A Discrete Variation of the Littlewood–Offord Problem

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Abstract

The Littlewood–Offord Problem concerns the number of subsums of a set of vectors that fall in a given convex set. We present a discrete variation of this problem where we estimate the number of subsums that are $(0,1)$-vectors. We then utilize this to find the maximum order of graphs with given rank or corank. The rank of a graph G is the rank of its adjacency matrix $A(G)$ and the corank of G is the rank of $A(G)+I$.

Mathematics Subject Classifications: 05C50, 05C75, 15A03

1 Introduction

1.1 Littlewood–Offord Problem and its variants

Littlewood and Offord [17] dealt with the following problem in studying the number of real zeros of random polynomials: given ℓ complex numbers of modulus at least 1, from all 2^ℓ subsums, at most how many can differ from each other by less than 1? They obtained the bound $\mathcal{O}\left(\frac{\log \ell}{\sqrt{\ell}}2^\ell\right)$, which was good enough for their purpose. Erdős [4] noticed that for real numbers, Sperner's theorem (stating that any family of subsets of an ℓ-set no two of which being comparable by inclusion has size at most $\binom{\ell}{\lfloor \frac{\ell}{2}\rfloor}$) implies a best possible bound. Suppose $x_1,\dots,x_\ell$ are real numbers of modulus at least 1. For $S \subset \{1,\dots,\ell\}$, set $x_S=\sum_{i\in S} x_i$. Then $|x_S - x_{S'}|<1$ implies that S and S' are not comparable by inclusion. So Sperner's theorem implies the following:

Theorem 1 (Erdős [4]). Let $x_1,\dots,x_\ell$ be real numbers with $|x_i|\geqslant 1$ for all i. Let Λ be an open interval of length 1. Then the total number of ℓ-tuples $(\epsilon_1,\dots,\epsilon_\ell)\in\{0,1\}^\ell$ with $\epsilon_1 x_1+\cdots+\epsilon_\ell x_\ell\in\Lambda$ is at most $\binom{\ell}{\lfloor \frac{\ell}{2}\rfloor}$.

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This bound is clearly best possible: if $x_1 = \cdots = x_\ell = 1$, then $\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$ of the subsums are equal to $\lfloor \frac{\ell}{2} \rfloor$. Kleitman [14] and Katona [13] proved that the bound $\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$ holds for sums of complex numbers as well. Later, Kleitman (settling a conjecture of Erdős [4]) proved that instead of complex numbers, vectors in a Hilbert space can be taken.

Theorem 2 (Kleitman [15]). Let $\mathbf{x}_1, \ldots, \mathbf{x}_\ell$ be vectors in a Hilbert space, each with length at least 1. Let Λ be an open ball of diameter 1. Then the total number of ℓ-tuples $(\epsilon_1, \ldots, \epsilon_\ell) \in \{0,1\}^\ell$ with $\epsilon_1 \mathbf{x}_1 + \cdots + \epsilon_\ell \mathbf{x}_\ell \in \Lambda$ is at most $\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$.

These results attracted the attention of many researchers and numerous variants of the Littlewood–Offord problem have been proposed and investigated. Tao and Vu [22] initiated a line of work known as inverse Littlewood–Offord theorems. This theory and its variants played a key role in estimating the singularity probability of random matrices (see, for instance, [6, 21, 22, 23]). The Littlewood–Offord type theorems has also arisen in other contexts. In [10, 24] a modular version of the Littlewood–Offord problem is considered with application to database security.

In the present paper, we address the following question:

Discrete Variation of Littlewood–Offord Problem. Given $\mathbf{x}_1, \ldots, \mathbf{x}_\ell \in \mathbb{R}^k$, consider all 2^ℓ subsums $\mathbf{x}_S = \sum_{i \in S} \mathbf{x}_i$ for $S \subseteq \{1, \ldots, \ell\}$. How many of these are $(0,1)$-vectors? In other words, among the 2^ℓ linear combinations of the columns of the matrix $\left[\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_\ell\ \right]$ with $0, 1$ coefficients, how many result in a $(0,1)$-vector?

We observe that (see Remark 5 below) it is enough to consider reduced matrices, that is, matrices with all distinct rows, each having at least two non-zero components. Throughout, we denote the set of all $(0,1)$-vectors of length ℓ by $\{0,1\}^\ell$.

Theorem 3. Let $\mathbf{x}_1, \ldots, \mathbf{x}_\ell \in \mathbb{R}^k$ such that the matrix $\left[\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_\ell\ \right]$ is reduced. Let $\Lambda = \{0,1\}^k$. Then the total number of ℓ-tuples $(\epsilon_1, \ldots, \epsilon_\ell) \in \{0,1\}^\ell$ with $\epsilon_1 \mathbf{x}_1 + \cdots + \epsilon_\ell \mathbf{x}_\ell \in \Lambda$ is at most $\frac{2^k+1}{2^{k+1}} \cdot 2^\ell$ if $k \leqslant \ell - 1$, and $2^{\ell-1}$ if $k \geqslant \ell$.

The proof of Theorem 3 as well as the characterization of the equality cases for $1 \leqslant k \leqslant \ell - 1$ will be given in Section 2. Our subsequent results in Section 3 delve into applications of Theorem 3 within the subject of rank-order problems in algebraic graph theory, a domain that is described in the next subsection.

1.2 Rank-order problems for graphs

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A(G)$ whose (i,j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *order* of G is the number of vertices of G. We denote the set of neighbors of a vertex v of G by $N(v)$. By *eigenvalues* and *rank* of G, we mean the eigenvalues and the rank of $A(G)$ over the reals. We denote the latter by $\mathrm{rank}(G)$.

Let μ be a graph eigenvalue. An extremal problem in algebraic graph theory asks for finding the maximum order n of a graph G where $\mathrm{rank}(A(G) - \mu I)$ is a given integer r. Rowlinson [19] showed that if $\mu \notin \{0, -1\}$, then $n < r + 2^r$. This was improved in [20] to

$n \leqslant \frac{1}{2}r(r+5)-2$. Bell and Rowlinson [2] finally proved that if $\mu \notin \{0,-1\}$, then either (i) $n \leqslant \frac{1}{2}r(r+1)$ or (ii) $\mu = 1$ and $G = K_2$ or $2K_2$.

