Graphs with Girth $2\ell + 1$ and Without Longer Odd Holes that Contain an Odd K_4-Subdivision

Rong Chen[a] Yidong Zhou[a]

Submitted: Jun 14, 2023; Accepted: Dec 29, 2023; Published: Feb 23, 2024

Abstract

We say that a graph G has an *odd K_4-subdivision* if some subgraph of G is isomorphic to a K_4-subdivision which if embedded in the plane the boundary of each of its faces has odd length and is an induced cycle of G. For a number $\ell \geqslant 2$, let $\mathcal{G}_\ell$ denote the family of graphs which have girth $2\ell + 1$ and have no odd hole with length greater than $2\ell + 1$. Wu, Xu and Xu conjectured that every graph in $\bigcup_{\ell \geqslant 2} \mathcal{G}_\ell$ is 3-colorable. Recently, Chudnovsky et al. and Wu et al., respectively, proved that every graph in $\mathcal{G}_2$ and $\mathcal{G}_3$ is 3-colorable. In this paper, we prove that no 4-vertex-critical graph in $\bigcup_{\ell \geqslant 5} \mathcal{G}_\ell$ has an odd K_4-subdivision. Using this result, Chen proved that all graphs in $\bigcup_{\ell \geqslant 5} \mathcal{G}_\ell$ are 3-colorable.

Keywords: chromatic number, odd holes

Mathematics Subject Classifications: 05C15, 05C17, 05C69

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. A *proper coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. A graph is *k-colorable* if it has a proper coloring using at most k colors. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable.

The *girth* of a graph G, denoted by $g(G)$, is the minimum length of cycles in G. A *hole* in a graph is an induced cycle of length at least four. An *odd hole* means a hole of odd length. For any integer $\ell \geqslant 2$, let $\mathcal{G}_\ell$ be the family of graphs that have girth $2\ell + 1$ and have no odd holes of length at least $2\ell + 3$. Robertson conjectured in [4] that the Petersen graph is the only graph in $\mathcal{G}_2$ that is 3-connected and internally 4-connected. Plummer and Zha [5] disproved Robertson's conjecture and proposed the conjecture that all 3-connected and internally 4-connected graphs in $\mathcal{G}_2$ have bounded chromatic numbers, and proposed

[a]Center for Discrete Mathematics, Fuzhou University, Fuzhou, P. R. China (rongchen@fzu.edu.cn, zoed98@126.com).

the strong conjecture that such graphs are 3-colorable. The first was proved by Xu, Yu, and Zha [9], who proved that all graphs in $\mathcal{G}_2$ are 4-colorable. The strong conjecture proposed by Plummer and Zha in [5] was solved by Chudnovsky and Seymour [2]. Wu, Xu, and Xu [7] showed that graphs in $\bigcup_{\ell \geqslant 2} \mathcal{G}_\ell$ are 4-colorable and conjectured

Conjecture 1. ([7], Conjecture 6.1.) For each integer $\ell \geqslant 2$, every graph in $\mathcal{G}_\ell$ is 3-colorable.

Wu, Xu and Xu [8] recently proved that Conjecture 1 holds for $\ell = 3$.

We say that a graph G has an *odd K_4-subdivision* if some subgraph of G is isomorphic to a K_4-subdivision which if embedded in the plane the boundary of each of its faces has odd length and is an induced cycle of G. Note that an odd K_4-subdivision of G maybe not induced. However, when $G \in \mathcal{G}_\ell$ for each integer $\ell \geqslant 2$, all odd K_4-subdivisions of G are induced by Lemma 8 (2). In this paper, we prove the following theorem.

Theorem 2. *No 4-vertex-critical graph in $\bigcup_{\ell \geqslant 5} \mathcal{G}_\ell$ has an odd K_4-subdivision.*

Using Theorem 2, Chen [1] proved that Conjecture 1 holds for all $\ell \geqslant 5$. Recently, following idea in this paper and [1], Wang and Wu [6] further proved that Conjecture 1 holds for $\ell = 4$.

2 Preliminaries

A *cycle* is a connected 2-regular graph. Let G be a graph. A vertex $v \in V(G)$ is called a *degree-k vertex* if it has exactly k neighbours. For any $U \subseteq V(G)$, let $G[U]$ be the subgraph of G induced on U. For subgraphs H and H' of G, set $|H| := |E(H)|$ and $H \Delta H' := E(H) \Delta E(H')$. Let $H \cup H'$ denote the subgraph of G whose vertex set is $V(H) \cup V(H')$ and edge set is $E(H) \cup E(H')$. Let $H \cap H'$ denote the subgraph of G with edge set $E(H) \cap E(H')$ and without isolated vertex. Let $N(H)$ be the set of vertices in $V(G) - V(H)$ that have a neighbour in H. Set $N[H] := N(H) \cup V(H)$.

