Demazure Product of Permutations and Hopping

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Abstract

The Demazure product (also called the 0-Hecke product or greedy product) is an associative operation on Coxeter groups with interesting properties and important applications. In this paper, we study permutation groups and present a way to compute the Demazure product of two permutations using only their one-line notation and not relying on reduced words. The algorithm starts from their usual product and then applies a new operator, which we call the hopping operator. We also give an analogous result for the group of signed permutations.

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1 Introduction

Coxeter groups play an important role in the representation theory of Lie groups and the geometry of their associated flag varieties and Schubert varieties. There is an interesting associative operation on Coxeter groups called the **Demazure product** [5] (also 0-**Hecke** or **greedy product**). In this paper, we study the Demazure product for two classes of Coxeter groups: permutations and signed permutations. Our main result is a method to compute this product using a new operator which we call the **hopping operator**, that allows us to compute the Demazure product without using reduced words, lengths, or simple transpositions. This result is stated for permutations in Theorem 12 and for signed permutations in Theorem 20.

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Let W be a Coxeter group with generating set S. It has relations of the the form

$$(st)^{m_{st}} = id, \quad s, t \in S \tag{1}$$

for some values $m_{st} \in \mathbb{Z}_{>0} \cup \{\infty\}$ where $m_{st} = 1$ if and only if $s = t$. Coxeter groups come equipped with a length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$ and a partial order $\leqslant$ called the Bruhat order. For more details on the basic properties of Coxeter groups, see [3]. The **Coxeter monoid** structure (also called the 0-**Iwahari-Hecke monoid** structure) on W is defined to be the monoid generated by S with a product $\star$ satisfying the Coxeter braid relations in Equation (1) for $s \neq t$ along with the relation $s \star s = s$ for all $s \in S$ (this new relation replaces $s^2 = id$ in the usual product). This monoid product what we refer to as the Demazure product and was first studied by Norton in [12] in the context of Hecke algebras. It is well known that, as sets, $W = \langle S, \star \rangle$. We say an expression $w = s_1 \cdots s_k$ is reduced if $\ell(w) = k$. Recall that if $\ell(w) = k$, then w cannot be expressed with fewer than k generators in S. We say $u \leqslant w$ in the **Bruhat order** if there exists a reduced word of u that is a subword of some reduced word of w ([3], Theorem 2.2.2). The next lemma records some basic facts about the Coxeter monoid.

Lemma 1. *[12, Lemma 1.3, Corollary 1.4] Let W be a Coxeter group with generating set S. Then the following are true:*

1. *Let $(s_1, \ldots, s_k)$ be a sequence of generators in S. Then*

$$s_1 \cdots s_k \leqslant s_1 \star \cdots \star s_k$$

with equality if and only if $(s_1, \ldots, s_k)$ is a reduced expression.

2. *For any $s \in S$ and $w \in W$,*

$$s \star w = \begin{cases} w & \text{if} \quad \ell(sw) < \ell(w) \\ sw & \text{if} \quad \ell(sw) > \ell(w). \end{cases}$$

As a consequence, there is a very nice interpretation of $w \star u$ for any $w, u \in W$ in terms of Bruhat intervals. Define the **Bruhat interval** $[u, w] := \{v \in W \mid u \leqslant v \leqslant w\}$. If $u = id$, then $[id, w]$ is called the **lower interval** of w.

Proposition 2. *[7, Lemma 1], [8, Proposition 8] For any $w, u \in W$, the lower interval*

$$[id, w \star u] = \{ab \mid a \in [id, w], b \in [id, u]\}.$$

We give an example of this phenomenon. The poset in Figure 1 is the Bruhat order of the symmetric group S_4 which has three simple generators s_1, s_2, s_3. The elements in the lower interval of $s_1s_2s_3s_2s_1 = s_1s_2s_3 \star s_2s_1$ are colored red. All elements in the lower interval $[id, s_1s_2s_3s_2s_1]$ can be written as $a \star b$ where $a \leqslant s_1s_2s_3$ and $b \leqslant s_2s_1$. For example, $s_2s_3s_1$ can be written as $s_2s_3 \star s_1$.

