Weak Total Coloring Conjecture and Hadwiger's Conjecture on Total Graphs

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Abstract

The total graph of a graph G, denoted by $T(G)$, is defined on the vertex set $V(G) \cup E(G)$ with $c_1, c_2 \in V(G) \cup E(G)$ adjacent whenever c_1 and c_2 are adjacent to (or incident on) each other in G. The total chromatic number $\chi''(G)$ of a graph G is defined to be the chromatic number of its total graph. The well-known Total Coloring Conjecture or *TCC* states that for every simple finite graph G having maximum degree $\Delta(G)$, $\chi''(G) \leqslant \Delta(G) + 2$. In this paper, we consider two ways to weaken *TCC*:

1. *Weak TCC:* This conjecture states that for a simple finite graph G, $\chi''(G) = \chi(T(G)) \leqslant \Delta(G)+3$. While *weak TCC* is known to be true for 4-colorable graphs, it has remained open for 5-colorable graphs. In this paper, we settle this long pending case.

2. *Hadwiger's Conjecture for total graphs:* We can restate *TCC* as a conjecture that proposes the existence of a strong χ-bounding function for the class of total graphs in the following way: If H is the total graph of a simple finite graph, then $\chi(H) \leqslant \omega(H) + 1$, where $\omega(H)$ is the clique number of H. A natural way to relax this question is to replace $\omega(H)$ by the Hadwiger number $\eta(H)$, the number of vertices in the largest clique minor of H. This leads to the Hadwiger's Conjecture (*HC*) for total graphs: if H is a total graph then $\chi(H) \leqslant \eta(H)$. We prove that this is true if H is the total graph of a graph with sufficiently large connectivity. It is known that *(European Journal of Combinatorics, 76, 159–174,2019)* if Hadwiger's Conjecture is proved for the squares of certain special class of split graphs, then it holds also for the general case. The class of total graphs turns out to be the squares of graphs obtained by a natural structural modification of this type of split graphs.

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1 Introduction

1.1 Total Coloring Conjecture

Let G be a simple finite graph. For a vertex $v \in V(G)$ in G, we define $N_G(v) := \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$ and $E_G(v) := \{e \in E(G) : e \text{ is incident on } v \text{ in } G\}$. The degree of a vertex $v \in V(G)$ is defined as $d(v) = |N_G(v)| = |E_G(v)|$. The maximum degree of G, $\Delta(G) = \max_{v \in V(G)} d(v)$.

A vertex coloring is called a *proper vertex-coloring* if no two adjacent vertices are assigned the same color. The minimum number of colors required to achieve a proper vertex-coloring of G is called the chromatic number of G, and is denoted by $\chi(G)$. Like the vertex-coloring problem, the problem of coloring the edges of a graph is also one that has received much attention in the literature. A coloring of the edges of a graph is called a *proper edge-coloring* if no two adjacent edges are assigned the same color. The minimum number of colors required in any proper edge-coloring of a graph G is called its *edge-chromatic number* or *chromatic index* and is denoted by $\chi'(G)$. The following theorem was proved by Vizing :

Theorem 1 (Vizing's Theorem)**.** *For a simple finite graph G, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$.*

Vizing's theorem suggests a classification of simple finite graphs into two classes. A graph G is said to be in *class I* if $\chi'(G) = \Delta(G)$ and in *class II* if $\chi'(G) = \Delta(G) + 1$.

The notion of line graphs allows us to view the edge coloring problem as a vertex coloring problem. Given a graph G, the line graph of G is the graph $L(G) = (V', E')$ where $V' = E(G)$ and $e_1, e_2 \in V'$ form an edge $e_1e_2 \in E'$ whenever $e_1$and e_2 are adjacent edges in G. Clearly $\chi'(G) = \chi(L(G))$. The clique number of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. Clearly $\omega(G) \leqslant \chi(G)$ for all graphs. It is easy to see that $\omega(L(G)) \geqslant \Delta(G)$, and therefore Vizing's theorem guarantees that the chromatic number of any line graph is at most 1 more than its clique number. Given a class of graphs $\mathcal{F}$, if there exists a function f such that $\forall G \in \mathcal{F}, \chi(G) \leqslant f(\omega(G))$, then $\mathcal{F}$ is a *χ-bounded class* and f is a *χ-bounding function* for $\mathcal{F}$. χ-boundedness of graph classes is an extensively studied topic, see the survey by Scott and Seymour [26] for further references. Thus Vizing's Theorem guarantees a very strong χ-bounding function for the class of line graphs.

After vertex and edge colorings, it was natural for graph theorists to consider coloring vertices and edges simultaneously. A *total coloring* of a graph G is a coloring of all its *elements*, that is, vertices and edges, such that no two adjacent elements are assigned the same color. (When we refer to two adjacent *elements* of a graph, they are either adjacent to each other or incident on each other as the case may be.) That is, it is an assignment c of colors to $V(G) \cup E(G)$ such that $c|_{V(G)}$ is a proper vertex-coloring, $c|_{E(G)}$ is a proper edge-coloring, and $c(uv) \notin \{c(u), c(v)\}$ for any edge $uv \in E(G)$. The minimum possible number of colors in any total coloring of G is called the *total chromatic number* of G, and is denoted by $\chi''(G)$.

The *total coloring conjecture* (or *TCC*) was proposed by Behzad [3] and Vizing [27] independently between 1964 and 1968.

Conjecture 2 (Total coloring conjecture)**.** For any simple, finite graph G,

$$\chi''(G) \leqslant \Delta(G) + 2$$

Theorem 3. *If we use multigraphs instead of simple graphs, the above statement may not hold. This is true of Theorem 1 also. In this paper, we consider only simple finite graphs.*

We may state weaker versions of Conjecture 2 by relaxing the upper bound on $\chi''(G)$ as follows:

Conjecture 4 (**(*k*)**-*TCC*)**.** Let $k \geqslant 2$ be a fixed positive integer. For any graph G,

$$\chi''(G) \leqslant \Delta(G) + k$$

Note 5. (2)-*TCC* is the same as Conjecture 2; that is, the total coloring conjecture. We shall refer to (3)-*TCC* as the *weak total coloring conjecture* or *weak TCC* for short.

Even though many researchers have examined *TCC* over the years, it remains unsolved to date and is considered one of the hardest open problems in graph coloring. Bollobás and Harris [6] proved that $\chi''(G) \leqslant \frac{11}{6}\Delta(G)$, when $\Delta(G)$ is sufficiently large. Later Kostochka [17] proved that $\chi''(G) \leqslant \frac{3}{2}\Delta(G)$, when $\Delta(G) \geqslant 6$. The best-known result to date for the general case was obtained by Molloy and Reed [19] :

Theorem 6. *$\chi''(G) \leqslant \Delta(G) + C$, where C is equal to 10^{26}.*

Theorem 7. *Molloy and Reed have mentioned in [19] that though they only show a value 10^{26} for the constant C, with much more effort it can be brought down to 500.*

While *TCC* remains unsolved for the general case, it is known to hold for some special classes of graphs. For example, it is easy to see that *TCC* is true for all complete graphs and bipartite graphs [4]. The case where the maximum degree of the graph is bounded above by some small positive integer is studied by many researchers [22, 28, 16]. The best known result in this direction is the following.

Theorem 8 ([15])**.** *TCC holds for all graphs G with maximum degree at most 5.*

Many researchers studied *TCC* for planar graphs [7, 32, 24]. The total coloring conjecture is known to hold for planar graphs except when the maximum degree is 6.

Theorem 9 ([24])**.** *Let G be a planar graph. If $\Delta(G) \neq 6$, then TCC holds for G, otherwise $\chi''(G) \leqslant \Delta(G) + 3$.*

Considering the difficulty level of *TCC*, it makes sense to study relaxations of *TCC*. We intend to consider two ways to weaken *TCC*. The first one is the most obvious way: Increase the upper bound appearing in the statement of *TCC*. This leads to *weak TCC* and (k)-*TCC*, stated in Conjecture 4. Now we will develop the background for the second one.

