Degree Factors with Red-Blue Coloring of Regular Graphs

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Abstract

Recently, motivated to control a distribution of the vertices having specified degree in a degree factor, the authors introduced a new problem in [Graphs Combin. **39** (2023) #85], which is a degree factor problem of graphs whose vertices are colored with red or blue. In this paper, we continue its research on regular graphs.

Among some results, our main theorem is the following: Let a, b and k be integers with $1 \leqslant a \leqslant k \leqslant b \leqslant k+a+1$, and let r be a sufficiently large integer compared to a, b and k. Let G be an r-regular graph. For every red-blue vertex coloring of G in which no two red vertices are adjacent, there exists a factor F of G such that $\deg_F(x) \in \{a, b\}$ for every red vertex x and $\deg_F(y) \in \{k, k+1\}$ for every blue vertex y.

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1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively, and $|G|$ denotes the order of G. Thus $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ and $\deg_G(v)$ denote the *neighborhood* and the *degree* of v in G, respectively. For two disjoint sets $X, Y \subseteq V(G)$, let $e_G(X, Y)$ denote the number of edges of G between X and Y. Let $\mathbb{Z}$ denote the set of integers. For a set $\mathbb{S} \subseteq \mathbb{Z}$, a graph G is called an $\mathbb{S}$-*graph* if $\deg_G(v) \in \mathbb{S}$ for every vertex $v \in V(G)$. An $\{r\}$-graph is said to be r-*regular*. Similarly, a spanning subgraph F of G is called an $\mathbb{S}$-*factor* of G if $\deg_F(v) \in \mathbb{S}$ for every vertex $v \in V(G)$.

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In particular, an $\{a,b\}$-factor F of G satisfies $\deg_F(v) = a$ or b for every vertex v of G. Moreover, a $\{k\}$-factor is briefly called a *k-factor*.

In this paper, we consider degree factors of regular graphs. We begin with some known theorems related to our results.

Theorem 1. *Let λ, r and k be integers with $1 \leqslant \lambda \leqslant r$ and $1 \leqslant k < r$. Let G be a λ-edge connected r-regular graph. If one of the following conditions holds, then G has a k-factor.*
(i) G is a bipartite graph (König [10]).
(ii) Both r and k are even (Petersen [13]).
(iii) r is even, k is odd, $|G|$ is even and $r/\lambda \leqslant k \leqslant r - r/\lambda$ (Gallai [6]).
(iv) r is odd, k is even and $k \leqslant r - r/\lambda$ (Gallai [6]).
(v) Both r and k are odd and $r/\lambda \leqslant k$ (Gallai [6] and Bäbler [2]).

As a corollary of Theorem 1(ii)–(v), we obtain the following.

Corollary 2. *Let λ, r and k be integers with $1 \leqslant \lambda \leqslant r$ and $r/\lambda \leqslant k \leqslant r - r/\lambda$. Let G be a λ-edge connected r-regular graph such that $k|G|$ is even. Then G has a k-factor.*

As a relaxation of k-factors, the existence of $\{k, k+1\}$-factors is also studied.

Theorem 3. *Let r and k be integers with $1 \leqslant k < r$. Let G be an r-regular graph. Then the following hold.*

(i) *G has a $\{k, k+1\}$-factor (Tutte [14]).*
(ii) *For a maximal independent set W of G, there exists a $\{k, k+1\}$-factor F of G such that $\deg_F(x) = k$ for every $x \in V(G) - W$, as well as a $\{k, k+1\}$-factor H such that $\deg_H(x) = k+1$ for every $x \in V(G) - W$ (Egawa and Kano [4]).*
(iii) *If $k \leqslant 2r/3 - 1$, then G has a $\{k, k+1\}$-factor each of whose components is regular (Kano [8]).*

Note that Theorems 1 and 3 and their proofs are found in [1]. Here we focus on Theorem 3(ii), which controls a distribution of the vertices of degree k or $k+1$ in a $\{k, k+1\}$-factor. Inspired by the result, new version problems were posed in [5] as follows: Let G be a graph. We color every vertex of G with red or blue, and let $R(G)$ and $B(G)$ be the set of red vertices and the set of blue vertices of G, respectively. For two sets $\mathbb{S}_R, \mathbb{S}_B \subseteq \mathbb{Z}$, a spanning subgraph F of G is called a *two-tone* $(\mathbb{S}_R, \mathbb{S}_B)$*-factor* of G if $\deg_F(x) \in \mathbb{S}_R$ for every $x \in R(G)$ and $\deg_F(y) \in \mathbb{S}_B$ for every $y \in B(G)$. Note that if G has no blue vertices (resp. no red vertices), then a two-tone $(\mathbb{S}_R, \mathbb{S}_B)$-factor becomes an $\mathbb{S}_R$-factor (resp. an $\mathbb{S}_B$-factor) of G.

Two-tone factors of cubic graphs were studied as follows (here, Theorem 4(i) was obtained in [7] as a byproduct of other results).

Theorem 4. *Let G be a connected cubic graph. For every red-blue vertex coloring of G, the following statements hold.*

(i) *If G is 3-edge connected and $|R(G)|$ is even, then G has a two-tone $(\{1\}, \{0,2\})$-factor (Kaiser [7]).*

(ii) *If G is claw-free and $|R(G)|$ is even, then G has a two-tone $(\{1\}, \{0, 2\})$-factor (Furuya and Kano [5]).*

(ii) *If G is 3-edge connected claw-free, $|R(G)|$ is even and the distance between any two red vertices is at least 3, then G has a two-tone $(\{1\}, \{2\})$-factor (Furuya and Kano [5]).*

In this paper, we continue the research of two-tone factors of regular graphs. Our first result is the following.

Theorem 5. *Let r and k be integers with $1 \leqslant k < r$, and let G be an r-regular graph. For every red-blue vertex coloring of G, G has a two-tone $(\{k-1, k\}, \{k, k+1\})$-factor.*

Note that Theorem 5 can be used to prove Theorems 1(i) and 3(i) as follows. For an r-regular graph G, if we color all vertices of G with blue, then by Theorem 5, we can find a $\{k, k+1\}$-factor of G, and so Theorem 3(i) is obtained. We next prove Theorem 1(i) by using Theorem 5. Let G be an r-regular bipartite graph with the bipartition (X, Y). We color all vertices in X with red and all vertices in Y with blue. Then by Theorem 5, G has a two-tone $(\{k-1, k\}, \{k, k+1\})$-factor F. By counting the number of edges of F, we can verify that F must be a k-factor of G, and hence Theorem 1(i) is obtained.

Next we add some distance conditions on red vertices, and prove the following two theorems.

