A Spectral Extremal Problem on Non-Bipartite Triangle-Free Graphs

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Abstract

A theorem of Nosal and Nikiforov states that if G is a triangle-free graph with m edges, then $\lambda(G) \leqslant \sqrt{m}$, where the equality holds if and only if G is a complete bipartite graph. A well-known spectral conjecture of Bollobás and Nikiforov [J. Combin. Theory Ser. B 97 (2007)] asserts that if G is a K_{r+1}-free graph with m edges, then $\lambda_1^2(G) + \lambda_2^2(G) \leqslant (1 - \frac{1}{r})2m$. Recently, Lin, Ning and Wu [Combin. Probab. Comput. 30 (2021)] confirmed the conjecture in the case $r = 2$. Using this base case, they proved further that $\lambda(G) \leqslant \sqrt{m-1}$ for every non-bipartite triangle-free graph G, with equality if and only if $m = 5$ and $G = C_5$. Moreover, Zhai and Shu [Discrete Math. 345 (2022)] presented an improvement by showing $\lambda(G) \leqslant \beta(m)$, where $\beta(m)$ is the largest root of $Z(x) := x^3 - x^2 - (m-2)x + m - 3$. The equality in Zhai–Shu's result holds only if m is odd and G is obtained from the complete bipartite graph $K_{2,\frac{m-1}{2}}$ by subdividing exactly one edge. Motivated by this observation, Zhai and Shu proposed a question to find a sharp bound when m is even. We shall solve this question by using a different method and characterize three kinds of spectral extremal graphs over all triangle-free non-bipartite graphs with even size. Our proof technique is mainly based on applying Cauchy interlacing theorem of eigenvalues of a graph, and with the aid of a triangle counting lemma in terms of both eigenvalues and the size of a graph.

Mathematics Subject Classifications: 05C35, 05C50

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We usually write n and m for the number of vertices and edges, respectively. One of the main problems of algebraic graph theory is to determine the combinatorial properties of a graph that are

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reflected from the algebraic properties of its associated matrices. Let G be a simple graph on n vertices. The *adjacency matrix* of G is defined as $A(G) = [a_{ij}]_{n\times n}$ where $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G, and $a_{ij} = 0$ otherwise. We say that G has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if these values are eigenvalues of the adjacency matrix $A(G)$. Let $\lambda(G)$ be the maximum value in absolute among all eigenvalues of G, which is known as the *spectral radius* of G.

1.1 The spectral extremal graph problems

A graph G is called *F-free* if it does not contain an isomorphic copy of F as a subgraph. Clearly, every bipartite graph is C_3-free. The *Turán number* of a graph F is the maximum number of edges in an n-vertex F-free graph, and it is usually denoted by $\mathrm{ex}(n, F)$. An F-free graph on n vertices with $\mathrm{ex}(n, F)$ edges is called an *extremal graph* for F. As is known to all, the Mantel theorem (see, e.g., [2]) asserts that if G is a triangle-free graph on n vertices, then

$$e(G) \leqslant \lfloor n^2/4 \rfloor, \tag{1}$$

where the equality holds if and only if G is the balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

There are numerous extensions and generalizations of Mantel's theorem; see [3, 5]. Especially, Turán (see, e.g., [2, pp. 294–301]) extended Mantel's theorem by showing that if G is a K_{r+1}-free graph on n vertices with maximum number of edges, then G is isomorphic to the graph $T_r(n)$, where $T_r(n)$ denotes the complete r-partite graph whose part sizes are as equal as possible. Each vertex part of $T_r(n)$ has size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. The graph $T_r(n)$ is usually called Turán's graph. Five alternative proofs of Turán's theorem are selected into THE BOOK[1] [1, p. 285]. Moreover, we refer the readers to the surveys [10, 41].

Spectral extremal graph theory, with its connections and applications to numerous other fields, has enjoyed tremendous growth in the past few decades. There is a rich history on the study of bounding the eigenvalues of a graph in terms of various parameters. For example, one can refer to [4] for spectral radius and cliques, [35] for independence number and eigenvalues, [44, 22] for eigenvalues of outerplanar and planar graphs, [8, 51] for excluding friendship graph, and [45, 52, 12] for excluding minors. It is a traditional problem to bound the spectral radius of a graph. Let G be a graph on n vertices with m edges. It is natural to ask how large the spectral radius $\lambda(G)$ may be. A well-known result states that

$$\lambda(G) < \sqrt{2m}. \tag{2}$$

This bound can be guaranteed by $\lambda(G)^2 < \sum_{i=1}^{n} \lambda_i^2 = \mathrm{Tr}(A^2(G)) = \sum_{i=1}^{n} d_i = 2m$. We recommend the readers to [13, 14, 32] for more extensions.