Just like we defined line graphs and used it to perceive the edge coloring problem on G as a vertex coloring problem on $L(G)$, we can define a similar structure in the context of total coloring. Given a graph $G = (V, E)$ the *total graph* of G is the graph $T(G) = (V'', E'')$ where $V'' = V \cup E$, the set of all elements of G, and $c_1c_2 \in E''$ whenever elements c_1,c_2 are adjacent in G. Given a class $\mathcal{F}$ of graphs, we define $T(\mathcal{F}) := \{T(G) : G \in \mathcal{F}\}$. We use $\mathcal{T}$ to denote the class of total graphs i.e. $\mathcal{T} = T(\mathcal{G})$, where $\mathcal{G}$ is the class of all simple finite graphs.

Note 10. We shall call vertices in $T(G)$ that correspond to vertices in the original graph G as v-vertices and vertices that correspond to edges in G as e-vertices. For ease of notation, we denote an element of G, that is, an edge or a vertex in G, and its corresponding vertex in $T(G)$, by the same letter. Thus, the v-vertex in $T(G)$ corresponding to $x \in V(G)$ is also denoted by x.

Observation 11. *Any total coloring of a graph G corresponds to a proper vertex coloring of $T(G)$ and vice-versa. Thus, the total chromatic number of G is equal to the chromatic number of $T(G)$ that is, $\chi''(G) = \chi(T(G))$.*

Observation 12. *For every vertex $x \in V(G)$, the v-vertex corresponding to x and the e-vertices corresponding to the edges in $E_G(x)$ form a clique of order $d_G(x) + 1$ in $T(G)$. It follows that there exists a clique of order $\Delta(G) + 1$ in $T(G)$. Therefore, $\Delta(G) + 1 \leqslant \omega(T(G)) \leqslant \chi(T(G)) = \chi''(G)$. In fact, it is not difficult to see that the clique number of the total graph of G, $\omega(T(G)) = \Delta(G) + 1$, when $\Delta(G) \geqslant 2$.*

It is easy to see that if $\Delta(G) \leqslant 1$, then $\chi''(G) = \omega(T(G))$. Thus, in view of observation 12, Conjecture 2 can be thought to be suggesting the existence of a very strong χ-bounding function for the class of total graphs, and can be restated as follows:

Conjecture 13 (Restatement of total coloring conjecture)**.** $\forall H \in \mathcal{T}, \chi(H) \leqslant \omega(H) + 1$

From the structural perspective, one obvious way to relax Conjecture 13 is to replace clique number $\omega(H)$ by Hadwiger number $\eta(H)$, the number of vertices in the largest clique minor of H. This directly leads us to the study of Hadwiger's Conjecture on Total Graphs, hereafter abbreviated as "*HC on $\mathcal{T}$*". (See the next section for the formal statement of Hadwiger's Conjecture.)

Problem 14 (Hadwiger's Conjecture on Total Graphs or "*HC* on $\boldsymbol{\mathcal{T}}$")**.**

$$\chi(H) \leqslant \eta(H), \forall H \in \mathcal{T}$$

Theorem 15. *Careful readers may object that if we replace $\omega(H)$ by $\eta(H)$ in Conjecture 13, the new inequality should be $\chi(H) \leqslant \eta(H) + 1, \forall H \in \mathcal{T}$. But we will show in Theorem 27 that Conjecture 13 indeed implies HC on $\mathcal{T}$ as stated in problem 14.*

Needless to say, Hadwiger's Conjecture is an even more celebrated and long-standing conjecture than *TCC* in graph theory, owing its origin to the Four Color Theorem itself. While the general case of Hadwiger's Conjecture remains unsolved, it was proved for several special classes of graphs. For example, Reed and Seymour [20] proved Hadwiger's Conjecture for line graphs more than 15 years ago. Chudnovsky and Fradkin [10] even generalized the result to quasi-line graphs. But to the best of our knowledge, for the class of total graphs, whose definition is similar in spirit to that of line graphs, Hadwiger's Conjecture has not yet been proven or studied.

1.2 Hadwiger's Conjecture: The connection between the general case and the case of total graphs

Given an edge $e = uv$ in G, the *contraction* of the edge e involves the following : deleting vertices u and v, introducing a new vertex w_e, and making w_e adjacent to all vertices in the set $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$. The new graph thus obtained is denoted by G/e.

A graph H is said to be a *minor* of G if a graph isomorphic to H can be obtained from G, by performing a sequence of operations involving only vertex deletions, edge deletions, and edge contractions. If G contains H as a minor, then we write $H \preceq G$.

The celebrated *Hadwiger's Conjecture* is a far-reaching generalization of the *Four Color theorem*. It was proposed by Hugo Hadwiger [13] in 1943.

Conjecture 16 (Hadwiger's Conjecture)**.** Given any graph G and $t > 0$,

$$\chi(G) \geqslant t \implies K_t \preceq G$$

In other words, every graph has either a clique minor on t vertices (K_t-minor) or a proper vertex-coloring using $(t - 1)$ colors.

For $t \leqslant 3$, Hadwiger's Conjecture is easy to prove. The $t = 4$ case was proved by Hadwiger himself in [13]. The Four Color Theorem was proved by Appel and Haken [1, 2] in 1977. Using a result of Wagner [29], it can be shown that the Four Color Theorem is equivalent to Hadwiger's Conjecture for $t = 5$. Robertson, Seymour, and Thomas [21] proved in 1993 that Hadwiger's Conjecture holds true for $t = 6$. The conjecture remains unsolved for $t \geqslant 7$. So far, Hadwiger's Conjecture has been proved for several classes of graphs; see [5, 10, 18, 20, 30, 31].

Now we describe a curious observation from [8]. Given a graph G, its square G^2 is defined on the vertex set $V(G)$ with u and v being adjacent in G^2 whenever the distance between u and v in G is at most 2. For a class of graphs $\mathcal{F}$, let $\mathcal{F}^2$ denote the set of graphs $\{G^2 : G \in \mathcal{F}\}$. Recall that a split graph is a graph whose vertex set can be partitioned into an independent set and a clique. Let $\mathcal{S}$ denote the special class of split graphs, whose vertex set is partitioned into an independent set A and a clique B with the following extra constraints: (1) Each vertex in B has exactly 2 neighbours in A. (2) There are no two vertices in B having the same neighbourhood in A.

The following fact is from [8]. (In [8], the following fact is not stated explicitly in this detail, but can be easily read out from the proof of Theorem 1.2, therein.)

Fact 17. *Proving the general case of Hadwiger's Conjecture is equivalent to proving HC for the class* $\mathcal{S}^2$.

Obviously proving *HC* for $\mathcal{S}^2$ is extremely difficult, despite its tantalizingly specialized appearance, since it is a reformulation of the general case. From this perspective, it is natural to consider classes of graphs that can be obtained by simple structural modifications of $\mathcal{S}$ and see how difficult it is to prove *HC* for the squares of such classes. One such modification is to assume that both A and B are independent sets, keeping everything else the same, i.e. the class of bipartite graphs with parts A and B with the extra constraints: (1) Each vertex in B has exactly 2 neighbors in A. (2) No two vertices in B have the same neighbourhood in A. In other words, get a bipartite graph from each split graph in $\mathcal{S}$ by converting the clique B to an independent set. Let this new class be denoted by $\hat{\mathcal{S}}$. How difficult is it to prove *HC* for $\hat{\mathcal{S}}^2$? Careful inspection reveals that $\hat{\mathcal{S}}$ is a familiar class of graphs, the sub-divided graphs.

