the neighbor of T in C_T, other than $f(T)$, we join according to the basic construction 2 the isomorphic cycles $f^i(ud^{\lambda_1}u^{k_2-1})C_T$, $0 \leqslant i \leqslant \lambda_1$ into a Hamiltonian cycle C' of $G\left(\bigcup_{i=0}^{\lambda_1} f^i(ud^{\lambda_1}u^{k_2-1})I(T)\right)$ (see Fig. 13 for $\lambda_1 = 4$). Clearly, since $s(P) = ud^{\lambda_1}u^{k_2-1}f(T)$, the edge $e(s(P))$ belongs to C'.

Let $e = ud^{\lambda_1}u^{k_2-1}e^i_{|T|}$, where $i = 2$ or 3, such that e belongs to the path in the cycle $ud^{\lambda_1}u^{k_2-1}C_T$ with endpoints the vertices P and $ud^{\lambda_1}u^{k_2-1}d^{|T|-2}ud$ that does not contain $s(P)$, and let g be the edge in this path adjacent to e. Then, by using the reverse procedure of the basic construction 2 for the pair $(g, f(g))$, we can split the cycle C' into



$\lambda_1 = 1$



$\lambda_1 \geqslant 3$

Figure 12: The construction of C_p for $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}$, for k_2 even, λ_1, λ_2 odd, $\lambda_2 \geqslant 3$.

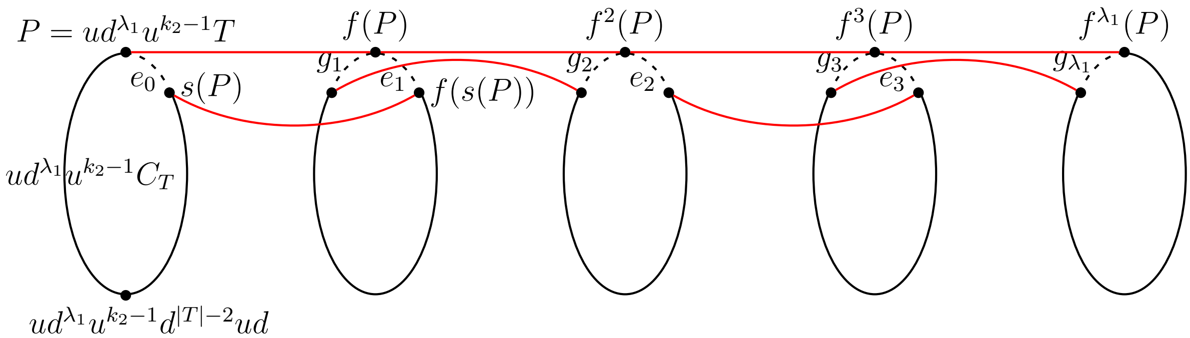


Figure 13: The construction of the cycle C' for $k_1 = 1$, k_2 even, $\rho \geqslant 3$ and λ_1 even ($\lambda_1 = 4$).

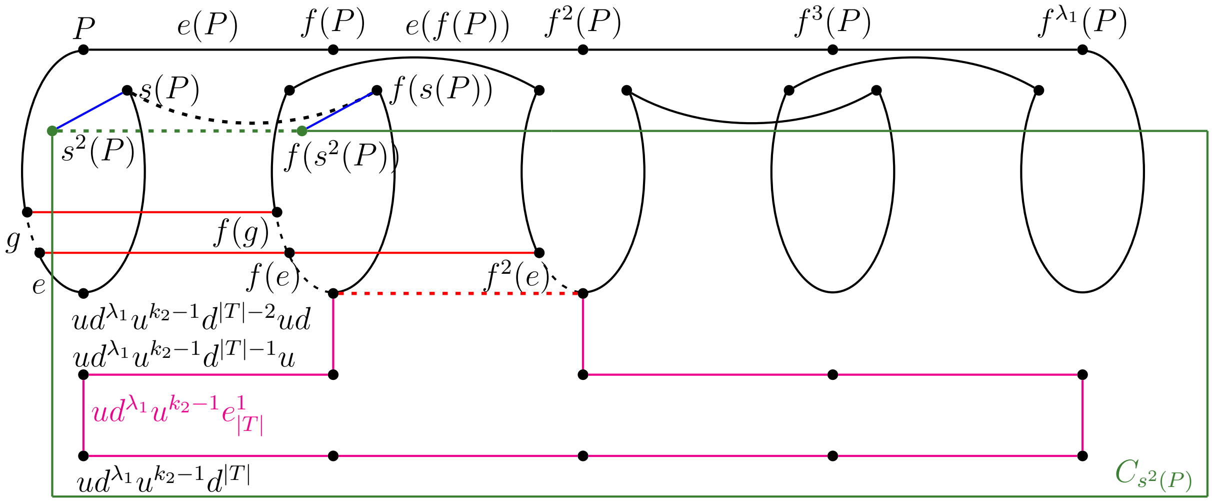


Figure 14: The construction of C_p for $k_1 = 1$, k_2 even, $\rho \geqslant 3$ and λ_1 even ($\lambda_1 = 4$).