As the above result suggests, $\mu = 0, -1$ are somewhat exceptional. We first discuss the case of $\mu = 0$. In general, the order of graphs G with a fixed $r = \operatorname{rank}(G)$ can be unbounded. In fact, the order of G can be increased without changing its rank by adding a new vertex v *twin* with a vertex u (i.e. with $N(u) = N(v)$) to G or adding isolated vertices. For this reason, only *reduced graphs*, that is, graphs with no isolated vertices and no twins are taking into account. For the reduced graphs with rank r, it is easily seen that the order is bounded above by $2^r - 1$. This bound is far from being sharp. Kotlov and Lovász [16] solved the problem asymptotically. They proved that any reduced graph of rank r has order $\mathcal{O}(2^{r/2})$ and, for every $r \geqslant 2$, they constructed a reduced graph of rank r and order

$$m(r) = \begin{cases} 2^{\frac{r}{2}+1} - 2 & r \text{ even}, \\ 5 \cdot 2^{\frac{r-3}{2}} - 2 & r \text{ odd}. \end{cases}$$

This is conjectured to be the precise value of the maximum order:

Conjecture 4 (Akbari, Cameron and Khosrovshahi [1]). The maximum order of a reduced graph with rank $r \geqslant 2$ is equal to $m(r)$.

Haemers and Peeters [11] proved Conjecture 4 for graphs containing an induced matching of size $r/2$ for even r or an induced subgraph consisting of a matching of size $(r-3)/2$ and a cycle of length 3 for odd r. Ghorbani, Mohammadian and Tayfeh-Rezaie [9] proved that if Conjecture 4 is valid for all reduced graphs of rank at most 46, then it is true in general. Further, they showed that the order of every reduced graph of rank r is at most $8m(r)+14$. This problem has been also investigated within specific families of graphs. In [7], it is proved that the maximum order of every reduced tree and bipartite graph of rank r is $3r/2-1$ and $2^{r/2}+r/2-1$, respectively. This value is shown to be $3 \cdot 2^{\lfloor r/2 \rfloor - 2} + \lfloor r/2 \rfloor$ for non-bipartite triangle-free graphs in [8].

For the other exceptional eigenvalue, namely $\mu = -1$, one should consider the rank of $A(G)+I$ which we call it the *corank* of G denoted by $\operatorname{corank}(G)$. Similar to the case of rank, the order of graphs with a fixed corank can be unbounded. In fact, in any graph G, adding a new vertex v *cotwin* with a vertex u (i.e. with $N(u) \cup \{u\} = N(v) \cup \{v\}$) to G, increases the order of G without changing its corank. Therefore, one should consider *coreduced* graphs, i.e. graphs with no cotwins. Similar to the case of rank, in [5], we showed that the order of coreduced graphs with corank r is $\mathcal{O}(2^{r/2})$. It was also shown that the order of any tree and bipartite graph of corank r is at most $2r-3$ and $2r-2$, respectively, and the order of any coreduced cotree (i.e. the complement of a tree) of corank r is at most $\lfloor 3r/2 - 2 \rfloor$.

As applications for our discrete variation of Littlewood–Offord Problem, we (i) determine the maximum order of a coreduced graph with a bipartite complement of given corank, and (ii) give a new proof for the result of [7] on the maximum order of a reduced bipartite graph of given rank. In both cases, we characterize the graphs achieving the maximum order. These results will be presented in Section 3.

2 Discrete Variation of Littlewood–Offord Problem

Our objective in this section is to prove Theorem 3. Some notation is in order. In the remainder of the paper all vectors are treated as "row vectors." Let $\mathbf{v}$ be a real vector. The *weight* of $\mathbf{v}$, denoted by $\mathrm{wt}(\mathbf{v})$, is the number of non-zero components of $\mathbf{v}$. Let A be a $k \times \ell$ matrix. We set

$$\Omega(A) := \left\{ \mathbf{b} \in \{0,1\}^{\ell} : \mathbf{b}A^{\top} \in \{0,1\}^{k} \right\}.$$

In other words, $\Omega(A)$ is the set of $(0,1)$-vectors $\mathbf{b}$ of length ℓ such that the linear combination of the columns of A with the coefficients from $\mathbf{b}$ gives a $(0,1)$-vector. As a discrete variation of Littlewood–Offord Problem, in this section we deal with estimating the size of $\Omega(A)$. We call a real matrix *reduced* if all its rows are distinct and have weight at least 2. Our main result is that if A is reduced, then $\Omega(A)$ has size at most $2^{\ell-1}$ for $k \geqslant \ell$, and $\frac{2^k+1}{2^{k+1}} \cdot 2^{\ell}$ for $k \leqslant \ell - 1$.

Remark 5. Here we justify the restriction to the reduced matrices. If $\mathbf{v}$ is vector of length ℓ, then $\Omega(\mathbf{v})$ is the set of all $\mathbf{b} \in \{0,1\}^{\ell}$ such that the inner product $\mathbf{v} \cdot \mathbf{b}$ is 0 or 1. Note that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are all the rows of A, then

$$\Omega(A) = \Omega(\mathbf{v}_1) \cap \cdots \cap \Omega(\mathbf{v}_k). \tag{1}$$

So deleting repeated rows does not alter $\Omega(A)$. If some $\mathbf{v}_i$ has weight one and its non-zero component is not 1, then $|\Omega(\mathbf{v}_i)| = 2^{\ell-1}$, and thus by (1), $|\Omega(A)| \leqslant 2^{\ell-1}$, so we are done. Otherwise, assume that any weight-one row $\mathbf{v}_i$ is a $(0,1)$-vector. In that case, $\Omega(\mathbf{v}_i) = \{0,1\}^{\ell}$. It follows that $\Omega(A) = \Omega(A')$ where A' is obtained from A be removing repeated rows as well as any row of weight at most 1.

As we shall see, our main problem on bounding $|\Omega(A)|$ for real matrices A, can be reduced to $(0,\pm1)$-matrices. So in the next few lemmas, we deal with matrices/vectors with $0, \pm1$ entries.

Lemma 6. Let $\mathbf{v}$ be a ±1-vector of length ℓ. If the number of 1's in $\mathbf{v}$ is k, then, $|\Omega(\mathbf{v})| = \binom{\ell+1}{k} \leqslant \binom{\ell+1}{\lfloor \frac{\ell+1}{2} \rfloor}$.