Theorem 6. *Let r and k be integers with $0 \leqslant k < r$. Let G be an r-regular graph. For every red-blue vertex coloring of G in which no two red vertices are adjacent, G has a two-tone $(\{k\}, \{k, k+1\})$-factor.*

Theorem 7. *Let r, a and k be integers with $0 \leqslant a \leqslant r$ and $0 \leqslant k < r$. Let G be an r-regular graph. For every red-blue vertex coloring of G in which the distance between any two red vertices is at least 3, G has a two-tone $(\{a\}, \{k, k+1\})$-factor.*

Note that Theorems 6 and 7 are also stronger versions of Theorem 3(i). Thus, considering the best possibility of Theorem 3(i), we cannot strengthen Theorems 6 and 7 by replacing "$\{k, k+1\}$" with "$\{k\}$".

Remark that if both r and k are even, then the distance condition in Theorem 6 is redundant (see Theorem 1(ii)). On the other hand, in any other cases, the distance condition in Theorem 6 is necessary. Let r and k be integers with $0 \leqslant k < r$, and suppose that at least one of r and k is odd. Then it is known that there exists a connected r-regular graph G having no k-factor (see [3]). We color all vertices of G with red. Then there are two adjacent red vertices and G has no two-tone $(\{k\}, \{k, k+1\})$-factor.

The distance condition in Theorem 7 is also necessary if $a \notin \{k, k+1\}$. Let r, a and k be integers with $0 \leqslant a \leqslant r$, $0 \leqslant k < r$ and $a \notin \{k, k+1\}$. Let G be an r-regular bipartite graph with the bipartition (X, Y). We color all vertices in X with red and all vertices in Y with blue. Then there are two red vertices at distance 2 and we can easily verify that G has no two-tone $(\{a\}, \{k, k+1\})$-factor.

Our next aim is to strongly control the degree of red vertices in Theorem 6. The following theorem is our main result.

Theorem 8. *Let a, b and k be integers with $1 \leqslant a \leqslant k \leqslant b \leqslant k+a+1$, and let r be a sufficiently large integer compared to a, b and k. Let G be an r-regular graph. Then for every red-blue vertex coloring of G in which no two red vertices are adjacent, G has a two-tone $(\{a,b\},\{k,k+1\})$-factor.*

Note that if r satisfies the following inequality, then Theorem 8 holds.

$$r \geqslant \max\left\{\frac{(k+1)(a+1)}{k+a+2-b}, \frac{(k+1)(b-1)}{a}, b+1\right\}.$$

Although we cannot drop the condition that $b \leqslant k+a+1$ in Theorem 8 for convenience of the proof, we do not know whether the condition is necessary or not. So we pose the following problem for readers.

Problem 9. Let a, b and k be integers with $1 \leqslant a \leqslant k$ and $a+k+2 \leqslant b \leqslant r$, and let r be a sufficiently large integer compared to a, b and k. Let G be an r-regular graph. Does G have a two-tone $(\{a,b\},\{k,k+1\})$-factor for every red-blue vertex coloring of G in which no two red vertices are adjacent?

We finally focus on another additional condition, which is the edge-connectivity, and prove the following theorem.

Theorem 10. *Let λ, r and k be integers with $1 \leqslant \lambda \leqslant r$ and $r/\lambda \leqslant k \leqslant r - r/\lambda$. Let G be a λ-edge connected r-regular graph. Then for every red-blue vertex coloring of G in which $|R(G)|$ is even if k is odd, and $|B(G)|$ is even if k is even, G has a two-tone $(\{k\},\{k-1,k+1\})$-factor.*

Let us remark that Theorem 10 is a generalization of Corollary 2 and Theorem 4(i). In Section 2, we give a stronger proposition than Theorem 5 (see Proposition 12), and by using the proposition, we prove Theorem 7. We prove Theorems 6 and 10 in Sections 3 and 4, respectively. In Section 5, we prove Theorem 8, whose proof is fairly long.

2 Proof of Theorems 5 and 7

Let G be a graph, and let $g, f : V(G) \to \mathbb{Z}$ be functions satisfying $g(v) \leqslant f(v)$ for all $v \in V(G)$. A spanning subgraph F of G is called a *(g,f)-factor* of G if $g(v) \leqslant \deg_F(v) \leqslant f(v)$ for all $v \in V(G)$. The following theorem plays a key rule in the proof of Theorems 5 and 7

Theorem 11 (Kano and Saito [9], see [1, Theorem 4.14])**.** *Let G be a graph. Let $g, f : V(G) \to \mathbb{Z}$ be functions, and let θ be a real number with $0 \leqslant \theta \leqslant 1$. If $g(v) \leqslant \theta \deg_G(v) \leqslant f(v)$ and $g(v) < f(v)$ for all $v \in V(G)$, then G has a (g,f)-factor.*

We start with the following proposition. Note that if we consider an $\{r'-1, r'\}$-graph G with no vertex of degree $r'-1$, then Proposition 12 implies Theorem 5.

Proposition 12. *Let r' and k be integers with $1 \leqslant k < r'$. Let G be an $\{r'-1, r'\}$-graph. Then for every red-blue vertex coloring of G in which every vertex of degree $r'-1$ is red, G has a two-tone $(\{k-1,k\},\{k,k+1\})$-factor.*

Proof. Define two functions $g, f : V(G) \to \mathbb{Z}$ as

$$g(v) = \begin{cases} k-1 & \text{if } v \in R(G) \\ k & \text{if } v \in B(G), \end{cases} \quad \text{and} \quad f(v) = \begin{cases} k & \text{if } v \in R(G) \\ k+1 & \text{if } v \in B(G). \end{cases}$$

Note that $g(v) < f(v)$ for all $v \in V(G)$. Let $\theta = k/r'$. For a vertex $v \in V(G)$, if $\deg_G(v) = r'-1$, then v is red and $g(v) = k-1 \leqslant \theta \deg_G(v) \leqslant k = f(v)$ because $k < r'$; and if $\deg_G(v) = r'$, then $g(v) \leqslant k = \theta \deg_G(v) \leqslant f(v)$. Hence by Theorem 11, G has a (g,f)-factor, which is the desired factor. □

Proof of Theorem 7. Let r, a, k and G be as in Theorem 7. For each $x \in R(G)$, take a set $S(x)$ of a edges of G incident with x. Let K be the spanning subgraph of G with $E(K) = \bigcup_{x \in R(G)} S(x)$. Since the distance between any two red vertices is at least 3, for each $y \in B(G)$, we have $|N_G(y) \cap R(G)| \leqslant 1$, and so $\deg_K(y) \in \{0,1\}$. If $k = 0$, then K is the desired factor. Thus we may assume that $k \geqslant 1$.