two cycles. It is easy to check that one of these cycles contains the edge $f(e)$ and the other contains the edge $f^2(e)$, so that, using the pair $(f(e), f^2(e))$, we can rejoin these two cycles to a Hamiltonian cycle with the same vertices with C' and also containing the edge $e(f(ud^{\lambda_1}u^{k_2-1})d^{|T|-2}du)$. (In Fig. 14, this cycle consists of all red, all non-dashed black edges and the edge $e(s(P))$.) Next, by substituting this edge with the path (colored magenta in Fig. 14) starting at $f(ud^{\lambda_1}u^{k_2-1})d^{|T|-2}du$, ending at $f^2(ud^{\lambda_1}u^{k_2-1})d^{|T|-2}du$ and passing through the vertices $f^i(ud^{\lambda_1}u^{k_2-1})d^{|T|-1}u$, $f^i(ud^{\lambda_1}u^{k_2-1})d^{|T|}$, $0 \leqslant i \leqslant \lambda_1$, we obtain the cycle C. Finally, using the parallel edges $e(s(P))$ of C and $e(s^2(P))$ of $C_{s^2(P)}$, we join the cycles C and $C_{s^2(P)}$ to a Hamiltonian cycle C_P of $G(P)$ (see Fig. 14).

Clearly, by the above construction, it follows that the edges $e(P)$, $e(f(P))$ and, for $k_2 = 2$, the edges $ud^{\lambda_1}ue^i_{|T|}$, $i \in [3]$, and $e(ud^{\lambda_1}ud^{|T|-1}u)$ belong to C. Furthermore, for $k_2 > 2$, the edges $ud^{\lambda_1}ue^1_{|T|}$, $e(ud^{\lambda_1}ud^{|T|-1}u)$ and either one of the edges $ud^{\lambda_1}ue^2_{|T|}$ or $ud^{\lambda_1}ue^3_{|T|}$ belong to $C_{s^2(P)}$, by the induction hypothesis. Thus, the cycle C_P satisfies the required properties (i.e., 1, 2, 3, 4 and 6).

ii) λ_1 odd. We first note that in this case the cycle $C_{s^2(P)}$ contains the edge $e = e(ud^{\lambda_1}u^{k_2-2}dud^{n-\lambda_1-k_2-1})$. Indeed, this follows from property 6, if $k_2 = 2$, and from property 7, if $k_2 > 2$.

We will construct a Hamiltonian cycle C of $G(I(P) \setminus I(s^2(P)))$ which contains the edge $e' = e(ud^{\lambda_1}u^{k_2-1}d^{n-\lambda_1-k_2})$. Then, using the pair (e, e') of parallel edges, we can join the cycles $C_{s^2(P)}$ and C according to the basic construction 2, into a Hamiltonian cycle C_P of $G(P)$. The construction of the cycle C is done by considering two further cases:

1) Assume that either $\lambda_2 = 1$ and $\rho \geqslant 4$, or $\lambda_2 \geqslant 2$. Clearly, we have that

$$[d^{|T|}, T] = [ud^{|T|-1}, T] \cup [d^{|T|}, f(T)], \qquad \text{where } T = ud^{\lambda_2}Q, f(T) = dud^{\lambda_2-1}Q.$$

Then, since in this case the path $ud^{\lambda_2-1}Q$ belongs to $\mathcal{K}$, by the induction hypothesis, there exists a Hamiltonian cycle $C_{f(T)}$ of $G(f(T))$.

Let A be the subset of $[d^{|T|}, f(T)]$ consisting of the paths that start with du, i.e., $A = f([ud^{|T|-1}, T])$. Clearly each vertex of A is adjacent in $C_{f(T)}$ to at least one vertex of A. For every path $S' \in A$, the path $S \in [ud^{|T|-1}, T]$ such that $f(S) = S'$ is called the inverse image (with respect to f) of S'. Firstly, we describe an algorithm for constructing a cycle from two copies of $[d^{|T|}, T]$, using the following steps:

S1. Sketch two copies of the cycle $C_{f(T)}$ and for each one of them spread a copy of the interval $[ud^{|T|-1}, T]$ so that every vertex $S \in [ud^{|T|-1}, T]$ is facing the vertex $f(S)$ of $C_{f(T)}$. Then, connect each copy of the cycle $C_{f(T)}$ with a copy of the path of vertices $(d^{|T|}, d^{|T|-1}u, d^{|T|-2}ud)$. In Fig. 15, the copies of the two cycles are presented as two concentric circles and the elements of $[ud^{|T|-1}, T]$ and A are marked with dots of the same color (black and blue).

S2. For each copy of the cycle, delete the edge $e^j_{|T|}$, where $j = 2$ or 3, lying on the left of vertex $d^{|T|-2}ud$.

S3. Delete alternatively edges with endpoints in A, starting from the clockwise first (resp. second) edge after $e^j_{|T|}$ in the first (resp. second) copy of $C_{f(T)}$. This way, no adjacent edges are deleted and if an edge is deleted in one copy, its corresponding edge in the other copy is not deleted.