Let P be an (x, y)-path and Q be a (y, z)-path. When P and Q are internally disjoint, let PQ denote the (x, z)-path $P \cup Q$. Evidently, PQ is a path when $x \neq z$, and PQ is a cycle when $x = z$. Let P^* denote the set of internal vertices of P. When $u, v \in V(P)$, let $P(u, v)$ denote the subpath of P with ends u, v. For simplicity, we will let $P^*(u, v)$ denote $(P(u, v))^*$.

A graph is *k-vertex-critical* if $\chi(G) = k$ but $\chi(G \setminus v) < k$ for each $v \in V(G)$. Dirac in [3] proved that every k-vertex-critical graph is $(k-1)$-edge-connected. Hence, we have

Lemma 3. *For each integer $k \geqslant 4$, each k-vertex-critical graph G has no 2-edge-cut.*

A *theta graph* is a graph that consists of a pair of distinct vertices joined by three internally disjoint paths. Let C be a hole of a graph G. A path P of G is a *chordal path* of C if $V(P^*) \cap V(C) = \emptyset$ and $C \cup P$ is an induced theta-subgraph of G. Lemma 4 will be frequently used.

Lemma 4. *Let $\ell \geqslant 2$ be an integer and C be an odd hole of a graph $G \in \mathcal{G}_\ell$. Let P be a chordal path of C, and P_1, P_2 be the internally disjoint paths of C that have the same ends as P. Assume that $|P|$ and $|P_1|$ have the same parity. If $|P_1| \neq 1$, then $|P_1| > |P_2|$ and all chordal paths of C with the same ends as P_1 have length $|P_1|$.*

Proof. Since $|C| = 2\ell + 1$, $|P_1| \neq 1$ and $|P|$ and $|P_1|$ have the same parity, $P \cup P_2$ is an odd hole. Moreover, since $g(G) = 2\ell + 1$ and all odd holes in G have length $2\ell + 1$, we have $\ell + 1 \leqslant |P_1| = |P|$ and $|P_2| \leqslant \ell$, so $|P_1| > |P_2|$ and all chordal paths of C with the same ends as P_1 have length $|P_1|$. □

Let P be a path with i vertices. If $G - V(P)$ is disconnected, then we say that P is a *P_i-cut*. Usually, a P_2-cut is also called a *K_2-cut*. Evidently, every k-vertex-critical graph has no K_2-cut. Chudnovsky and Seymour in [2] proved that every 4-vertex-critical graph G in $\mathcal{G}_2$ has no P_3-cut. Using the same argument as [2], Wu et al. [8] extend this result to graphs in $\bigcup_{\ell \geqslant 2} \mathcal{G}_\ell$. Since the paper [8] does not include a proof of Lemma 5, we give a proof here for completeness.

Lemma 5. *([8]) For any number $\ell \geqslant 2$, every 4-vertex-critical graph in $\mathcal{G}_\ell$ has neither a K_2-cut nor a P_3-cut.*

Proof. It is well-known that every k-vertex-critical graph has no clique as a cut. Hence, it suffice to show that every 4-vertex-critical graph in $\mathcal{G}_\ell$ has no P_3-cut. Let $G \in \mathcal{G}_\ell$ be a 4-vertex-critical graph. Assume to the contrary that $P = v_1v_2v_3$ is a path such that $G \backslash \{v_1, v_2, v_3\}$ is disconnected. Since G has no K_3 as its cut, $v_1v_3 \notin E(G)$. Let A_1 be the a component of $G \backslash \{v_1, v_2, v_3\}$, and let A_2 be the union of all other components. Set $G_i := G[A_i \cup \{v_1, v_2, v_3\}]$ for $i = 1, 2$. Since G is 4-vertex-critical, both G_1 and G_2 are 3-colorable. Let $\phi_i : V(G_i) \to \{1, 2, 3\}$ be a 3-coloring for $i = 1, 2$. By symmetry we may assume that $\phi_i(v_1) = 1$ and $\phi_i(v_2) = 2$ for $i = 1, 2$. Thus $\phi_1(v_3), \phi_2(v_3) \in \{1, 3\}$. If $\phi_1(v_3) = \phi_2(v_3)$, then G is 3-colorable, which is a contradiction. Thus by symmetry we may assume that $\phi_1(v_3) = 1$ and $\phi_2(v_3) = 3$. Let H_1 be the subgraph of G_1 induced on the set of vertices $v \in V(G_1)$ with $\phi_1(v) \in \{1, 3\}$. If v_1, v_3 belong to different components of H_1, then by exchanging colors in the component containing v_3, we obtain another 3-coloring of G_1 that can be combined with ϕ_2 to show that G is 3-colorable. So v_1, v_3 belong to the same component of H_1. Then there is an induced (v_1, v_3)-path P_1 in H_1 having even length as $\phi_1(v_1) = 1 = \phi_1(v_3)$. Similarly, there is an induced (v_1, v_3)-path P_2 in G_2 having odd length as $\phi_2(v_1) = 1$ and $\phi_2(v_3) = 3$. Moreover, since PP_1, PP_2 are cycles of G and $g(G) = 2\ell + 1$, we have $|P_1| \geqslant 2\ell - 1$ and $|P_2| \geqslant 2\ell$, so $P_1 \cup P_2$ is an odd hole of G of length at least $4\ell - 1$, which is a contradiction as $G \in \mathcal{G}_\ell$. □