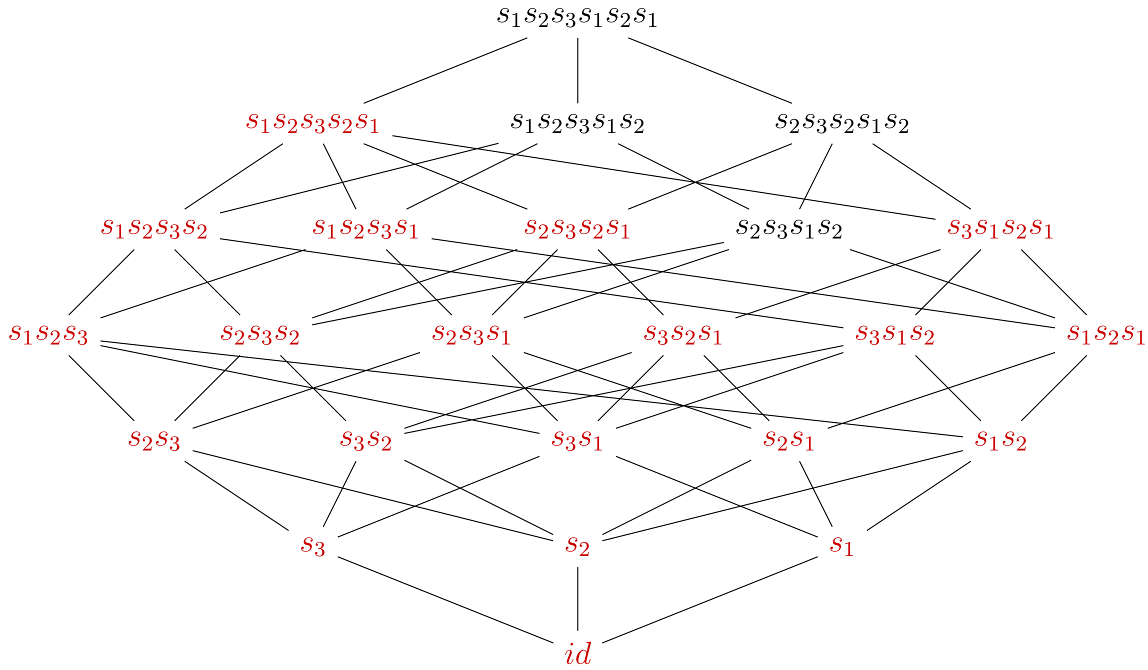

Figure 1: The Bruhat order of S_4 and the lower interval of $s_1s_2s_3s_2s_1$ in red.

This product has been used and studied in various fields that depend on Coxeter groups [6], [9], [10], [13], [14], [15]. For example, in Lie theory, the Demazure product naturally arises in the study of BN pairs and reductive groups. Specifically, the relation on Borel double orbit closures is given by

$$\overline{BwBuB} = \overline{B(w \star u)B}.$$

While the product $w \star u$ has been well studied for many years, it is difficult to calculate using reduced expressions of w and u. In this paper, we present a new method to compute the Demazure product of two permutations in the symmetric group using their one-line notation. This algorithm starts with the usual product of permutations and a brand new operation we call **hopping**. We state this result in Theorem 12. In Section 4, we prove an analogous result for signed permutations which is stated in Theorem 20. We remark that other "reduced word free" characterizations of the Demazure product exist for permutations. In [4, Fact 2.4], Chan and Pflueger give a characterization of the Demazure product for permutations in terms of rank functions. In [14], Pflueger later shows that this characterization of the Demazure product extends to the larger classes of almost-sign-preserving permutations. In [17], a recursive algorithm for computing the Demazure product starting with the Monge matrix (rank matrix) of a permutation and applying tropical operations.

2 The hopping operator

In this section, we focus on the permutation group or symmetric group S_n. These groups are also known as Coxeter groups of type A. The group S_n is a Coxeter group with simple generating set $S = \{s_1, \ldots, s_{n-1}\}$ satisfying the relations $s_i^2 = id$ and

$$(s_i s_j)^2 = id \text{ if } |i-j| > 1 \text{ and } (s_i s_{i+1})^3 = id. \tag{2}$$

The generator s_i corresponds to the simple transposition $(i, i+1)$ and for $w \in S_n$, let $w = w(1)w(2)\cdots w(n)$ denote the permutation in one-line notation. We denote the set of integers $[\![n]\!] := \{1, 2, \ldots, n\}$ and use the notation $L := [a_1, \ldots, a_k]$ to denote an ordered subset of $[\![n]\!]$ (without repetition). Note that with this notation, $[a] \neq [\![a]\!]$. We define a new operator on permutations called the **hopping operator**.

Definition 3. For $t \in [\![n]\!]$ and L an ordered subset of $[\![n]\!]$, the **hopping operator**

$$h_{t,L} : S_n \to S_n$$

acts on a permutation w according to the following algorithm: Scan to the right (within the one-line notation of w) of t and look for the element furthest to the right in L that is greater than t. If it exists, swap t and that element, replace w with the resulting permutation, and repeat. The algorithm ends when there are no elements of L within w to the right of t.

For example, take $n = 8$ and $w = 891726435$. Then $h_{1,[2,3,4,5,6,7,8]}(w) = 897625431$ is obtained by the following process:

$$89\mathbf{17}26435 \to 897\mathbf{1}2\mathbf{6}435 \to 89762\mathbf{1}43\mathbf{5} \to 897625431.$$

For another example, we have $h_{1,[3,6,5,7,2]}(w) = 892756431$ and is obtained by the following process:

$$89\mathbf{172}6435 \to 8927\mathbf{1}643\mathbf{5} \to 892756431.$$

Remark 4. There are some trivial manipulations that we can do to the ordered subset L in a hopping operator and keep the operator the same. Fix $w \in S_n$, $t \in [\![n]\!]$ and L to be an ordered subset of $[\![n]\!]$.

- (truncate) Let L' denote the ordered subset obtained from L by removing all elements smaller than t or that do not appear to the right of t within w. Then we have $h_{t,L} = h_{t,L'}$.
- (split/merge) Suppose that L is the concatenation of two ordered subsets L_1 and L_2. Then $h_{t,L} = h_{t,L_1} h_{t,L_2}$.

For any ordered subset $L \subseteq [\![n]\!]$, let $w(L) \subseteq [\![n]\!]$ denote the ordered list obtained by w acting on the elements of L. While a permutation may not preserve L, it can be viewed as an operator on ordered subsets of $[\![n]\!]$ preserving size.