S4. For each S_1 in a copy of $[ud^{|T|-1}, T]$, add the edge $\{S_1, S_2\}$ (resp. $\{S_1, f(S_1)\}$) if there exists (resp. does not exist) $S_2 \in [ud^{|T|-1}, T]$ such that $\{f(S_2), f(S_1)\}$ is not a deleted edge.

S5. Connect each vertex of the first copy having degree 1 in the graph obtained after the application of the previous steps, with its corresponding vertex in the second copy.

To see that the resulting graph is a cycle, we can decompose the graph obtained after the application of the first two steps into subgraphs consisting of either two copies of a maximal path of vertices not in A, or two copies of a maximal path of vertices in A and their inverse images. Then, each subgraph is transformed into a cycle, by applying the remaining steps to it. The resulting cycles can then be joined according to the basic construction 2 into a single cycle. This join recovers the edges deleted in the decomposition stage, so that this description is equivalent to the above algorithm and their results are the same.

The resulting cycle is depicted in Fig. 15, where the edges added at steps S4 and S5 are marked by green and red color respectively. Also note that each vertex belonging to $[ud^{|T|-1}, T]$ is adjacent to its corresponding vertex in the other copy. This is an important property that will be used in the sequel.

Next, by applying the above construction for the two copies $f^{i-1}(ud^{\lambda_1}u^{k_2-1})[d^{|T|}, T]$ and $f^{i}(ud^{\lambda_1}u^{k_2-1})[d^{|T|}, T]$, where $i \in [\lambda_1]$, i odd, we obtain a Hamiltonian cycle C_i of $G(f^{i-1}(ud^{\lambda_1}u^{k_2-1})[d^{|T|}, T] \cup f^{i}(ud^{\lambda_1}u^{k_2-1})[d^{|T|}, T])$.

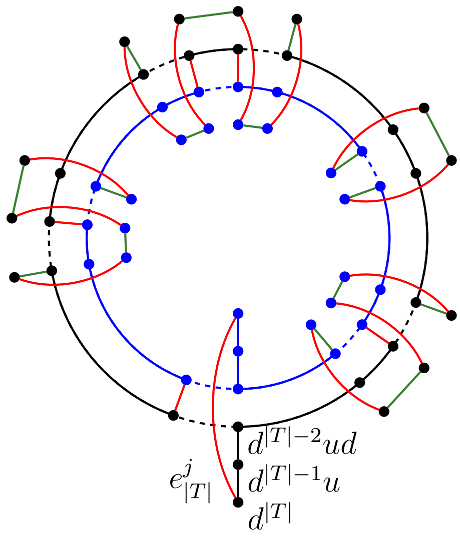


Figure 15: The cycle generated by two copies of $[d^{|T|}, T]$.

It is easy to see that for each $i \in [\lambda_1]$ odd, the edges e_{i-1} and e_i belong to C_i, where $e_i = f^i(ud^{\lambda_1}u^{k_2-1})\{d^{|T|}, d^{|T|-1}u\}$, $i \in [0, \lambda_1]$. Then, for $\lambda_1 > 1$, using the pairs of parallel edges (e_i, e_{i+1}), where $i \in [\lambda_1 - 2]$ odd, we can join according to the basic construction 2 all cycles C_i into the desired cycle C (consisting of all red, all black non-dashed edges and e' in Fig. 16). Finally, for $\lambda_1 = 1$, we set $C = C_1$.

It is clear from the construction of C_1 that the cycle C contains the edge $e' = e(ud^{\lambda_1}u^{k_2-1}d^{n-\lambda_1-k_2})$ and every edge of the form $e(S)$, where $S \in I(P)$ starts with $ud^{\lambda_1}u^{k_2}$. In particular, the edges $e(P)$ and $e(ud^{\lambda_1}u^{k_2}dud^{n-\lambda_1-k_2-3})$ belong to C. Moreover, for $k_2 = 2$ (resp. $k_2 > 2$) the edge $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ and either one of the edges $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ or $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$ belong to C_1 (resp. $C_{s^2(P)}$). Thus, the cycle C_P satisfies the required properties 1, 2, 3 and 7.

2) Assume that $\lambda_2 = 1$ and $\rho = 3$, i.e., $P = ud^{\lambda_1}u^{k_2}du^{k_3}d^{\lambda_3}$. We note that in this case k_3, λ_3 must be even. Indeed, since $P \in \mathcal{K}$, by Proposition 5, we have that $|P|_d = \lambda_1 + 1 + \lambda_3$ is even, which gives that λ_3 is even. Furthermore, since $\lambda_3 \neq 1$, by Theorem 7, the length of P is odd, so that k_3 is also even.