Let $G' = G - R(G)$. Then $\deg_{G'}(v) \in \{r-1, r\}$ for all $v \in V(G')$, i.e., G' is an $\{r-1,r\}$-graph. We recolor every vertex of G' with red or blue so that $R(G') = \{v \in V(G') : \deg_K(v) = 1\}$ and $B(G') = V(G') - R(G')$. Note that for a vertex $v \in V(G')$, if $\deg_{G'}(v) = r-1$, then $\deg_K(v) = 1$, and hence v is colored with red. By Proposition 12 with $r' = r$, G' has a two-tone $(\{k-1,k\},\{k,k+1\})$-factor H. Let F be the spanning subgraph of G with $E(F) = E(K) \cup E(H)$. For a vertex $v \in V(G)$, if $v \in R(G)$, then $\deg_F(v) = \deg_K(v) = a$; if $v \in B(G)$ and $\deg_K(v) = 1$, then $\deg_F(v) = \deg_K(v) + \deg_H(v) \in \{1 + (k-1), 1 + k\}$; if $v \in B(G)$ and $\deg_K(v) = 0$, then $\deg_F(v) = \deg_K(v) + \deg_H(v) \in \{0 + k, 0 + (k+1)\}$. Consequently, F is the desired factor. □

3 Proof of Theorems 6

Let G be a graph. For an integer-valued function h defined on $V(G)$ and a set $X \subseteq V(G)$, we briefly write

$$h(X) := \sum_{v \in X} h(v).$$

In particular, we define $\deg_G(X)$ by letting $\deg_G(X) = \sum_{v \in X} \deg_G(v)$. A criterion for a graph to have an (g,f)-factor is given in the following theorem.

Theorem 13 (Lovász [11], see [1, Theorem 4.1])**.** *Let G be a graph. Let $g, f : V(G) \to \mathbb{Z}$ be functions satisfying $g(v) \leqslant f(v)$ for all $v \in V(G)$. Then G has a (g,f)-factor if and only if for all pairs of disjoint sets $S, T \subseteq V(G)$,*

$$\gamma(S,T) := f(S) + \deg_G(T) - g(T) - e_G(S,T) - q^*(S,T) \geqslant 0,$$

where $q^*(S,T)$ denotes the number of components D of $G - (S \cup T)$ satisfying

$$g(u) = f(u) \ \text{ for all } \ u \in V(D), \quad \text{and} \quad f(D) + e_G(D,T) \equiv 1 \pmod 2. \qquad (1)$$

A component D of $G-(S\cup T)$ satisfying (1) is called a *q^*-odd component* of $G-(S\cup T)$. We now prove Theorem 6.

Proof of Theorem 6. Let r, k and G be as in Theorem 6. If $k=0$, then the spanning subgraph of G having no edge is the desired factor. Thus we may assume that $k \geqslant 1$.

Define two functions $g, f : V(G) \to \mathbb{Z}$ as

$$g(v)=k \quad \text{and} \quad f(v)=\begin{cases} k & \text{if } v \in R(G) \\ k+1 & \text{if } v \in B(G). \end{cases}$$

Then G has a two-tone $(\{k\},\{k,k+1\})$-factor if and only if G has a (g,f)-factor. Hence by Theorem 13, it suffices to show that $\gamma(S,T)=f(S)+\deg_G(T)-g(T)-e_G(S,T)-q^*(S,T) \geqslant 0$ for all pairs of disjoint sets $S,T \subseteq V(G)$.

Since $r \geqslant 1$, every component of G contains at least one blue vertex. This together with the fact that $g(y)<f(y)$ for all $y \in B(G)$ leads to $q^*(\emptyset,\emptyset)=0$. Hence for the case of $S=T=\emptyset$, we have $\gamma(\emptyset,\emptyset)=-q^*(\emptyset,\emptyset)=0$. Thus we may assume that $S \cup T \neq \emptyset$.

Let $D_1, D_2, \ldots, D_m$ be the q^*-odd components of $G-(S\cup T)$, where $m=q^*(S,T)$. Since D_i satisfies (1), $V(D_i) \subseteq R(G)$. Since no two red vertices are adjacent, this implies that D_i consists of exactly one red vertex. Hence $e_G(S\cup T, D_i)=r$. Let $\theta=k/r$. Then $0<\theta<1$, and we have

$$\begin{aligned}
\gamma(S,T) &= f(S)+\deg_G(T)-g(T)-e_G(S,T)-q^*(S,T) \\
&\geqslant k|S|+\deg_G(T)-k|T|-e_G(S,T)-m \\
&= \frac{k}{r}\deg_G(S)+\left(1-\frac{k}{r}\right)\deg_G(T)-e_G(S,T)-m \\
&\geqslant \theta\left(\sum_{1\leqslant i\leqslant m} e_G(S,D_i)+e_G(S,T)\right) \\
&\quad +(1-\theta)\left(\sum_{1\leqslant i\leqslant m} e_G(T,D_i)+e_G(T,S)\right)-e_G(S,T)-m \\
&= \sum_{1\leqslant i\leqslant m}\Big(\theta e_G(S,D_i)+(1-\theta)e_G(T,D_i)-1\Big). \qquad (2)
\end{aligned}$$

For each $i \in \{1,2,\ldots,m\}$, let $\varphi_i=\theta e_G(S,D_i)+(1-\theta)e_G(T,D_i)-1$. By the conditions that $1 \leqslant k < r$ and $\theta=k/r$, we easily obtain that $1 \leqslant \theta r$ and $1 \leqslant (1-\theta)r$.

If $e_G(S,D_i) \geqslant 1$ and $e_G(T,D_i) \geqslant 1$, then $\varphi_i \geqslant \theta+(1-\theta)-1=0$. If $e_G(S,D_i)=0$, then $e_G(T,D_i)=e_G(S\cup T,D_i)=r$, and hence $\varphi_i=(1-\theta)e_G(T,D_i)-1=(1-\theta)r-1 \geqslant 0$. If $e_G(T,D_i)=0$, then $e_G(S,D_i)=e_G(S\cup T,D_i)=r$, and hence $\varphi_i=\theta e_G(S,D_i)-1=\theta r-1 \geqslant 0$. In either case, we have $\varphi_i \geqslant 0$. Hence by (2), $\gamma(S,T) \geqslant \sum_{1\leqslant i\leqslant m}\varphi_i \geqslant 0$. This completes the proof of Theorem 6. □

4 Proof of Theorem 10

Let G be a graph, and let $g, f : V(G) \to \mathbb{Z}$ be functions satisfying $g(v) \leqslant f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. A spanning subgraph F of G is called a *parity (g, f)-factor* of G if $g(v) \leqslant \deg_F(v) \leqslant f(v)$ and $\deg_F(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. A criterion for a graph to have a parity (g, f)-factor is given in the following theorem.

Theorem 14 (Lovász [12] (see [1, Theorem 6.1])**.** *Let G be a graph. Let $g, f : V(G) \to \mathbb{Z}$ be functions satisfying $g(v) \leqslant f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. Then G has a parity (g, f)-factor if and only if for all pairs of disjoint sets $S, T \subseteq V(G)$,*

$$\eta(S, T) := f(S) + \deg_G(T) - g(T) - e_G(S, T) - q(S, T) \geqslant 0,$$

where $q(S, T)$ denotes the number of components D of $G - (S \cup T)$ satisfying

$$f(D) + e_G(D, T) \equiv 1 \pmod 2. \tag{3}$$

A component D of $G-(S\cup T)$ satisfying (3) is called a *q-odd component* of $G-(S\cup T)$.