Lemma 6. *Let $\ell \geqslant 2$ be an integer and x, y be non-adjacent vertices of a graph $G \in \mathcal{G}_\ell$. Let P be an induced (x, y)-path of G. If $|P| \leqslant \ell$ and all induced (x, y)-paths have length $|P|$, then no block of G contains two non-adjacent vertices in $V(P)$. In particular, each vertex in P^* is a cut-vertex of G.*

Proof. Assume not. Then there is a block B of G containing two consecutive edges of P. Let Q be an induced path in B with ends in $V(P)$ and with $V(P) \cap V(Q^*) = \emptyset$.

Since every pair of edges in a 2-connected graph is contained in a cycle, such a Q exists. Without loss of generality we may further assume that Q is chosen with $|Q|$ as small as possible. Let C be the unique cycle in $P \cup Q$. Then $C \Delta P$ is an (x, y)-path. Since Q is induced, the ends of Q are not adjacent. Moreover, since Q is chosen with $|Q|$ as small as possible, $C \Delta P$ is an induced (x, y)-path, so $|C \Delta P| = |P| \leqslant \ell$ by the assumption of the lemma. Hence, $|C| \leqslant 2\ell$, contrary to the fact $g(G) = 2\ell + 1$. □

Lemma 7. *Let $\ell \geqslant 4$ be an integer and x, y be non-adjacent vertices of a graph $G \in \mathcal{G}_\ell$. Let X be a vertex cut of G with $\{x, y\} \subseteq X \subseteq N[\{x, y\}]$, and G_1 be an induced subgraph of G whose vertex set consists of X and the vertex set of a component of $G - X$. If all induced (x, y)-paths in G_1 have length k with $4 \leqslant k \leqslant \ell$, then G has a degree-2 vertex, a K_1-cut, or a K_2-cut.*

Proof. Assume that G has no degree-2 vertices. Let P be an induced (x, y)-path in G_1. Let uvw be a subpath of P^*. Such uvw exists as $k \geqslant 4$. By the definition of G_1, we have $v \notin X$, so $N_G[v] = N_{G_1}[v]$, which implies $d_{G_1}(v) \geqslant 3$. By applying Lemma 6 to G_1, there is a block B of G_1 such that either $V(B) \cap V(P) = \{v\}$, or B is not isomorphic to K_2 and $V(B) \cap V(P)$ is $\{u, v\}$ or $\{u, v\}$. When the first case happens, since $X \subseteq N[\{x, y\}]$, $x, y \notin B$ and $v \notin X$, we have $X \cap V(B) = \emptyset$, for otherwise $P(x, v)$ or $P(v, y)$ is contained in a cycle of $P \cup B$, so the vertex v is a cut-vertex of G as X is a vertex cut of G. When the latter case happens, by symmetry we may assume that $V(B) \cap V(P) = \{u, v\}$. Since B is a block of G_1, $X \subseteq N[\{x, y\}]$ and uvw is a subpath of P^*, we have $V(B) \cap X = \{u\} \cap X$, so $\{u, v\}$ is a K_2-cut of G_1 and G. □