We consider the path $R = ud^{\lambda_1}u^{k_2}du^{k_3-2}du^2d^{\lambda_3-1}$. Clearly, since $R \in \mathcal{K}$, with $R \leqslant P$, and R satisfies the conditions of the previous case, there exists a Hamiltonian cycle C' of $G(I(R) \setminus I(s^2(R)))$ which contains the edges $e' = e(ud^{\lambda_1}u^{k_2-1}d^{n-\lambda_1-k_2})$ and $e(ud^{\lambda_1}u^{k_2}dud^{n-\lambda_1-k_2-3})$. We can easily check that the set $I(P) \setminus I(s^2(P))$ is written as a disjoint union

$$I(P) \setminus I(s^2(P)) = (I(R) \setminus I(s^2(R))) \cup \mathcal{Z}, \qquad \text{where } \mathcal{Z} = \bigcup_{\substack{i \in [0,\lambda_1] \\ j \in [0,\lambda_3+1]}} \{P_{j,i}, Q_{j,i}\} \text{ and}$$

$$P_{j,i} = f^i(ud^{\lambda_1})u^{k_2}du^{k_3-1}f^j(ud^{\lambda_3}), \qquad Q_{j,i} = f^i(ud^{\lambda_1})u^{k_2-1}du^{k_3}f^j(ud^{\lambda_3}).$$

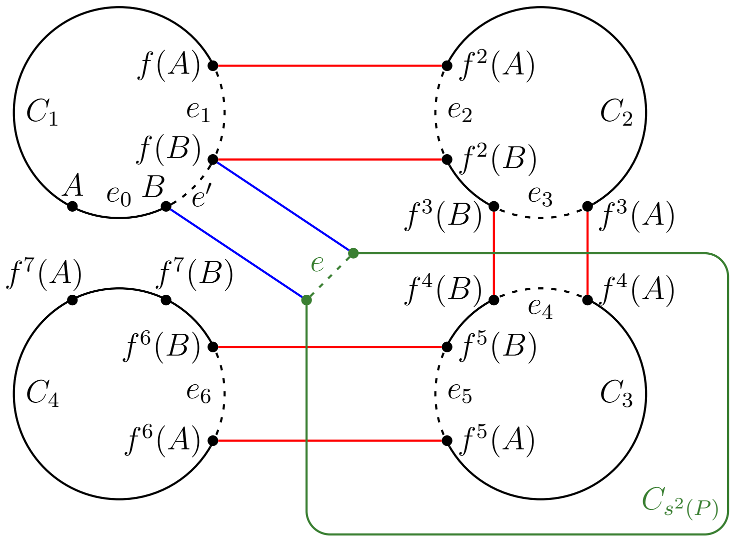


Figure 16: The construction of C and C_p for $k_1 = 1$, k_2 even, λ_1 odd ($\lambda_1 = 7$) and either $\rho \geqslant 3$ and $\lambda_1 \geqslant 2$ or $\rho \geqslant 4$ and $\lambda_1 = 1$, where $A = ud^{\lambda_1}u^{k_2-1}d^{|T|-1}u$ and $B = ud^{\lambda_1}u^{k_2-1}d^{|T|}$.

We note that $f(P_{j,i}) = P_{j,i+1}$, $f(Q_{j,i}) = Q_{j,i+1}$ and $s(P_{j,i}) = Q_{j,i}$, for all i, j. In particular, since $P = P_{0,0}$, we have that $f(P) = P_{0,1}$ and $s(P) = Q_{0,0}$. We will construct a Hamiltonian cycle $C_{\mathcal{Z}}$ of $G(\mathcal{Z})$ as follows:

First, for each $i \in [0, \lambda_1]$, we construct the vertical cycles

$$C_i = (P_{0,i}, P_{1,i}, \ldots, P_{\lambda_3,i}, P_{\lambda_3+1,i}, Q_{\lambda_3+1,i}, Q_{\lambda_3,i}, \ldots, Q_{1,i}, Q_{0,i}, P_{0,i})$$

and then we join them to a cycle $C_{\mathcal{Z}}$, according to the basic construction 2, using the pairs of parallel edges

$$(\{P_{0,i}, P_{1,i}\}, \{P_{0,i+1}, P_{1,i+1}\}), i \in [0, \lambda_1 - 1], i \text{ is even} \quad \text{and}$$

$$(\{Q_{0,i}, Q_{1,i}\}, \{Q_{0,i+1}, Q_{1,i+1}\}), i \in [1, \lambda_1 - 2], i \text{ is odd.}$$

Next, using the parallel edges $e(P_{1,0}) = e(ud^{\lambda_1}u^{k_2}du^{k_3-1}dud^{\lambda_3-1})$ of $C_{\mathcal{Z}}$ and $e(R) = e(ud^{\lambda_1}u^{k_2}du^{k_3-2}du^2d^{\lambda_3-1})$ of C', we can join the cycles C' and $C_{\mathcal{Z}}$ into the desired cycle C of $G(I(P) \setminus I(s^2(P)))$, which contains the edge e' (see Fig. 17, where C consists of all non-dashed black, red and blue edges and the edge e'). Clearly, the edge $e(P)$ belongs to $C_{\mathcal{Z}}$ and the edge $e(ud^{\lambda_1}u^{k_2}dud^{n-\lambda_1-k_2-3})$ belongs to C'. Moreover, for $k_2 = 2$ (resp. $k_2 > 2$) the edge $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ and either $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ or $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$ belongs to C' (resp. $C_{s^2(P)}$). Thus, the cycle C_P satisfies also in this case the required properties 1, 2, 3, and 7.