A graph G is a subdivided graph if it can be obtained from another graph H by subdividing each edge uv in H; that is, replacing uv by a path uwv, where w is a new vertex. Now if we consider the set of newly introduced vertices as A and the original vertices of H as B, it is easy to see that $G \in \hat{\mathcal{S}}$. The converse is also true: it is easy to verify that if a graph G belongs to $\hat{\mathcal{S}}$, then there exists another graph H such that G is obtained by sub-diving all the edges of H. Interestingly, the class of squares of subdivided graphs is the same as the class of total graphs.

Fact 18. *If H is a graph and G is obtained by subdividing every edge of H, then the graph G^2 is isomorphic to the total graph $T(H)$ of H. In other words, the class of total graphs $\mathcal{T} = \hat{\mathcal{S}}^2$.*

Considering the structural closeness of $\hat{\mathcal{S}}$ and $\mathcal{S}$ and the fact from [8] that *HC* on $\mathcal{S}^2$ is equivalent to the general case of *HC*, it is not very surprising that proving Hadwiger's Conjecture for total graphs turns out be challenging.

Theorem 19. *Let G be a graph and let $T(G)$ be its total graph. Denote the set of v-vertices and the set of e-vertices in $T(G)$ by V and E, respectively. Note that the subgraph of $T(G)$ induced by V is isomorphic to the original graph G, the subgraph of $T(G)$ induced by E is isomorphic to the line graph $L(G)$ of G, and the bipartite graph induced by the edges between the sets V and E is precisely the subdivided graph $S(G)$ of G.*

1.3 Our contributions to Hadwiger's Conjecture on total graphs

In Section 2 and Section 3, we explore the difficulty level of *HC on* $\mathcal{T}$ in comparison with *TCC*. Our first result is Theorem 27 which states that if *TCC* is true for a class $\mathcal{F}$ of graphs that is closed under taking subgraphs, then Hadwiger's Conjecture holds true for the class $T(\mathcal{F})$. Taking $\mathcal{F} = \mathcal{G}$, the class of simple finite graphs, we infer that *TCC* implies *HC on* $\mathcal{T}$, thus justifying our claim that problem 14 is a relaxation of Conjecture 2 and its restatement, Conjecture 13. But is *TCC* indeed a stronger statement than *HC on* $\mathcal{T}$ in the sense that some hypothesis weaker than *TCC* implies *HC on* $\mathcal{T}$? Our next result confirms this.

In Theorem 30, we show that if *weak TCC* (see Conjecture 4) is true for a class $\mathcal{F}$ of graphs that is closed under taking subgraphs, then Hadwiger's Conjecture is true for the class $T(\mathcal{F})$. Thus the weaker hypothesis *weak TCC* implies *HC on* $\mathcal{T}$; therefore *HC on* $\mathcal{T}$ is strictly easier than *TCC*. A graph G is t-total critical if G is connected and $\forall e \in E(G), \chi''(G-e) < \chi''(G) = t$. We show that for every $(\Delta(G)+3)$-total critical graph G, $T(G)$ has a clique minor of order $\Delta(G)+3$.

Can we prove that the weaker hypothesis (k)-*TCC* (for some fixed positive integer $k \geqslant 4$; see Conjecture 4) implies *HC on* $\mathcal{T}$? We do not know the answer, but our approach of showing the existence of a $\Delta(G)+k$ clique minor in $T(G)$, assuming that G is $(\Delta+k)$-total critical, does not seem to work when $k \geqslant 4$. More sophisticated approaches may be needed to answer this question.

However, if we have the additional guarantee that the graph G has a high vertex connectivity, then Hadwiger's Conjecture will hold for $T(G)$ if (k)-*TCC* is true for G. We show in Theorem 37 that if (k)-*TCC* is true, then Hadwiger's Conjecture is true for $T(\mathcal{F})$, where $\mathcal{F} := \{G : \kappa(G) \geqslant 2k-1\}$ (here, $\kappa(G)$ denotes the vertex-connectivity of G). By combining this with the upper bound of Molloy and Reed [19], we get that there exists a constant C' such that Hadwiger's Conjecture is true for $T(\mathcal{F})$, where $\mathcal{F} := \{G : \kappa(G) \geqslant C'\}$. Here $C' = 2C-1$, where C is the constant from Theorem 6; in [19] Molloy and Reed proved $C = 10^{26}$, but mention that with more detailed analysis, C can be brought down to 500.

1.4 Our contributions to the total coloring literature

In this section we concentrate on the first (and the more straightforward) relaxation of *TCC* mentioned earlier: *weak TCC*.

The first reason that motivated us to go through the known literature on total coloring to find out the important special classes $\mathcal{F}$ of graphs for which *weak TCC* is known to hold (while *TCC* may still remain unsolved), is the promise of Theorem 30 that *HC* is true for $T(\mathcal{F})$ for such $\mathcal{F}$. We realized that there are some important graph classes in this category, for example, planar graphs. Unfortunately, for general graphs, it seems the chance of either *TCC* or *weak TCC* getting proven in the near future is very low.

Is it possible that *weak TCC* is a more reasonable conjecture than *TCC*? It is remarkable that another very well-known and widely believed conjecture, the *list coloring conjecture*, implies the *weak TCC*, whereas *TCC* seems beyond its reach. *TCC* is indeed a bold conjecture, in the sense that at times it makes us think that it is not unreasonable to look for counterexamples. For example, despite intense research for decades, the conjecture could not be proved for planar graphs with a maximum degree 6, though it has been found to hold good for all other cases of planar graphs. On the other hand, it is very easy to prove weak *TCC* for all planar graphs: *If* G *is a planar graph, then* $\chi''(G) \leqslant \Delta(G)+3$. Moreover, *weak TCC* is in fact true for a much wider class, namely 4-colorable graphs, which properly includes planar graphs, whereas proving *TCC* on 4-colorable graphs seems to be extremely difficult. This led us to survey the status of *weak TCC* on 5-colorable graphs, and try to fill the research gap therein.

Weak TCC can be proved for 4-colorable graphs (that is, when $\chi(G) \leqslant 4$) using the following well-known argument: We color the vertices of the graph using the colors $\{1,2,3,4\}$. Applying Vizing's theorem we can color the edges of the graph using $\Delta+1$ colors from $\{3,4,\ldots,\Delta(G)+3\}$. We uncolor all edges with colors 3 or 4. Note that for every uncolored edge, there exist at least two colors in $\{1,2,3,4\}$ that are not the colors assigned to its endpoints. Let us associate with each uncolored edge the list containing these two colors. It is easy to see that the subgraph induced by the uncolored edges consists only of paths and even cycles, and is therefore 2-edge-choosable. This proves *weak TCC* for 4-colorable graphs. As a special case, planar graphs satisfy *weak TCC* since by the four color theorem all planar graphs are 4-colorable.

As mentioned earlier this naturally leads to the question of whether *weak TCC* can be proved for k-colorable graphs with $k \geqslant 5$. It is not the case that researchers who worked on total coloring failed to notice this question altogether. In fact, from the early days of research in total coloring, researchers have tried to find upper bounds for total chromatic number of a graph in terms of its (vertex) chromatic number.

Theorem 20. *If we go through the mainstream literature on total coloring, apart from efforts to bring the upper bound closer to the conjectured $\Delta+2$, there were efforts to study TCC for graph classes defined by their maximum degree, i.e. $\Delta \leqslant 4$, $\Delta \leqslant 5$ etc. Obviously, the other two parameters for such a study were the chromatic number $\chi(G)$ and the chromatic index $\chi'(G)$. The latter is essentially Δ, due to Vizing's Theorem. Studying TCC for graphs of bounded chromatic number was the next natural option after bounded maximum degree graphs.*

The most important result in this direction was proved by Hind [14].

Theorem 21 (Hind)**.** *For every graph G,*

$$\chi''(G) \leqslant \chi'(G) + 2\left\lceil \sqrt{\chi(G)} \right\rceil \leqslant \Delta(G) + 1 + 2\left\lceil \sqrt{\chi(G)} \right\rceil .$$

We note that for small values of $\chi(G)$ (such as $\leqslant 9$), Theorem 21 implies $\chi''(G) \leqslant \Delta(G)+7$. Naturally many researchers have attempted to improve this result using adaptations of the technique used in Hind's proof. One such result was presented by Sánchez-Arroyo [23] and a slightly better bound was obtained by Chew [9].