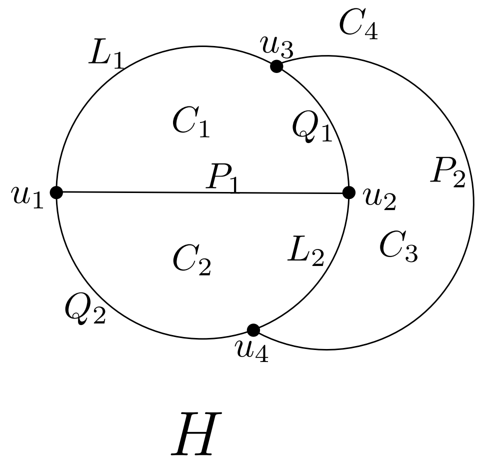

Figure 1: u_1, u_2, u_3, u_4 are the degree-3 vertices of H. All faces C_1, C_2, C_3, C_4 of H are odd holes. $\{P_1, P_2\}$, $\{Q_1, Q_2\}$, $\{L_1, L_2\}$ are the pairs of vertex disjoint arrises of H.

Let H be a graph that is isomorphic to a subdivision of K_4, and let P be a path of H whose ends are degree-3 vertices in H. If P^* contains no degree-3 vertex of H, then we say that P is an *arris* of H. Evidently, there are exactly six arrises of H. See Figure 1.

Lemma 8. *For any integer $\ell \geqslant 2$, if a graph $G \in \mathcal{G}_\ell$ has an an odd K_4-subdivision H, then the following statements hold.*

(1) Each pair of vertex disjoint arrises in H have the same length and their lengths are at most ℓ.

(2) H is an induced subgraph of G.

(3) When $\ell \geqslant 3$, every vertex in $V(G) - V(H)$ has at most one neighbour in $V(H)$.

Proof. Without loss of generality we may assume that H is pictured as the graph in Figure 1. First, we prove that (1) is true. Assume that $|P_1| > |P_2|$. Since C_1 and C_4 are odd holes, $|Q_1| < |Q_2|$. Hence, $|P_2 \cup Q_1 \cup L_2| < |P_1 \cup Q_2 \cup L_2|$, which is a contradiction to the fact that C_2 and C_3 are both odd holes. So $|P_1| = |P_2|$. By symmetry each pair of vertex-disjoint arrises have the same length. Moreover, since $C_1 \Delta C_2$ is an even cycle with length at least $2\ell + 2$, we have $|P_1| \leqslant \ell$. By symmetry we have $|Q_1|, |L_1| \leqslant \ell$. So (1) holds.

Secondly, we prove that (2) is true. Suppose not. Since odd holes have no chord, by symmetry we may assume that there is an edge st in G with $s \in V(P_1^*)$ and $t \in V(P_2^*)$. On one hand, since $P_1(u_1, s)stP_2(t, u_4)Q_2$ and $P_1(u_2, s)stP_2(t, u_3)Q_1$ are cycles, by (1) we have

$$|P_1| + |P_2| + |Q_1| + |Q_2| + 2 = 2(|P_1| + |Q_1| + 1) \geqslant 2(2\ell + 1).$$

On the other hand, since $|P_1|, |Q_1| \leqslant \ell$ by (1), we have $|P_1| = |Q_1| = \ell$, implying that $|L_1| = |L_2| = 1$. Moreover, by the symmetry between L_1, L_2 and Q_1, Q_2, we have $|Q_1| = |Q_2| = 1$, which is a contradiction as $|Q_1| = \ell \geqslant 2$. So (2) holds.

Finally, we prove that (3) is true. Suppose to the contrary that some vertex $x \in V(G) - V(H)$ has at least two neighbours in $V(H)$. Since a vertex not in an odd hole can not have two neighbours in the odd hole, x has exactly two neighbours x_1, x_2 in $V(H)$. By symmetry we may further assume that $x_1 \in V(P_1^*)$ and $x_2 \in V(P_2^*)$. Since $C_1' = P_1(u_1, x_1)x_1xx_2P_2(x_2, u_3)L_1$ and $C_2' = P_1(u_1, x_1)x_1xx_2P_2(x_2, u_4)Q_2$ are cycles whose lengths have different parity,

$$|C_1'| + |C_2'| = 2\ell + 1 + 2(2 + |P_1(u_1, x_1)|) \geqslant 4\ell + 3.$$

Hence, $|P_1(u_1, x_1)| = \ell - 1$ and $x_1u_2 \in E(H)$ as $|P_1| \leqslant \ell$ by (1). This implies that $u_2x_1xx_2$ is a chordal path of C_3 with length 3, which is a contradiction to Lemma 4 as $\ell \geqslant 3$. □

By Lemma 8 (1), all odd K_4-subdivisions of a graph $G \in \mathcal{G}_\ell$ have exactly $4\ell + 2$ edges for each number $\ell \geqslant 2$.