It is also a popular problem to study the extremal structure for graphs with given number of edges. For example, it is not difficult to show that if G has m edges, then G contains at most $\frac{\sqrt{2}}{3} m^{3/2}$ triangles; see, e.g., [2, p. 304] and [7]. In addition, it is an

[1] Paul Erdős liked to talk about THE BOOK, in which God maintains the perfect proofs for mathematical theorems, and he also said that you need not believe in God but you should believe in THE BOOK.

instrumental topic to study the interplay between these two problems mentioned-above. More precisely, one can investigate the largest eigenvalue of the adjacency matrix in a triangle-free graph with given number of edges[2]. Dating back to 1970, Nosal [40], and later Nikiforov [32, 35] independently obtained such a result.

Theorem 1 (Nosal [40], Nikiforov [32, 35])**.** *Let G be a graph with m edges. If G is triangle-free, then*

$$\lambda(G) \leqslant \sqrt{m}, \tag{3}$$

where the equality holds if and only if G is a complete bipartite graph.

Mantel's theorem in (1) can be derived from (3). Indeed, using Rayleigh's inequality, we have $\frac{2m}{n} \leqslant \lambda(G) \leqslant \sqrt{m}$, which yields $m \leqslant \lfloor n^2/4 \rfloor$. Thus, Theorem 1 could be viewed as a spectral version of Mantel's theorem. Moreover, Theorem 1 implies a result of Lovász and Pelikán [27], which asserts that if G is a tree on n vertices, then $\lambda(G) \leqslant \sqrt{n-1}$, with equality if and only if $G = K_{1,n-1}$.

Inequality (3) impulsed the great interests of studying the maximum spectral radius for F-free graphs with given number of edges, see [32, 35] for K_{r+1}-free graphs, [34, 50, 46] for C_4-free graphs, [49] for $K_{2,r+1}$-free graphs, [49, 31] for C_5-free or C_6-free graphs, [29] for C_7-free graphs, [43, 9, 26] for $C_4^{\triangle}$-free or $C_5^{\triangle}$-free graphs, where $C_k^{\triangle}$ is a graph on $k+1$ vertices obtained from C_k and C_3 by sharing a common edge; see [37] for B_k-free graphs, where B_k denotes the book graph consisting of k triangles sharing a common edge, [21] for F_2-free graphs with given number of edges, where F_2 is the friendship graph consisting of two triangles intersecting in a common vertex, [38, 39] for counting the number of C_3 and C_4. We refer the readers to the surveys [36, 18] and references therein.

In particular, Bollobás and Nikiforov [4] posed the following nice conjecture.

Conjecture 2 (Bollobás–Nikiforov, 2007)**.** Let G be a K_{r+1}-free graph of order at least $r+1$ with m edges. Then

$$\lambda_1^2(G) + \lambda_2^2(G) \leqslant 2m\Big(1 - \frac{1}{r}\Big).$$

Recently, Lin, Ning and Wu [23] confirmed the base case $r = 2$; see, e.g., [37, 17] for related results. Furthermore, the base case leads to Theorem 3 in next section.

1.2 The non-bipartite triangle-free graphs

The extremal graphs determined in Theorem 1 are the complete bipartite graphs. Excepting the largest extremal graphs, the second largest extremal graphs were extensively studied over the past years. In this paper, we will pay attention mainly to the spectral extremal problems for non-bipartite triangle-free graphs with given number of edges. Using the inequalities from majorization theory, Lin, Ning and Wu [23] confirmed the triangle case in Conjecture 2, and then they proved the following result.

[2]Note that when we consider the result on a graph with respect to the given number of edges, we shall ignore the possible isolated vertices if there are no confusions.

Theorem 3 (Lin–Ning–Wu, 2021)**.** *Let G be a triangle-free graph with m edges. If G is non-bipartite, then*

$$\lambda(G) \leqslant \sqrt{m-1},$$

where the equality holds if and only if $m = 5$ and $G = C_5$.