Proof of Theorem 10. Let λ, r, k and G be as in Theorem 10. Define two functions $g, f : V(G) \to \mathbb{Z}$ as

$$g(v) = \begin{cases} k & \text{if } v \in R(G) \\ k-1 & \text{if } v \in B(G), \end{cases} \quad \text{and} \quad f(v) = \begin{cases} k & \text{if } v \in R(G) \\ k+1 & \text{if } v \in B(G). \end{cases}$$

Then G has a two-tone $(\{k\}, \{k-1, k+1\})$-factor if and only if G has a parity (g, f)-factor. Hence by Theorem 14, it suffices to show that $\eta(S, T) = f(S) + \deg_G(T) - g(T) - e_G(S, T) - q(S, T) \geqslant 0$ for all pairs of disjoint sets $S, T \subseteq V(G)$.

By the assumption on the cardinality of $R(G)$ and $B(G)$, we have $f(V(G)) = k|R(G)| + (k+1)|B(G)| \equiv 0 \pmod 2$. Since G is connected, this implies that for the case of $S = T = \emptyset$, $\eta(\emptyset, \emptyset) = -q(\emptyset, \emptyset) = 0$. Thus we may assume that $S \cup T \neq \emptyset$.

Let $D_1, D_2, \ldots, D_m$ be the q-odd components of $G - (S \cup T)$, where $m = q(S, T)$. Since G is λ-edge connected, $e_G(S \cup T, D_i) \geqslant \lambda$ for every $i \in \{1, 2, \ldots, m\}$. Let $\theta = k/r$. Then $0 < \theta < 1$, and we have

$$\begin{aligned} \eta(S, T) &= f(S) + \deg_G(T) - g(T) - e_G(S, T) - q(S, T) \\ &\geqslant k|S| + \deg_G(T) - k|T| - e_G(S, T) - m \\ &= \frac{k}{r} \deg_G(S) + \left(1 - \frac{k}{r}\right) \deg_G(T) - e_G(S, T) - m \\ &\geqslant \theta \left(\sum_{1 \leqslant i \leqslant m} e_G(S, D_i) + e_G(S, T) \right) \\ &\quad + (1-\theta) \left(\sum_{1 \leqslant i \leqslant m} e_G(T, D_i) + e_G(T, S) \right) - e_G(S, T) - m \\ &= \sum_{1 \leqslant i \leqslant m} \Big(\theta e_G(S, D_i) + (1-\theta) e_G(T, D_i) - 1 \Big). \end{aligned} \tag{4}$$

For each $i \in \{1, 2, \ldots, m\}$, let $\varphi_i = \theta e_G(S, D_i) + (1-\theta)e_G(T, D_i) - 1$. By the conditions that $r/\lambda \leqslant k \leqslant r(1-1/\lambda)$ and $\theta = k/r$, we obtain $1 \leqslant \theta\lambda$ and $1 \leqslant (1-\theta)\lambda$ since $k \leqslant r(1-1/\lambda)$ implies that $\theta \leqslant 1 - 1/\lambda$ and $1/\lambda \leqslant 1 - \theta$.

If $e_G(S, D_i) \geqslant 1$ and $e_G(T, D_i) \geqslant 1$, then $\varphi_i \geqslant \theta + (1-\theta) - 1 = 0$. If $e_G(S, D_i) = 0$, then $e_G(T, D_i) = e_G(S \cup T, D_i) \geqslant \lambda$, and hence $\varphi_i = (1-\theta)e_G(T, D_i) - 1 \geqslant (1-\theta)\lambda - 1 \geqslant 0$. If $e_G(T, D_i) = 0$, then $e_G(S, D_i) = e_G(S \cup T, D_i) \geqslant \lambda$, and hence $\varphi_i = \theta e_G(S, D_i) - 1 \geqslant \theta\lambda - 1 \geqslant 0$. In either case, we have $\varphi_i \geqslant 0$. Hence by (4), $\eta(S, T) \geqslant \sum_{1 \leqslant i \leqslant m} \varphi_i \geqslant 0$. This completes the proof of Theorem 10. □

5 Proof of Theorem 8

Let a, b, k, r and G be as in Theorem 8. Since r is sufficiently large, we may assume that $r \geqslant \max\{(k+1)(a+1)/(k+a+2-b), (k+1)(b-1)/a, b+1\}$. If $b = k$, then by Theorem 6, G has a two-tone $(\{b\}, \{k, k+1\})$-factor. Thus we may assume that $k+1 \leqslant b$. In particular, $a < b$. If Y is a subset of a set X, then we often write $X - Y$ for $X \setminus Y$.

Let $R = R(G)$ and $B = B(G)$. By Theorem 6, there exists a two-tone $(\{a\}, \{a, a+1\})$-factor H of G. Since $a \leqslant k$, H is also a two-tone $(\{a, b\}, \{a, a+1, \ldots, k+1\})$-factor of G. Let m be the maximum value of $\min\{\deg_F(y) : y \in B\}$ among all two-tone $(\{a, b\}, \{a, a+1, \ldots, k+1\})$-factors F of G. Namely,

$$m = \max\Big\{ \min\big\{ \deg_F(y) : y \in B \big\} : F \text{ is a two-tone } (\{a, b\}, \{a, a+1, \ldots, k+1\})\text{-factor of } G \Big\}.$$

Let $\mathcal{F}$ be the family of two-tone $(\{a, b\}, \{a, a+1, \ldots, k+1\})$-factors F of G that satisfies (i) $\min\{\deg_F(y) : y \in B\} = m$ and (ii) $|\{y \in B : \deg_F(y) = m\}|$ is as small as possible.

If $m \geqslant k$, then $\min\{\deg_F(y) : y \in B\} \geqslant k$ for $F \in \mathcal{F}$, and hence F is a two-tone $(\{a, b\}, \{k, k+1\})$-factor of G, as desired. Thus, by way of contradiction, suppose that $m \leqslant k - 1$.

Now we give some notations and terminologies. Let $F \in \mathcal{F}$. Define

$$\begin{aligned} R_F(i) &= \{x \in R : \deg_F(x) = i\} \quad \text{for } i \in \{a, b\}, \text{ and} \\ B_F(j) &= \{y \in B : \deg_F(y) = j\} \quad \text{for } j \in \{m, m+1, \ldots, k+1\}. \end{aligned}$$

Then $B_F(m) \neq \emptyset$, and $|B_F(m)|$ is minimum. For two vertices $y \in B_F(m)$ and $v \in V(G)$, a path P is called an *alternating (F, y, v)-path* if the following three conditions hold. For convenience, we allow the case of $y = v$, that is, P is a (F, y, y)-path consisting of one vertex y and having length 0.