Proof. With no loss of generality, we may assume that $\mathbf{v} = (1, \ldots, 1, -1, \ldots, -1)$, where the number of 1's is k. Let $\mathbf{b} = (b_1, \ldots, b_\ell) \in \Omega(\mathbf{v})$ and $\mathbf{b}' = (1-b_1, \ldots, 1-b_k, b_{k+1}, \ldots, b_\ell)$. Assume that $\mathrm{wt}((b_1, \ldots, b_k)) = s$ and $\mathrm{wt}((b_{k+1}, \ldots, b_\ell)) = t$. Hence $\mathrm{wt}(\mathbf{b}') = k - s + t$. We have $s - t = \mathbf{b} \cdot \mathbf{v} \in \{0,1\}$ and hence $\mathrm{wt}(\mathbf{b}') \in \{k, k-1\}$. So the number of different $\mathbf{b}'$ (and so the number of different $\mathbf{b} \in \Omega(\mathbf{v})$) is equal to $\binom{\ell}{k-1} + \binom{\ell}{k} = \binom{\ell+1}{k}$. We know that $\binom{\ell+1}{k} \leqslant \binom{\ell+1}{\lfloor \frac{\ell+1}{2} \rfloor}$, so the proof is complete. □

Given a matrix A, we denote its submatrix consisting of all the non-zero columns by A^*. If A^* is obtained by removing j zero columns, then it is clear that

$$|\Omega(A)| = 2^{j} \cdot |\Omega(A^*)|. \tag{2}$$

We say that the matrix A' is *equivalent* with A and write $A' \simeq A$, if A can be transformed into A' by row and/or column permutations. It is observed that

$$|\Omega(A')| = |\Omega(A)|.$$

From (1), it is also clear that if the matrix B is obtained by removing some of the rows of A, then

$$|\Omega(A)| \leqslant |\Omega(B)|.$$

We denote the all 1's and all 0's vectors by $\mathbf{1}$ and $\mathbf{0}$, respectively.

Lemma 7. Let A be a $k \times (k+2)$ matrix of the form

$$\left[\begin{array}{cc|cccc} \pm 1 & \pm 1 & \pm 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \pm 1 & \pm 1 & 0 & \dots & \pm 1 & 0 \\ \pm 1 & \pm 1 & 0 & \dots & 0 & a \end{array}\right], \tag{3}$$

where $a \in \{0, \pm 1\}$. Then $|\Omega(A)| \leqslant 2^{k+1} + 2$ and the equality holds if and only if A is of the form

$$A_1 = \left[\begin{array}{cc|c|c} 1 & 1 & & 0 \\ \vdots & \vdots & -I_{k-1} & \vdots \\ 1 & 1 & & 0 \\ \hline 1 & 1 & \mathbf{0} & b \end{array}\right], \quad A_2 = \left[\begin{array}{cc|c|c} a_1 & -a_1 & & 0 \\ \vdots & \vdots & I_{k-1} & \vdots \\ a_{k-1} & -a_{k-1} & & 0 \\ \hline 1 & -1 & \mathbf{0} & c \end{array}\right], \tag{4}$$

where $a_i \in \{1, -1\}$, $b \in \{0, -1\}$ and $c \in \{0, 1\}$.

Proof. If in some row of A with weight 3 there are not two 1's, then by Lemma 6 and (2), $|\Omega(A)| \leqslant \binom{4}{1} \cdot 2^{k-1} = 2^{k+1}$ and we are done. So assume that in any row of A with weight 3, there are exactly two 1's. First, suppose that in the right block of A there exist two entries with different signs. Then A contains a $2 \times (k+2)$ submatrix B with

$$B^* = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}.$$

We see that

$$\Omega(B^*) = \{0000, 0010, 0110, 0111, 1000, 1001, 1101, 1111\}.$$

Thus $|\Omega(A)| \leqslant |\Omega(B)| = |\Omega(B^*)| \cdot 2^{k-2} = 2^{k+1}$, and so we are done. Hence, we assume that in the right block of A all the non-zero entries have the same sign. It follows that A is of the form either A_1 or A_2. We have

$$\Omega(A_1) = \begin{cases} \{\mathbf{0}, \mathbf{0}1\} \cup \left(\{01, 10\} \times \{0,1\}^k\right) & \text{if } b = 0, \\ \{\mathbf{0}, \mathbf{1}\} \cup \left(\{01, 10\} \times \{0,1\}^k\right) & \text{if } b = -1. \end{cases}$$

For A_2, consider the $(0,1)$-vectors $\mathbf{b} = \frac{1}{2}(1-a_1, \ldots, 1-a_{k-1})$ and $\mathbf{b}' = \frac{1}{2}(1+a_1, \ldots, 1+a_{k-1})$. Then

$$\Omega(A_2) = \begin{cases} \{10\mathbf{b}0, 10\mathbf{b}1\} \cup \big(\{00, 11\} \times \{0,1\}^k\big) & \text{if } c = 0, \\ \{10\mathbf{b}0, 01\mathbf{b}'1\} \cup \big(\{00, 11\} \times \{0,1\}^k\big) & \text{if } c = 1. \end{cases}$$

Therefore, $|\Omega(A_1)| = |\Omega(A_2)| = 2^{k+1} + 2$. □

Similar to Lemma 7, the following can be obtained.

Lemma 8. Let A be $k \times (k+1)$ matrix of the form

$$\left[\begin{array}{c|ccc} \pm 1 & \pm 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pm 1 & 0 & \ldots & \pm 1 \end{array}\right]. \tag{5}$$

Then $|\Omega(A)| \leqslant 2^k + 1$. The equality holds if and only if A is one of the following matrices:

$$A_3 = \left[\begin{array}{c|c} 1 & \\ \vdots & -I_k \\ 1 & \end{array}\right], \quad A_4 = \left[\begin{array}{c|c} \pm 1 & \\ \vdots & I_k \\ \pm 1 & \end{array}\right]. \tag{6}$$

We also need the following lemma on $(0, \pm 1)$-matrices with two or three rows.

Lemma 9. Let A be a $k \times s$ reduced $(0, \pm 1)$-matrix and t be the maximum weight of the rows of A.

(i) If $k = 2$, $t = 6, 7$ and $s \leqslant 14$, then $|\Omega(A)| \leqslant 2^{s-1}$.

(ii) If $k = 2$, $t = 4, 5$ and $s \leqslant 10$, then $|\Omega(A)| < \frac{5}{8} \cdot 2^s$.

(iii) If $k = 2$, $t = 3$, $s \leqslant 6$, and A^* is not equivalent with

$$B_0 = \begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & 0 \\ \pm 1 & \pm 1 & 0 & a \end{bmatrix}, \tag{7}$$

where $a \in \{0, \pm 1\}$, then $|\Omega(A)| \leqslant \frac{9}{16} \cdot 2^s$.

(iv) If $k = 3$, $t = 4, 5$ and $s \leqslant 15$, then $|\Omega(A)| \leqslant 2^{s-1}$.

(v) If $k = 3$, $t = 3$, $s \leqslant 9$, and A^* is not equivalent with the matrix given in (3), then $|\Omega(A)| \leqslant 2^{s-1}$.