Case E $k_1 = 1$, k_2 odd. In this case, we can easily check, using Theorem 7, that λ_1, λ_2 have the same parity. If $\rho = 2$, then P has even length and, according to Theorem 7, it ends with ud, so that $\lambda_2 = 1$ and this case is already covered in case C. Therefore, it

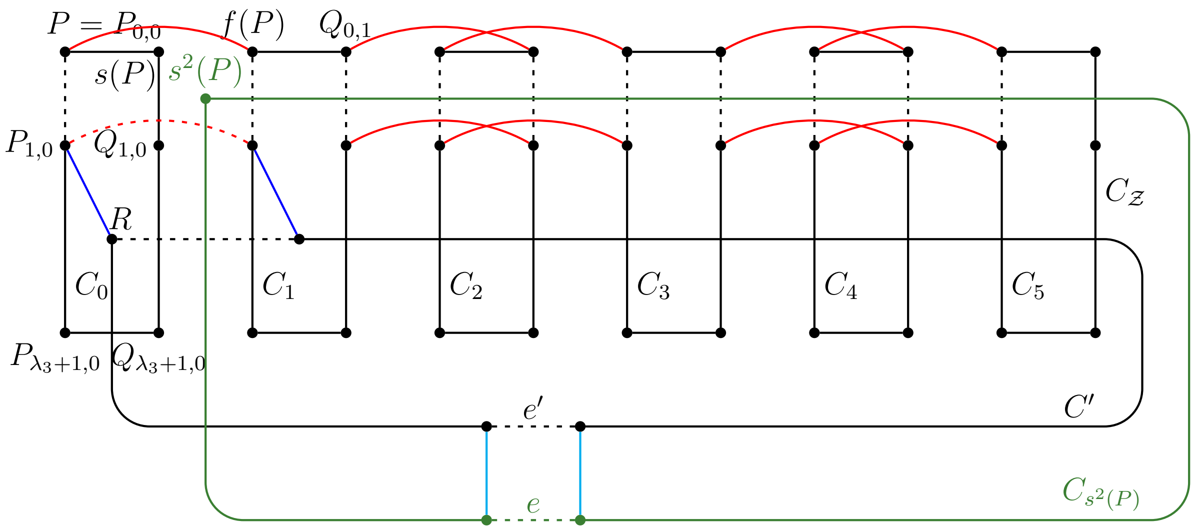


Figure 17: The construction of C and C_p for $k_1 = 1$, k_2 even, λ_1 odd ($\lambda_1 = 5$), $\lambda_2 = 1$ and $\rho = 3$.

is enough to consider that $\rho \geqslant 3$, i.e., $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}Q$, where $Q = u^{k_3}d^{\lambda_3}\cdots u^{k_\rho}d^{\lambda_\rho}$. In this case, the path

$$s(P) = \begin{cases} ud^{\lambda_1+1}ud^{\lambda_2-1}Q, & \text{if } k_2 = 1 \\ ud^{\lambda_1}u^{k_2-1}dud^{\lambda_2-1}Q, & \text{if } k_2 > 1 \end{cases}$$

belongs to $\mathcal{K}_n$, so that, by the induction hypothesis, there exists a Hamiltonian cycle $C_{s(P)}$ of $G(s(P))$ which satisfies the associated properties.

We will construct a Hamiltonian cycle C of $G(I(P) \setminus I(s(P)))$ which contains the edges $e(P)$, $e(f(P))$ whenever λ_1 is even, $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ whenever λ_1 is even and $k_2 = 1$, and the edge $e' = e(ud^{\lambda_1}u^{k_2}d^{n-\lambda_1-k_2-1})$ whenever λ_1 is odd. We note that, for λ_1 even, if $k_2 > 1$, then by property 4 it follows that the edge $e(f(s(P)))$ belongs to $C_{s(P)}$, whereas if $k_2 = 1$, then by property 1 it follows that the edge $ud^{\lambda_1+1}ue^1_{n-\lambda_1-3}$ belongs to $C_{s(P)}$. Moreover, for λ_1 odd, by property 7, if $k_2 > 1$, and by property 6, if $k_2 = 1$, it follows that the edge $e = e(ud^{\lambda_1}u^{k_2-1}dud^{n-\lambda_1-k_2-2})$ belongs to $C_{s(P)}$. Then, the required Hamiltonian cycle C_P of $G(P)$ is created by joining the cycles C and $C_{s(P)}$, according to the basic construction 2, using the pairs $(e(f(P)), e(f(s(P))))$, for λ_1 even and $k_2 > 1$, $(ud^{\lambda_1}ue^1_{n-\lambda_1-2}, ud^{\lambda_1+1}ue^1_{n-\lambda_1-3})$, for λ_1 even and $k_2 = 1$, and (e', e) for λ_1 odd.

It remains to construct the cycle C and show that the induced cycle C_P satisfies the required properties. For this, we consider two cases.