(A1) y and v are the end-vertices of P.
(A2) if $y \neq v$, then the edge of P incident with y belongs to $E(G) - E(F)$, and
(A3) the edges of P are alternately in $E(G) - E(F)$ and in $E(F)$.

For $F \in \mathcal{F}$, we further define some sets of vertices related to alternating paths.

- Let V_F^{even} (resp. V_F^{odd}) be the set of vertices $v \in V(G)$ such that there exists an alternating (F, y, v)-path of even length (resp. odd length) for some $y \in B_F(m)$.
- For each $i \in \{a, b\}$, let $R_F^{\text{even}}(i) = R_F(i) \cap V_F^{\text{even}}$ and $R_F^{\text{odd}}(i) = R_F(i) \cap V_F^{\text{odd}}$.
- For each $j \in \{m, m+1, \dots, k+1\}$, let $B_F^{\text{even}}(j) = B_F(j) \cap V_F^{\text{even}}$ and $B_F^{\text{odd}}(j) = B_F(j) \cap V_F^{\text{odd}}$.

We restate that for a vertex $y \in B_F(m)$, there is an alternating (F, y, y)-path in G, and so $B_F^{\text{even}}(m) = B_F(m)$.

Claim 15. *Let $F \in \mathcal{F}$. Then*

(i) $V_F^{\text{even}} = R_F^{\text{even}}(a) \cup R_F^{\text{even}}(b) \cup B_F^{\text{even}}(m) \cup B_F^{\text{even}}(m+1)$.

(ii) $V_F^{\text{odd}} = R_F^{\text{odd}}(a) \cup R_F^{\text{odd}}(b) \cup B_F^{\text{odd}}(k+1)$.

Proof of (i). Suppose that there exists an alternating (F, y_1, y_2)-path P of even length for $y_1 \in B_F(m)$ and $y_2 \in B_F(j)$ with $j \in \{m+2, m+3, \dots, k+1\}$. Then the spanning subgraph F_1 of G defined by

$$E(F_1) = (E(F) \cup E(P)) - (E(F) \cap E(P))$$

satisfies $\deg_{F_1}(y_1) = \deg_F(y_1) + 1 = m + 1$, $\deg_{F_1}(y_2) = \deg_F(y_2) - 1 \in \{m+1, m+2, \dots, k\}$, and $\deg_{F_1}(v) = \deg_F(v)$ for every $v \in V(G) - \{y_1, y_2\}$. Hence F_1 is a two-tone $(\{a, b\}, \{m, m+1, \dots, k+1\})$-factor of G and $|B_{F_1}(m)| < |B_F(m)|$, which contradicts the fact that $F \in \mathcal{F}$ (or the definition of m). Thus $B_F^{\text{even}}(j) = \emptyset$ for all $j \in \{m+2, m+3, \dots, k+1\}$, which proves (i).

Proof of (ii). Suppose that there exists an alternating (F, y_1, y_2)-path P of odd length for $y_1 \in B_F(m)$ and $y_2 \in B_F(j)$ with $j \in \{m, m+1, \dots, k\}$. Then the spanning subgraph F_2 of G defined by $E(F_2) = (E(F) \cup E(P)) - (E(F) \cap E(P))$ satisfies $\deg_{F_2}(y_1) = \deg_F(y_1) + 1 = m + 1$, $\deg_{F_2}(y_2) = \deg_F(y_2) + 1 \in \{m+1, m+2, \dots, k+1\}$, and $\deg_{F_2}(v) = \deg_F(v)$ for every $v \in V(G) - \{y_1, y_2\}$. Hence F_2 is a two-tone $(\{a, b\}, \{m, m+1, \dots, k+1\})$-factor of G and $|B_{F_2}(m)| < |B_F(m)|$, which contradicts the fact that $F \in \mathcal{F}$ (or the definition of m). Thus $B_F^{\text{odd}}(j) = \emptyset$ for all $j \in \{m, m+1, \dots, k\}$, which proves (ii). $\square$

Claim 16. $V_F^{\text{even}} \cap V_F^{\text{odd}} = \emptyset$ *for* $F \in \mathcal{F}$.

Proof. Suppose that there exists a vertex $z \in V_F^{\text{even}} \cap V_F^{\text{odd}}$. Then for some $y_1, y_2 \in B_F(m)$, there exist an alternating (F, y_1, z)-path Q_1 of even length and an alternating (F, y_2, z)-path Q_2 of odd length. Since $m \leqslant k - 1$, we have $(B_F(m) \cup B_F(m+1)) \cap B_F(k+1) = \emptyset$. This together with Claim 15 implies that $z \in R$. Then $|V(Q_1)| \geqslant 3$.

Write $N_{Q_1}(z) = \{w_1\}$ and $N_{Q_2}(z) = \{w_2\}$. Note that $zw_1 \in E(F)$ and $zw_2 \in E(G) - E(F)$. Since $Q_1 - z$ is an alternating (F, y_1, w_1)-path of odd length and $Q_2 - z$ is an alternating (F, y_2, w_2)-path of even length, we have $w_1 \in V_F^{\text{odd}}$ and $w_2 \in V_F^{\text{even}}$. Since R is an independent set of G, this together with Claim 15 implies that $w_1 \in B_F^{\text{odd}}(k+1)$ and $w_2 \in B_F^{\text{even}}(m) \cup B_F^{\text{even}}(m+1)$.

Since $zw_1 \in E(F)$, if $w_1 \notin V(Q_2)$, then $Q_2 + zw_1$ is an alternating (F, y_2, w_1)-path of even length, and hence $w_1 \in B_F^{\text{even}}(k+1)$, which contradicts Claim 15. Thus $w_1 \in V(Q_2)$.

Let Q_2' be the subpath of Q_2 connecting y_2 and w_1. Since $w_1 \in B_F(k+1)$ and $B_F^{\text{even}}(k+1) = \emptyset$ by Claim 15, Q_2' is an alternating (F, y_2, w_1)-path of odd length. In particular, the unique edge of Q_2' incident with w_1 belongs to $E(G) - E(F)$. Since $zw_1 \in E(F)$ and $zw_2 \in E(G) - E(F)$, we have $w_1 \neq w_2$. Then $Q_2' + \{w_1 z, zw_2\}$ is an alternating (F, y_2, w_2)-path of odd length, and hence $w_2 \in B_F(k+1)$ by Claim 15. On the other hand, $Q_2 - z$ is an alternating (F, y_2, w_2)-path of even length, and so $w_2 \in B_F(m) \cup B_F(m+1)$ by Claim 15. This is a contradiction since $m+1 < k+1$. Therefore, the claim holds. □

Claim 17. *Let $F \in \mathcal{F}$. Then*

(i) $N_{G-E(F)}(v) \subseteq V_F^{\text{odd}}$ *for every* $v \in V_F^{\text{even}}$*, and*

(ii) $N_F(v) \subseteq V_F^{\text{even}}$ *for every* $v \in V_F^{\text{odd}}$.