We verified Lemma 9 by performing an exhaustive computer search.[1] As it may not be clear from the statement, we discuss here why such a search is feasible. As an instance,

[1] The Python code of the program is available at: https://wp.kntu.ac.ir/ghorbani/ComputFiles/PythonCode.txt

we give an enumeration on the total number of inner products required to verify the part (i) of the lemma with $t = 7$. Let $\mathbf{v}$ be the first row of A of weight 7 and d be the number of 1's in $\mathbf{v}$. If $d \neq 4$, then by Lemma 6, $|\Omega(\mathbf{v}^*)| \leqslant \binom{8}{3} < 2^6$ implying that $|\Omega(\mathbf{v})| \leqslant |\Omega(\mathbf{v}^*)| \cdot 2^{s-7} < 2^{s-1}$, and we are done. So let $d = 4$. Then A is equivalent with a matrix of the form

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & c_1 & c_2 & c_3 & c_4 \end{bmatrix},$$

where $a_1 \leqslant a_2 \leqslant a_3$, $b_1 \leqslant \cdots \leqslant b_7$ and $c_1 \leqslant \cdots \leqslant c_4$. Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, \ldots, b_7)$ and $\mathbf{c} = (c_1, \ldots, c_4)$. We must have $2 \leqslant \mathrm{wt}(\mathbf{a}) + \mathrm{wt}(\mathbf{b}) + \mathrm{wt}(\mathbf{c}) \leqslant 7$. If $\mathrm{wt}(\mathbf{b}) = 7$, then $\mathrm{wt}(\mathbf{a}) = \mathrm{wt}(\mathbf{c}) = 0$ and thus $|\Omega(A)| = |\Omega(\mathbf{v}^*)| \cdot |\Omega(\mathbf{b})| \leqslant \binom{8}{4}^2 < 2^{13}$, and we are done. So $\mathrm{wt}(\mathbf{b}) \leqslant 6$. Suppose that $\mathrm{wt}(\mathbf{a}) = i$, $\mathrm{wt}(\mathbf{b}) = j$ and $\mathrm{wt}(\mathbf{c}) = r$. Given that the components of these vectors are increasing, the numbers of choices for $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are $i+1, j+1$, and $r+1$, respectively. We have $0 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 6$, and $0 \leqslant k \leqslant 4$. Furthermore, since $i + j + k \leqslant 7$, we must have $j \leqslant 7 - i$ and $k \leqslant 7 - i - j$. Taking into account these conditions on i, j, k, it follows that the number of different choices for the second row of A is at most

$$\sum_{i=0}^{3}(i+1) \sum_{j=0}^{\min(6,7-i)} (j+1) \sum_{r=0}^{\min(4,7-i-j)} (r+1) = 1267.$$

Now, for any choice of A we should compute $\mathbf{x}A^\top$ for any $\mathbf{x} \in \{0,1\}^{14}$. Since A^* has $j+7$ columns, it suffices to compute $\mathbf{x}A^{*\top}$ for any $\mathbf{x} \in \{0,1\}^{j+7}$. It turns out that the total number of required inner products to verify the assertion is at most

$$2\sum_{i=0}^{3}(i+1) \sum_{j=0}^{\min(6,7-i)} 2^{j+7}(j+1) \sum_{r=0}^{\min(4,7-i-j)} (r+1) = 3035648,$$

which shows the feasibility of the exhaustive search.

We are now prepared to prove the main result of the paper. For convenience, we repeat Theorem 3 here, including the equality cases.

Theorem 10. If A is a $k \times \ell$ reduced matrix, then

$$|\Omega(A)| \leqslant \begin{cases} \frac{2^k+1}{2^{k+1}} \cdot 2^\ell & k \leqslant \ell - 1, \\ 2^{\ell-1} & k \geqslant \ell. \end{cases}$$

For $1 \leqslant k \leqslant \ell - 1$, the equality holds if and only if A^* is equivalent with one of the matrices A_1, A_2, A_3, A_4 given in (4) and (6).

Proof. We first show that if A has an entry other than $0, \pm 1$, then we are done. To see this, with no loss of generality, assume that $\mathbf{v} = (v_1, v_2, \ldots, v_\ell)$, with $v_1 \notin \{0, \pm 1\}$, is some row of A. Let $\mathbf{a} = (1, a_2, \ldots, a_\ell) \in \{0,1\}^\ell$ and $\mathbf{a}' = (0, a_2, \ldots, a_\ell)$. We claim that at most one of $\mathbf{a}$ and $\mathbf{a}'$ belong to $\Omega(\mathbf{v})$, since otherwise

$$|v_1| = |\mathbf{a} \cdot \mathbf{v} - \mathbf{a}' \cdot \mathbf{v}| \in \{0, 1\},$$

which is a contradiction. Thus, at most one of $\mathbf{a}$ or $\mathbf{a}'$ belong to $\Omega(\mathbf{v})$. This implies that $|\Omega(A)| \leqslant |\Omega(\mathbf{v})| \leqslant 2^{\ell-1}$. So we may assume that all the entries of A are $0, \pm 1$.

Assume that the row $\mathbf{v}$ with $\mathrm{wt}(\mathbf{v}) = t$ has the largest weight among the rows of A. By Lemma 6 and (2), we have $|\Omega(\mathbf{v})| \leqslant \binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} 2^{\ell-t}$. For $t \geqslant 8$, by induction, we have $\binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} < 2^{t-1}$. Hence if $t \geqslant 8$, then $|\Omega(A)| \leqslant |\Omega(\mathbf{v})| < 2^{\ell-1}$, and we are done. Therefore, we suppose that $t \leqslant 7$. We consider the following four cases.

Case 1. $k = 1$

Since $k = 1$, and A is a reduced matrix, the weight of each row of A is at least two. Thus, $k \leqslant \ell - 1$, which means that we only need to show that $|\Omega(A)| \leqslant \frac{3}{4} \cdot 2^{\ell}$.

As $t \geqslant 2$, we have $\binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} \leqslant \frac{3}{4} \cdot 2^{t}$ with equality for $t = 2, 3$. Now, from Lemma 6 it follows that $|\Omega(A)| \leqslant |\Omega(\mathbf{v})| \leqslant \binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} 2^{\ell-t} \leqslant \frac{3}{4} \cdot 2^{\ell}$. The equality holds if and only if $t = 2, 3$ which agrees with the equality cases of the theorem.