3 Proof of Theorem 2

Let H_1, H_2 be vertex disjoint induced subgraphs of a graph G. An induced (v_1, v_2)-path P is a *direct connection* linking H_1 and H_2 if v_i is the only vertex in $V(P)$ having a neighbour in $V(H_i)$ for each $i \in \{1, 2\}$. Evidently, $V(P) \cap V(H_1 \cup H_2) = \emptyset$ and the set of internal vertices of each shortest path joining H_1 and H_2 induces a direct connection linking H_1 and H_2.

For convenience, Theorem 2 is restated here in another way.

Theorem 9. *Let $\ell \geqslant 5$ be an integer, and G be a graph in $\mathcal{G}_\ell$. If G is 4-vertex-critical, then G has no odd K_4-subdivisions.*

Proof. Suppose not. Let H be a subgraph of G that is isomorphic to an odd K_4-subdivision and pictured as the graph in Figure 1. By Lemma 8 (2), H is an induced subgraph of G. By Lemma 8 (1), we have

$$|P_1| = |P_2| \leqslant \ell, \quad |Q_1| = |Q_2| \leqslant \ell, \quad \text{and} \quad |L_1| = |L_2| \leqslant \ell. \tag{3.1}$$

Without loss of generality we may assume that P_1, P_2 are longest arrises in H.

Let e, f be the edges of P_2 incident with u_3, u_4, respectively. Since G is 4-vertex-critical, $\{e, f\}$ is not an edge-cut of G by Lemma 3, so there is a direct connection P in $G\backslash\{e, f\}$ linking P_2^* and $H - V(P_2^*)$. Let v_1, v_2 be the ends of P with v_2 having a neighbour in P_2^* and v_1 having a neighbour in $H - V(P_2^*)$. By Lemma 8 (3), both v_1 and v_2 have a unique neighbour in $V(H)$. Let x, y be the neighbours of v_1 and v_2 in $V(H)$, respectively. That is, $x \in V(H) - V(P_2^*)$ and $y \in V(P_2^*)$. Set $P' := xv_1Pv_2y$. Since H is an induced subgraph of G, so is $H \cup P'$.

9.1. $x \notin \{u_1, u_2\}$.

Subproof. Assume not. By symmetry we assume that $x = u_1$. Set $C_4' = L_1P'P_2(y, u_3)$. Since C_4 is an odd hole, by symmetry we may assume that C_4' is an even hole and $C_4 \Delta C_4'$ is an odd hole. Since $P'P_2(y, u_3)$ is a chordal path of C_1, by (3.1) and Lemma 4, we have $|L_1| = 1$. So $|P_1| = |Q_1| = \ell$ by (3.1) again. Since $P'P_2(y, u_4)$ is a chordal path of C_2 and $C_4 \Delta C_4'$ is an odd hole, $|P'P_2(y, u_4)| = |P_1L_2| = \ell + 1$ by (3.1) and Lemma 4 again. Moreover, since $|P_2| = \ell$ and $|L_1| = 1$, we have $|C_4'| \leqslant 2\ell$, which is not possible. So $x \neq u_1$. □

Set $d(H) := |P_1| - \min\{|Q_1|, |L_1|\}$. We say that $d(H)$ is the *difference* of H. Without loss of generality we may assume that among all odd K_4-subdivisions, H is chosen with difference as small as possible.

9.2. $x \notin V(P_1)$.

Subproof. Suppose to the contrary that $x \in V(P_1)$. Then $x \in V(P_1^*)$ by 9.1. Without loss of generality we may assume that $|L_1| \geqslant |Q_1|$. Set $C_2' = Q_2P_1(u_1, x)P'P_2(y, u_4)$. Since C_4 is an odd hole, either C_2' or $C_4 \Delta C_2'$ is an odd hole. Suppose that $C_4 \Delta C_2'$ is an odd hole. Since $C_1 \cup C_3 \cup P'$ is an odd K_4-subdivision, by Lemma 8 (1) and (3.1), we have $|P'| = |Q_1|$, $|P_1(u_1, x)| = |P_2(u_4, y)|$, and $|P_1(u_2, x)| = |P_2(u_3, y)|$. So C_2' is an even hole of length $2(|Q_2| + |P_1(u_1, x)|)$ by (3.1) again, implying $|L_1| + |P_1(u_1, x)| \geqslant |Q_2| + |P_1(u_1, x)| \geqslant \ell + 1$ as $|L_1| \geqslant |Q_1|$. Then $|C_4 \Delta C_2' \Delta C_1| = 2|P_1(x, u_2)Q_1| \leqslant 2\ell$, contrary to the fact $g(G) = 2\ell + 1$. So C_2' is an odd hole.

