Sharp Degree Bounds for Fake Weighted Projective Spaces

Andreas Bäuerle[a]

Submitted: Nov 28, 2022; Accepted: Jan 27, 2024; Published: Feb 23, 2024

Abstract

We give sharp upper bounds on the anticanonical degree of fake weighted projective spaces, only depending on the dimension and the Gorenstein index.

Mathematics Subject Classifications: 14M25, 52B20

1 Introduction

A d-dimensional *fake weighted projective space* is a quotient $X = (\mathbb{C}^{d+1}\backslash\{0\})/G$ by a diagonal action of $G := \mathbb{C}^*\times\Gamma$, where Γ is a finite abelian group and the factor $\mathbb{C}^*$ acts via positive weights. Any fake weighted projective space X is normal, $\mathbb{Q}$-factorial, of Picard number one and is a Fano variety, i.e. its anticanonical divisor $-\mathcal{K}$ is ample. Apart from the classical projective spaces, all fake weighted projective spaces are singular, but have at most abelian quotient singularities.

Fake weighted projective spaces form an interesting example class for the general question of effectively bounding geometric data of a Fano variety in terms of its singularities. For instance, Kasprzyk [12] bounds the order of the torsion part of the divisor class group of a fake weighted projective space X and in [1] the authors give a sharp bound on this invariant provided that X has at most canonical singularities. Another invariant of the singularities is the *Gorenstein index*, i.e. the minimal positive integer ι such that $\iota\mathcal{K}$ is Cartier. In the case of Gorenstein index $\iota = 1$, Nill [13] provides a bound for the *degree* of a d-dimensional fake weighted projective space X, i.e. the self intersection number $(-\mathcal{K})^d$ of its anticanonical divisor.

In the present paper, we extend Nill's bound to higher Gorenstein indices. For any $d \geqslant 2$ define a $(d+1)$-tuple of positive integers by

$$Q_{\iota,d} := \left(\frac{2t_{\iota,d}}{s_{\iota,1}}, \ldots, \frac{2t_{\iota,d}}{s_{\iota,d-1}}, 1, 1\right), \quad s_{\iota,k} := \iota\, s_{\iota,1}\cdots s_{\iota,k-1} + 1, \quad t_{\iota,k} := \iota\, s_{\iota,1}\cdots s_{\iota,k-1},$$

[a] Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany (`baeuerle@math.uni-tuebingen.de`).

where $s_{\iota,1} := \iota + 1$. For $\iota = 1, 2, 3$ the beginning of the sequences $(s_{\iota,k})_k$ are the following

$$(s_{1,k})_k = 2, 3, 7, \ldots \qquad (s_{2,k})_k = 3, 7, 43, \ldots \qquad (s_{3,k})_k = 4, 13, 157, \ldots$$

Moreover, the corresponding $(d+1)$-tuples $Q_{\iota,d}$ for $d = 2, 3$ are given by:

$$\begin{aligned} Q_{1,2} &= (2,1,1), & Q_{2,2} &= (4,1,1), & Q_{3,2} &= (6,1,1), \\ Q_{1,3} &= (6,4,1,1), & Q_{2,3} &= (28,12,1,1), & Q_{3,3} &= (78,24,1,1). \end{aligned}$$

Our main result provides sharp upper bounds on the degree $(-\mathcal{K})^d$ in terms of the Gorenstein index and lists the cases attaining these bounds:

Theorem 1. *The anticanonical degree of any d-dimensional fake weighted projective space X of Gorenstein index ι is bounded from above according to the following table.*

d	1	2	2	3	3	$\geqslant 4$
ι	$\geqslant 1$	1	$\geqslant 2$	1	$\geqslant 2$	$\geqslant 1$
bound on $(-\mathcal{K})^d$	2	9	$\frac{2(\iota+1)^2}{\iota}$	72	$\frac{2\,t_{\iota,3}^2}{\iota^4}$	$\frac{2\,t_{\iota,d}^2}{\iota^{d+1}}$
attained exactly by	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}(2\iota,1,1)$	$\mathbb{P}(3,1,1,1)$, $\mathbb{P}(6,4,1,1)$	$\mathbb{P}(Q_{\iota,3})$	$\mathbb{P}(Q_{\iota,d})$

Equality on the degree holds if and only if X is isomorphic to one of the weighted projective spaces in the last row of the table.

Theorem 1 provides sharp upper degree bounds for fake weighted projective spaces. Lower degree bounds for given dimension and Gorenstein index should exist due to the fact that there are only finitely many isomorphy classes of fake weighted projective spaces with fixed dimension and Gorenstein index, see [5] for a classification procedure; it would be nice to have a closed formula for lower degree bounds like in Theorem 1. Leaving the class of fake weighted projective spaces, in [3] the authors provide sharp upper degree bounds for any toric Fano variety, only depending on it's dimension, provided it has at most canonical singularities.

The article is organized as follows. Section 2 provides basic properties of fake weighted projective spaces. In Section 3 we assign to any d-dimensional fake weighted projective space of Gorenstein index ι a certain partition of $1/\iota$ into $d+1$ unit fractions and give a formula to compute the anticanonical degree in terms of the denominators of these unit fractions. Section 4 contains the number theoretic part of the proof of Theorem 1. In Section 5 we complete the proof of the main result. This amounts to constructing a weighted projective space of given dimension d and Gorenstein index ι whose unit fraction partition of $1/\iota$ meets a maximality condition.

2 Fake weighted projective spaces

We recall basic properties of fake weighted projective spaces and fix our notation, see also [13, Sec. 3]. The reader is assumed to be familiar with the very basics of toric geometry [8, 9]. Throughout the article N is a rank d lattice for some $d \in \mathbb{Z}_{\geqslant 2}$. Its dual lattice is denoted by $M = \mathrm{Hom}(N, \mathbb{Z})$ with pairing $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}$. We write $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Polytopes $P \subseteq N_{\mathbb{R}}$ are assumed to be full dimensional. The *normalized volume* of a d-dimensional polytope P is $\mathrm{Vol}(P) = d!\,\mathrm{vol}(P)$, where $\mathrm{vol}(P)$ denotes its euclidean volume. Suppose the origin $\mathbf{0} \in N_{\mathbb{R}}$ is contained in the interior of P. Then the *dual* of P is the polytope

$$P^* \ := \ \{u \in M_{\mathbb{R}};\ \langle u, v \rangle \geqslant -1 \text{ for all } v \in P\} \ \subseteq \ M_{\mathbb{R}}.$$