Theorem 22 (Chew)**.** *For any connected multigraph G,*

$$\chi''(G) \leqslant \begin{cases} \chi'(G) + \left\lceil \frac{\chi(G)}{3} \right\rceil + 1 & \textit{if } \chi(G) \equiv 2 \pmod 3 \\ \chi'(G) + \left\lfloor \frac{\chi(G)}{3} \right\rfloor + 1 & \textit{otherwise} \end{cases}$$

This implies that $\chi''(G) \leqslant \chi'(G)+3$, for $\chi(G) \leqslant 5$. Although *weak TCC* for *class-I* 5-colorable graphs follows from this result, *weak TCC* is not known to hold for the entire class of 5-colorable graphs to date. In Theorem 40, we prove the following long-pending result:

Weak TCC is true for 5-colorable graphs.

Theorem 23. *The method used in our proof is completely different from the proofs of Theorem 21 and Theorem 22. Proving TCC for* 5*-colorable graphs is likely to be a considerably harder problem. In fact, TCC remains to be proved even for* 4*-colorable graphs. As mentioned before the* $\Delta = 6$ *case for planar graphs is still open even after decades of research.*

2 Hadwiger's Conjecture and total coloring

Definition 24 (total-critical graph)**.** A graph G is said to be *t-total-critical* if G is connected, $\chi''(G) = t$ and $\chi''(G - e) \leqslant t - 1$, for any edge e of G.

Note that any graph that contains at least one edge has a total chromatic number of at least 3.

Lemma 25. *If a graph H is t-total-critical, where $t \geqslant \Delta(H) + 2$, then H has no cut-vertices.*

Proof. Let H be a t-total-critical graph, where $t \geqslant \Delta(H) + 2$. Note that it follows from the definition of a t-total critical graph that H is connected. Let us choose a vertex v in H with $d_H(v) = d$ (say). We claim that v is not a cut-vertex in H. Otherwise if v is a cut-vertex then $2 \leqslant d \leqslant \Delta(H)$. We can assume that for some $s < d$, s neighbours of v say, $v_1, v_2, \ldots, v_s$ lie in one connected component C_1, while the remaining $d - s$ neighbours, $v_{s+1}, \ldots, v_d$ lie in the remaining connected components $C_2, \ldots, C_k$ ($k \geqslant 2$) of $H - v$. Let the edge vv_i be denoted by e_i, for all $i \leqslant d$. Define $T_1 := H[V(C_1) \cup \{v\}]$ and $T_2 := H - V(C_1)$. Both T_1 and T_2 can be total colored with $t - 1$ colors since H is t-total-critical.

For $i \in [2]$, consider a total coloring $\varphi_i : V(T_i) \cup E(T_i) \to [t-1]$ of T_i. By permuting colors if necessary, we can assume that $\varphi_1(v) = \varphi_2(v)$ and $\varphi_1(E_{T_1}(v)) \cap \varphi_2(E_{T_2}(v)) = \emptyset$. But now, $\varphi_1 \cup \varphi_2$ is a total coloring of H using at most $t - 1$ colors, a contradiction. □

Lemma 26. *Let G be a $(\Delta(G) + k)$-total-critical graph on at least* 3 *vertices, for some $2 \leqslant k \leqslant \Delta(G)$. Then the minimum degree of G is at least k.*

Proof. Assume that there exists a vertex $w \in V(G)$ with $d_G(w) \leqslant k-1$, and let $\Delta(G) = \Delta$. Since G has at least three vertices and there are no cut-vertices in G (lemma 25), w cannot have degree 1. If $k = 2$, then this is already a contradiction. Hence we assume that $k \geqslant 3$.

Let the neighbours of w be $v_1, v_2, \ldots, v_j$, where $2 \leqslant j \leqslant k - 1$. Call the edge $v_i w$ as e_i, for each $i \in [j-1]$ and $v_j w$ as f. Then $G - f$ has a total coloring using $\Delta + k - 1$ colors, which we shall assume to be the set $[\Delta + k - 1]$.

Assume that v_i is assigned color α_i for each $i \in [j]$ and the edge e_i is assigned color β_i for each $i \in [j-1]$. Call the color assigned to w as c.

If $\alpha_j = c$, pick a color $c' \in \{1, 2, \ldots, \Delta+k-1\} \setminus (\{\alpha_i : i \in [j-1]\} \cup \{\beta_i : i \in [j-1]\} \cup \{c\})$ (such a c' exists because $k \leqslant \Delta$ implies $\Delta + k - 1 > 2(k-2) + 1$), and assign it to the vertex w. Now, the vertices w and v_j have different colors on them. Else if $\alpha_j \neq c$, let $c' := c$.

Since $d_G(v_j) \leqslant \Delta$, and therefore $d_{G-f}(v_j) \leqslant \Delta - 1$, there exists a list L of at least $(k-1)$ colors that are not assigned to v_j or its incident edges. If at least one of the colors in L is not in $\{c'\} \cup \{\beta_i : i \in [j-1]\}$, assign it to the edge f giving G a total coloring using at most $(\Delta + k - 1)$ colors. Otherwise since $|L| \geqslant k-1$ and w has at most $k-2$ incident edges in $G - f$, we have the set of colors that appear on w and its incident edges is exactly the list L; that is, $\{c'\} \cup \{\beta_i : i \in [j-1]\} = L$. Choose a color $c'' \in \{1, 2, \ldots, \Delta + k - 1\} \setminus (\{c', \alpha_j\} \cup \{\alpha_i : i \in [j-1]\} \cup \{\beta_i : i \in [j-1]\})$ (such a c'' exists as $j \leqslant k - 1 < \Delta$ implies $\Delta + k - 1 > 2(k-2) + 2$), and assign it to the vertex w and then assign color c' to the edge f, giving G a total coloring using at most $(\Delta + k - 1)$ colors.

This is a contradiction as $\chi''(G) = \Delta + k$ by hypothesis. Hence, $d_G(w) \geqslant k$ for each vertex $w \in G$. □

Theorem 27. *Let $\mathcal{F}$ be a class of graphs that is closed under the operation of taking subgraphs. If TCC is true for $\mathcal{F}$, then Hadwiger's Conjecture holds for the class $T(\mathcal{F})$.*

Proof. Assume that TCC holds for all graphs in $\mathcal{F}$. Let $G \in \mathcal{F}$ and $\Delta := \Delta(G)$. Therefore by observation 12, $\Delta + 1 \leqslant \chi''(G) \leqslant \Delta + 2$. Note that the last inequality follows because TCC holds for $\mathcal{F}$.

Case 1: $\chi''(G) = \Delta + 1$.

$T(G)$ contains a clique of size $\Delta + 1$ (observation 12). Therefore, this case is trivial.

Case 2: $\chi''(G) = \Delta + 2$.

Let H be a $(\Delta + 2)$-total critical subgraph of G. We then have $\chi''(H) = \Delta + 2$. It is clear that $\Delta(H) = \Delta$, otherwise since H also belongs to $\mathcal{F}$, TCC would imply that $\chi''(H) \leqslant \Delta + 1$. Choose a vertex $v_\Delta \in H$ such that $d_H(v_\Delta) = \Delta$. Let $e_1, e_2, \ldots, e_\Delta$ be the edges incident on v_Δ in H. From lemma 25 we know $H - v_\Delta$ is connected. This implies that the subgraph X of $T(G)$ induced by the elements of $H - v_\Delta$ is connected. In $T(G)$, contract the subgraph X into a single vertex w. Since the subgraph X contains the v-vertex corresponding to every neighbour of v_Δ, the vertex w is now adjacent to the e-vertices $e_1, \ldots, e_\Delta$ and the v-vertex v_Δ, forming a $(\Delta + 2)$-clique minor in $T(G)$. □

3 Weak total coloring conjecture

The reader may recall the weaker versions of TCC obtained by relaxing the upper bound on $\chi''(G)$, namely (k)-TCC and *weak TCC*, which are stated as Conjecture 4. In this section, we will study these weaker versions in detail. We first make a general observation about graphs that do not contain cut-vertices.