Since $C_2 \cup C'_2 \cup C_3$ is an odd K_4-subdivision, it follows from Lemma 8 (1) and (3.1) that

$$|P'| = |L_2|, \ |P_1(u_1, x)| = |P_2(u_3, y)|, \text{ and } |P_1(u_2, x)| = |P_2(u_4, y)|. \tag{3.2}$$

Then $|C_2 \Delta C'_2 \Delta C_1| = 2|L_1| + 2\ell + 1$. Since $C_2 \Delta C'_2 \Delta C_1$ is not an odd hole,

$$1 \in \{|Q_2|, |P_1(u_2, x)|, |P_2(u_3, y)|\}. \tag{3.3}$$

When $|P_1(u_2, x)| = 1$, since $|C_2 \Delta C'_2 \Delta C_1| = 2|L_1| + 2\ell + 1$ and $g(G) = 2\ell + 1$, we have $|L_1| = |P'| = \ell$ by (3.2), implying $|P_1| = \ell$ and $|Q_1| = 1$ as P_1, P_2 are longest arrises in H. Hence, $d(H) = \ell - 1$. Then $G[V(C_1 \cup C'_2 \cup P_2)]$ is an odd K_4-subdivision with difference $\ell - 2$, which is a contradiction to the choice of H. So $|P_1(u_2, x)| \geqslant 2$. Assume that $|Q_2| = 1$. Then $|L_1| = |P_1| = \ell$ by (3.1). Since $|P_1(u_2, x)| \geqslant 2$, the graph $G[V(C'_2 \cup C_2 \cup C_3)]$ is an odd K_4-subdivision whose difference is at most $\ell - 2$, which is a contradiction to the choice of H as $d(H) = \ell - 1$. So $|Q_2| \geqslant 2$. Then $yu_3 \in E(H)$ by (3.3), implying $xu_1 \in E(H)$ by (3.2). Hence, $|C_4 \Delta C'_2| = 2 + 2|L_1|$ by (3.1) and (3.2), and so $|L_1| = \ell$ by (3.1) again. Since $|P_1| \geqslant |L_1|$, we have $|P_1| = \ell$ and $|Q_1| = 1$ by (3.1), which is a contradiction as $|Q_2| \geqslant 2$. □

9.3. *When $x \in \{u_3, u_4\}$, the vertices x and y are adjacent, that is, $xy \in \{e, f\}$.*

Subproof. By symmetry we may assume that $x = u_3$. Assume to the contrary that x, y are not adjacent. Set $C'_3 = P'P_2(y, u_3)$. Since P' is a chordal path of C_3, we have that C'_3 is an odd hole by Lemma 4 and (3.1). Since $C'_3 \Delta C_3$ is an even hole, $|Q_1| = |L_2| = 1$ by (3.1) and Lemma 4 again. Then $|P_1| = 2\ell - 1 > \ell$, which is a contradiction to (3.1). So $e = xy$. □

9.4. *When $x \in V(L_1^*)$, we have that $|Q_1| = 1$, $|P_1| = |L_1| = \ell$, $|P'| = 2\ell - 1$, $xu_3, yu_3 \in E(H)$ and xu_3yP' is an odd hole.*

Subproof. Set $C'_4 = L_1(x, u_1)Q_2P_2(u_4, y)P'$. We claim that $C_4 \Delta C'_4$ is an odd hole. Assume to the contrary that $C_4 \Delta C'_4$ is an even hole. Since $x \neq u_3$, the subgraph $C_1 \cup (C_4 \Delta C'_4)$ is an induced theta subgraph. Hence, $xu_3 \in E(H)$ by (3.1) and Lemma 4. Similarly, $yu_3 \in E(H)$. Since P' is a chordal path of C_4, we get a contradiction to Lemma 4. So the claim holds, implying that C'_4 is an even hole.

Since $x \neq u_3$, the graph $C_2 \cup C'_4$ is an induced theta subgraph of G. Moreover, since C'_4 is an even hole, $|Q_2| = 1$ by (3.1) and Lemma 4. Hence, $|P_1| = |L_1| = \ell$ by (3.1) again. Assume that y, u_3 are not adjacent. Since $C_1 \cup C'_4$ is an induced theta subgraph of G, we have $xu_1 \in E(H)$, implying $|P(x, u_3)| = \ell - 1$. Since P' is a chordal path of C_4 and $C_4 \Delta C'_4$ is an odd hole, $yu_3 \in E(H)$ by Lemma 4, a contradiction. Hence, $yu_3 \in E(H)$. By symmetry we have $xu_3 \in E(H)$. This proves 9.4. □