For a facet F of P we denote by $u_F \in M_{\mathbb{R}}$ the unique linear form with $\langle u_F, v \rangle = -1$ for all $v \in F$. We have

$$P^* \ = \ \mathrm{conv}(\, u_F;\ F \text{ facet of } P\,), \qquad P \ = \ \{v \in N_{\mathbb{R}};\ \langle u_F, v \rangle \geqslant -1,\ F \text{ facet of } P\}.$$

A *lattice polytope* $P \subseteq N_{\mathbb{R}}$ is a polytope whose vertices are lattice points in N. An *IP-polytope* is a lattice polytope that contains the origin $\mathbf{0} \in N_{\mathbb{R}}$ in its interior. A *Fano* polytope is an IP-polytope whose vertices are primitive lattice points. We regard two lattice polytopes $P \subseteq N_{\mathbb{R}}$ and $P' \subseteq N'_{\mathbb{R}}$ as isomorphic if there is a lattice isomorphism $\varphi \colon N \to N'$ mapping P bijectively to P'.

For an elementary proof of the following Proposition we refer to [10, Sec. 2].

Proposition 2. *The fake weighted projective spaces are precisely the toric varieties $X = X(P)$ associated to the face fan of Fano simplices $P \subseteq N_{\mathbb{R}}$.*

Example 3. As a running example, we consider the two-dimensional Fano simplex P with the vertices

$$v_0 \ = \ (1, 0), \qquad v_1 \ = \ (1, 4), \qquad v_2 \ = \ (-7, -12).$$

The corresponding fake weighted projective plane $X = X(P)$ has the divisor class group

$$\mathrm{Cl}(X) \ \cong \ \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Under this isomorphism the classes of the three torus-invariant divisors D_0, D_1, D_2 of X are given by

$$[D_0] \ = \ (4, \bar{3}), \qquad [D_1] \ = \ (3, \bar{1}), \qquad [D_2] \ = \ (1, \bar{0}).$$

Denote by $C(4) \subseteq \mathbb{C}$ the group of 4-th roots of unity. The variety X can be realized as the quotient of $\mathbb{C}^3 \backslash \{0\}$ by the action of $G = \mathbb{C}^* \times C(4)$ given by

$$(t, \eta) \cdot (z_0, z_1, z_2) \ = \ (t^4 \eta^3 z_0, t^3 \eta z_1, t z_2).$$

Two fake weighted projective spaces are isomorphic if and only if the corresponding Fano simplices are isomorphic. The weighted projective spaces among them correspond to Fano simplices whose vertices generate the lattice. Many geometric properties of a fake weighted projective space can be read off the corresponding simplex. Here we focus our attention on the Gorenstein index and the anticanonical degree.

Definition 4. The *index* of an IP-polytope $P \subseteq N_{\mathbb{R}}$ is the positive integer

$$\iota_P \ := \ \min\left(k \in \mathbb{Z}_{\geqslant 1};\ kP^* \text{ is a lattice polytope} \right).$$

Lemma 5. *The Gorenstein index of any fake weighted projective space $X = X(P)$ coincides with the index ι_P of the corresponding Fano simplex $P \subseteq N_{\mathbb{R}}$.*

Proof. The dual polytope P^* coincides with the polytope $P_{(-\mathcal{K})}$ associated to $-\mathcal{K}$:

$$P_{(-\mathcal{K})} \ = \ \operatorname{conv}(\, m \in M_{\mathbb{R}};\ \chi^m \in \Gamma(X, \mathcal{O}_X(-\mathcal{K}))\,).$$

The assertion thus follows from [8, Thm. 4.2.8]. □

Lemma 6. *See for instance* [9, p. 111]*. Let $X = X(P)$ a d-dimensional fake weighted projective space. Then we have $(-\mathcal{K}_X)^d \ = \ \operatorname{Vol}(P^*)$.*

Example 7. We continue Example 3. The dual of P is the rational simplex P^* with the vertices

$$u_0 \ = \ \left(1, -\frac{1}{2}\right), \qquad u_1 \ = \ \left(-1, \frac{2}{3}\right), \qquad u_2 \ = \ (-1, 0)\,.$$

Thus P has index $\iota_P = 6$. The group of Cartier divisor classes of X is the intersection of the subgroups of $\operatorname{Cl}(X)$ generated by the torus-invariant divisor classes:

$$\langle [D_0] \rangle \cap \langle [D_1] \rangle \cap \langle [D_2] \rangle \ = \ \langle (48, \bar{0}) \rangle \ \subseteq \ \operatorname{Cl}(X) \ = \ \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

An anticanonical divisor of X is given by the sum of the torus-invariant divisors. In $\operatorname{Cl}(X)$ we have

$$[-\mathcal{K}] \ = \ [D_0] + [D_1] + [D_2] \ = \ (8, \bar{0}).$$

The 6-fold of $\mathcal{K}$ is the smallest multiple that is Cartier. Thus X has Gorenstein index $\iota = 6$.

Any weighted projective space $\mathbb{P}(q_0, \ldots, q_d)$ is up to an isomorphism uniquely determined by its weights $(q_0, \ldots, q_d)$. More generally we assign weights to any IP-simplex $P \subseteq N_{\mathbb{R}}$.

Definition 8. See [7,13]. A *weight system Q (of length d)* is a $(d{+}1)$-tuple $Q = (q_0, \ldots, q_d)$ of positive integers. We call

$$|Q| \ := \ q_0 + \cdots + q_d, \qquad \lambda_Q \ := \ \gcd(Q), \qquad Q_{\mathrm{red}} := Q/\lambda_Q$$

the *total weight*, the *factor* and the *reduction* of Q, respectively. A weight system Q is called *reduced* if it coincides with its reduction and it is called *well-formed* if $\gcd(q_j;\ j = 0, \ldots, d,\ j \neq i) = 1$ holds for all $i = 0, \ldots, d$.