Proof of (i). Suppose that there exists a vertex $u \in N_{G-E(F)}(v) \setminus V_F^{\text{odd}}$ for some $v \in V_F^{\text{even}}$. If $v \in B_F(m)$, then vu is an alternating (F, v, u)-path of length one, and so $u \in V_F^{odd}$, which is a contradiction. Thus $v \notin B_F(m)$.

Let P be an alternating (F, y, v)-path of even length for some $y \in B_F(m)$. Then $|V(P)| \geqslant 3$ and the unique edge of P incident with v belongs to $E(F)$. Since $vu \in E(G) - E(F)$, if $u \notin V(P)$, then $P + vu$ is an alternating (F, y, u)-path of odd length, and hence $u \in V_F^{\text{odd}}$, which is a contradiction. Thus $u \in V(P)$. If $u = y$, then yv is an alternating (F, y, v)-path of length one, and so $v \in V_F^{odd}$, which contradicts $v \in V_F^{even}$. Thus $u \neq y$.

Let P' be the subpath of P connecting y and u. Since $u \notin V_F^{\text{odd}}$, P' is an alternating (F, y, u)-path of even length. Since $uv \in E(G) - E(F)$, $P' + uv$ is an alternating (F, y, v)-path of odd length, and hence $v \in V_F^{\text{even}} \cap V_F^{\text{odd}}$, which contradicts Claim 16.

Proof of (ii). Suppose that there exists a vertex $u \in N_F(v) \setminus V_F^{\text{even}}$ for some $v \in V_F^{\text{odd}}$. Then there exists an alternating (F, y, v)-path P of odd length for some $y \in B_F(m)$. Note that $|V(P)| \geqslant 2$ and the unique edge of P incident with v belongs to $E(G) - E(F)$. Since $vu \in E(F)$, if $u \notin V(P)$, then $P + vu$ is an alternating (F, y, u)-path of even length, and hence $u \in V_F^{\text{even}}$, which is a contradiction. Thus $u \in V(P)$. Let P' be the subpath of P connecting y and u. Since $u \notin V_F^{\text{even}}$, P' is an alternating (F, y, u)-path of odd length. In particular, $|V(P')| \geqslant 2$ and the unique edge of P' incident with u belongs to $E(G) - E(F)$. Since $uv \in E(F)$, $P' + uv$ is an alternating (F, y, v)-path of even length, and hence $v \in V_F^{\text{even}} \cap V_F^{\text{odd}}$, which contradicts Claim 16. □

Choose an element F_0 of $\mathcal{F}$ so that $|R_{F_0}^{\text{even}}(b)| + |R_{F_0}^{\text{odd}}(a)|$ is as small as possible.

Claim 18. (i) $e_{F_0}(\{x\}, B_{F_0}(k+1)) \leqslant b - a - 1$ *for every* $x \in R_{F_0}^{\text{even}}(b)$.

(ii) $e_{G-E(F_0)}(\{x\}, B_{F_0}(m) \cup B_{F_0}(m+1)) \leqslant b - a - 1$ *for every* $x \in R_{F_0}^{\text{odd}}(a)$.

Proof of (i). Suppose that $e_{F_0}(\{x\}, B_{F_0}(k+1)) \geqslant b - a$ for some $x \in R_{F_0}^{\text{even}}(b)$. Take $z_1, z_2, \ldots, z_{b-a} \in N_{F_0}(x) \cap B_{F_0}(k+1)$. Let $F_1 = F_0 - \{xz_i : i \in \{1, 2, \ldots, b-a\}\}$. Then $\deg_{F_1}(x) = \deg_{F_0}(x) - (b-a) = a$, $\deg_{F_1}(z_i) = \deg_{F_0}(z_i) - 1 = k$ $(> m)$ for every $i \in \{1, 2, \ldots, b-a\}$, and $\deg_{F_1}(v) = \deg_{F_0}(v)$ for every $v \in V(G) - \{x, z_i : i \in \{1, 2, \ldots, b-a\}\}$. In particular, every red vertex $u \neq x$ satisfies $\deg_{F_1}(u) = \deg_{F_0}(u)$ since u and x are not adjacent. Hence F_1 is a two-tone $(\{a, b\}, \{m, m+1, \ldots, k+1\})$-factor, $\min\{\deg_{F_1}(y) : y \in B\} = m$ and $B_{F_1}(m) = B_{F_0}(m)$. This implies that $F_1 \in \mathcal{F}$ and $x \in R_{F_1}(a)$.

Now we show that

$$x \notin R_{F_1}^{\text{odd}}(a). \tag{5}$$

Suppose that $x \in R_{F_1}^{\text{odd}}(a)$. Then there exists an alternating (F_1, y, x)-path P of odd length for some $y \in B_{F_1}(m)$. Recall that $x \in R_{F_0}^{\text{even}}(b)$. If P is an alternating (F_0, y, x)-path, then $x \in V_{F_0}^{\text{even}} \cap V_{F_0}^{\text{odd}}$, which contradicts Claim 16. Thus P is not an alternating (F_0, y, x)-path. By the definition of F_1, the unique edge of P incident with x is $z_s x$ for some $s \in \{1, 2, \ldots, b-a\}$. Then $P - x$ is an alternating (F_0, y, z_s)-path of even length, and hence $z_s \in B_{F_0}^{\text{even}}(k+1)$, which contradicts Claim 15. Thus (5) holds.

Suppose that there exists a vertex $x' \in (R_{F_1}^{\text{even}}(b) \cup R_{F_1}^{\text{odd}}(a)) \setminus (R_{F_0}^{\text{even}}(b) \cup R_{F_0}^{\text{odd}}(a))$. By $x \in R_{F_0}^{\text{even}}(b)$, we have $x' \neq x$, and so $\deg_{F_1}(x') = \deg_{F_0}(x')$ as x' is a red vertex. By the choice of x', there exists an alternating (F_1, y', x')-path Q for some $y' \in B_{F_1}(m)$ that is not an alternating (F_0, y', x')-path. Since $F_1 = F_0 - \{xz_i : i \in \{1, 2, \ldots, b-a\}\}$, this implies that $xz_t \in E(Q)$ for some $t \in \{1, 2, \ldots, b-a\}$. Let Q' be the subpath of Q connecting y' and x. By (5), Q' is an alternating (F_1, y', x)-path of even length. Since $xz_t \in E(G) - E(F_1)$, $z_t \notin V(Q')$. Hence $Q' + xz_t$ is an alternating (F_1, y', z_t)-path of odd length, and hence $z_t \in B_{F_1}^{\text{odd}}(k)$, which contradicts Claim 15. Thus $R_{F_1}^{\text{even}}(b) \cup R_{F_1}^{\text{odd}}(a) \subseteq R_{F_0}^{\text{even}}(b) \cup R_{F_0}^{\text{odd}}(a)$. Since $x \in R_{F_0}^{\text{even}}(b) \setminus (R_{F_1}^{\text{even}}(b) \cup R_{F_1}^{\text{odd}}(a))$ by $\deg_{F_1}(x) = a$ and (5), we have $|R_{F_1}^{\text{even}}(b)| + |R_{F_1}^{\text{odd}}(a)| < |R_{F_0}^{\text{even}}(b)| + |R_{F_0}^{\text{odd}}(a)|$, which contradicts the choice of F_0.