The upper bound in Theorem 3 is not sharp for $m > 5$. Motivated by this observation, Zhai and Shu [50] provided a further improvement on Theorem 3. For every integer $m \geqslant 3$, we denote by $\beta(m)$ the largest root of

$$Z(x) := x^3 - x^2 - (m-2)x + m - 3. \tag{4}$$

If m is odd, then we define $SK_{2,\frac{m-1}{2}}$ as the graph obtained from the complete bipartite graph $K_{2,\frac{m-1}{2}}$ by subdividing an edge; see Figure 1 for two drawings. Clearly, $SK_{2,\frac{m-1}{2}}$ is a triangle-free graph with m edges, and it is non-bipartite as it contains a copy of C_5. By computations, we know that $\beta(m)$ is the spectral radius of $SK_{2,\frac{m-1}{2}}$.

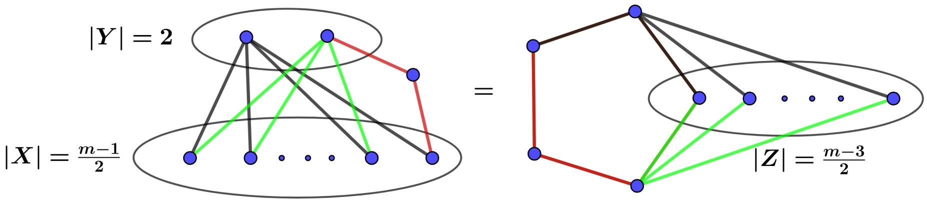


Figure 1: m is odd and the graph $SK_{2,\frac{m-1}{2}}$.

The improvement of Zhai and Shu [50] on Theorem 3 can be stated as below.

Theorem 4 (Zhai–Shu, 2022)**.** *Let G be a graph of size m. If G is triangle-free and non-bipartite, then*

$$\lambda(G) \leqslant \beta(m),$$

with equality if and only if $G = SK_{2,\frac{m-1}{2}}$.

Indeed, the result of Zhai and Shu improved Theorem 3. It was proved in [50, Lemma 2.2] that for every $m \geqslant 6$,

$$\sqrt{m-2} < \beta(m) < \sqrt{m-1}. \tag{5}$$

The original proof of Zhai and Shu [50] for Theorem 4 is technical and based on the use of the Perron components. Subsequently, Li and Peng [19] provided an alternative proof by applying Cauchy interlacing theorem. We remark that $\lim_{m\to\infty}(\beta(m) - \sqrt{m-2}) = 0$. In addition, Wang [46] improved Theorem 4 slightly by determining all the graphs with size m whenever it is a non-bipartite triangle-free graph satisfying $\lambda(G) \geqslant \sqrt{m-2}$.

1.3 A question of Zhai and Shu

The upper bound in Theorem 4 could be attained only if m is odd, since the extremal graph $SK_{2,\frac{m-1}{2}}$ is well-defined only in this case. Thus, it is interesting to determine the spectral extremal graph when m is even. Zhai and Shu in [50, Question 2.1] proposed the following question formally.

Question 5 (Zhai–Shu [50])**.** For even m, what is the extremal graph attaining the maximum spectral radius over all triangle-free non-bipartite graphs with m edges?

In this paper, we shall solve this question and determine the spectral extremal graphs. Although Question 5 seems to be another side of Theorem 4, we would like to point out that the even case is actually more difficult and different, and the original method is ineffective in this case.

Definition 6 (Spectral extremal graphs)**.** Suppose that $m \in 2\mathbb{N}$. Let L_m be the graph obtained from the subdivision $SK_{2,\frac{m-2}{2}}$ by hanging an edge on a vertex with the maximum degree. If $\frac{m-3}{3}$ is a positive integer, then we define Y_m as the graph obtained from C_5 by blowing up a vertex to an independent set $I_{\frac{m-3}{3}}$ on $\frac{m-3}{3}$ vertices, then adding a new vertex, and joining this vertex to all vertices of $I_{\frac{m-3}{3}}$. If $\frac{m-4}{3}$ is a positive integer, then we write T_m for the graph obtained from C_5 by blowing up two adjacent vertices to independent sets $I_{\frac{m-4}{3}}$ and I_2, respectively, where $I_{\frac{m-4}{3}}$ and I_2 form a complete bipartite graph; see Figure 2.