For any $i < j$, let us define the transposition

$$\tau_{i,j} := s_i s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i.$$

In other words, $\tau_{i,j}$ is the transposition that swaps i and j. It is easy to see that

$$s_i \tau_{i,j} s_i = \tau_{i+1,j} \tag{3}$$

for any $1 \leqslant i \leqslant n-1$ and $i+1 < j \leqslant n$. Hopping operators satisfy the following commuting relation with simple transpositions:

Lemma 5. *Let $w \in S_n$. For any $i > t$ and ordered subset $L \subseteq [\![n]\!]$, we have*

$$s_i h_{t,L}(w) = h_{t,s_i(L)}(s_i w).$$

Proof. We will prove $h_{t,L} = s_i h_{t,s_i(L)} s_i$ by induction on the length of L. If $|L| = 1$, then $L = [\ell]$ for some $\ell \in [\![n]\!]$. We have two cases based on whether or not ℓ precedes t in the one-line notation of w. First, if ℓ precedes t, then $h_{t,[\ell]}(w) = w$ and $h_{t,s_i([\ell])}(s_i w) = s_i w$. Otherwise, if t precedes ℓ, we consider two subcases. If $\ell \notin \{i, i+1\}$, then $\tau_{t,\ell} = \tau_{t,s_i(\ell)}$ and $\tau_{t,\ell}$ commute with s_i. This implies that $h_{t,[\ell]} = \tau_{t,\ell}$ and $s_i h_{t,s_i([\ell])} s_i = s_i \tau_{t,s_i(\ell)} s_i = \tau_{t,s_i(\ell)}$ are the same. If $\ell \in \{i, i+1\}$, then the lemma follows from Equation (3).

Now assume for the sake of induction that the lemma is true for ordered subsets of sizes smaller than L. We split L into L' and $[\ell]$, where ℓ is the last element of L. We have $h_{t,[\ell]} = s_i h_{t,s_i([\ell])} s_i$ and $h_{t,L'} = s_i h_{t,s_i(L')} s_i$ from the induction hypothesis. Combining these equations using Remark 4, we get

$$h_{t,L} = h_{t,L'} h_{t,[\ell]} = s_i h_{t,s_i(L')} h_{t,s_i([\ell])} s_i = s_i h_{t,s_i(L)} s_i.$$

This completes the proof. $\square$

For example, let $w = 514632$. We compare the action of $s_3 h_{1,[2,3,4,5]}$ and $h_{1,[2,4,3,5]} s_3$ on w. First, we have

$$514632 \xrightarrow[h_{1,[2,3,4,5]}]{} 543621 \xrightarrow[s_3]{} 534621.$$

On the other hand, we get

$$514632 \xrightarrow[s_3]{} 513642 \xrightarrow[h_{1,[2,4,3,5]}]{} 534621,$$

yielding the same result as Lemma 5.

Notice that the above Lemma 5 only works when $i > t$. In the case $i = t$, we have a result below that is similar in flavor but slightly different:

Lemma 6. *Let $w \in S_n$. For any $1 \leqslant i \leqslant n-1$ and L an ordered subset of $[\![n]\!] \setminus [\![i+1]\!] = \{i+2, \ldots, n\}$, we have*

$$s_i h_{i+1,L}(w) = h_{i,L}(s_i w).$$

Proof. We will prove $s_i h_{i+1,L} s_i = h_{i,L}$ by induction on the length of L. First, it is easy to check that when $|L| = 1$, the lemma follows from Equation (3). Now assume for the sake of induction that the lemma is true for ordered subsets of size smaller than L. We split L into L' and ℓ, where ℓ is the last element of L. We have $s_i h_{i+1,L'} s_i = h_{i,L'}$ and $s_i h_{i+1,[\ell]} s_i = h_{i,[\ell]}$ from the induction hypothesis. Using Remark 4, we get

$$h_{i,L} = h_{i,L'} h_{i,[\ell]} = s_i h_{i+1,L'} s_i s_i h_{i+1,[\ell]} s_i = s_i h_{i+1,L} s_i.$$

This completes the proof. $\square$

In the next lemma, we describe how the Demazure operator and the hopping operator interact with each other.

Lemma 7. *For any* $1 \leqslant i \leqslant n-1$ *and* $w \in S_n$, *we have*

$$s_i \star w = h_{i,[i+1]}(s_i w).$$

Proof. We have two cases based on whether or not $i+1$ comes after i in w. First, if $i+1$ appears before i in w, then $s_i \star w = w$, and in $s_i w$ we have i appearing before $i+1$. So $h_{i,[i+1]}(s_i w) = s_i(s_i w) = w$. Otherwise, if $i+1$ appears after i in w, then $s_i \star w = s_i w$, and in $s_i w$ we have $i+1$ appearing before i so the hopping operator $h_{i,[i+1]}$ makes no changes to $s_i w$. $\square$

Throughout the paper, we will denote the product of a sequence of consecutive simple transpositions by

$$\mathrm{C}_{a,b} := s_a s_{a+1} \cdots s_{a+b-1}.$$

If $b = 0$, then $\mathrm{C}_{a,0}$ is the identity. The operator $\mathrm{C}_{a,b}$ acts on a permutation w by mapping each of $a, a+1, \dots, a+b$ to $a+1, \dots, a+b, a$ respectively. In other words, it is a cyclic shift of the elements $a, a+1, \dots, a+b$ by one. For example, in S_8, we have $\mathrm{C}_{2,4} = s_2 s_3 s_4 s_5$ corresponding to the permutation 13456278. From Lemma 5 we immediately get the following corollary:

Corollary 8. *Let* $w \in S_n$. *For any* $a > t$ *and ordered subset* $L \subseteq [\![n]\!]$, *we have*

$$\mathrm{C}_{a,b}\, h_{t,L}(w) = h_{t,\mathrm{C}_{a,b}(L)}\, \mathrm{C}_{a,b}(w).$$

The Demazure product with $\mathrm{C}_{a,b}$ can be described using the usual product on permutations and a hopping operator as follows:

Proposition 9. *For any* $1 \leqslant i \leqslant j \leqslant n-1$ *and* $w \in S_n$, *we have*

$$\mathrm{C}_{i,j-i+1} \star w = h_{i,[i+1,\dots,j+1]}(\mathrm{C}_{i,j-i+1}\, w). \tag{4}$$

Proof. We use strong induction on $j-i$. The $j-i=0$ case follows from Lemma 7. Assume for sake of induction that we have

$$\mathrm{C}_{i+1,j-(i+1)+1} \star w = h_{i+1,[i+2,\dots,j+1]}(\mathrm{C}_{i+1,j-(i+1)+1}\, w) \tag{5}$$

for any permutation w. We denote the expression given in Equation (5) as α. Then the left-hand side of Equation (4) can be expressed as $s_i \star \alpha$, which in turn is equal to $h_{i,[i+1]}(s_i\alpha)$ from Lemma 7. It suffices to prove

$$h_{i,[i+1]}s_ih_{i+1,[i+2,\dots,j+1]}(\mathrm{C}_{i+1,j-(i+1)+1}\, w) = h_{i,[i+1,\dots,j+1]}(s_i\, \mathrm{C}_{i+1,j-(i+1)+1}\, w).$$

Therefore our goal is to show $h_{i,[i+1]}s_ih_{i+1,[i+2,\dots,j+1]}s_i = h_{i,[i+1,\dots,j+1]}$. Remark 4 implies that we can split $h_{i,[i+1,\dots,j+1]} = h_{i,[i+1]}h_{i,[i+2,\dots,j+1]}$. Now the Proposition follows from Lemma 6.

$\square$

For example, let $w = 124567893 = s_3s_4s_5s_6s_7s_8 = \mathrm{C}_{3,6}$ and $v = 891726435$. The above proposition implies that $w \star v = h_{3,[4,5,6,7,8,9]}(wv)$. Starting with the usual product $wv = 931827546$, we get

$$9\mathbf{3}\mathbf{1}\mathbf{8}27546 \to 981\mathbf{3}\mathbf{2}\mathbf{7}546 \to 98172\mathbf{3}54\mathbf{6} \to 981726543.$$

Thus $w \star v = 981726543$.

Definition 10. Given $w \in S_n$, we let $w \nwarrow a$ stand for the subword of w obtained by restricting to the subword strictly left of a, then removing all elements smaller than a.

For example, take $w = 891726435$. We have $w \nwarrow 2 = 897$, obtained by taking the subword strictly left of two and then removing elements smaller than two. Similarly, we get $w \nwarrow 4 = 8976$, again obtained by taking the subword strictly left of four and then removing elements smaller than four.

Notice that if $w = s_is_{i+1}\cdots s_j$, then $w \nwarrow i = [i+1, i+2, \dots, j+1]$. Proposition 9 implies that if $w = s_is_{i+1}\cdots s_j$, then $w \star v = h_{i,w\nwarrow i}(wv)$.

3 The main result

In this section we give a formula for the Demazure product between two arbitrary permutations by writing one of the permutations as a product $\mathrm{C}_{i,j}$'s and then carefully iterating Proposition 9.

Definition 11. Given a word w on a set of elements in $[\![n]\!]$ and a subset $L \subseteq [\![n]\!]$, we use $w|_L$ to denote the subword obtained from w by only taking the entries contained in L.

Theorem 12. *For any $w, v \in S_n$, we have $w \star v = h_{n-1,w\nwarrow n-1}\cdots h_{2,w\nwarrow 2}h_{1,w\nwarrow 1}(wv)$.*

Proof. Let $(j_1, \dots, j_{n-1})$ denote the inversion sequence of w (see [16, Chapter 1.3]). In other words, j_i denotes the number of inversions in w of the form (k, i). It is easy to check that

$$w = \mathrm{C}_{n-1,j_{n-1}}\cdots \mathrm{C}_{2,j_2}\,\mathrm{C}_{1,j_1}\,. \tag{6}$$

The reason we use this decomposition is that for each i, we have

$$(\mathrm{C}_{n-1,j_{n-1}}\cdots \mathrm{C}_{i,j_i})|_{\{i+1,\dots,n\}} = w|_{\{i+1,\dots,n\}}.$$

To see this, notice that w is obtained from the expression in Equation 6, and $\mathrm{C}_{i-1,j_{i-1}} \cdots \mathrm{C}_{1,j_1}$ does not change the ordering between $i, \ldots, n$.