E1 $\rho \geqslant 4$. In this case, $Q \in \mathcal{K}$, so that by the induction hypothesis there exists a Hamiltonian cycle C_Q of $G(Q)$ which satisfies the required properties. Then, using the equality

$$I(P) \setminus I(s(P)) = \bigcup_{i=0}^{\lambda_1} f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{|Q|}, Q]$$

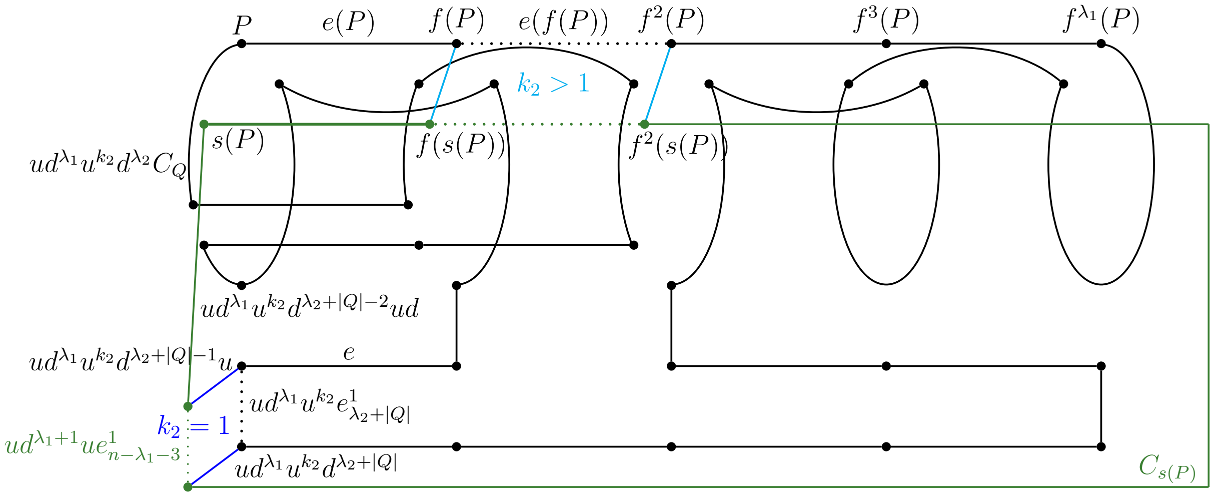


Figure 18: The construction of C_p for $k_1 = 1$, k_2 odd, $\rho \geqslant 4$, λ_1 even ($\lambda_1 = 4$). The two blue (resp. cyan) edges are used in the case where $k_2 = 1$ (resp. $k_2 > 1$).

and the isomorphic cycles $f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}C_Q$, $i \in [0, \lambda_1]$, we will construct the cycle C. For this, we further consider two subcases.

i) λ_1 (and λ_2) even. In this case, the construction of C is the same as in case D2(i) (see Fig. 14) and is omitted. Clearly, from this construction, it follows that C contains the edges $e(P)$, $e(f(P))$ and, for $k_2 = 1$, also the edges $e(ud^{\lambda_1}ud^{n-\lambda_1-2})$ and $ud^{\lambda_1}ue^i_{n-\lambda_1-2}$, $i \in [3]$. Furthermore, for $k_2 > 1$, by the induction hypothesis, the cycle $C_{s(P)}$ contains the edges $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$, $e(ud^{\lambda_1}ud^{n-\lambda_1-2})$ and either one of the edges $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ or $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$. Thus, the cycle C_P satisfies the required properties 1 (for $k_2 \geqslant 2$), 2, 3 and 6 (see Fig. 18).

ii) λ_1 (and λ_2) odd. The construction of the cycle C in this case is similar to the construction used in case D2(ii) (first case). First, we describe an algorithm for constructing a cycle from two copies of the interval $[d^{|Q|}, Q]$, using the following steps:

S1. Sketch two copies of the cycle C_Q, each one of them connected to a copy of the path of vertices $(d^{|Q|}, d^{|Q|-1}u, d^{|Q|-2}ud)$. In Fig. 19, the two copies of the two cycles are presented as two concentric circles.

S2. Delete alternatively half of the edges of the first (resp. second) copy of C_Q, starting from the clockwise first (resp. second) edge after the vertex $d^{|Q|-2}ud$.

S3. In the graph obtained from the previous two steps, connect each vertex in $[d^{|Q|}, Q] \setminus \{d^{|Q|-2}ud, d^{|Q|-1}u\}$ from the one copy with its corresponding vertex in the other copy. It is easy to check that the resulting graph is a cycle such that every two equal vertices in $[d^{|Q|}, Q] \setminus \{d^{|Q|-2}ud, d^{|Q|-1}u\}$ are adjacent.

Next, by applying the above construction for the two copies $f^{i-1}(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{|Q|}, Q]$ and $f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{|Q|}, Q]$, where $i \in [\lambda_1]$, i odd, we obtain a Hamiltonian cycle C_i of $G(f^{i-1}(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{|Q|}, Q] \cup f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{|Q|}, Q])$.