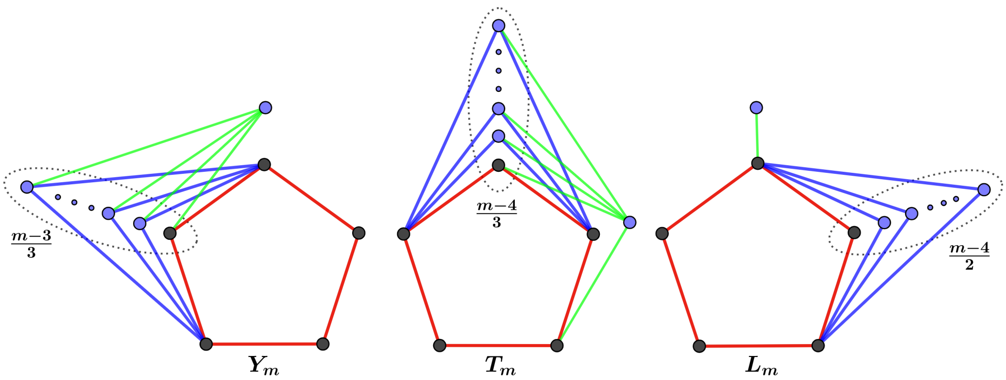


Figure 2: Extremal graphs in Theorem 7.

Theorem 7 (Main result)**.** *Let m be even and $m \geqslant 4.7 \times 10^5$. Suppose that G is a triangle-free graph with m edges and G is non-bipartite.*
(a) If $m = 3t$ for some $t \in \mathbb{N}$, then $\lambda(G) \leqslant \lambda(Y_m)$, with equality if and only if $G = Y_m$.
(b) If $m = 3t+1$ for some $t \in \mathbb{N}$, then $\lambda(G) \leqslant \lambda(T_m)$, with equality if and only if $G = T_m$.
(c) If $m = 3t+2$ for some $t \in \mathbb{N}$, then $\lambda(G) \leqslant \lambda(L_m)$, with equality if and only if $G = L_m$.

The construction of L_m is natural. Nevertheless, it is not apparent to find Y_m and T_m. There are some analogous results that the extremal graphs depend on the parity of the size m in the literature. For example, the C_5-free or C_6-free spectral extremal graphs with m edges are determined in [49] when m is odd, and later in [31] when m is even. Moreover, the $C_4^{\triangle}$-free or $C_5^{\triangle}$-free spectral extremal graphs are determined in [43] for odd m, and subsequently in [9, 26] for even m. In addition, the results of Nikiforov [33], Zhai and Wang [48] showed that the C_4-free spectral extremal graphs with given order n also rely on the parity of n. In a nutshell, for large size m, there is a common phenomenon that the extremal graphs in two cases are extremely similar, that is, the extremal graph in the even case is always constructed from that in the odd case by handing an edge to a vertex with maximum degree. Surprisingly, the extremal graphs in our conclusion break down this common phenomenon and show a new structure of the extremal graphs.

Outline of the paper. In Section 2, we shall present some lemmas, which shows that the spectral radius of L_m is smaller than that of Y_m if $\frac{m}{3} \in \mathbb{N}$, as well as that of T_m if $\frac{m-1}{3} \in \mathbb{N}$. Moreover, we will provide the estimations on both $\lambda(L_m)$ and $\beta(m)$. In Section 3, we will show some forbidden induced subgraphs, which helps us to characterize the local structure of the desired extremal graph. In Section 4, we present the proof of Theorem 7. Our proof of Theorem 7 is quite different from that of Theorem 4 in [50]. The techniques used in our proof borrows some ideas from Lin, Ning and Wu [23] as well as Ning and Zhai [38]. We shall apply Cauchy's interlacing theorem and a triangle counting result, which make full use of the information of all eigenvalues of a graph. In Section 5, we conclude this paper with some possible open problems for interested readers.

Notations. We shall follow the standard notation in [6] and consider only simple and undirected graphs. Let $N(v)$ be the set of neighbors of a vertex v, and $d(v)$ be the degree of v. For a subset $S \subseteq V(G)$, we write $e(S)$ for the number of edges with two endpoints in S, and $N_S(v) = N(v) \cap S$ for the set of neighbors of v in S. Let K_{r+1} be the complete graph on $r+1$ vertices, and $K_{s,t}$ be the complete bipartite graph with parts of sizes s and t. Let I_k be an independent set on k vertices. We write C_n and P_n for the cycle and path on n vertices, respectively. Given graphs G and H, we write $G \cup H$ for the union of G and H. In other words, $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For simplicity, we write kG for the union of k copies of G. We denote by $t(G)$ the number of triangles in G.