Case 2. $k = 2$

In this case, we need to show that for $\ell = 2$, $|\Omega(A)| \leqslant 2$, and for $\ell \geqslant 3$, $|\Omega(A)| \leqslant \frac{5}{8} \cdot 2^{\ell}$. (The only possibility for A in the case $\ell = 2$ is that A is equivalent to the matrix B_1 below.)

First, assume that $t = 2$. Then, A^* is equivalent with one of

$$B_1 = \begin{bmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 \end{bmatrix}.$$

It is easy to check that at most two vectors from $\{0,1\}^2$ can belong to $\Omega(B_1)$, that is, $|\Omega(B_1)| \leqslant 2$. So if $A^* \simeq B_1$, then $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-2} \leqslant 2^{\ell-1}$, implying the result. We have $|\Omega(B_2)| = |\Omega((\pm 1, \pm 1))|^2 \leqslant 9$. Thus, if $A^* \simeq B_2$, then

$$|\Omega(A)| = |\Omega(B_2)| \cdot 2^{\ell-4} = \frac{9}{16} \cdot 2^{\ell} < \frac{5}{8} \cdot 2^{\ell},$$

and we are done. Finally, let $A^* \simeq B_3$. By Lemma 8, $|\Omega(B_3)| \leqslant 5$. It follows that $|\Omega(A)| \leqslant \frac{5}{8} \cdot 2^{\ell}$ and the equality holds if and only if A^* is equivalent with A_3 or A_4 of (6).

If $t = 3$, then A^* has $s \leqslant 6$ columns because the weight of the second row of A is at most t. If A^* is not equivalent with B_0 of (7), then Lemma 9 (iii) implies that $|\Omega(A^*)| \leqslant \frac{9}{16} \cdot 2^{s}$ and thus $|\Omega(A)| \leqslant \frac{9}{16} \cdot 2^{\ell} < \frac{5}{8} \cdot 2^{\ell}$. If $A^* \simeq B_0$, then $s = 4$ and by Lemma 7, $|\Omega(A^*)| \leqslant 10$. It follows that

$$|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-4} \leqslant \frac{5}{8} \cdot 2^{\ell},$$

and the equality holds if and only if A^* is equivalent with either A_1 or A_2 of (4).

If $t = 4, 5$, then A^* has $s \leqslant 10$ columns. By Lemma 9 (ii), $|\Omega(A^*)| < \frac{5}{8} \cdot 2^{s}$. It follows that

$$|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-s} < \frac{5}{8} \cdot 2^{\ell}.$$

If $t = 6, 7$, then $\binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} = \frac{35}{64} \cdot 2^{t} < \frac{5}{8} \cdot 2^{t}$. Then by Lemma 6, $|\Omega(A)| \leqslant |\Omega(\mathbf{v})| \leqslant \binom{t+1}{\lfloor \frac{t+1}{2} \rfloor} 2^{\ell-t} < \frac{5}{8} \cdot 2^{\ell}$.

Case 3. $k = 3$

In this case, we need to show that for $\ell = 2, 3$, $|\Omega(A)| \leqslant 2^{\ell-1}$, and for $\ell \geqslant 4$, $|\Omega(A)| \leqslant \frac{9}{16} \cdot 2^{\ell}$.

First, let $t = 2$. Comparing the $2 \times \ell$ submatrices of A with B_1, B_2, B_3 of Case 2, we see that A satisfies in one of the following three cases.

(i) For some $2 \times \ell$ submatrix B of A, we have $B^* \simeq B_1$. Thus $|\Omega(A)| \leqslant |\Omega(B)| \leqslant 2^{\ell-1}$.

(ii) For all $2 \times \ell$ submatrices B of A, we have $B^* \simeq B_3$. Then A^* is equivalent either with the matrix given in (5), or with

$$\begin{bmatrix} \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 \\ 0 & \pm 1 & \pm 1 \end{bmatrix}. \tag{8}$$

If the former occurs, then by Lemma 8, $|\Omega(A)| \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^{\ell} = \frac{9}{16} \cdot 2^{\ell}$ and the equality holds if and only if A^* is equivalent with A_3 or A_4 of (6). So assume that A^* is equivalent with (8). If some 2×3 submatrix B of A^* is equivalent to neither of A_3, A_4 of (6), then by Lemma 8, $|\Omega(A^*)| \leqslant |\Omega(B)| \leqslant 4$. It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-3} \leqslant 2^{\ell-1}$, as desired. Otherwise, A^* is equivalent with either of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then it can be easily checked that $|\Omega(A^*)| = 4$ and thus $|\Omega(A)| \leqslant 4 \cdot 2^{\ell-3} = 2^{\ell-1}$, and we are done.

(iii) A has two $2 \times \ell$ submatrices that are either both equivalent with B_2, or one is equivalent with B_2 and the other one with B_3. It turns out that A^* is equivalent with either of

$$\begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} \pm 1 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & \pm 1 \\ 0 & \pm 1 & \pm 1 & 0 \end{bmatrix}.$$

For the first one, we have $|\Omega(A^*)| \leqslant |\Omega((\pm 1, \pm 1))|^3 \leqslant 27$, and thus $|\Omega(A)| \leqslant |\Omega(A^*)| \cdot 2^{\ell-6} < 2^{\ell-1}$. For the second one, $|\Omega(A^*)| \leqslant |\Omega((\pm 1, \pm 1))| \cdot |\Omega(B_3)| \leqslant 15$, and thus $|\Omega(A)| \leqslant |\Omega(A^*)| \cdot 2^{\ell-5} < 2^{\ell-1}$. For the third one, if we have $|\Omega(A^*)| \leqslant 8$, then it will follow that $|\Omega(A)| \leqslant 2^{\ell-1}$. Otherwise, $|\Omega(A^*)| \geqslant 9$. On the other hand, $\Omega(A^*) \subseteq \Omega(B_2)$. Since $|\Omega(B_2)| \leqslant 9$, it follows that $\Omega(A^*) = \Omega(B_2)$. This in turn implies that $\Omega(B_2) \subseteq \Omega(\mathbf{x})$ where $\mathbf{x} = (0, \pm 1, \pm 1, 0)$. At least one of 0100 or 1100 and at least one of 0010 or 0011 belong to $\Omega(B_2)$. This implies that $\mathbf{x} = (0, 1, 1, 0)$. Also $\Omega(B_2)$ contains a vector of the form $*11*$. Such a vector cannot belong to $\Omega(\mathbf{x})$, a contradiction.