Definition 9. See [7, 13]. To any IP-simplex $P \subseteq N_{\mathbb{R}}$ with vertices $v_0, \ldots, v_d \in N$ we associate a weight system by

$$Q_P \;:=\; (q_0, \ldots, q_d), \qquad q_i \;:=\; |\det(\, v_j;\ j = 0, \ldots, d,\ j \neq i\,)|.$$

Weight systems of isomorphic IP-simplices coincide up to order. The reduction $(Q_P)_{\mathrm{red}}$ is the unique reduced weight system $(q_0, \ldots, q_d)$ satisfying

$$\sum_{i=0}^{d} q_i v_i \;=\; 0.$$

Moreover, if P is Fano, then $(Q_P)_{\mathrm{red}}$ is well-formed. Following the naming convention of [13] we call $\lambda_P := [N : N_P]$ the *factor* of the IP-simplex $P \subseteq N_{\mathbb{R}}$, where $N_P \subseteq N$ is the sublattice generated by the vertices of P. In [1, 12] it is called the *multiplicity* of P and denoted by $\operatorname{mult} P$.

Example 10. For the two-dimensional Fano simplex P from Example 3 and Example 7 we have

$$Q_P \;=\; (16, 12, 4), \qquad |Q_P| \;=\; 32, \qquad \lambda_{Q_P} \;=\; 4, \qquad (Q_P)_{\mathrm{red}} \;=\; (4, 3, 1).$$

For the sublattice $N_P \subseteq \mathbb{Z}^2$, generated by the vertices of P, and it's index we have

$$N_P \;=\; \langle\, (1,0),\, (0,4)\,\rangle, \qquad \lambda_P \;=\; [\mathbb{Z}^2 : N_P] \;=\; 4.$$

Lemma 11. *See* [7, Lemma 2.4]. *For any IP-simplex P we have $\lambda_P = \lambda_{Q_P}$.*

If $P \subseteq N_{\mathbb{R}}$ is a Fano simplex then its factor λ_P coincides with the order of the torsion part of $\mathrm{Cl}(X(P))$. In particular $X(P)$ is a weighted projective space if and only if Q_P is reduced. The following Theorem is a reformulation of [7, 4.5–4.7]. Compare also [4, Thm. 5.4.5] and [6, Prop. 2].

Theorem 12. *To any well-formed weight system Q of length d there exists a d-dimensional Fano simplex $P_Q \subseteq N_{\mathbb{R}}$, unique up to an isomorphism, with $Q_{P_Q} = Q$. Any fake weighted projective space $X = X(P)$ with $(Q_P)_{\mathrm{red}} = Q$ is isomorphic to the quotient of the weighted projective space $\mathbb{P}(Q)$ by the action of the finite group N/N_P corresponding to the inclusion $N_P \subseteq N$.*

As an immediate consequence of Theorem 12 we can relate the Gorenstein index and the anticanonical degree of a fake weighted projective space $X(P)$ to those of the weighted projective space $\mathbb{P}((Q_P)_{\mathrm{red}})$.

Corollary 13. *Let $X = X(P)$ a d-dimensional fake weighted projective space and let $X' = \mathbb{P}((Q_P)_{\mathrm{red}})$ the corresponding weighted projective space. Then the Gorenstein index of X is a multiple of the Gorenstein index of X'. Moreover we have $\lambda_P(-\mathcal{K}_X)^d = (-\mathcal{K}_{X'})^d$. In particular, $(-\mathcal{K}_X)^d = (-\mathcal{K}_{X'})^d$ holds if and only if X is isomorphic to X'.*

Proof. By Theorem 12 there is a square matrix H in a lattice basis of N with determinant λ_P such that $P = HP_Q$ holds. Dualizing yields $P_Q^* = H^*P^*$, where H^* denotes the transpose of H. Applying Lemma 5 and Lemma 6 yields the assertions. □

Example 14. We continue Example 10. The vertices v_0', v_1', v_2' of the Fano simplex P' associated with the weighted projective plane $X' = \mathbb{P}(4,3,1) = \mathbb{P}((Q_P)_{\mathrm{red}})$ and the vertices u_0', u_1', u_2' of it's dual simplex $(P')^*$ are given by

$$v_0' \;=\; (1,0), \qquad v_1' \;=\; (0,1), \qquad v_2' \;=\; (-4,-3),$$
$$u_0' \;=\; (1,-1), \qquad u_1' \;=\; \left(-1, \frac{5}{3}\right), \qquad u_2' \;=\; (-1,-1).$$

Thus P' has index $\iota_{P'} = 3$. The indices of P and P' satisfy $\iota_P = 6 = 2 \cdot 3 = 2\,\iota_{P'}$. The simplex P is the image of P' under the linear map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix

$$H \;=\; \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}.$$

We can recover $X = X(P)$ as the quotient of $\mathbb{P}(4,3,1)$ by the action of the group $C(4)$ of 4-th roots of unity given in homogeneous coordinates by

$$\eta \cdot [z_0, z_1, z_2] \;=\; [\eta^3 z_0, \eta z_1, z_2].$$

Using Lemma 6, for the degrees of X and X' we obtain

$$(-\mathcal{K}_{X'})^2 \;=\; \mathrm{Vol}((P')^*) \;=\; \frac{16}{3} \;=\; 4 \cdot \frac{4}{3} \;=\; \lambda_P \mathrm{Vol}(P^*) \;=\; \lambda_P(-\mathcal{K}_X)^2.$$

3 Unit fraction partitions

To any d-dimensional IP-simplex $P \subseteq N_{\mathbb{R}}$ of index ι we assign a partition of $1/\iota$ into a sum of $d+1$ unit fractions. The main result of this section is Proposition 17 where we present a formula to compute the normalized volume of the dual polytope P^* in terms of the denominators of these unit fractions.

Definition 15. Let $\iota \in \mathbb{Z}_{\geqslant 1}$. A tuple $A = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ is called a *uf-partition of ι of length n* if the following holds:

$$\frac{1}{\iota} \;=\; \sum_{k=1}^{n} \frac{1}{a_k}.$$

A tuple $A = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ is called a *uf-partition* if it is a uf-partition of ι for some $\iota \in \mathbb{Z}_{\geqslant 1}$.

Proposition 16. *Let $P \subseteq N_{\mathbb{R}}$ a d-dimensional IP-simplex of index ι with weight system $Q_P = (q_0, \dots, q_d)$. Then*

$$A(P) \;:=\; \left(\frac{\iota|Q_P|}{q_0}, \dots, \frac{\iota|Q_P|}{q_d}\right)$$

is a uf-partition of ι of length $d+1$. We call it the uf-partition of ι associated to P.