Claim 28. *Suppose that G is a 2-connected graph and v is a vertex in G. Then there exists a vertex $w \in N(v)$ such that $\{w, v\}$ is not a separator of G.*

Proof. Suppose for contradiction that each set in $\{\{v, w\}\}_{w \in N(v)}$ is a separator of G. Since G has no cut-vertices, we can conclude that each set in $\{\{v, w\}\}_{w \in N(v)}$ is a minimum separator of G. For $w \in N(v)$, let $F(v, w)$ be the number of vertices in the smallest component in $G - \{v, w\}$. Choose $w \in N(v)$ such that $F(v, w)$ is as small as possible. Let S be the smallest component in $G - \{v, w\}$. Since $\{v, w\}$ is a minimum separator, v has a neighbor in each component of $G - \{v, w\}$, and so also in S. Let $x \in S \cap N(v)$. Now consider the graph $G - \{v, x\}$. By our assumption, $\{v, x\}$ is a separator of G, and hence $G - \{v, x\}$ is not connected. Since w has a neighbor in each connected component of $G - \{v, w\}$ (as $\{v, w\}$ is a minimum separator), the vertex w together with the vertices in all the components of $G - \{v, w\}$ other than S now lie in one connected component of $G - \{v, x\}$. Thus the other connected components of $G - \{v, x\}$ must all be subgraphs of S. Since there is at least one other connected component and it cannot contain $x \in S$, we can conclude that there is a connected component of $G - \{v, x\}$ that is smaller than S. This contradicts the choice of w and S. $\square$

Lemma 29. *Let H be a $(\Delta + 3)$-total-critical graph having maximum degree Δ. Then $\eta(T(H)) \geqslant \Delta + 3$.*

Proof. Since H is $(\Delta + 3)$-total-critical, we have that H is connected. Choose a vertex v_Δ in H, with $d_H(v_\Delta) = \Delta$. By lemma 25, H does not contain any cut-vertices. Then by Claim 28, there exists a $w \in N_H(v_\Delta)$, such that $\{v_\Delta, w\}$ is not a separator in H. Recall that *TCC* holds for all graphs with maximum degree at most 5 (Theorem 8). Therefore, $\Delta \geqslant 6$ since $\chi''(H) = \Delta + 3$ by assumption. It follows from lemma 26 that $\delta(H) \geqslant 3$. Let e' and e'' be the e-vertices in $T(H)$ corresponding to two edges incident with w in $H - \{v_\Delta\}$.

Our goal is to construct a clique minor of order $\Delta + 3$ in $T(H)$. For this we use the following *branch sets*:

1. The set S^0 consisting of the v-vertices that correspond to $\{v_\Delta, w\}$. Note that S^0 is connected in $T(H)$ and can be contracted to a vertex z.

2. The singleton $\{e\}$ for each e-vertex $e \in T(H)$ that corresponds to an edge in $E_H(v_\Delta)$. Let S^1 denote the set of these Δ e-vertices.

3. $S^2 = S' \cup \{e'\}$, where S' is the set of e-vertices corresponding to the edges in $E(H \setminus \{v_\Delta, w\})$. Note that S^2 is connected and can be contracted to a single vertex y.

4. $S^3 = S'' \cup \{e''\}$, where S'' is the set of v-vertices corresponding to the vertices in $V(H \setminus \{v_\Delta, w\})$. S^3 is also connected and can be contracted to a single vertex x.

It is straightforward to see that the e-vertices in S^1 along with x, y and z form the branch sets of a $K_{\Delta+3}$ minor in $T(H)$.

$\square$

Theorem 30. *Let $\mathcal{F}$ be a class of graphs that is closed under the operation of taking subgraphs. If weak TCC is true for $\mathcal{F}$, then Hadwiger's Conjecture holds for the class $T(\mathcal{F})$.*

Proof. Let G be a graph in $\mathcal{F}$. Let $\Delta = \Delta(G)$. Since *weak TCC* holds for G, we know that $\chi''(G) \leqslant \Delta + 3$. Together with observation12, this implies that $\chi''(G) \in \{\Delta+1, \Delta+2, \Delta+3\}$.

Case 1: $\chi''(G) = \Delta + 3$

Take a $(\Delta{+}3)$-total-critical subgraph H of G. Clearly, $\Delta(H) = \Delta$; otherwise if $\Delta(H) < \Delta$, since H also belongs to $\mathcal{F}$, H would have a total coloring using at most $(\Delta+2)$ colors by *weak TCC*, which is a contradiction. Now by lemma 29, $T(H)$ contains a clique minor of order $\Delta + 3$. As $T(H)$ is a subgraph of $T(G)$, we are done.

Case 2: $\chi''(G) = \Delta + 2$

Let H be a $(\Delta+2)$-total critical subgraph of G. Since H, being a subgraph of G, belongs to $\mathcal{F}$, and therefore satisfies *weak TCC*, we have that $\Delta(H) \geqslant \Delta - 1$. As observed earlier, we assume that H is connected, since if not, we can just take the only component of H that is not an isolated vertex to be H. If $\Delta(H) = \Delta$, then choose a vertex v_Δ of H such that $d_H(v_\Delta) = \Delta$. By lemma 25, we know that $H - v_\Delta$ is connected. Then in $T(H)$, we can contract the v-vertices corresponding to vertices in $V(H) - v_\Delta$ to a single vertex z (as in the proof of Theorem 27) so that the e-vertices corresponding to the edges incident on v_Δ, the v-vertex corresponding to v_Δ, and the vertex z form a clique of order $\Delta + 2$ in the resulting graph. On the other hand, if $\Delta(H) = \Delta - 1$, then since H is $(\Delta(H) + 3)$-total-critical, we have by lemma 29 that $T(H)$ contains a clique minor of order $(\Delta - 1) + 3 = \Delta + 2$.

Case 3: $\chi''(G) = \Delta + 1$

As we can see from observation12, this case is trivial. ☐

Corollary 31. *Hadwiger's Conjecture holds true for the total graphs of all planar graphs.*

Given a graph G with a list L_e of colors assigned to each edge e, a *list edge coloring* is a proper coloring of $E(G)$ such that each edge e is assigned a color from the list L_e. A graph G is said to be *k-edge-choosable* if a list edge coloring of G exists for any assignment of lists of size k to the edges of G. The smallest positive integer k such that G is k-edge-choosable is called the *edge choosability* of G, which is denoted as $ch'(G)$. The following is a well-known fact.

Lemma 32. *For any graph G, $\chi''(G) \leqslant ch'(G) + 2$.*

The *list coloring conjecture* is as follows :

Conjecture 33 (List coloring conjecture)**.** For any graph G, $ch'(G) = \chi'(G)$.

Theorem 34. *Let $\mathcal{F}$ be a class of graphs closed under the operation of taking subgraphs. If $\mathcal{F}$ satisfies the list coloring conjecture, then Hadwiger's Conjecture holds for the class $T(\mathcal{F})$.*

Proof. Since $\mathcal{F}$ satisfies the list coloring conjecture, for each $G \in \mathcal{F}$, $\chi''(G) \leqslant ch'(G)+2 = \chi'(G) + 2 \leqslant \Delta(G) + 3$ (by lemma 32 and Theorem 1), that is, $\mathcal{F}$ satisfies *weak TCC*. The theorem then follows from Theorem 30. ☐

Thus, the validity of the list coloring conjecture would imply that Hadwiger's Conjecture is true for all total graphs.