Next, let $t = 3$. Since the weight of each row of A is at most t, A^* has $s \leqslant 9$ columns. If A^* is not equivalent with the matrix given in (3), then by Lemma 9 (v), $|\Omega(A^*)| \leqslant 2^{s-1}$. It follows that $|\Omega(A)| = |\Omega(A^*)| \cdot 2^{\ell-s} \leqslant 2^{\ell-1}$, as desired. Otherwise, by Lemma 7, $|\Omega(A)| \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^\ell = \frac{9}{16} \cdot 2^\ell$ and the equality holds if and only if A^* is equivalent with A_1 or A_2 of (4).

If $t = 4, 5$, then A^* has $s \leqslant 15$ columns. By Lemma 9 (iv), $|\Omega(A^*)| \leqslant 2^{s-1}$. It follows that $|\Omega(A)| \leqslant |\Omega(A^*)| \cdot 2^{\ell-s} \leqslant 2^{\ell-1}$, and we are done.

If $t = 6, 7$, in a similar manner as above we are done by Lemma 9 (i).

Case 4. $k \geqslant 4$

First let $t = 2$. If A^* is equivalent with the matrix given in (5), then by Lemma 8, $|\Omega(A)| \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^\ell$ and the equality holds if and only if A^* is equivalent with A_3 or A_4 of (6). Otherwise, as shown in Case 3, for some $3 \times \ell$ submatrix B of A we have $|\Omega(B)| \leqslant 2^{\ell-1}$, and so we are done.

If $t = 3$, then we are done similarly as for $t = 2$.

If $4 \leqslant t \leqslant 7$, then we are done by Lemma 9 as in Case 3. □

3 Applications

In this section, we present two applications for our result on the discrete variation of Littlewood–Offord Problem. We first give a new proof for the result of [7] on the maximum order of a reduced bipartite graph with a given rank. Then we present another application on finding the maximum order of a coreduced *cobipartite* graph (i.e. the complement of a bipartite graph) with a given corank.

We need further notation. Let G be a bipartite graph. Then its adjacency matrix can be put in the form:

$$A(G) = \left[\begin{array}{c|c} O & B \\ \hline B^\top & O \end{array}\right].$$

We call $B = B(G)$ a *bipartite adjacency matrix* of G. When G is connected, this is unique up to permutations of rows and columns. We denote the $\ell \times 2^\ell$ matrix whose columns consist of all $(0, 1)$-vectors of length ℓ by $\mathbb{B}_\ell$. The bipartite graph G with $B(G) = \mathbb{B}_\ell$ is denoted by $\mathcal{B}_\ell$. The graph $\mathcal{B}_\ell$ is in fact the *incidence graph* of $[\ell] := \{1, \dots, \ell\}$ versus $\mathcal{P}([\ell])$, the power set of $[\ell]$. We also denote the column space and the row space of a matrix M by $\mathrm{Col}(M)$ and $\mathrm{Row}(M)$, respectively.

3.1 Bipartite graphs

The graph $\mathcal{B}_\ell$ has an isolated vertex. We denote the resulting graph by removing this isolated vertex by $\mathcal{B}'_\ell$. So $\mathcal{B}'_\ell$ is a reduced bipartite graph of rank 2ℓ and order $2^\ell + \ell - 1$.

As the first application of Theorem 10, we give a new proof for the following theorem from [7].

Theorem 11. Let G be a reduced bipartite graph of order n and rank r. Then $n \leqslant 2^{r/2} + r/2 - 1$ and the equality holds if and only if G is isomorphic to $\mathcal{B}'_{r/2}$.

Proof. Let $B = B(G)$ be a $p \times q$ matrix with rank ℓ. We have $r = 2\ell$. We can assume that $p \leqslant q$. First, suppose that $p = \ell$. Since G is a reduced graph, B has no two identical columns nor a zero column. Thus $q \leqslant 2^\ell - 1$ with equality if and only if B is equal to the matrix $\mathbb{B}_\ell$ whose zero column is removed. It follows that $n = p + q \leqslant 2^\ell + \ell - 1$ with equality if and only if G is isomorphic to $\mathcal{B}'_\ell$.

Now, assume that $p = \ell + k$ with $k \geqslant 1$. By performing column-elementary operations, we can find a basis for $\mathrm{Col}(B)$ as follows (a permutation of the rows might be also necessary):

$$W = \left[\frac{I_\ell}{C_{k\times\ell}}\right].$$

Since G is a reduced graph, W has no two identical rows and no zero row. This implies that C is a reduced matrix. Any column of B is a non-zero $(0,1)$-vector, so it is generated by a linear combination of the columns of W if the corresponding vector of coefficients belong to $\Omega(W)\backslash\{\mathbf{0}\}$. It turns out that $q \leqslant |\Omega(W)|-1$. It is also clear that $\Omega(W) = \Omega(C)$. If $k \geqslant \ell$, by Theorem 10, $|\Omega(C)| \leqslant 2^{\ell-1}$ and then as $p \leqslant q$, we have $n = p + q \leqslant 2q \leqslant 2(|\Omega(C)| - 1) < 2^\ell$, so we are done. Hence, assume that $k \leqslant \ell - 1$. By Theorem 10, $|\Omega(C)| \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^\ell$, and thus $n \leqslant \ell + k + \frac{2^k+1}{2^{k+1}} \cdot 2^\ell - 1$. If $\ell = 2$, then $k = 1$, and so $p = \ell + k = 3$ and $q \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^\ell - 1 = 2$, which is impossible. Hence, $\ell \geqslant 3$. Note that $k + \frac{2^k+1}{2^{k+1}} \cdot 2^\ell$ is maximized at $k = 1$. Thus $k + \frac{2^k+1}{2^{k+1}} \cdot 2^\ell \leqslant 1 + \frac{3}{4} \cdot 2^\ell < 2^\ell$ for $\ell \geqslant 3$. Therefore, $n < 2^\ell + \ell - 1$, which completes the proof. □

3.2 Cobipartite graphs

As the second application of Theorem 10, we determine the maximum order of coreduced cobipartite graphs with a given corank and characterize the graphs achieving the maximum order.

From known relations between ranks of matrix sums (see the item 0.4.5 (d) in [12, p. 13]), we obtain the following:

Lemma 12. For a symmetric matrix M, $\mathrm{rank}(M + J) = \mathrm{rank}(M) + 1$ if and only if $\mathbf{1} \notin \mathrm{Row}(M)$.

The following lemma is crucial for the proof of the main result of this section.