Proof. The entries of $A(P)$ are positive. We show that they are integers. Denote by $v_0, \dots, v_d \in N$ the vertices of P. For $0 \leqslant i \leqslant d$ let $F_i = \mathrm{conv}(v_0, \dots, \hat{v}_i, \dots, v_d)$ the i-th facet of P, where $\hat{v}_i$ means that v_i is omitted. For all $i = 0, \dots, d$ we have

$$0 \;=\; \sum_{j=0}^{d} q_j \langle \iota u_{F_i}, v_j \rangle \;=\; \langle \iota u_{F_i}, v_i \rangle q_i - \iota \sum_{\substack{j=0, \\ j \neq i}}^{d} q_j \;=\; (\langle \iota u_{F_i}, v_i \rangle + 1) q_i - \iota |Q_P|.$$

By definition of the index, $\iota u_{F_i} \in M$ holds. Thus q_i divides $\iota|Q_P|$, which means that $A(P)$ consists of integers. Now summing over the reciprocals of $A(P)$ we see that it is in fact a uf-partition of ι. $\square$

Proposition 17. *For any d-dimensional IP-simplex $P \subseteq N_{\mathbb{R}}$ with associated uf-partition $A(P) = (a_0, \dots, a_d)$ of ι_P we have*

$$\lambda_P \mathrm{Vol}(\iota_P P^*) \;=\; \frac{a_0 \cdots a_d}{\mathrm{lcm}(a_0, \dots, a_d)}.$$

Example 18. We continue Example 14. The Fano simplex P has index $\iota_P = 6$ and weight system $Q_P = (16, 12, 4)$. It's uf-partition is given by

$$A(P) \;=\; (12, 16, 48).$$

This is a uf-partition of $\iota_P = 6$. Indeed, we have

$$\frac{1}{6} \;=\; \frac{1}{12} + \frac{1}{16} + \frac{1}{48}.$$

With respect to the formula given in Proposition 17, we have

$$\lambda_P \mathrm{Vol}(\iota_P P^*) \;=\; \lambda_P \iota_P^2 \mathrm{Vol}(P^*) \;=\; 4 \cdot 6^2 \cdot \frac{4}{3} \;=\; \frac{12 \cdot 16 \cdot 48}{48} \;=\; \frac{a_0 a_1 a_2}{\mathrm{lcm}(a_0, a_1, a_2)}.$$

Proposition 17 extends [13, Prop. 4.5.5] to the case of non-reflexive IP-simplices. For the proof of Proposition 17 and in preparation for the proof of the main result we extend Batyrev's correspondence between weight systems of reflexive simplices and uf-partitions of $\iota = 1$ given in [4, Thm. 5.4.3] to the case of higher indices.

Definition 19. The *index* of a weight system $Q = (q_0, \dots, q_d)$ is the positive integer

$$\iota_Q \;:=\; \min(\, k \in \mathbb{Z}_{\geqslant 1};\; q_i \mid k\,|Q| \text{ for all } i = 0, \dots, d\,).$$

Definition 20. For a uf-partition $A = (a_1, \dots, a_n)$ of ι we call

$$t_A \;:=\; \mathrm{lcm}(a_1, \dots, a_n), \qquad \lambda_A \;:=\; \gcd(\iota, a_1, \dots, a_n), \qquad A_{\mathrm{red}} \;:=\; A/\lambda_A$$

the *total weight*, the *factor* and the *reduction* of A, respectively. A uf-partition A is called *reduced* if it coincides with its reduction and it is called *well-formed* if $a_i \mid \mathrm{lcm}(a_j\,;\, j \neq i)$ holds for all $i = 1, \dots, n$.

Proposition 21. *Let $Q = (q_0, \ldots, q_d)$ any weight system of length d and index ι and let $A = (a_0, \ldots, a_d)$ any uf-partition of length $d+1$. Then the following hold:*

(i) $A(Q) := (\iota|Q|/q_0, \ldots, \iota|Q|/q_d)$ *is a reduced uf-partition of ι of length $d+1$.*

(ii) $Q(A) := (t_A/a_0, \ldots, t_A/a_d)$ *is a reduced weight system of length d.*

(iii) $Q(A(Q)) = Q_{\mathrm{red}}$ *and* $A(Q(A)) = A_{\mathrm{red}}$ *hold and this correspondence respects well-formedness.*

Example 22. We continue Example 18. We have the weight system and the uf-partition of $\iota_P = 6$:

$$Q \;=\; Q(P) \;=\; (16, 12, 4), \qquad A \;=\; A(P) \;=\; (12, 16, 48).$$

The weight system Q has index $\iota_Q = 3$. Total weight, factor and reduction of A are given by

$$t_A \;=\; 48, \qquad \lambda_A \;=\; 2, \qquad A_{\mathrm{red}} \;=\; (6, 8, 24).$$

With respect to Proposition 21, we obtain the uf-partition and the weight system

$$A(Q) \;=\; (6, 8, 24) \;=\; A_{\mathrm{red}}, \qquad Q(A) \;=\; (4, 3, 1) \;=\; Q_{\mathrm{red}}.$$

For the proof of Proposition 21 we need the following Lemmas.

Lemma 23. *For $\iota, a_1, \ldots, a_n \in \mathbb{Z}$ set*

$$G(\iota; a_1, \ldots, a_n) \;:=\; \begin{bmatrix} (a_1 - \iota) & -\iota & \ldots & -\iota \\ -\iota & (a_2 - \iota) & \ddots & \vdots \\ & & \ddots & \\ \vdots & \ddots & (a_{n-1} - \iota) & -\iota \\ -\iota & \ldots & -\iota & (a_n - \iota) \end{bmatrix}.$$

Then

$$\det(G(\iota; a_1, \ldots, a_n)) \;=\; a_1 \cdots a_n - \iota \sum_{i=1}^{n} \prod_{j \neq i} a_j.$$

Proof. We prove the Lemma by induction on n. The cases $n = 1$ and $n = 2$ are verified by direct computation. Let $n \geqslant 3$. Subtracting the second to last row of $G := G(\iota; a_1, \ldots, a_n)$ from the last row, we obtain

$$\det(G) \;=\; a_n \det(G') + a_{n-1} \det(G''),$$

where $G' = G(\iota; a_1, \ldots, a_{n-1})$ and $G'' = G(\iota; a_1, \ldots, a_{n-2}, 0)$. By the induction hypothesis we have

$$\det(G') \;=\; a_1 \cdots a_{n-1} - \iota \sum_{i=1}^{n-1} \prod_{j \neq i} a_j, \qquad \det(G'') \;=\; -\iota a_1 \cdots a_{n-2}.$$