4 Connectivity and Hadwiger's Conjecture for total graphs

Given a partition of the vertex set of a graph, an edge of the graph is said to be a *cross-edge* if its endpoints belong to different sets of the partition.

Our next result makes use of the following well-known theorem :

Theorem 35 (Tutte; Nash-Williams; See [11])**.** *A multigraph contains k edge-disjoint spanning trees if and only if every partition P of its vertex set contains at least $k(|P|-1)$ cross-edges.*

Corollary 36 (See [11])**.** *Every $2k$-edge-connected multigraph G has k edge-disjoint spanning trees.*

Theorem 37. *Let G be a $(2k-1)$-connected graph. Then, the total graph $T(G)$ contains a clique minor of order $\Delta + k$, where $\Delta = \Delta(G)$.*

Proof. Since the statement is trivially true when $k = 1$ (by observation 12), we may assume that $k \geqslant 2$. Let v_Δ be a vertex of maximum degree in G. Let the set of e-vertices in $T(G)$ corresponding to the edges in $E_G(v_\Delta)$ be denoted by Z. Note that Z induces a Δ-clique (call it K_Z) in $T(G)$. Let H be the graph obtained from G by removing v_Δ. We will show that there exist k disjoint but pair-wise adjacent connected induced subgraphs of $T(H)$, so that by contracting these induced subgraphs we get k more vertices that can be added to the existing Δ-clique K_Z thus achieving the desired $(\Delta + k)$-clique minor in $T(G)$.

Since G is $(2k-1)$-connected, the vertex-connectivity of H is at least $2k-2$. It follows that edge-connectivity of H is also at least $2k-2$. Hence, by Corollary 36, H has $(k-1)$ pair-wise edge-disjoint spanning trees.

Let $T_1, \dots, T_{k-1}$ be the pair-wise edge-disjoint spanning trees in H. Let E'_i be the set of e-vertices in $T(G)$ corresponding to the edges in $E(T_i)$, for $i \in [k-1]$. Clearly the induced subgraph of $T(G)$ on E'_i is isomorphic to the line graph of T_i and is connected and therefore, can be contracted to a single vertex, say y_i. Let $Y := \{y_i : i \in [k-1]\}$. The set of v-vertices of $T(G)$ excluding v_Δ is also connected since they induce a subgraph isomorphic to H itself, and can be contracted to a single vertex v^*. Let F be the graph so obtained. Since $T_1, \dots, T_{k-1}$ are all spanning trees of H, it follows that $y_i y_j \in E(F)$ for distinct $i, j \in [k-1]$ and also that $v^* y_i \in E(F)$ for all $i \in [k-1]$. Further, it can be seen that in F, each $y_i \in Y$ and v^* are adjacent to all the e-vertices in Z. Therefore, $Y \cup Z \cup \{v^*\}$ forms a clique of order $\Delta + k$ in F, implying that $T(G)$ contains a $(\Delta+k)$-clique minor. □

Corollary 38. *If a $(2k-1)$-connected graph G satisfies (k)-TCC where $k \geqslant 2$, then Hadwiger's Conjecture is true for $T(G)$.*

Proof. Since G satisfies (k)-TCC, we have $\chi(T(G)) = \chi''(G) \leqslant \Delta(G)+k$. By Theorem 37, $\eta(T(G)) \geqslant \Delta(G) + k$. Thus Hadwiger's Conjecture is true for $T(G)$. $\square$

By Theorem 6, we have that (C)-TCC is true, where C is the constant in Theorem 6. We thus have the following corollary.

Corollary 39. *If G is a $(2C-1)$-connected graph, then $T(G)$ satisfies Hadwiger's Conjecture, where C is the constant from Theorem 6.*

5 Chromatic number and total coloring

Notational notes for this section. Given a coloring of the vertices and edges of a graph, we call a vertex v or an edge e which is assigned color c, a c-vertex, or a c-edge. Given a set of colors C, we call a vertex v or an edge e having a color from C a C-vertex or a C-edge, respectively. We say that the vertex v "sees" the color c if v is a c-vertex, or v has a c-edge incident on it. Note that the situation that a c-vertex u is adjacent to v is not referred to by the phrase, "v sees the color c". For a vertex v, we define $S_\varphi(v)$ to be the set of colors that are "seen" by the vertex v with respect to a given coloring φ. For positive integers i, j such that $i \leqslant j$, we denote by $[i, j]$ the set $\{i, i+1, \ldots, j\}$. A *trail* in a graph is a sequence of vertices $v_1v_2 \ldots v_k$ such that for $i \in [k-1]$, v_iv_{i+1} is an edge of the graph, and the edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ are all pairwise distinct. Note that a vertex can appear more than once on a trail. Given two subsets of edges A and B, we say that a trail $v_1v_2 \ldots v_k$ in the graph is an (A, B)-trail if $v_iv_{i+1} \in A$ when i is odd and $v_iv_{i+1} \in B$ when i is even.

Theorem 40. *Let G be a simple finite connected graph such that $\chi(G) \leqslant 5$. Then $\chi''(G) \leqslant \Delta(G) + 3$.*

Proof. By the known results, we may assume that $\chi(G) \geqslant 5$ and so $\Delta(G) \geqslant 4$. We begin by coloring $V(G)$ with colors in $[5]$, and $E(G)$ with colors in $[3, \Delta+3]$, where $\Delta := \Delta(G)$. Recall that such an edge coloring is possible by Theorem 1. Note that this coloring of $V(G) \cup E(G)$, which we shall denote by α, need not be a total coloring.

Clearly, in this coloring, no two adjacent vertices have the same color, and no two adjacent edges have the same color. But there could exist edges of the form $e = uv$ such that $\alpha(e) = \alpha(v)$ or $\alpha(e) = \alpha(u)$. Note that if that happens, then $\alpha(e) \in \{3, 4, 5\}$. We shall call such an edge a "conflicting edge".

We now create another coloring $\pi : V(G) \cup E(G) \to [\Delta+3]$, by slightly modifying the coloring α if necessary, so that the following property holds for π.

Property A. There is no 5-vertex v having a 3-edge, 4-edge and 5-edge incident on it.

The coloring α can be transformed into such a coloring π by repeatedly doing the following: Consider each 5-vertex v having a 3-edge, 4-edge, and 5-edge incident on it. Since v can now see at most $\Delta - 3$ colors from $[6, \Delta+3]$, there exists a color $c \in [6, \Delta+3]$ that is not seen by v. We assign the color c to v. Note that no neighbors of v have the color c since the 5-vertices of α formed an independent set and the color c was not assigned to any vertex in α.

We call this coloring π the "original coloring" of G. We emphasize that:

(i) $\pi|_{V(G)}$ and $\pi|_{E(G)}$ are proper colorings of $V(G)$ and $E(G)$, respectively.

(ii) π satisfies Property A.

(iii) Any conflicting edge in π is also a conflicting edge in α. Thus, if e is a conflicting edge in π, then $\pi(e) \in \{3,4,5\}$.

Our goal now is to start with the coloring π of G and then recolor each conflicting edge so that it is no longer a conflicting edge in the resulting coloring. We will do so in three *phases* and in each phase, we select a color $i \in \{3,4,5\}$ and recolor the conflicting i-edges one by one. Our recoloring strategy will rely on the following lemma.

Lemma 41. *Let φ be a coloring of $V(G) \cup E(G)$ at a certain stage of the recoloring process. Let $i \in \{3,4,5\}$ and let $A_i = [i-1]$. Suppose φ satisfies the following properties:*

(a) $\varphi|_{V(G)} = \pi|_{V(G)}$,

(b) $\varphi|_{E(G)}$ *is a proper edge coloring of G,*

(c) there are no conflicting A_i-edges in G (see the notational notes at the beginning of this section for the definition of A_i-edges),

(d) every A_i-edge in G has original color in $\{3,4,\ldots,i\}$,

(e) every A_i-edge in G that has original color i has an endpoint colored i, that is, it was originally a conflicting i-edge, and

(f) if e' is a conflicting i-edge in G, then it has original color i, that is, $\pi(e') = \varphi(e') = i$.