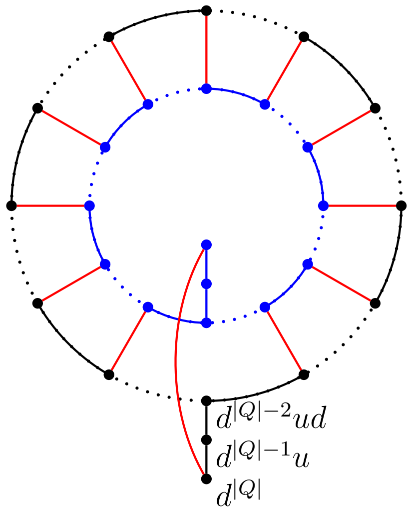


Figure 19: The cycle generated by two copies $[d^{|Q|}, Q]$.

It is easy to see that for each $i \in [\lambda_i]$ odd, the edges e_{i-1} and e_i belong to C_i, where $e_i = f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}\{d^{|Q|}, d^{|Q|-1}u\}$, $i \in [0, \lambda_1]$. Then, for $\lambda_1 > 1$, using the pairs of parallel edges (e_i, e_{i+1}), where $i \in [\lambda_1 - 2]$ odd, we can join according to the basic construction 2 all cycles C_i into the desired cycle C. Finally, for $\lambda_1 = 1$, we set $C = C_1$. A figure for this construction is omitted, since it is almost identical to Fig. 16. The only difference is that now the green circle represents $C_{s(P)}$ and $A = ud^{\lambda_1}u^{k_2}d^{\lambda_2}d^{|Q|-1}u$, $B = ud^{\lambda_1}u^{k_2}d^{\lambda_2}d^{|Q|}$.

It is clear from the construction of C_1 that $e(S)$ belongs to C_1 for every $S \in I(P) \setminus ud^{\lambda_1}u^{k_2}d^{\lambda_2}\{d^{|Q|-2}ud, d^{|Q|-1}u\}$ starting with $ud^{\lambda_1}u^{k_2}d^{\lambda_2}$. In particular, the edges $e(P)$ and $e' = e(ud^{\lambda_1}u^{k_2}d^{|Q|+\lambda_2})$ belong to C. Moreover, for $k_2 = 1$ (resp. $k_2 > 1$) the edge $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ and either one of the edges $ud^{\lambda_1}ue^2_{n-\lambda_1-2}$ or $ud^{\lambda_1}ue^3_{n-\lambda_1-2}$ belong to C_1 (resp. $C_{s(P)}$). Thus, the cycle C_P satisfies the required properties 1, 2 and 3.

E2 $\rho = 3$, i.e., $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}u^{k_3}d^{\lambda_3}$. We note that in this case k_3 must be odd and λ_3 even. Indeed, since $P \in \mathcal{K}_n$, by Proposition 5, we have that $|P|_d = \lambda_1 + \lambda_2 + \lambda_3$ is even, so that since $\lambda_1 + \lambda_2$ is even, we deduce that λ_3 is also even. Then, since $\lambda_3 \neq 1$, by Theorem 7, we have that $|P|$ is odd, which gives that k_3 is odd.

Assume first that $k_3 > 1$ and let $R = ud^{\lambda_1}u^{k_2}d^{\lambda_2}u^{k_3-2}du^2d^{\lambda_3-1}$. Then, $R \in \mathcal{K}_n$, $R \leqslant P$ and R has four peaks, so that it satisfies the conditions of the previous case. It follows that there exists a Hamiltonian cycle C' of $G(I(R) \setminus I(s(R)))$ which contains the edge $e(R)$, the edge $e(f(R))$ for λ_1 even, the edge $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ for λ_1 even and $k_2 = 1$, and the edge $e' = e(ud^{\lambda_1}u^{k_2}d^{n-\lambda_1-k_2-1})$ for λ_1 odd. Moreover, since

$$I(P) \setminus I(s(P)) = \bigcup_{i=0}^{\lambda_1} f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{k_3+\lambda_3}, u^{k_3}d^{\lambda_3}] \quad \text{and}$$

$$I(R) \setminus I(s(R)) = \bigcup_{i=0}^{\lambda_1} f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}[d^{k_3+\lambda_3}, u^{k_3-2}du^2d^{\lambda_3-1}],$$