Plugging these into the equation for $\det(G)$ yields the assertion. $\square$

Lemma 24. *For any uf-partition $(a_1, \ldots, a_n)$ of ι and any $1 \leqslant k < n$ we have*

$$\det(G(\iota; a_1, \ldots, a_k)) \;\geqslant\; 1.$$

Proof. For any $1 \leqslant k < n$ we have $1/a_1 + \cdots + 1/a_k < 1/\iota$. Multiplying both sides by $\iota\, a_1 \cdots a_k$ and subtracting the left hand side we obtain

$$0 \;<\; a_1 \cdots a_k - \iota \sum_{i=1}^{k} \prod_{j \neq i} a_j \;=\; \det(G(\iota; a_1, \ldots, a_k)).$$

Since the determinant of $G(\iota; a_1, \ldots, a_k)$ is an integer, it must be at least one. □

Proof of Proposition 21. We prove (i). The weight system Q is of index ι, so q_i divides $\iota\,|Q|$. Hence $A(Q)$ consists of positive integers. Summing over the reciprocals of $A(Q)$ shows that it is a uf-partition of ι. Assume $A(Q)$ is not reduced and let A' its reduction. Then A' is a uf-partition of ι' for some $\iota' < \iota$. This means that each q_i divides $\iota'|Q|$, which contradicts the minimality of the index ι of Q. Thus $A(Q)$ is reduced. Item (ii) follows directly from the definition of t_A. We prove (iii). Let $Q = (q_0, \ldots, q_d)$ a weight system of length d and index ι and write $A(Q) = (a_0, \ldots, a_d)$. To show that $Q(A(Q)) = Q_{\mathrm{red}}$ holds we consider the matrix $G = G(\iota; a_0, \ldots, a_d)$ as defined in Lemma 23. Both Q and $Q(A(Q))$ are contained in its kernel and the latter weight system is reduced. So it suffices to show that G is of rank d. This follows from Lemma 24, as the minor of G, obtained by deleting the last row and column, equals $\det(G(\iota; a_0, \ldots, a_{d-1}))$. Now let $A = (a_0, \ldots, a_d)$ a uf-partition of ι of length $d+1$. Write $Q(A) = (q_0, \ldots, q_d)$ and let $A(Q) = (a'_0, \ldots, a'_d)$. This is a uf-partition of ι_Q. Note that each q_i divides $\iota\,|Q|$ as well as $\iota_Q|Q|$. The minimality of the index of Q implies that ι_Q divides ι. With $\lambda := \iota/\iota_Q$ we obtain

$$\lambda a'_i \;=\; \frac{\iota}{\iota_Q}\frac{\iota_Q|Q|}{q_i} \;=\; \frac{\iota\, t_{A(Q)} a_i}{\iota\, t_{A(Q)}} \;=\; a_i,$$

which yields $A(Q) = \lambda A'$. As A' is reduced, we obtain $A(Q)_{\mathrm{red}} = A'$. For the last assertion in (iii) let $Q = (q_0, \ldots, q_d)$ a reduced weight system of length d and write $A(Q) = (a_0, \ldots, a_d)$. By the first part of item (iii) we have $q_i = t_{A(Q)}/a_i$. The weight system Q is well-formed if and only if for all $i = 0, \ldots, d$ we have

$$\prod_{j \neq i} a_j \;=\; t_{A(Q)} \gcd\left(\prod_{k \neq i,j} a_k;\; j \neq i\right).$$

This in turn is equivalent to the well-formedness of $A(Q)$. □

Corollary 25. *For any d-dimensional IP-simplex $P \subseteq N_{\mathbb{R}}$ we have $A(P)_{\mathrm{red}} = A(Q_P)$ and $\iota_P|Q_P| = \lambda_P t_{A(P)}$.*

Proof. For the first assertion note that the uf-partitions $A(P)$ and $A(Q_P)$ only differ by the factor ι_P/ι_{Q_P}. Moreover $A(Q_P)$ is reduced by Proposition 21 (i). The second assertion follows from the identity $|(Q_P)_{\mathrm{red}}| = t_{A(Q_P)}/\iota_{Q_P} = t_{A(P)}/\iota_P$. □

Proof of Proposition 17. The normalized volume of an IP-simplex equals the total weight of its associated weight system. Thus $\mathrm{Vol}(\iota_P P^*) = |Q_{\iota_P P^*}|$ holds. By [13, Prop. 3.6] the total weights $|Q_{\iota_P P^*}|$ and $|Q_P|$ are related by the identity

$$|Q_{\iota_P P^*}| \;=\; \frac{\iota_P^d |Q_P|^d}{q_0 \cdots q_d},$$

where $Q_P = (q_0, \dots, q_d)$. Moreover by Corollary 25 we have $\lambda_P = \iota_P |Q_P| / t_{A(P)}$. Multiplying the normalized volume by the factor λ_P then yields the assertion:

$$\lambda_P \mathrm{Vol}(\iota_P P^*) \;=\; \lambda_P \frac{\iota_P^d |Q_P|^d}{q_0 \cdots q_d} \;=\; \frac{1}{t_{A(P)}} \frac{\iota_P^{d+1} |Q_P|^{d+1}}{q_0 \cdots q_d} \;=\; \frac{a_0 \cdots a_d}{\mathrm{lcm}(a_0, \dots, a_d)}.$$

□

4 Sharp bounds for uf-partitions

The main result of this section is Proposition 27, which constitutes the number theoretic part of the proof of Theorem 1. The Lemmas thereafter are preparation for the proof of Proposition 27.

Definition 26. For any $\iota \in \mathbb{Z}_{\geqslant 1}$ we define a sequence $S_\iota = (s_{\iota,1}, s_{\iota,2}, \dots)$ of positive integers by

$$s_{\iota,1} \;:=\; \iota + 1, \qquad s_{\iota,k+1} \;:=\; s_{\iota,k}(s_{\iota,k} - 1) + 1.$$

Moreover, for any $k \in \mathbb{Z}_{\geqslant 1}$ we set $t_{\iota,k} := s_{\iota,k} - 1$. We denote by $\mathrm{syl}_{\iota,n}$ the uf-partition of ι of length n given by

$$\mathrm{syl}_{\iota,n} \;:=\; (s_{\iota,1}, \dots, s_{\iota,n-2}, 2\,t_{\iota,n-1}, 2\,t_{\iota,n-1}).$$

Following the naming convention in [13] we call $\mathrm{syl}_{\iota,n}$ the *enlarged Sylvester partition of* ι *of length* n.