9.5. *Assume that P' has the structure stated as in 9.4. Then no vertex in $V(G) - V(H \cup P')$ has two neighbours in $H \cup P'$.*

Subproof. Assume to the contrary that some vertex $a \in V(G) - V(H \cup P')$ has two neighbours a_1, a_2 in $H \cup P'$. Since no vertex has two neighbours in an odd hole, it follows from Lemma 8 (3) and 9.4 that a has exactly two neighbours in $H \cup P'$ with $a_1 \in V(H) - \{x, y, u_3\}$ and $a_2 \in V(P)$. When $xa_2 \notin E(P')$, let Q be the unique (y, a_1)-path in $G[V(P) \cup \{a_1, y\}]$. Since Q^* is a direct connection in $G \backslash \{e, f\}$ linking P_2^* and $H - V(P_2^*)$, by 9.2-9.4 and the symmetry between P' and Q, we have $a_1 \in \{x, u_3\}$, contrary to the fact $a_1 \in V(H) - \{x, y, u_3\}$. So $xa_2 \in E(P')$. Moreover, since $|P_1| = |L_1| = \ell \geqslant 5$ and $g(G) = 2\ell + 1$, we have $a_1 \notin V(P_1)$. Let u_1' be the neighbour of u_1 in L_1. When $a_1 \in V(C_2) - V(P_1)$, since aa_2 is a direct connection in $G \backslash \{u_1u_1', u_3x\}$ linking L_1^* and $H - V(L_1^*)$, which is not possible by 9.4 and the symmetry between P_2 and L_1. So $a \in V(L_1^*)$. Then xa_2aa_1 is a chordal path of C_1 with length 3, contrary to Lemma 4. □

9.6. $x \in \{u_3, u_4\}$ *and* $xy \in \{e, f\}$.

Subproof. By 9.1-9.3, it suffices to show that $x \notin V(L_1^* \cup L_2^* \cup Q_1^* \cup Q_2^*)$. Assume not. By symmetry we may assume that $x \in V(L_1^*)$. By 9.4, we have that

$$xu_3 \in E(L_1),\ e = yu_3,\ |P'| = 2\ell - 1,\ |P_1| = |L_1| = \ell,\ \text{and } |Q_1| = 1.$$

Since no 4-vertex-critical graph has a P_3-cut by Lemma 5, to prove that 9.6 is true, it suffice to show that $\{x, y, u_3\}$ is a P_3-cut of G. Suppose not. Let R be a shortest induced path in $G - \{x, y, u_3\}$ linking P and $H - \{x, y, u_3\}$. Let s and t be the ends of R with $s \in V(P)$. By 9.5, $|R| \geqslant 3$ and no vertex in $V(H \cup P') - \{x, y, u_3, s, t\}$ has a neighbour in R^*.

We claim that $t \notin V(L_1 \cup P_2) - \{x, y, u_3\}$. Suppose not. By symmetry we may assume that $t \in V(L_1) - \{x, u_3\}$. Let R_1 be the induced (y, t)-path in $G[V(P' \cup R) - \{x\}]$. When u_3 has no neighbour in R_1^*, set $R_2 := R_1$ and $C := R_2L_1(t, u_3)u_3y$. When u_3 has a neighbour in R_1^*, let $t' \in V(R_1^*)$ be a neighbour of u_3 closest to t and set $R_2 := u_3t'R_1(t', t)$ and $C := R_2L_1(t, u_3)$. Note that $C_4 \Delta C$ is a hole, although C maybe not a hole. Since $C \Delta C_1 \Delta C_2$ is an odd hole with length at least $2\ell + 3$ when C is an odd cycle, to prove the claim, it suffices to show that $|C|$ is odd. When x has a neighbour in R_2^*, since $|R_2| \geqslant 2\ell$ by (3.1) and the fact that $g(G) = 2\ell + 1$, the subgraph $C_4 \Delta C$ is an even hole, which implies that C is an odd cycle. So we may assume that x has no neighbour in R_2^*. When u_3 is an end of R_2, since R_2 is a chordal path of C_1, it follows from Lemma 4 and (3.1) that C is an odd hole. When y is an end of R_2 and u_3yR_2 is a chordal path of C_1, for the similar reason, C is an odd hole. Hence, we may assume that y is an end of R_2 and u_3yR_2 is not a chordal path of C_1, implying $xs \in E(P')$ and $s \in V(R_2)$. Since $P \subset R_2$, we have $|R_2| > 2\ell$, so $C_4 \Delta C$ is an even hole, implying that C is an odd cycle. Hence, this proves the claim.