Proof of (ii). Suppose that $e_{G-E(F_0)}(\{x\}, B_{F_0}(m) \cup B_{F_0}(m+1)) \geqslant b-a$ for some $x \in R_{F_0}^{\text{odd}}(a)$. Take $w_1, w_2, \ldots, w_{b-a} \in N_{G-E(F_0)}(x) \cap (B_{F_0}(m) \cup B_{F_0}(m+1))$. Let $F_2 = F_0 + \{xw_i : i \in \{1, 2, \ldots, b-a\}\}$. Then $\deg_{F_2}(x) = \deg_{F_0}(x) + (b-a) = b$, $\deg_{F_2}(w_i) = \deg_{F_0}(w_i) + 1 \in \{m+1, m+2\} \subseteq \{m+1, m+2, \ldots, k+1\}$ for every $i \in \{1, 2, \ldots, b-a\}$, and $\deg_{F_2}(v) = \deg_{F_0}(v)$ for every $v \in V(G) - \{x, w_i : i \in \{1, 2, \ldots, b-a\}\}$. In particular, every red vertex $u \neq x$ satisfies $\deg_{F_2}(u) = \deg_{F_0}(u)$ since u and x are not adjacent. Hence F_2 is a two-tone $(\{a, b\}, \{m, m+1, \ldots, k+1\})$-factor, $\min\{\deg_{F_2}(y) : y \in B\} \geqslant m$ and $|B_{F_2}(m)| \leqslant |B_{F_0}(m)|$. Considering the definitions of m and $\mathcal{F}$, this forces $\min\{\deg_{F_2}(y) : y \in B\} = m$ and $|B_{F_2}(m)| = |B_{F_0}(m)|$, and so $F_2 \in \mathcal{F}$. If $w_i \in B_{F_2}(m+1)$ for some $i \in \{1, 2, \ldots, b-a\}$, which is equivalent to $w_i \in B_{F_0}(m)$, then $|B_{F_2}(m)| < |B_{F_0}(m)|$, which is a contradiction. Thus

$$w_i \notin B_{F_0}(m) \cup B_{F_2}(m+1) \text{ for every } i \in \{1, 2, \ldots, b-a\}. \tag{6}$$

Now we show that

$$x \notin R_{F_2}^{\text{even}}(b). \tag{7}$$

Suppose that $x \in R_{F_2}^{\text{even}}(b)$. Then there exists an alternating (F_2, y, x)-path P of even length for some $y \in B_{F_2}(m)$. Recall that $x \in R_{F_0}^{\text{odd}}(a)$. If P is an alternating (F_0, y, x)-path, then $x \in V_{F_0}^{\text{even}} \cap V_{F_0}^{\text{odd}}$, which contradicts Claim 16. Thus P is not an alternating

(F_0, y, x)-path. By the definition of F_2, the unique edge of P incident with x is $w_s x$ for some $s \in \{1, 2, \ldots, b-a\}$. Then $P - x$ is an alternating (F_0, y, w_s)-path of odd length, and hence by (6), $w_s \in B^{\text{odd}}_{F_0}(m+1)$, which contradicts Claim 15. Thus (7) holds.

Suppose that there exists a vertex $x' \in (R^{\text{even}}_{F_2}(b) \cup R^{\text{odd}}_{F_2}(a)) \setminus (R^{\text{even}}_{F_0}(b) \cup R^{\text{odd}}_{F_0}(a))$. By $x \in R^{\text{odd}}_{F_0}(a)$, we have $x' \neq x$, and so $\deg_{F_2}(x') = \deg_{F_0}(x')$ as x' is a red vertex. There exists an alternating (F_2, y', x')-path Q for some $y' \in B_{F_2}(m)$ that is not an alternating (F_0, y', x')-path. Since $F_2 = F_0 + \{xw_i : i \in \{1, 2, \ldots, b-a\}\}$, this implies that $xw_t \in E(Q)$ for some $t \in \{1, 2, \ldots, b-a\}$. Let Q' be the subpath of Q connecting y' and x. By (7), Q' is an alternating (F_2, y', x)-path of odd length. Since $xw_t \in E(F_2)$, we have $w_t \notin V(Q')$. Hence $Q' + xw_t$ is an alternating (F_2, y', w_t)-path of even length. This together with (6) leads to $w_t \in B^{\text{even}}_{F_2}(m+2)$, which contradicts Claim 15. Thus $R^{\text{even}}_{F_2}(b) \cup R^{\text{odd}}_{F_2}(a) \subseteq R^{\text{even}}_{F_0}(b) \cup R^{\text{odd}}_{F_0}(a)$. Since $x \in R^{\text{odd}}_{F_0}(a) \setminus (R^{\text{even}}_{F_2}(b) \cup R^{\text{odd}}_{F_2}(a))$ by $\deg_{F_2}(x) = b$ and (7), we have $|R^{\text{even}}_{F_2}(b)| + |R^{\text{odd}}_{F_2}(a)| < |R^{\text{even}}_{F_0}(b)| + |R^{\text{odd}}_{F_0}(a)|$, which contradicts the choice of F_0. $\square$

Recall that no two red vertices are adjacent in G. Hence by Claims 15, 16, 17(ii) and 18(i), we have

$$\begin{aligned}
a|R^{\text{odd}}_{F_0}(a)| &+ b|R^{\text{odd}}_{F_0}(b)| + (k+1)|B^{\text{odd}}_{F_0}(k+1)| \\
&= e_{F_0}(R^{\text{odd}}_{F_0}(a), V^{\text{even}}_{F_0}) + e_{F_0}(R^{\text{odd}}_{F_0}(b), V^{\text{even}}_{F_0}) + e_{F_0}(B^{\text{odd}}_{F_0}(k+1), V^{\text{even}}_{F_0}) \\
&= e_{F_0}(V^{\text{odd}}_{F_0}, V^{\text{even}}_{F_0}) \\
&= e_{F_0}(V^{\text{odd}}_{F_0}, R^{\text{even}}_{F_0}(a)) + e_{F_0}(V^{\text{odd}}_{F_0}, R^{\text{even}}_{F_0}(b)) + e_{F_0}(V^{\text{odd}}_{F_0}, B^{\text{even}}_{F_0}(m)) \\
&\quad + e_{F_0}(V^{\text{odd}}_{F_0}, B^{\text{even}}_{F_0}(m+1)) \\
&\leqslant a|R^{\text{even}}_{F_0}(a)| + (b-a-1)|R^{\text{even}}_{F_0}(b)| + m|B^{\text{even}}_{F_0}(m)| \\
&\quad + (m+1)|B^{\text{even}}_{F_0}(m+1)|. \qquad (8)
\end{aligned}$$