Lemma 13. Let B be a $p \times q$ $(0,1)$-matrix with $p \leqslant q$, $\mathrm{rank}(B) = \ell$ and $\mathbf{1} \in \mathrm{Row}(B)$. Also assume that B has no two identical columns or rows nor a zero row. If $p+q \geqslant 2^{\ell-1}+\ell-1$ and $\ell \geqslant 6$, then B is a submatrix of

$$\begin{bmatrix} \mathbb{B}_{\ell-1} \\ \hline \mathbf{1} \\ \hline J - \mathbb{B}_{\ell-1} \end{bmatrix}, \tag{9}$$

with a single exception in the case that $\ell = 6$, $p + q = 2^{\ell-1} + \ell - 1$, and the columns of

B are generated by

$$\left[\begin{array}{c} I_6 \\ \hline \mathbf{x} \\ \hline \mathbf{1} \\ \hline J_6 - I_6 \\ \hline \mathbf{1} - \mathbf{x} \end{array}\right], \tag{10}$$

for some vector $\mathbf{x}$ of weight 2 or 3.

Proof. We first construct a new matrix from B as follows: if $\mathbf{1}$ is not already a row of B, we add it to the rows. Additionally, for any row $\mathbf{x} \neq \mathbf{1}$ of B, if $\mathbf{1} - \mathbf{x}$ is not a row, we add that as well. We call the resulting matrix B'. The matrix B' is of the following form:

$$B' = \left[\begin{array}{c} B_0 \\ \hline \mathbf{1} \\ \hline J - B_0 \end{array}\right],$$

where B_0 consists of the rows of B' whose first component is zero. As B' is obtained by adding some rows to B, it follows that $\operatorname{rank}(B') \geqslant \operatorname{rank}(B)$. However, each row of B' can be expressed as a linear combination of $\mathbf{1}$ and some row of B. Since $\mathbf{1} \in \operatorname{Row}(B)$, this implies $\operatorname{Row}(B') \subseteq \operatorname{Row}(B)$, leading to $\operatorname{rank}(B') = \operatorname{rank}(B) = \ell$. Given that $\mathbf{1} \notin \operatorname{Row}(B_0)$ and every row of B' can be formed through a linear combination of the rows of B_0 and $\mathbf{1}$, we conclude that $\operatorname{rank}(B_0) = \ell - 1$. Our assumption on B guarantees that B_0 has no two identical columns/rows and no zero rows. If B_0 has $\ell - 1$ rows, then B_0 is a submatrix of $\mathbb{B}_{\ell-1}$, and we are done. Therefore, assume that B_0 has $\ell - 1 + k$ rows for some $k \geqslant 1$. So, $p \leqslant 2\ell + 2k - 1$. By performing column-elementary operations and possibly permuting the rows, we can assume that B_0 has a basis of the form

$$\left[\begin{array}{c} I_{\ell-1} \\ \hline C_{k \times (\ell-1)} \end{array}\right].$$

This basis has no identical rows nor a zero row. This implies that C is a reduced matrix. Every column of B belongs to $\{A\mathbf{b}^\top : \mathbf{b} \in \Omega(C)\}$. So $q \leqslant |\Omega(C)|$. If $k \geqslant \ell - 1$, then by Theorem 10, $|\Omega(C)| \leqslant 2^{\ell-2}$. Thus $p + q \leqslant 2q \leqslant 2|\Omega(C)| \leqslant 2^{\ell-1}$, which is a contradiction. Hence, assume that $1 \leqslant k \leqslant \ell - 2$. By Theorem 10, we have $|\Omega(C)| \leqslant \frac{2^k+1}{2^{k+1}} \cdot 2^{\ell-1}$, and so

$$p + q \leqslant f := 2\ell + 2k - 1 + \frac{2^k + 1}{2^{k+1}} \cdot 2^{\ell-1}.$$

If $\ell = 6$ and $2 \leqslant k \leqslant 4$, by direct computation one can verify that $f < 2^{\ell-1} + \ell - 1$. For $\ell = 6$ and $k = 1$, we have $f = 2^{\ell-1} + \ell - 1$. This implies that $q = |\Omega(C)| = \frac{3}{4} \cdot 2^{\ell-1}$. By the cases of equality in Theorem 10, C should consists of a vector of weight 2 or 3, and thus $\operatorname{Col}(B)$ has a basis of the form (10). If $\ell \geqslant 7$, $2k + \frac{2^k+1}{2^{k+1}} \cdot 2^{\ell-1}$ is maximized at $k = 1$. Therefore,

$$f \leqslant 2\ell + 1 + \frac{3}{4} \cdot 2^{\ell-1} < 2^{\ell-1} + \ell - 1,$$

from which the result follows. □

We denote the bipartite graph G with

$$B(G) = \left[\frac{\mathbb{B}_\ell}{J - \mathbb{B}_\ell}\right],$$

by $\mathcal{D}_\ell$. In other words, $\mathcal{D}_\ell$ is a bipartite graph with parts $\{1, 1', \dots, \ell, \ell'\}$ and $\mathcal{P}([\ell])$, such that each $S \in \mathcal{P}([\ell])$ has the ℓ neighbors $\{i : i \in S\} \cup \{j' : j \in [\ell] \setminus S\}$. As an instance, $\mathcal{D}_3$ is depicted in Figure 1.

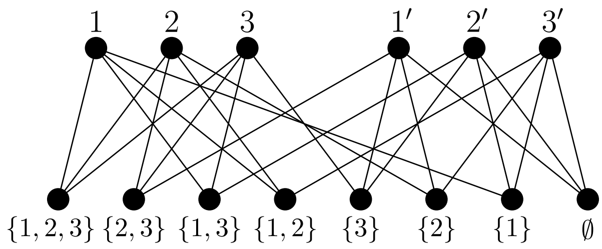

Figure 1: The graph $\mathcal{D}_3$.

Now, we are in a position to prove the main result of this section. Recall that the complement of a graph G is denoted by $\overline{G}$.

Theorem 14. If $\overline{G}$ is a coreduced cobipartite graph with order n and corank r, then

$$n \leqslant \begin{cases} 2^{\frac{r}{2}-1} + r - 2 & r \text{ even}, \\ 2^{\frac{r-1}{2}} + \frac{r-1}{2} & r \text{ odd}. \end{cases}$$

The equality holds if and only if G is isomorphic to $\mathcal{D}_{\frac{r}{2}-1}$ for even r, and to $\mathcal{B}_{\frac{r-1}{2}}$ for odd r.