Further, let e be a conflicting i-edge in G. Then G can be recolored to a coloring ψ so that e is no longer a conflicting edge in ψ, and the conditions (a) – (f) hold for this new coloring ψ too.

Proof. Let $e = v_0v_1$, where $\varphi(e) = \varphi(v_1) = i$. By (f), we know that the original color of e is i.

If there exists a color in $A_i \setminus (S_\varphi(v_0) \cup S_\varphi(v_1))$, we can recolor e with that color to get ψ. It is easy to verify that all the properties (a) – (f) hold for ψ.

So we shall assume that $A_i \subseteq S_\varphi(v_0) \cup S_\varphi(v_1)$. Let X be the set of edges incident on v_0 that have original color in $[3,i]$. Since $|[3,i]| = i-2$ and $\pi|_{E(G)}$ is a proper edge coloring, we have $|X| \leqslant i-2$. Since v_0v_1 has original color i, we have $v_0v_1 \in X$, which implies that $|X \setminus \{v_0v_1\}| \leqslant i-3$. Since $\varphi(v_0v_1) = i$, it is not an A_i-edge. Thus there are at most $i-3$ A_i-edges in X. Since every A_i-edge has original color in $[3,i]$ by (d), we know that every A_i-edge incident on v_0 is in X. It follows that there are at most $i-3$ A_i-edges incident on v_0. Considering the possibility that $\varphi(v_0)$ also may be in A_i, we have $|S_\varphi(v_0) \cap A_i| \leqslant i-2$.

This implies that there exists a color $\gamma \in A_i \setminus S_\varphi(v_0)$. As $A_i \subseteq S_\varphi(v_0) \cup S_\varphi(v_1)$, we have $\gamma \in S_\varphi(v_1)$. Therefore, as $\varphi(v_1) = i$, there exists an edge v_1v_2 such that $\varphi(v_1v_2) = \gamma$. As $\gamma \notin S_\varphi(v_0)$, we have $v_1v_2 \neq v_0v_1$.

We call an edge f in G an $\bar{i}$-edge if $\varphi(f) \in A_i \setminus \{\gamma\}$ and the original color of f is i. By (e), we know that one of the endpoints of every $\bar{i}$-edge is colored i. Let Γ and $\bar{I}$ be the set of all γ- and $\bar{i}$-edges in G, respectively.

Now consider a maximal $(\Gamma, \bar{I})$-trail $P = v_1v_2 \ldots v_k$, where $k \geqslant 2$, starting with the edge v_1v_2. It is easy to see that every edge on P is an A_i-edge. Then we have the following:

(1) For each odd $t \in \{1, 2, \ldots, k\}$, $\varphi(v_t) = i$.

Note that $\varphi(v_1) = i$. Suppose that for some odd $t \in \{1, 2, \ldots, k-2\}$, we have $\varphi(v_t) = i$. Recall that $\varphi|_{V(G)} = \pi|_{V(G)}$ and that $\pi|_{V(G)}$ is a proper vertex coloring of G. Then clearly $\varphi(v_{t+1}) \neq i$, and since $v_{t+1}v_{t+2}$ is an $\bar{i}$-edge and an endpoint of every $\bar{i}$-edge must be colored i, we have $\varphi(v_{t+2}) = i$.

(2) v_0P is a path.

Otherwise, some vertex must be repeated while traversing v_0P. Let v be the first such vertex. If $v \neq v_0$, then it either has two edges having original color i or two γ-edges incident on it; as both $\varphi|_{E(G)}$ and $\pi|_{E(G)}$ are proper edge colorings of G, this is a contradiction. Now, let $v = v_0$. Recall that by choice of γ, the vertex v_0 has no incident γ-edge. Then, since v_0v_1 has original color i, there must be two incident edges with original color i at v_0; another contradiction. It follows that v_0P is a path.

Note that the last vertex v_k on P satisfies the following properties:

(3a) If $v_{k-1}v_k$ is an $\bar{i}$-edge, $\gamma \notin S_\varphi(v_k)$.

(3b) If $v_{k-1}v_k$ is a γ-edge, then there exists $c' \in A_i \setminus S_\varphi(v_k)$. (Note that $c' \neq \gamma$.)

To see this, note that in the first case, we have that k is odd, and then by (1), $\varphi(v_k) = i \neq \gamma$. By the maximality of P, we have that v_k has no incident γ-edges. In the second case, suppose $A_i \subseteq S_\varphi(v_k)$. This happens only if, for each $c \in A_i \setminus \{\varphi(v_k)\}$, v_k has an c-edge incident on it. By property (d) and the fact that $\pi|_{E(G)}$ is a proper edge coloring, these $i-2$ edges have a distinct original color from the set $[3, i]$. Since $|[3, i]| = i - 2$, we can conclude that v_k has an incident edge v_kw with original color i and $\varphi(v_kw) \in A_i$. Is it possible that $w = v_{k-1}$? Recall that in this case, $k-1$ is odd and $v_{k-2}v_{k-1}$ is an i-edge (when $k = 2$) or an $\bar{i}$-edge (when $k > 2$). As $\pi|_{E(G)}$ is a proper edge-coloring, $v_{k-1}v_k$ cannot have original color i and so $w \neq v_{k-1}$. As $\varphi(v_{k-1}v_k) = \gamma$, we have $\varphi(v_kw) \neq \gamma$, and so v_kw is an $\bar{i}$-edge. This means v_k has an incident $\bar{i}$-edge outside of P. This violates the maximality of the trail P. So there exists a color $c' \in A_i \setminus S_\varphi(v_k)$.

Now we recolor the edges on v_0P by the following color shift strategy :

- For each $1 \leqslant t \leqslant k-1$, assign $\varphi(v_tv_{t+1})$ to the preceding edge $v_{t-1}v_t$ by setting $\psi(v_{t-1}v_t) := \varphi(v_tv_{t+1})$.
- If the last edge $v_{k-1}v_k$ on P is an $\bar{i}$-edge, set $\psi(v_{k-1}v_k) := \gamma$.

- If $v_{k-1}v_k$ is a γ-edge, pick a color $c' \in A_i \setminus S_\varphi(v_k)$ and set $\psi(v_{k-1}v_k) := c'$. Such a c' exists by (3b).

Finally, set $\psi(v) := \varphi(v)$, for each vertex $v \in V(G)$, and $\psi(e) := \varphi(e)$, for each edge $e \in E(G) \setminus E(v_0P)$.

It is easy to see that properties (a), (d), (e), and (f) hold for this new coloring ψ: We have not recolored any vertex, therefore (a) holds; Other than v_0v_1, all the edges that we have recolored are A_i-edges (with respect to φ) and they remain A_i-edges with respect to ψ as well; The only new A_i-edge with respect to ψ is $e = v_0v_1$. This is consistent with (d) and (e); the set of conflicting i-edges with respect to ψ is the set of conflicting i-edges with respect to φ minus v_0v_1, therefore (f) holds. We shall now proceed to show that the color shift strategy produces no new conflicts, ensuring that ψ satisfies properties (b) and (c).

We first show that for any $t \in \{1, 2, \ldots, k-1\}$, we have $\psi(v_{t-1}v_t) \neq \psi(v_tv_{t+1})$. For $t \leqslant k-2$, we know that $\psi(v_{t-1}v_t) = \varphi(v_tv_{t+1})$, $\psi(v_tv_{t+1}) = \varphi(v_{t+1}v_{t+2})$, and that $\varphi(v_tv_{t+1}) \neq \varphi(v_{t+1}v_{t+2})$ (as φ satisfies (b)). It follows that $\psi(v_{t-1}v_t) \neq \psi(v_tv_{t+1})$. Next, if $t = k-1$, then $\psi(v_{k-2}v_{k-1}) = \varphi(v_{k-1}v_k)$ and by (3a) and (3b), $\psi(v_{k-1}v_k) \notin S_\varphi(v_k)$. Since $\varphi(v_{k-1}v_k) \in S_\varphi(v_k)$, this means that $\psi(v_{k-2}v_{k-1}) \neq \psi(v_{k-1}v_k)$. Thus in ψ, no two consecutive edges of v_0P have the same color.