Proposition 27. *Let* $\iota \in \mathbb{Z}_{\geqslant 1}$ *and* $n \geqslant 3$. *Assume* $(\iota, n) \neq (1, 3)$. *For any uf-partition* $A = (a_1, \dots, a_n)$ *of* ι *with* $a_1 \leqslant \cdots \leqslant a_n$ *we have the two inequalities*

$$\frac{a_1 \cdots a_n}{\mathrm{lcm}(a_1, \dots, a_n)} \;\leqslant\; a_1 \cdots a_{n-1} \;\leqslant\; \frac{2\,t_{\iota,n-1}^2}{\iota}.$$

The right inequality holds with equality if and only if one of the following holds:

- $(\iota, n) = (2, 3)$ *and* $A = (6, 6, 6)$;
- $(\iota, n) = (1, 4)$ *and* $A = (2, 6, 6, 6)$;
- A *is the enlarged Sylvester partition* $\mathrm{syl}_{\iota,n}$.

Proposition 27 is an extension of [13, Thm. 5.1.3] to uf-partitions of $\iota \geqslant 2$. There Nill expands the techniques of Izhboldin and Kurliandchik presented in [11], see also [2]. Here we modify Nill's arguments to incorporate the cases for $\iota \geqslant 2$. Let $\iota, n \in \mathbb{Z}_{\geqslant 1}$. We denote by $A_\iota^n \subseteq \mathbb{R}^n$ the compact set of all tuples $x \in \mathbb{R}^n$ that satisfy the following three conditions:

(A1) $x_1 \geqslant \cdots \geqslant x_n \geqslant 0$.

(A2) $x_1 + \cdots + x_n = 1/\iota$.

(A3) $x_1 \cdots x_k \leqslant \iota(x_{k+1} + \cdots + x_n)$ for all $k = 1, \ldots, n-1$.

Lemma 28. *For any uf-partition $A = (a_1, \ldots, a_n)$ of ι with $a_1 \leqslant \cdots \leqslant a_n$ the tuple $(1/a_1, \ldots, 1/a_n)$ is contained in A_ι^n.*

Proof. The tuple $(1/a_1, \ldots, 1/a_n)$ fulfills conditions (A1) and (A2). For the third condition let $1 \leqslant k \leqslant n-1$. Then we have

$$\iota\left(\frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}\right) \;=\; 1 - \iota\left(\frac{1}{a_1} + \cdots + \frac{1}{a_k}\right) \;=\; \frac{a_1 \cdots a_k - \iota\left(\sum_{j=1}^k \prod_{i \neq j} a_i\right)}{a_1 \cdots a_k}.$$

The numerator on the right hand side is at least one by Lemma 24. This yields the desired inequality. □

For the proof of Proposition 27 we need the following Lemmas 29 to 31. They extend [13, Lemmas 5.4, 5.6] to uf-partitions of $\iota \geqslant 2$.

Lemma 29. *Let $\iota \in \mathbb{Z}_{\geqslant 1}$, $n \in \mathbb{Z}_{\geqslant 1}$ and $1 \leqslant r \leqslant n$. Assume $(\iota, n, r) \neq (1, 2, 2)$. Then we have*

$$(r+1)^r \, t_{\iota, n-r+1}^{r+1} \;\leqslant\; 2\, t_{\iota,n}^2.$$

Equality holds if and only if either $r = 1$ or $(\iota, n, r) = (1, 3, 2)$ or $(\iota, n, r) = (2, 2, 2)$.

Proof. We prove the Lemma by induction on n and r. The case $r = 1$ is clear. Let $r \geqslant 2$. The cases $n = 2$ and $n = 3$ are verified by direct computation. Let $n \geqslant 4$. Then for any $2 \leqslant r \leqslant n$ we have $s_{\iota,n-1} > (r+1)^2/r$. Furthermore, for any $k \in \mathbb{Z}_{\geqslant 1}$ we have $s_{\iota,k} > (r+1)/r$. Combining these two inequalities, we obtain:

$$r\left(\frac{r+1}{r}\right)^r \;<\; s_{\iota,n-r+1} \cdots s_{\iota,n-1}.$$

Moreover $t_{\iota,n-1}^2 < t_{\iota,n}$ holds for all (ι, n). Now by the induction hypothesis the assertion is true for $(\iota, n-1, r-1)$, ie. $r^{r-1}\, t_{\iota,n-r+1}^r \leqslant 2\, t_{\iota,n-1}^2$ holds. Combining this with the previous inequalities, we obtain:

$$(r+1)^r \, t_{\iota,n-r+1}^{r+1} \;\leqslant\; 2\, t_{\iota,n-1}^2\, r \left(\frac{r+1}{r}\right)^r t_{\iota,n-r+1} \;<\; 2\, t_{\iota,n-1}^2\, t_{\iota,n} \;<\; 2\, t_{\iota,n}^2.$$ □

Lemma 30. *Let $n \geqslant 3$ and let $y \in A_\iota^n$ minimizing the product $y_1 \cdots y_{n-1}$. Denote by $i_0 \in \{1, \dots, n\}$ the least index such that $y_{i_0} = y_n$ holds. Then the following hold:*

(i) $i_0 \leqslant n-1$.