Furthermore, it follows from Claims 15, 16, 17(i) and 18(ii) that

$$\begin{aligned}
(r-a)|R^{\text{even}}_{F_0}(a)| &+ (r-b)|R^{\text{even}}_{F_0}(b)| + (r-m)|B^{\text{even}}_{F_0}(m)| + (r-m-1)|B^{\text{even}}_{F_0}(m+1)| \\
&= e_{G-E(F_0)}(R^{\text{even}}_{F_0}(a), V^{\text{odd}}_{F_0}) + e_{G-E(F_0)}(R^{\text{even}}_{F_0}(b), V^{\text{odd}}_{F_0}) \\
&\quad + e_{G-E(F_0)}(B^{\text{even}}_{F_0}(m), V^{\text{odd}}_{F_0}) + e_{G-E(F_0)}(B^{\text{even}}_{F_0}(m+1), V^{\text{odd}}_{F_0}) \\
&= e_{G-E(F_0)}(V^{\text{even}}_{F_0}, V^{\text{odd}}_{F_0}) \\
&= e_{G-E(F_0)}(V^{\text{even}}_{F_0}, R^{\text{odd}}_{F_0}(a)) + e_{G-E(F_0)}(V^{\text{even}}_{F_0}, R^{\text{odd}}_{F_0}(b)) \\
&\quad + e_{G-E(F_0)}(V^{\text{even}}_{F_0}, B^{\text{odd}}_{F_0}(k+1)) \\
&\leqslant (b-a-1)|R^{\text{odd}}_{F_0}(a)| + (r-b)|R^{\text{odd}}_{F_0}(b)| + (r-k-1)|B^{\text{odd}}_{F_0}(k+1)|. \qquad (9)
\end{aligned}$$

Then, for any two positive real numbers α and β, we have

$$\begin{aligned}0 &\geqslant \alpha\Big(\big(\text{the left side of (8)}\big) - \big(\text{the right side of (8)}\big)\Big)\\ &\quad + \beta\Big(\big(\text{the left side of (9)}\big) - \big(\text{the right side of (9)}\big)\Big)\\ &= \big(\alpha a - \beta(b-a-1)\big)|R^{\text{odd}}_{F_0}(a)| + \big(\alpha b - \beta(r-b)\big)|R^{\text{odd}}_{F_0}(b)|\\ &\quad + \big(\alpha(k+1) - \beta(r-k-1)\big)|B^{\text{odd}}_{F_0}(k+1)|\\ &\quad + \big(-\alpha a + \beta(r-a)\big)|R^{\text{even}}_{F_0}(a)| + \big(-\alpha(b-a-1) + \beta(r-b)\big)|R^{\text{even}}_{F_0}(b)|\\ &\quad + \big(-\alpha m + \beta(r-m)\big)|B^{\text{even}}_{F_0}(m)|\\ &\quad + \big(-\alpha(m+1) + \beta(r-m-1)\big)|B^{\text{even}}_{F_0}(m+1)|. \qquad (10)\end{aligned}$$

Put $\alpha = r-k-1$ and $\beta = k+1$. Recall that $k+1 \leqslant b \leqslant k+a+1$, $r \geqslant (k+1)(a+1)/(k+a+2-b)$ and $r \geqslant (k+1)(b-1)/a$. Then we obtain

$$\begin{aligned}\alpha a - \beta(b-a-1) &= (r-k-1)a - (k+1)(b-a-1)\\ &= ra - (k+1)(b-1) \geqslant 0, \qquad (11)\end{aligned}$$

$$\alpha b - \beta(r-b) = (r-k-1)b - (k+1)(r-b) = r(b-k-1) \geqslant 0, \qquad (12)$$

$$\alpha(k+1) - \beta(r-k-1) = (r-k-1)(k+1) - (k+1)(r-k-1) = 0, \qquad (13)$$

$$-\alpha a + \beta(r-a) = -(r-k-1)a + (k+1)(r-a) = r(k+1-a) > 0, \qquad (14)$$

$$\begin{aligned}-\alpha(b-a-1) + \beta(r-b) &= -(r-k-1)(b-a-1) + (k+1)(r-b)\\ &= r(k+a+2-b) - (k+1)(a+1) \geqslant 0, \qquad (15)\end{aligned}$$

$$-\alpha m + \beta(r-m) = -(r-k-1)m + (k+1)(r-m) = r(k+1-m) > 0, \qquad (16)$$

and

$$\begin{aligned}-\alpha(m+1) + \beta(r-m-1) &= -(r-k-1)(m+1) + (k+1)(r-m-1)\\ &= r(k-m) > 0. \qquad (17)\end{aligned}$$

Since $B^{\text{even}}_{F_0}(m) = B_{F_0}(m) \neq \emptyset$, it follows from (11)–(17) that

$$0 \geqslant (-\alpha m + \beta(r-m))|B^{\text{even}}_{F_0}(m)| > 0,$$

which is a contradiction.

This completes the proof of Theorem 8. □

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References

[1] J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, **LNM 2031**, Springer (2011).

[2] D. Bäbler, Über die Zerlegung regulärer Streckenkomplexe ungerader Ordnung, *Comment. Math. Helv.*, **10** (1937) 275–287.

[3] B. Bollobás, A. Saito and N.C. Wormald, Regular factors of regular graphs, *J. Graph Theory*, **9** (1985) 97–103.

[4] Y. Egawa and M. Kano, Sufficient conditions for graphs to have (g, f)-factors, *Discrete Math.*, **151** (1996) 87–90.

[5] M. Furuya and M. Kano, Factors with red-blue coloring of claw-free graphs and cubic graphs, *Graphs Combin.*, **39** (2023) #85.

[6] T. Gallai, On factorisation of graphs, *Acta Math. Acad. Sci. Hungar.*, **1** (1950) 133–153.

[7] T. Kaiser, Disjoint T-paths in tough graphs, *J. Graph Theory*, **59** (2008) 1–10.

[8] M. Kano, Factors of regular graphs, *J. Combin. Theory Ser. B*, **41** (1986) 27–36.

[9] M. Kano and A. Saito, $[a, b]$-factors of graphs, *Discrete Math.*, **47** (1983) 113–116.

[10] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* **77** (1916) 453–465.

[11] L. Lovász, Subgraphs with prescribed valencies, *J. Combin. Theory*, **8** (1970) 391–416.

[12] L. Lovász, The factorization of graphs. II, *Acta Math. Acad. Sci. Hungar.*, **23** (1972) 223–246.

[13] J. Petersen, Die Theorie der regulären graphs, *Acta Math.* **15** (1891) 193–220.

[14] W.T. Tutte, The subgraph problem, *Ann. Discrete Math.*, **3** (1978) 289–295.