By symmetry we may therefore assume that $t \in V(P_1) - \{u_1\}$. Let R_1 be the induced (y, t)-path in $G[V(P' \cup R) - \{x\}]$. By 9.1 and 9.2, either $xs \in E(P')$ and y has no neighbour in R or some vertex in $\{x, u_3\}$ has a neighbour in R_1^*. No matter which case happens, we have $|R_1| \geqslant 2\ell$. That is, $R_1P_2(y, u_4)$ is a chordal path of C_2 with length at least $3\ell - 1$, which is a contradiction to Lemma 4 as t, u_4 are non-adjacent. Hence, $\{x, y, u_3\}$ is a P_3-cut of G. □

By 9.6, there is a minimal vertex cut X of G with $\{u_3, u_4\} \subseteq X \subset N[\{u_3, u_4\}]$ and $\{u_3, u_4\} = X \cap V(H)$. Let G_1 be the induced subgraph of G whose vertex set consists of X and the vertex set of the component of $G - X$ containing P_2^*. Since $\ell \geqslant 5$, we have $|P_2| \geqslant 4$ by (3.1). If all induced (u_3, u_4)-paths in G_1 have length $|P_2|$, by Lemma 7, G has a degree-2 vertex, a K_1-cut or a K_2-cut, which is not possible as G is 4-vertex-critical. Hence, to finish the proof of Theorem 9, it suffices to show that all induced (u_3, u_4)-paths in G_1 have length $|P_2|$.

Let Q be an arbitrary induced (u_3, u_4)-path in G_1. When $|L_1| \geqslant 2$, since QQ_2 is a chordal path of C_1 by Lemma 8 (3) and the definition of G_1, we have $|QQ_2| = |Q_1P_1|$ by Lemma 4, so $|Q| = |P_1|$ by (3.1). Hence, by (3.1) we may assume that $|L_1| = 1$ and $|Q_1| = |P_1| = \ell$. Since Q_1L_2 is an induced (u_3, u_4)-path of length $\ell + 1$, either $|Q| = |P_2| = \ell$ or $|Q| \geqslant \ell + 1$ and $|Q|$ has the same parity as $\ell + 1$. Assume that the latter case happens. Without loss of generality we may further assume that Q is chosen with length at least $\ell + 1$ and with $|P_2 \cup Q|$ as small as possible. Since $|Q|$ and $|P_2|$ have different parity, $P_2 \cup Q$ is not bipartite. Moreover, by the choice of Q, the subgraph $P_2 \cup Q$ contains a unique cycle C and $|C|$ is odd. Since $Q = P_2 \Delta C$ is an induced path of length at least $\ell + 1$, we have $|C \cap Q| > |C \cap P_2| \geqslant 2$. So $C \Delta C_3 \Delta C_1$ is an odd hole of length at least $2\ell + 3$, which is not possible. □

Acknowledgements

The authors thank the referees for their careful reading of this manuscript and pointing out an error in our original version. This research was partially supported by grants from the National Natural Sciences Foundation of China (No. 11971111).

References

[1] R. Chen, Graphs with girth $2\ell + 1$ and without longer odd holes are 3-colorable, arXiv:2301.00112, 2023.

[2] M. Chudnovsky, P. Seymour, Proof of a conjecture of Plummer and Zha, *J. Graph Theory*, 103:437-450, 2023.

[3] G. Dirac, The structure of k-chromatic graphs, *Fund. Math.*, 40:#P42.55, 1953.

[4] D. Nelson, M. Plummer, N. Robertson, X. Zha, On a conjecture concerning the Petersen graph, *The Electronic J. of Combin.*, 18:#P20, 2011.

[5] M. Plummer, X. Zha, On a conjecture concerning the Petersen Graph: Part II, *The Electronic J. of Combin.*, 21:#P1.34, 2014.

[6] Y. Wang, R. Wu, Graphs with girth 9 and without longer odd holes are 3-colorable, arXiv:2307.01460, 2023.

[7] D. Wu, B. Xu, Y. Xu, On coloring of graphs of girth $2\ell + 1$ without longer odd holes (in Chinese), *Sci Sin Math.*, 53:103-120, 2023.

[8] D. Wu, B. Xu, Y. Xu, The chromatic number of heptagraphs, arXiv:2206.01400v1, 2022.

[9] B. Xu, G. Yu, X. Zha, A note on chromatic number and induced odd cycles, *The Electronic J. of Combin.*, 24(4):#P4.32, 2017.