(ii) *For any $1 \leqslant k \leqslant i_0 - 2$ we have $y_k = 1/s_{\iota,k}$.*

Proof. We prove (i). Assume $y_{n-1} > y_n$. Choose $0 < \epsilon < (y_{n-1} - y_n)/2$. Then the tuple

$$(\tilde{y}_1, \dots, \tilde{y}_n) \;=\; (y_1, \dots, y_{n-2}, y_{n-1} - \epsilon, y_n + \epsilon).$$

is contained in A_ι^n and $\tilde{y}_1 \cdots \tilde{y}_{n-1} < y_1 \cdots y_{n-1}$ holds, contradicting the minimality of y. Thus $y_{n-1} = y_n$ holds. We prove (ii). For this we first show that $y_k > y_{k+1}$ and $y_1 \cdots y_k = \iota(y_{k+1} + \cdots + y_n)$ holds for any $1 \leqslant k \leqslant i_0 - 2$. Assume on the contrary that $y_k = y_{k+1}$ holds for some $1 \leqslant k \leqslant i_0 - 2$. Then there are i, j with $1 \leqslant i \leqslant k < j < i_0$ and

$$y_{i-1} \;>\; y_i \;=\; \dots \;=\; y_k \;=\; \dots \;=\; y_j \;>\; y_{j+1}.$$

Note that in this case $y_1 \cdots y_k \;<\; \iota\,(y_{k+1} + \cdots + y_n)$ holds. Otherwise we could write $0 = y_k(\iota - y_1 \cdots y_{k-1}) + \iota(y_{k+2} + \cdots + y_n)$, but the right hand side is strictly positive. We can thus find $\epsilon > 0$ such that the tuple

$$(\tilde{y}_1, \dots, \tilde{y}_n) \;=\; (y_1, \dots, y_{i-1}, y_i + \epsilon, y_{i+1}, \dots, y_{j-1}, y_j - \epsilon, y_{j+1}, \dots, y_n)$$

is contained in A_ι^n. For the product of the first $n-1$ entries of $\tilde{y}$ we have

$$\tilde{y}_1 \cdots \tilde{y}_{n-1} \;=\; y_1 \cdots y_{n-1} \left(1 - \frac{\epsilon^2}{y_i y_j}\right) \;<\; y_1 \cdots y_{n-1},$$

contradicting the minimality of y. Hence $y_k > y_{k+1}$ holds for $k = 1, \dots, i_0 - 2$. Now assume that $y_1 \cdots y_k < \iota(y_{k+1} + \cdots + y_n)$ holds. Again there is $\epsilon > 0$ such that the tuple

$$(\tilde{y}_1, \dots, \tilde{y}_n) \;=\; (y_1, \dots, y_{k-1}, y_k + \epsilon, y_{k+1} - \epsilon, y_{k+2}, \dots, y_n)$$

is contained in A_ι^n, leading to the same contradiction as before. Hence $y_1 \cdots y_k$ equals $\iota(y_{k+1} + \cdots + y_n)$ for $k = 1, \dots, i_0 - 2$. Using these identities we can compute y_k. We have $y_1 = \iota(y_2 + \cdots + y_n) = 1 - \iota y_1$. Solving this for y_1 we obtain $y_1 = 1/(\iota + 1) = 1/s_{\iota,1}$. Proceeding in this way with the remaining identities yields $y_k = 1/s_{\iota,k}$ for all $1 \leqslant k \leqslant i_0 - 2$. □

Lemma 31. *Let $n \geqslant 3$ and $\iota \in \mathbb{Z}_{\geqslant 1}$. Assume $(\iota, n) \neq (1, 3)$ and let $x \in A_\iota^n$. Then we have*

$$x_1 \cdots x_{n-1} \geqslant \frac{\iota}{2\, t_{\iota,n-1}^2}.$$

Equality holds if and only if one of the following holds:

- $(\iota, n) = (2, 3)$ *and* $(x_1, x_2, x_3) = (1/6, 1/6, 1/6)$.

- $(\iota, n) = (1,4)$ *and* $(x_1, x_2, x_3, x_4) = (1/2, 1/6, 1/6, 1/6)$.
- $(1/x_1, \ldots, 1/x_n)$ *is the enlarged Sylvester partition* $\mathrm{syl}_{\iota,n}$.

Proof. Let $y \in A_\iota^n$ minimizing the product $y_1 \cdots y_{n-1}$. By Lemma 28 the tuple of reciprocals of the enlarged Sylvester partition $\mathrm{syl}_{\iota,n}$ is contained in A_ι^n. Hence

$$y_1 \cdots y_{n-1} \ \leqslant \ \frac{1}{s_{\iota,1}} \cdots \frac{1}{s_{\iota,n-2}} \cdot \frac{1}{2t_{\iota,n-1}} \ = \ \frac{\iota}{2\, t_{\iota,n-1}^2}$$

holds. Let $i_0 \in \{1, \ldots, n\}$ the least index with $y_{i_0} = y_n$. By Lemma 30 the index i_0 is at most $n-1$. Set $r := n - i_0$. We distinguish three cases.

Case 1. Assume $i_0 = 1$. Then $r = n-1$ and $y_k = 1/(\iota n)$ holds for all $k = 1, \ldots, n$. We obtain

$$\frac{\iota}{2\, t_{\iota,n-1}^2} \ \geqslant \ y_1 \cdots y_{n-1} \ = \ \frac{1}{(\iota n)^{n-1}} \ = \ \frac{1}{(r+1)^r t_{\iota,n-r}^r}.$$

Comparing this to Lemma 29 for the case $r = n-1$, we see that this is only possible for $(\iota, n, r) = (2,3,2)$ and $(y_1, y_2, y_3) = (1/6, 1/6, 1/6)$ and in this case equality holds.

Case 2. Assume $i_0 = 2$. Then $r = n-2$ and $y_1 > y_2 = \cdots = y_n$ holds. By condition (A2) we can express y_1 in terms of y_n via $y_1 = 1/\iota - (n-1)y_n$. Using this identity, together with condition (A3), we obtain an interval of possible values for y_n. On this interval we define a continuous function f by

$$f(y_n) \ := \ y_1 \cdots y_{n-1} \ = \ \left(\frac{1}{\iota} - (n-1)y_n\right) y_n^{n-2}, \qquad y_n \ \in \ \left[\frac{1}{(r+1)t_{\iota,n-r}}, \frac{1}{\iota n}\right).$$

The function f is monotone increasing. It thus attains its minimum on the lower boundary of the interval. We obtain

$$\frac{\iota}{2\, t_{\iota,n-1}^2} \ \geqslant \ y_1 \cdots y_{n-1} \ = \ f(y_n) \ \geqslant \ \frac{\iota}{(r+1)^r t_{\iota,n-r}^{r+1}}$$

Comparing this to Lemma 29 for the case $r = n-2$, this is only possible for (ι, n) equal to $(1,4)$ and $(y_1, y_2, y_3, y_4) = (1/2, 1/6, 1/6, 1/6)$, or $n = 3$ and $(1/y_1, 1/y_2, 1/y_3) = \mathrm{syl}_{\iota,3}$. In both cases equality holds.