From Proposition 9, we get that

$$w \star v = \cdots h_{i,[i+1,\ldots,i+j_i]}\, \mathrm{C}_{i,j_i} \cdots h_{2,[3,\ldots,2+j_2]}\, \mathrm{C}_{2,j_2} (h_{1,[2,\ldots,1+j_1]}\, \mathrm{C}_{1,j_1}\, v).$$

Note that if $j_i = 0$, then we set the corresponding hopping operator to the identity. To the left of $h_{i,[i+1,\ldots,i+j_i]}$ we have some other hopping operators and $\mathrm{C}_{n-1,j_{n-1}}, \ldots, \mathrm{C}_{i+1,j_{i+1}}$ in the above expression. Our strategy is to move each hopping operator to the left, passing through all $\mathrm{C}_{a,b}$'s, using Corollary 8. We start from the leftmost hopping operator, and eventually all hopping operators will be at the prefix of the expression describing $w \star v$. With this in mind, to prove the claim, we need to show that for each $i < n$, we get

$$\mathrm{C}_{n-1,j_{n-1}} \cdots \mathrm{C}_{i+1,j_{i+1}}\, h_{i,[i+1,\ldots,i+j_i]} = h_{i,w\nwarrow i}\, \mathrm{C}_{n-1,j_{n-1}} \cdots \mathrm{C}_{i+1,j_{i+1}}\,.$$

Let α denote the permutation $\mathrm{C}_{n-1,j_{n-1}} \cdots \mathrm{C}_{i+1,j_{i+1}}$. From Corollary 8 we get

$$\alpha h_{i,[i+1,\ldots,i+j_i]} = h_{i,\alpha([i+1,\ldots,i+j_i])}\alpha.$$

As in the observation following Equation 6, we have $\alpha([i+1,\ldots,i+j_i]) = w\nwarrow i$, which finishes the proof. $\square$

For example, let $w = 6541723$ and $v = 5436217$. The usual product of these two permutations is $wv = 7142563$. The Demazure product $w \star v$ corresponds to the sequence of hopping operators

$$h_{5,[6]} h_{4,[6,5]} h_{3,[6,5,4,7]} h_{2,[6,5,4,7]} h_{1,[6,5,4]}$$

acting on the usual product wv. Applying each of the hopping operators in order to wv, we get

$$7142563 \xrightarrow[h_{1,[6,5,4]}]{} 7452613 \xrightarrow[h_{2,[6,5,4,7]}]{} 7456213$$
$$\xrightarrow[h_{3,[6,5,4,7]}]{} 7456213 \xrightarrow[h_{4,[6,5]}]{} 7564213 \xrightarrow[h_{5,[6]}]{} 7654213.$$

This gives us $6541723 \star 5436217 = 7654213$.

4 Signed permutations

In this section, we prove an analogue of Theorem 12 for the group of signed permutations, also known as Coxeter groups of type B (or equivalently, type C). Signed permutations can be viewed as a permutation subgroup of S_{2n}. Let $\{s'_1, \ldots, s'_{2n-1}\}$ be the simple generators of the permutation group S_{2n}. We define B_n to be the subgroup of S_{2n} generated by $S := \{s_1, \ldots, s_n\}$ where

$$s_i := s'_i\, s'_{2n-i} \text{ for } 1 \leqslant i < n \text{ and } \quad s_n := s'_n. \tag{7}$$

We use the convention from [1] when working with type B simple transpositions. As a Coxeter group, the generators $s_1, \ldots, s_{n-1}$ of B_n satisfy the same relations as in type A (see Equation (2)) with the last generator s_n satisfying:

$$(s_i s_n)^2 = id \quad \text{for } 1 \leqslant i < n-1 \text{ and } (s_{n-1}s_n)^4 = id.$$

Similar to the symmetric group, the elements of the Coxeter group B_n can be expressed using a decorated one-line notation as follows:

Definition 13. A **signed permutation** of type B_n is a permutation of $[\![n]\!]$ along with a sign of $+$ or $-$ attached to each number.

For example, the signed permutation $[+4, -2, +3, -1] = [4, -2, 3, -1]$ is an element of B_4. For notational simplicity we drop the "$+$" signs from the one-line notation. The generator $s_i \in B_n$ corresponds to the simple transposition swapping i and $i+1$ if $i < n$ and s_n to the transposition swapping n with $-n$. The convention we use here is slightly different from that of Bjorner and Brenti in [3] in the sense that s_n plays the role of s_0. The product structure on signed permutations is just the usual composition of permutations with the added condition that $w(-i) = -w(i)$. Let $\pm[\![n]\!]$ denote the set $[\![n]\!] \cup -[\![n]\!]$, where $-[\![n]\!] := \{-1, \ldots, -n\}$. We impose the total ordering on $\pm[\![n]\!]$ given by:

$$1 < 2 < \cdots < n < -n < \cdots < -2 < -1.$$

By **unfolding** of a signed permutation $w \in B_n$ we mean the following: to the right of w, attach a reverse ordered copy of w with the signs flipped to get a permutation of $\pm[\![n]\!]$. The unfolding map respects the embedding of B_n as a subgroup of S_{2n} given above. Specifically, if we replace $-[\![n]\!]$ with $\{n+1, \ldots, 2n\}$, then the unfolding map assigns to each signed permutation in B_n a standard permutation in S_{2n}. For example the unfolding of $[4, -2, 3, -1]$ is

$$[4, -2, 3, -1, 1, -3, 2, -4]$$

and the corresponding permutation of $[\![8]\!]$ is $[4, 7, 3, 8, 1, 6, 2, 5]$. Conversely, given a permutation of of $\pm[\![n]\!]$ where the i-th entry is the opposite sign of $(2n+1-i)$-th entry, we can **fold** the permutation to get a signed permutation on $[\![n]\!]$. In this section, we will slightly abuse notation and identify a signed permutation of B_n with its unfolding in S_{2n}. When referring to the generators of S_{2n}, we set

$$s'_{-i} := s'_{2n-i}$$

and hence $s_i := s'_i s'_{-i}$ for any $i < n$.