Proof. Suppose that $\overline{G}$ is a coreduced cobipartite graph with corank r and the maximum possible order n. Let $\overline{A} = A(\overline{G})$ and $A = A(G)$. Also let $B = B(G)$ be a $p \times q$ matrix. So, $n = p + q$. With no loss of generality, assume that $p \leqslant q$. Since $\overline{G}$ is a coreduced graph, G has no twins. So B has no identical rows/columns. Note that G might have an isolated vertex. In which case, we can assume that the isolated vertex lies in the larger part of G, that is, B has a zero column rather than a zero row. Recall that $r = \operatorname{rank}(\overline{A} + I)$. So from $\overline{A} + I = J - A$, it follows that

$$r - 1 \leqslant \operatorname{rank}(A) = 2 \operatorname{rank}(B) \leqslant r + 1. \tag{11}$$

We verified the result for $r \leqslant 10$ by a computer search. This is done by implementing an algorithm from [3] (see also [1]) for constructing coreduced graphs of a fixed corank r. For a given r, the input of the algorithm is the set of coreduced graphs with both order and corank equal to r (which was generated by using McKay database of small graphs [18]) and the output of the algorithm is the set of all coreduced graphs of corank r. So in what follows, we assume that $r \geqslant 11$.

First suppose that $r = 2\ell$ is even and so $\ell \geqslant 6$. From (11) it follows that $\mathrm{rank}(A) = r$. Hence, by Lemma 12, $\mathbf{1} \in \mathrm{Row}(A)$. It follows that $\mathbf{1}_q \in \mathrm{Row}(B)$ and $\mathbf{1}_p^\top \in \mathrm{Col}(B)$. If $n = p + q < 2^{\ell-1} + 2\ell - 2$, there is nothing to prove. Hence, we assume that $p + q \geqslant 2^{\ell-1} + 2\ell - 2$. So B satisfies the conditions of Lemma 13, and thus it is a submatrix of the matrix C given in (9). However, $\mathbf{1}^\top \notin \mathrm{Col}(C)$ because $\mathrm{Col}(C)$ has the following basis:

$$\left[\begin{array}{c|c} \mathbf{0}^\top & I_{\ell-1} \\ \hline 1 & \mathbf{1}_{\ell-1} \\ \hline \mathbf{1}^\top & J_{\ell-1} - I_{\ell-1} \end{array}\right], \tag{12}$$

and it is clear that such a basis cannot generate $\mathbf{1}^\top$. Therefore, B must have at least one row or one column less than C. This shows that $n \leqslant 2^{\ell-1} + 2\ell - 2$. If we remove the $\mathbf{1}$ row of C, then the resulting matrix is $B(\mathcal{D}_{\ell-1})$. So $G = \mathcal{D}_{\ell-1}$, as desired. To finish the proof, we show that if one deletes any other row or any column from C, then $\mathbf{1}^\top$ does not belong to the column space of the resulting matrix. If we remove a row other than $\mathbf{1}$ from C to obtain C', then the restriction of (12) to C' forms a basis for $\mathrm{Col}(C')$. Again such a basis does not generate $\mathbf{1}^\top$. A similar argument works in the case that C' is obtained by removing one column from C.

Next, suppose that $r = 2\ell - 1$ is odd and so $\ell \geqslant 6$. Let $n \geqslant 2^{\ell-1} + \ell - 1$. To establish the theorem, it suffices to show that G is isomorphic to $\mathcal{B}_{\ell-1}$. By (11), we have $\mathrm{rank}(A) = 2\ell - 2$ or 2ℓ. If $\mathrm{rank}(A) = 2\ell - 2$, then we have necessarily $B = \mathbb{B}_{\ell-1}$, that is $G = \mathcal{B}_{\ell-1}$ and we are done. So in what follows, we assume that $\mathrm{rank}(A) = 2\ell$, i.e. $\mathrm{rank}(B) = \ell$. Given that $A = J - (\overline{A} + I)$, we have $\mathrm{rank}(J - (\overline{A} + I)) = r + 1 = \mathrm{rank}(-(\overline{A} + I)) + 1$. By invoking Lemma 12, this implies that $\mathbf{1} \notin \mathrm{Row}(\overline{A} + I)$. Furthermore, since $\mathrm{rank}(J - A) < \mathrm{rank}(-A)$, another application of Lemma 12 establishes that $\mathbf{1} \in \mathrm{Row}(A)$, implying $\mathbf{1} \in \mathrm{Row}(B)$. Given this and the condition $n \geqslant 2^{\ell-1} + \ell - 1$, the criteria outlined in Lemma 13 are satisfied. Consequently, $\mathrm{Col}(B)$ has a basis of the form (10) or B is a submatrix of (9). If the former occurs, then $\mathbf{1}^\top \notin \mathrm{Col}(B)$, which implies $\mathbf{1} \notin \mathrm{Row}(A)$, leading to a contradiction. Therefore, B is a submatrix of (9). Note that $\mathbf{1}$ cannot be a row of B. Otherwise, similar to the case of even r, we observe that $\mathbf{1}^\top \notin \mathrm{Col}(B)$, resulting in $\mathbf{1} \notin \mathrm{Row}(A)$, which is a contradiction. Now, we make use of the fact that $\mathbf{1} \notin \mathrm{Row}(\overline{A} + I)$. We have

$$\overline{A} + I = \left[\begin{array}{c|c} J & J - B \\ \hline J - B^\top & J \end{array}\right].$$

We claim that if some vector $\mathbf{x}$ is a row of B, then $\mathbf{1} - \mathbf{x}$ is not a row of B. If this fails, then we can obtain $\left[\begin{array}{c|c} 2\mathbf{1}_p & \mathbf{1}_q \end{array}\right]$ as sum of two rows of $\left[\begin{array}{c|c} J & J - B \end{array}\right]$. Also, as B has more than $2^{\ell-2}$ columns, it contains some two columns of the forms $\mathbf{y}^\top$ and $\mathbf{1}^\top - \mathbf{y}^\top$. The two corresponding rows in $\left[\begin{array}{c|c} J - B^\top & J \end{array}\right]$ sum up to $\left[\begin{array}{c|c} \mathbf{1}_p & 2\mathbf{1}_q \end{array}\right]$. It turns out that $\mathbf{1}_n = \frac{1}{3}\left[\begin{array}{c|c} 2\mathbf{1}_p & \mathbf{1}_q \end{array}\right] + \frac{1}{3}\left[\begin{array}{c|c} \mathbf{1}_p & 2\mathbf{1}_q \end{array}\right] \in \mathrm{Row}(\overline{A} + I)$, again a contradiction. This proves the claim. So we have established that B is a submatrix of (9) such that $\mathbf{1}_q$ is not a row of B and if $\mathbf{x}$ is a row of B, then $\mathbf{1} - \mathbf{x}$ is not a row of B. It follows that B has at most $\ell - 1$ rows. This is a contradiction because $\mathrm{rank}(B) = \ell$. This means that the case $\mathrm{rank}(A) = 2\ell$ is impossible, and the proof is complete. □

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