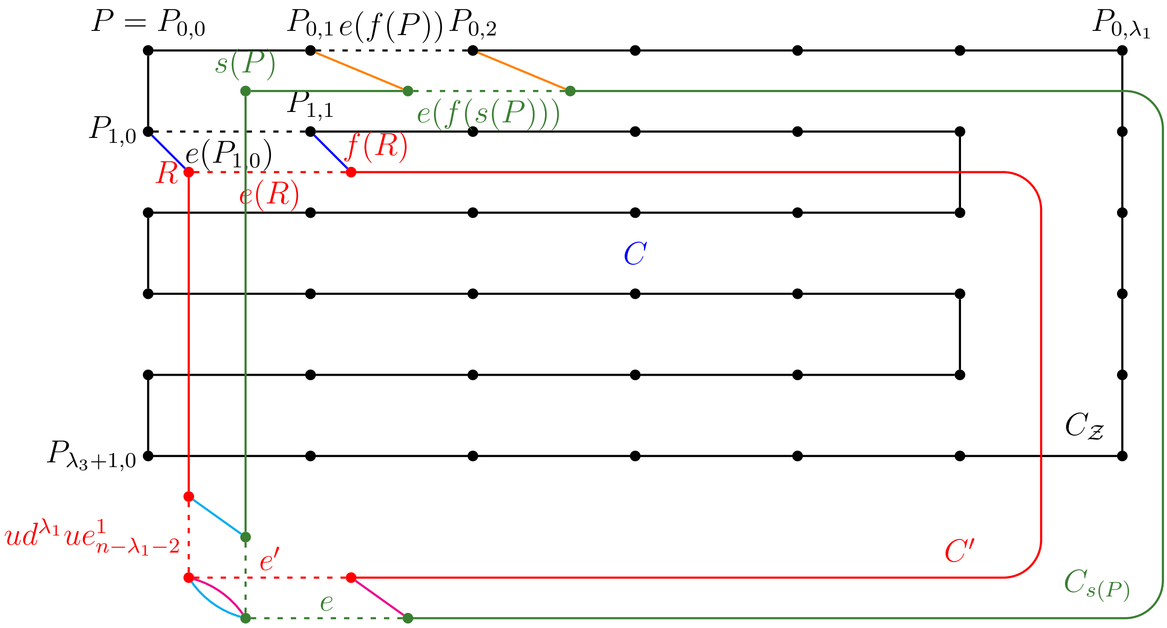


Figure 20: The cycle C_P for $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}u^{k_3}d^{\lambda_3}$, k_2 odd, $k_3 > 1$, $\lambda_1 > 1$, is constructed by joining the cycles C and $C_{s(P)}$ according to the three cases: *i)* λ_1 even and $k_2 > 1$, *ii)* λ_1 even and $k_2 = 1$, *iii)* λ_1 odd.

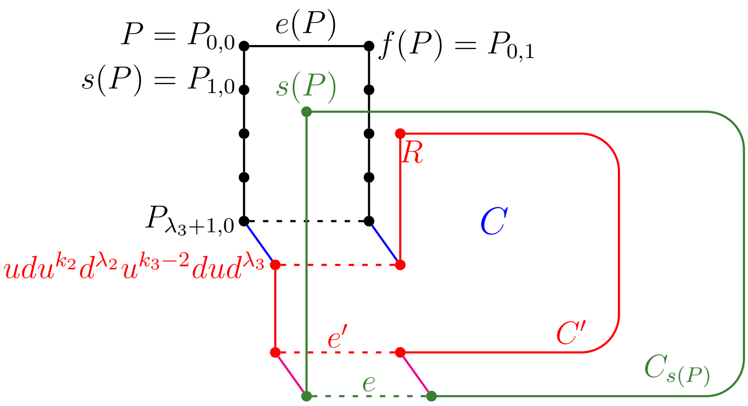


Figure 21: The construction of C_P for $P = ud^{\lambda_1}u^{k_2}d^{\lambda_2}u^{k_3}d^{\lambda_3}$, k_2 odd, $k_3 > 1$, $\lambda_1 = 1$.

we obtain that

$$I(P) \setminus I(s(P)) = (I(R) \setminus I(s(R))) \cup \mathcal{Z}, \tag{6}$$

where

$$\mathcal{Z} = \{P_{j,i} : j \in [0, \lambda_3 + 1], i \in [0, \lambda_1]\} \quad \text{and} \quad P_{j,i} = f^i(ud^{\lambda_1})u^{k_2}d^{\lambda_2}u^{k_3-1}f^j(ud^{\lambda_3}).$$

Following the construction used in case D1(i) for $\lambda_1 > 1$ and in case D1(ii) for $\lambda_1 = 1$, we obtain a Hamiltonian cycle $C_{\mathcal{Z}}$ of $G(\mathcal{Z})$. Then, we join the cycles C' and $C_{\mathcal{Z}}$ into a cycle C, according to the basic construction 2, using the parallel edges $e(P_{1,0}) \in C_{\mathcal{Z}}$, $e(R) \in C'$ for $\lambda_1 > 1$ (see Fig. 20), and $e(P_{\lambda_3+1,0}) \in C_{\mathcal{Z}}$, $e(udu^{k_2}d^{\lambda_2}u^{k_3-2}dud^{\lambda_3}) \in C'$ for $\lambda_1 = 1$ (see Fig. 21). In view of equality (6), C is a Hamiltonian cycle of $G(I(P) \setminus I(s(P)))$.

For $k_3 = 1$, we have that $I(P) \setminus I(s(P)) = \mathcal{Z}$ and we take $C = C_{\mathcal{Z}}$.

We can easily check that for both cases the cycle C contains the edge $e(f(P))$ for $\lambda_1 > 1$, the edge $ud^{\lambda_1}ue^1_{n-\lambda_1-2}$ for λ_1 even, $k_2 = 1$, and the edge $e' = e(ud^{\lambda_1}u^{k_2}d^{n-\lambda_1-k_2-1})$ for λ_1 odd. Thus, the cycle C satisfies the required conditions in order to be joined with the cycle $C_{s(P)}$, to obtain the cycle C_P.

Finally, also in this case, it is easy to check that the cycle C_P satisfies the required properties 1, 2, 3, 4, 5 and 6. □

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