Lemma 14. *For any signed permutations $w, v \in B_n$, we have*

$$w \star v = \operatorname{fold}(\operatorname{unfold}(w) \star \operatorname{unfold}(v)). \tag{8}$$

Proof. First observe that Equation (8) holds if we replace $\star$ with the group product since the unfolding map corresponds to the embedding of B_n into S_{2n}. We proceed by induction on the length of w and will use ℓ_B, ℓ_A to denote length in the Coxeter groups B_n and S_{2n} respectively. First, suppose that $w = s_i$ where $i < n$. Then $\text{unfold}(w) = s'_i s'_{-i}$ with respect to the embedding of B_n into S_{2n}. Since s'_i commutes with s'_{-i}, we have that $\ell_B(s_i v) = \ell_B(v) - 1$ if and only if $\ell_A(\text{unfold}(s_i v)) = \ell_A(\text{unfold}(v)) - 2$. Lemma 1 part (2) implies

$$s_i \star v = \text{fold}(\text{unfold}(s_i \star v)) = \text{fold}(\text{unfold}(s_i) \star \text{unfold}(v)).$$

A similar argument holds when $w = s_n$ and $\text{unfold}(w) = s'_n$. This proves the lemma in the case when $\ell_B(w) = 1$. Now suppose that $\ell_B(w) > 1$ and write $w = sw'$ for some $s \in S$ and $w' \in B_n$ where $\ell_B(w) = \ell_B(w') + 1$. By induction we get

$$w \star v = sw' \star v = s \star (w' \star v) = s \star \text{fold}(\text{unfold}(w') \star \text{unfold}(v)).$$

The inductive base case above implies

$$\begin{aligned} s \star \text{fold}(\text{unfold}(w') \star \text{unfold}(v)) &= \text{fold}(\text{unfold}(s) \star \text{unfold}(w') \star \text{unfold}(v)) \\ &= \text{fold}(\text{unfold}(w) \star \text{unfold}(v)). \end{aligned}$$

This completes the proof. $\square$

Next, we define a hopping operator for B_n analogous to Definition 3 for S_n.

Definition 15. Let $t \in \pm[\![n]\!]$ and L an ordered subset $\pm[\![n]\!]$ (without repetition). The **hopping operator**

$$h_{t,L} : B_n \to B_n$$

acts on a signed permutation w by the following algorithm: scan to the right (within the unfolding of w) of t and look for the element furthest to the right in L that is greater than t. If it exists, say q, then swap t and q and also swap $-t$ with $-q$ (unless $t = -q$). Replace w with the resulting unfolded signed permutation and repeat. The algorithm ends when there are no elements of L within w to the right of t.

For example, let $w = [2, 3, 5, -1, 4]$ with $t = 1$ and $L = [-2, -3, 4]$. We calculate the hopping $h_{1,[-2,-3,4]}(w)$. First we unfold w, which gives

$$\text{unfold}(w) = [2, 3, 5, -1, 4, -4, 1, -5, -3, -2].$$

To the right of 1 we have $[-5, -3, -2]$. We first swap 1 for -3, since -3 is the rightmost element of L that exists here. This gives us $[2, -1, 5, 3, 4, -4, -3, -5, 1, -2]$. After that, we again scan to the right of 1 to find $[-2]$. Then we swap 1 with -2, to get $[-1, 2, 5, 3, 4, -4, -3, -5, -2, 1]$. So, the signed permutation we end up with is $[-1, 2, 5, 3, 4]$.

Similar to the type A case, hopping operators satisfy a commuting relation with simple transpositions as in Lemma 18. We omit the proof, since it is analogous to that of Lemma 18. Similar to the type A case, we let B_n act on sublists of $\pm[n]$ via the corresponding signed permutation.

Lemma 16. *Let $w \in B_n$. For any $1 \leqslant t < i \leqslant n$ and ordered subset $L \subseteq \pm[\![n]\!]$ we have*

$$s_i h_{t,L}(w) = h_{t,s_i(L)}(s_i w).$$

Recall that for S_n, we defined $\mathrm{C}_{a,b} := s_a \cdots s_{a+b-1}$ and used the fact that any permutation naturally decomposes into a product of $\mathrm{C}_{a,b}$'s (see Equation (6)). For the type B_n case, we will define the analogous product of simple generators

$$\mathrm{C}^B_{a,b} := s_a \cdots s_{a+b-1}$$

where for any $j \geqslant 1$, we set $s_{n+j} := s_{n-j}$. Note that if $a \leqslant n$, then $1 \leqslant b \leqslant 2n - a$. For example, in B_7, we have

$$\mathrm{C}^B_{5,6} = s_5 s_6 s_7 s_8 s_9 s_{10} = s_5 s_6 s_7 s_6 s_5 s_4.$$

As a signed permutation, the product $\mathrm{C}^B_{a,b}$ corresponds to unfolding the identity permutation $[1, 2, \ldots, n]$ and shifting a to the right by b positions, then placing $-a$ in the mirrored position. For example, in B_7 we have

$$\mathrm{C}^B_{5,6} = [1, 2, 3, -5, 4, 6, 7, -7, -6, -4, 5, -3, -2, -1] = [1, 2, 3, -5, 4, 6, 7].$$

Notice that when $b \leqslant 2n - 2a + 1$, the support of $\mathrm{C}^B_{a,b}$ does not involve $s_1, \ldots, s_{a-1}$. From this, an immediate corollary of Lemma 16 is the following.