Case 3. Assume $i_0 \geqslant 3$. Since $y_{n-1} = y_n$ holds, this case only appears for $n \geqslant 4$. We have $1 \leqslant r \leqslant n-3$. By Lemma 30 we have $y_k = 1/s_{\iota,k}$ for all $1 \leqslant k \leqslant i_0 - 2$. Similar to the second case we use conditions (A2) and (A3) to express y_{i_0-1} in terms of y_n and determine an interval of possible values for y_n:

$$y_{i_0-1} \ = \ \frac{1}{t_{\iota,n-r-1}} - (r+1)\, y_n, \qquad y_n \ \in \ \left[\frac{1}{(r+1)t_{\iota,n-r}}, \frac{1}{(r+2)t_{\iota,n-r-1}}\right).$$

Again, we define a continuous function on that interval by $f := y_1 \cdots y_{n-1}$. It is monotone increasing up to some point in its domain and then it is monotone decreasing. It thus attains its minimum at the boundary. We obtain:

$$\frac{\iota}{2\, t_{\iota,n-1}^2} \ \geqslant \ y_1 \cdots y_{n-1} \ \geqslant \ \min\left(\frac{\iota}{(r+1)^r t_{\iota,n-r}^r}, \frac{\iota}{(r+2)^{r+1} t_{\iota,n-r-1}^{r+1}}\right).$$

Comparing this to Lemma 29 for $1 \leqslant r \leqslant n-3$, this is only possible for $r = 1$ and $y_n = 1/(2t_{\iota,n-1})$. Hence $(1/y_1, \ldots, 1/y_n)$ is the enlarged Sylvester partition $\mathrm{syl}_{\iota,n}$ and in this case equality holds. $\square$

Proof of Proposition 27. Let $A = (a_1, \ldots, a_n)$ a uf-partition of ι with non-decreasing entries. The first inequality in Proposition 27 is due to the fact that a_n divides $\mathrm{lcm}(a_1, \ldots, a_n)$. By Lemma 28 the tuple $x = (1/a_1, \ldots, 1/a_n)$ is contained in A_ι^n. The second inequality and the assertions thereafter follow immediately from Lemma 31. $\square$

5 Proof of the main result

We state and prove the main result of the article.

Definition 32. For any $d \geqslant 2$ and any $\iota \in \mathbb{Z}_{\geqslant 1}$ we denote by $Q_{\iota,d}$ the well-formed weight system

$$Q_{\iota,d} \ := \ Q(\mathrm{syl}_{\iota,d+1}) \ = \ \left(\frac{2t_{\iota,d}}{s_{\iota,1}}, \ldots, \frac{2t_{\iota,d}}{s_{\iota,d-1}}, 1, 1 \right),$$

where $t_{\iota,d}$ and $s_{\iota,k}$ are defined in Definition 26.

Theorem 33. *The anticanonical degree of any d-dimensional fake weighted projective space X of Gorenstein index ι is bounded from above according to the following table.*

d	1	2	2	3	3	$\geqslant 4$
ι	$\geqslant 1$	1	$\geqslant 2$	1	$\geqslant 2$	$\geqslant 1$
bound on $(-\mathcal{K}_X)^d$	2	9	$\frac{2(\iota+1)^2}{\iota}$	72	$\frac{2t_{\iota,3}^2}{\iota^4}$	$\frac{2t_{\iota,d}^2}{\iota^{d+1}}$
attained exactly by	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}(2\iota, 1, 1)$	$\mathbb{P}(3,1,1,1)$, $\mathbb{P}(6,4,1,1)$	$\mathbb{P}(Q_{\iota,3})$	$\mathbb{P}(Q_{\iota,d})$

Equality on the degree holds if and only if X is isomorphic to one of the weighted projective spaces in the last row of the table.

Proof. Let X a d-dimensional fake weighted projective space of Gorenstein index ι. Let $P \subseteq N_\mathbb{R}$ a d-dimensional Fano simplex with $X(P) \cong X$. Then P has index ι. Let $A := A(P) = (a_0, \ldots, a_d)$ the uf-partition of ι associated to P. We may assume that A is ordered non-decreasingly. By Lemma 6 and Proposition 17 we have

$$(-\mathcal{K}_X)^d \ = \ \mathrm{Vol}(P^*) \ = \ \frac{1}{\iota^d}\mathrm{Vol}(\iota P^*) \ \leqslant \ \frac{1}{\iota^d}\frac{a_0 \cdots a_d}{\mathrm{lcm}(a_0, \ldots, a_d)}.$$

For $d = 1$ there is only one fake weighted projective space, namely $\mathbb{P}^1$, which has anticanonical degree $-\mathcal{K}_{\mathbb{P}^1} = 2$. Let $d \geqslant 2$. In case $\iota = 1$ and $d = 2$ the right hand side of the inequality is bounded from above by 9 and $\mathbb{P}^2$ is the only Gorenstein fake weighted projective plane whose degree attains that value, see [13, Ex. 4.7]. If $(\iota, d) \neq (1,2)$, then Proposition 27 provides the upper bound

$$(-\mathcal{K}_X)^d \leqslant \frac{1}{\iota^d}\frac{a_0 \cdots a_d}{\operatorname{lcm}(a_0, \dots, a_d)} \leqslant \frac{2t_{\iota,d}^2}{\iota^{d+1}}.$$

Equality in the first case holds if and only if X is a weighted projective space, see Corollary 13. By Proposition 27 equality in the second case holds if and only if one of the following holds:

(i) $(\iota, d) = (2,2)$ and $A = (6,6,6)$.

(ii) $(\iota, d) = (1,3)$ and $A = (2,6,6,6)$.

(iii) $A = \mathrm{syl}_{\iota,d+1}$.

Note that the uf-partition in (i) is not reduced. In particular, there is no weighted projective plane $X(P)$ of Gorenstein index 2 with $A(P) = (6,6,6)$. The uf-partitions in (ii) and (iii) are reduced and well-formed. By Theorem 12 and Proposition 21 the uf-partition $A = (2,6,6,6)$ corresponds to the three-dimensional Gorenstein weighted projective space $X = \mathbb{P}(3,1,1,1)$ and the uf-partition $A = \mathrm{syl}_{\iota,d+1}$ corresponds to the d-dimensional weighted projective space $X = \mathbb{P}(Q_{\iota,d})$. □

Acknowledgements

The author is grateful to the referee for helpful remarks and valuable suggestions.

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