Since we are recoloring only the edges on v_0P, if (b) or (c) is violated in ψ, there must be an edge on v_0P such that it conflicts either with one of its end-points or with an adjacent edge. Let t be any index such that v_tv_{t+1} is such an edge with respect to ψ. Let $c = \psi(v_tv_{t+1})$. We will say that the edge v_tv_{t+1} has a conflict at the end-point v_{t+1}, if either $\psi(v_{t+1}) = c$ or v_{t+1} has an incident edge $v_{t+1}u \neq v_tv_{t+1}$ with $\psi(v_{t+1}u) = c$. Otherwise, the conflict has to be at the end-point v_t.

We claim that v_tv_{t+1} cannot have a conflict at the end-point v_{t+1}. If $t = k-1$, this is obvious from (3a) and (3b). For $t \leqslant k-2$, $\psi(v_tv_{t+1}) = \varphi(v_{t+1}v_{t+2}) = c$. Suppose that $\psi(v_{t+1}) = c$. Then since we are recoloring only edges, we have $\varphi(v_{t+1}) = c$, which contradicts the fact that φ satisfies (c). Next, suppose that there exists an edge $v_{t+1}u \neq v_tv_{t+1}$ such that $\psi(v_{t+1}u) = c$. Since no two consecutive edges of v_0P have the same color in ψ, we know that $u \neq v_{t+2}$. Then $v_{t+1}u$ is not an edge of v_0P which means that $\varphi(v_{t+1}u) = \psi(v_{t+1}u) = c$, which implies that there are two edges with color c incident at v_{t+1} in φ. This contradicts the fact that φ satisfied (b). Therefore, in ψ, the edge v_tv_{t+1} cannot have a conflict at the end-point v_{t+1}.

Hence, v_tv_{t+1} has a conflict at the end-point v_t. We consider the two possible cases separately.

Case 1: $\psi(v_tv_{t+1}) = \gamma$.

In this case the edge v_tv_{t+1} is an $\bar{i}$-edge (if $t > 0$) or i-edge (if $t = 0$) before the recoloring. Note that this means that $t+1$ is odd. If $t = 0$, then by the choice of γ, we have $\gamma \notin S_\varphi(v_0)$ and thus the edge v_0v_1 has no conflict at the end-point v_0 in ψ. For $t \geqslant 2$, since $\varphi(v_{t-1}v_t) = \gamma$, and φ satisfied (c) and (b), we know that $\psi(v_t) = \varphi(v_t) \neq \gamma$ and that no edge incident at v_t other than $v_{t-1}v_t, v_tv_{t+1}$ has the color γ in ψ. Since in

ψ, no two consecutive edges on v_0P have the same color, $\psi(v_{t-1}v_t) \neq \gamma$, and so we can conclude that v_tv_{t+1} has no conflict at the end-point v_t in ψ in this case.

Case 2: $\psi(v_tv_{t+1}) \neq \gamma$.

In this case, $(\varphi(v_{t+1}v_{t+2}) = \psi(v_tv_{t+1})) \neq \gamma$, which implies that $v_{t+1}v_{t+2}$ is an $\bar{i}$-edge before the recoloring. Then t is odd and $\varphi(v_tv_{t+1}) = \gamma$. By (1), $\psi(v_t) = \varphi(v_t) = i$. Since $\psi(v_tv_{t+1}) \in A_i$ and v_tv_{t+1} has a conflict at the end-point v_t, this implies that there exists some edge $v_tx \neq v_tv_{t+1}$ such that $\psi(v_tx) = \psi(v_tv_{t+1})$. As no two consecutive edges on v_0P have the same color in ψ, we have that $x \neq v_{t-1}$. Then $\varphi(v_tx) = \psi(v_tx) = \psi(v_tv_{t+1})$. Thus, the three edges $v_{t-1}v_t$, v_tx, and v_tv_{t+1} are all A_i-edges with respect to φ, and by property (d) have original colors in $[3, i]$. Since $\pi|_{E(G)}$ is a proper edge-coloring, all these three edges should have distinct colors with respect to π. This can happen only if $|[3, i]| \geqslant 3$, which happens only if $i = 5$. Then with respect to π, v_t is a 5-vertex with a 3-edge, a 4-edge, and a 5-edge all incident on it. This contradicts the fact that π satisfies Property A.

Thus, we conclude that in ψ, no edge on v_0P has a conflict at either of its end-points, which also implies that $e = v_0v_1$ is not a conflicting edge in ψ. Thus ψ satisfies (b), and since e is the only new A_i-edge in ψ, property (c) holds for ψ. This completes the proof. □

With the help of the above lemma, we can start with the original coloring π and proceed with the recoloring in three *phases*. In each phase, we recolor all the conflicting i-edges by invoking lemma 41 first with $i = 3$ in phase I, then $i = 4$ in phase II, and finally $i = 5$ in phase III. Note that after the phase corresponding to i, there are no conflicting i-edges in G. Therefore, after the three phases there are no conflicts in G. Thus, we have a total coloring of G using colors from $\{1, 2, 3, \ldots, \Delta + 3\}$ implying that, $\chi''(G) \leqslant \Delta + 3$. □

6 Open problems

1. Corollary 39 states that Hadwiger's Conjecture is true for graphs with high vertex-connectivity, that is, at least C' where C' is a constant.

   It is interesting to see whether a similar statement can be made with respect to edge-connectivity instead of vertex-connectivity. Note that Theorem 35 uses only edge-connectivity. Hence, it would be natural to expect Corollary 39 to be expressed in terms of edge-connectivity. Unfortunately, we could not find a way to do it.

   Of course, it would be most desirable to prove Hadwiger's Conjecture for all total graphs dropping any constraints.

2. In Theorem 30, we show that if *weak TCC* holds for all graphs, then Hadwiger's Conjecture is true for all total graphs. However, improving this result to get the following remains challenging.

"If (k)-*TCC* is true for all graphs, then Hadwiger's Conjecture is true for all total graphs."

for some fixed positive integer $k > 3$. We think that the case when $k = 4$ itself would be interesting for it may require a new idea compared to the one used for proving the $k = 3$ case.

3. Since we could not prove Hadwiger's Conjecture for all total graphs, it is interesting to try to prove Hadwiger's Conjecture for total graphs of specialized graph classes for which *weak TCC* is not yet known to hold.

4. It remains hard to prove *weak TCC* for the general case. In Theorem 40, we proved it for 5-colorable graphs. The current best upper bound on the total chromatic number of k-colorable graphs when $6 \leqslant k \leqslant 8$ is $\Delta + 4$. Proving *weak TCC* for k-colorable graphs even for small values of k, say $k = 6, 7, \ldots$, remains open.

5. In the proof of Theorem 40, $\chi'(G)$ colors are used to do the initial proper edge coloring and an additional two colors are used to get a proper vertex coloring. This suggests the question whether it is possible, when $\chi'(G) = \Delta(G)$ (that is, when G is *class I*), to obtain the bound $\chi''(G) \leqslant \Delta(G)+2$, instead of the $\Delta(G)+3$ bound that we proved in Theorem 40. Unfortunately, our proof fails to achieve this since even when dealing with *class I* graphs, we need one extra color, the $(\Delta + 3)$-th color, to establish the Property A for the original coloring. Property A is crucial in phase III of the recoloring process, that is when conflicting 5-edges are recolored (see proof of lemma 41).

   It would be interesting to prove $\chi''(G) \leqslant \Delta(G) + 2$ when G is a *class I* graph with chromatic number 5, possibly by some clever tweaking of the proof of Theorem 40.

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