Corollary 17. *Let $w \in B_n$ with $1 \leqslant t < a \leqslant n$ and $1 \leqslant b \leqslant 2n-2a+1$. For any ordered subset $L \subseteq \pm[\![n]\!]$, we have*

$$\mathrm{C}^B_{a,b}\, h_{t,L}(w) = h_{t,\mathrm{C}^B_{a,b}(L)}(\mathrm{C}^B_{a,b}\, w).$$

As in the type A case, the Demazure product with $\mathrm{C}^B_{a,b}$ can be described using the usual composition product on signed permutations and the hopping operator given in Definition 15. Recall that we identified s_{n+j} with s_{n-j} for $1 \leqslant j < n$. Similarly, we use $n + j$ to denote $-(n + 1 - j)$ for $1 \leqslant j < n$ when we are dealing with elements of $\pm[\![n]\!]$.

Lemma 18. *Let $v \in B_n$. For any $1 \leqslant i \leqslant n$, we have*

$$s_i \star v = h_{i,[i+1]}(s_i v).$$

Proof. In this proof, let $h^A_{i,L}$ denote the hopping operator given in Definition 3 acting on the permutation group S_{2n} and $h^B_{i,L}$ denote the hopping operator given in Definition 15 acting on $B_n \subseteq S_{2n}$. If $i < n$, then $s_i = s'_{-i} s'_i$ and by Lemma 14, we have

$$\mathrm{unfold}(s_i \star v) = s'_{-i} s'_i \star \mathrm{unfold}(v).$$

Proposition 9 implies

$$(s'_{-i} s_i{}') \star \mathrm{unfold}(v) = h^A_{-(i+1),[-i]} h^A_{i,[i+1]} s'_{-i} s_i{}'\, \mathrm{unfold}(v).$$

Note that swapping i with $i+1$ in $h^A_{i,[i+1]}$ mirrors swapping $-(i+1)$ with $-i$ in $h^A_{-(i+1),[-i]}$. Hence Lemma 14 implies

$$\mathrm{fold}((s'_{-i}s_i{}') \star \mathrm{unfold}(v)) = h^B_{i,[i+1]}(s_i v).$$

In the case of $i = n$, note that $s_n \star v = v$ if the sign of n in v is negative and $s_n \star v = s_n v$ otherwise. From this it follows that $s_n \star v = h_{n,[-n]}(s_n v)$. Since $n+1$ stands for $-n$ in our convention, the proof is complete. □

Proposition 19. *Let $w \in B_n$. For any $1 \leqslant i \leqslant j \leqslant 2n-1$, we have*

$$\mathrm{C}^B_{i,j-i+1} \star w = h_{i,[i+1,\ldots,j+1]}(\mathrm{C}^B_{i,j-i+1}\, w).$$

Proof. First, consider the case $i > n$. We have

$$\mathrm{C}^B_{i,j-i+1} = s_i \cdots s_j = (s'_i \cdots s'_j)(s'_{-i} \cdots s'_{-j}) = \mathrm{C}_{i,j-i+1}\, \mathrm{C}_{-i,-(j-i+1)}\,.$$

The first portion $(s'_i \cdots s'_j)$ acts on the negative entries in the unfolding of w, whereas the second portion $(s'_{-i} \cdots s'_{-j})$ acts on the positive entries of w, mirroring what happens for the negative entries. From Proposition 9, we have

$$\mathrm{C}_{-i,-(j-i+1)} \star \mathrm{unfold}(w)|_{[\![n]\!]} = h_{-i,[-(i+1),\ldots,-(j+1)]}\, \mathrm{C}_{-i,-(j-i+1)}\, \mathrm{unfold}(w)|_{[\![n]\!]}.$$

Since what happens for the negative entries of w mirrors that of what happens for positive entries of w, we get

$$\mathrm{C}_{i,(j-i+1)} \star \mathrm{unfold}(w)|_{-[\![n]\!]} = h_{i,[(i+1),\ldots,(j+1)]}\, \mathrm{C}_{i,(j-i+1)}\, \mathrm{unfold}(w)|_{-[\![n]\!]}.$$

Now using Lemma 14, we get the desired proposition.

Next, we resolve the remaining case $i \leqslant n$ by using induction on $n-i$. First, when $n-i=-1$, the proposition follows from the above case. Suppose that $n-i \geqslant 0$ and suppose for the sake of induction that we have the proposition is true for all i' such that $n-i' < n-i$. We start by analyzing the expression $s_i \star ((s_{i+1} \cdots s_j) \star v)$. From the induction hypothesis, we have that

$$(s_{i+1} \cdots s_j) \star v = h_{i+1,[i+2,\ldots,j+1]}(s_{i+1} \cdots s_{j+1} v).$$

By Lemma 18, in order to prove the claim, it suffices to show $h_{i,[i+1]} s_i h_{i+1,[i+2,\ldots,j+1]} = h_{i,[i+1,\ldots,j+1]} s_i$. Lemma 16 implies

$$s_i h_{i+1,[i+2,\ldots,j+1]} s_i = h_{i,[i+2,\ldots,j+1]}$$

and together with merging $h_{i,[i+1]} h_{i,[i+2,\ldots,j+1]}$ into $h_{i,[i+1,\ldots,j+1]}$ by Remark 4, we get our desired result. □

We now give an analogue of Definition 10 for signed permutations. For any $w \in B_n$ and $i > 0$, define $w \nwarrow i$ to be the subword of unfold(w) obtained be restricting to numbers to the left of i that are either greater than i, or less or equal to $-i$. For example, if $w = [-5, 3, 1, -2, 4]$, then

$$\text{unfold}(w) = [-5, 3, 1, -2, 4, -4, 2, -1, -3, 5].$$

In this case we have $w \nwarrow 1 = [-5, 3]$ and $w \nwarrow 2 = [-5, 3, -2, 4, -4]$.

Theorem 20. *For any $w, v \in B_n$, we have $w \star v = h_{n-1,w\nwarrow n-1} h_{2,w\nwarrow 2} h_{1,w\nwarrow 1}(wv)$.*

Proof. We show that the signed permutation w has a unique decomposition

$$w = \mathrm{C}^B_{n,j_n} \cdots \mathrm{C}^B_{2,j_2}\, \mathrm{C}^B_{1,j_1} \tag{9}$$

where j_i is the cardinality of $w\nwarrow i$. We let w^i denote $\mathrm{C}^B_{n,j_n} \cdots \mathrm{C}^B_{i,j_i}$.

We use induction on $n-i$ to show that $w^i|_{\pm\{i,\ldots,n\}} = w|_{\pm\{i,\ldots,n\}}$. In the base case, when $n - i = 0$, we only look at the ordering between n and $-n$. We have $j_n = 1$ if and only if $-n$ appears before n in w, and hence we get the desired result. Now assume for the sake of induction that we have the equality for all $i' > i$. Within the unfolding of $w|_{\pm\{i,\ldots,n\}}$, the entry i has exactly $w\nwarrow i$ many elements to its left. Therefore, to obtain $w|_{\pm\{i,\ldots,n\}}$ from the unfolding of $w^{i+1}|_{\pm\{i,\ldots,n\}}$, we need to move i exactly $j_i = |w\nwarrow i|$ many times to the right, which corresponds to multiplying C^B_{i,j_i} to the right of $w^{i+1}|_{\pm\{i,\ldots,n\}}$. Since $w^{i+1}\,\mathrm{C}^B_{i,j_i} = w^i$, we have completed the proof of the decomposition.

The theorem now follows an analogue of the proof of Theorem 12 where we use Proposition 19 and Corollary 17 instead of Proposition 9 and Corollary 8. $\square$

We give an example of computing the Demazure product on signed permutations using Theorem 20. Let $w = [-5, 3, 1, -2, 4]$ and $v = [-4, 2, -1, -3, 5]$ in B_5. The unfolding of w is $[-5, 3, 1, -2, 4, -4, 2, -1, -3, 5]$. The decomposition we get is

$$w = \mathrm{C}^B_{5,1}\, \mathrm{C}^B_{4,1}\, \mathrm{C}^B_{3,1}\, \mathrm{C}^B_{2,5}\, \mathrm{C}^B_{1,2}\,.$$

We start with the usual product $wv = [2, 3, 5, -1, 4]$. We apply the sequence of hopping operators

$$h_{5,[-5]} h_{4,[-5]} h_{3,[-5]} h_{2,[-5,3,-2,4,-4]} h_{1,[-5,3]}$$

to wv giving:

$$\begin{aligned}
&[2, 3, 5, -1, 4, -4, 1, -5, -3, -2] \xrightarrow[h_{1,[-5,3]}]{} [2, 3, -1, 5, 4, -4, -5, 1, -3, -2] \xrightarrow[h_{2,[-5,3,-2,4,-4]}]{} \\
&[-2, 3, -1, 5, -4, 4, -5, 1, -3, 2] \xrightarrow[h_{3,[-5]}]{} [-2, -5, -1, -3, -4, 4, 3, 1, 5, 2] \xrightarrow[h_{4,[-5]}]{} \\
&[-2, -5, -1, -3, -4, 4, 3, 1, 5, 2] \xrightarrow[h_{5,[-5]}]{} [-2, -5, -1, -3, -4, 4, 3, 1, 5, 2]
\end{aligned}$$

Theorem 20 implies that the Demazure product $w\star v = [-2, -5, -1, -3, -4]$. We conclude this section with some questions.

Question 21. Can the Demazure product for Coxeter groups of type D be described similarly to the case of type B?

Question 22. In [2], Billey and Weaver give a "one-line notation" algorithm to compute the maximal element in the intersection of a lower interval with an arbitrary coset of a maximal parabolic subgroup of type A. In [13], there is an alternate algorithm to compute the maximal element using the Demazure product (this second formula is for any Coxeter group and parabolic subgroup). Is there a way to apply Theorem 12 to recover the algorithm in [2]? If so, is there a generalization of the algorithm in [2] to the case where the parabolic subgroup is not maximal? or to the case of signed permutations? We note that the existence of such a maximal element for any Coxeter group W and a parabolic subgroup W_J was established in [11].

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