2 Preliminaries and outline of the proof

In this section, we will give the estimation on the spectral radius of L_m. Note that L_m exists whenever m is even, while Y_m and T_m are well-defined only if $m \pmod 3$ is 0 or 1, respectively. We will show that Y_m and T_m have larger spectral radius than L_m. In addition, we will introduce Cauchy interlacing theorem, a triangle counting result in terms of eigenvalues, and an operation of graphs which increases the spectral radius strictly. Before showing the proof of Theorem 7, we will illustrate the key ideas of our proof, and then we outline the main steps of the framework.

2.1 Bounds on the spectral radius of extremal graphs

By computations, we can obtain that $\lambda(Y_m)$ is the largest root of

$$Y(x) := x^4 - x^3 + (2-m)x^2 + (m-3)x + \tfrac{m}{3} - 1. \tag{6}$$

Similarly, $\lambda(T_m)$ is the largest root of

$$T(x) := x^5 - mx^3 + \tfrac{7m-22}{3}x + \tfrac{16-4m}{3}, \tag{7}$$

and $\lambda(L_m)$ is the largest root of the polynomial

$$L(x) := x^6 - mx^4 + (\tfrac{5m}{2} - 7)x^2 + (4-m)x + 2 - \tfrac{m}{2}. \tag{8}$$

Lemma 8. *If $m \in \{6, 8, 10\}$, then $\lambda(L_m) > \sqrt{m-2}$. If $m \geqslant 12$ is even, then*

$$\sqrt{m-2.5} < \lambda(L_m) < \sqrt{m-2}.$$

Moreover, we have $\lambda(L_6) \approx 2.1149$, $\lambda(L_8) \approx 2.4938$ and $\lambda(L_{10}) \approx 2.8424$.

Proof. The case $m \in \{6, 8, 10\}$ is straightforward. Next, we shall consider the case $m \geqslant 12$. By a direct computation, it is easy to verify that

$$L(\sqrt{m-2.5}) = -(1.25 + \sqrt{m-2.5})m + 4\sqrt{m-2.5} + 3.875 < 0,$$

which gives $\lambda(L_m) > \sqrt{m-2.5}$. Moreover, we have

$$L(\sqrt{m-2}) = \frac{1}{2}\left(m^2 - (9 + 2\sqrt{m-2})m + 8(2 + \sqrt{m-2})\right) > 0.$$

Furthermore, we have $L'(x) := \frac{\mathrm{d}}{\mathrm{d}x}L(x) = 6x^5 - 4mx^3 + (5m-14)x - m + 4$. By calculations, one can check that $L'(\sqrt{m-2}) > 0$ and $L'(x) \geqslant 0$ for every $x \geqslant \sqrt{m-2}$, which yields $L(x) > L(\sqrt{m-2}) > 0$ for every $x > \sqrt{m-2}$. Thus $\lambda(L_m) < \sqrt{m-2}$. □

Lemma 9. *If $m \geqslant 38$ is even and $m = 3t$ for some $t \in \mathbb{N}^*$, then*

$$\lambda(L_m) < \lambda(Y_m).$$

Proof. We know from (6) that $\lambda(Y_m)$ is the largest root of

$$Y(x) = x^4 - x^3 + (2-m)x^2 + (m-3)x + \tfrac{m-3}{3}.$$

By calculations, we can verify that

$$L(x) - x^2Y(x) = x^5 - 2x^4 + (3-m)x^3 + (\tfrac{13m}{6} - 6)x^2 + (4-m)x + 2 - \tfrac{m}{2},$$

and for every $m \geqslant 38$, we have

$$L(x) - x^2Y(x)\Big|_{x=\sqrt{m-3}} = \frac{m^2}{6} - m\sqrt{m-3} - m + 4\sqrt{m-3} + 2 > 0.$$

Moreover, we can show that $\frac{\mathrm{d}}{\mathrm{d}x}(L(x) - x^2Y(x)) > 0$ for every $x \geqslant \sqrt{m-3}$. Thus, it follows that $L(x) > x^2Y(x)$ for every $x \geqslant \sqrt{m-3}$. So $\lambda(L_m) < \lambda(Y_m)$, as needed. □

Lemma 10. *If $m \geqslant 10$ is even and $m = 3t + 1$ for some $t \in \mathbb{N}^*$, then*

$$\lambda(L_m) < \lambda(T_m).$$

Proof. Recall in (7) that $\lambda(T_m)$ is the largest root of $T(x)$. It is sufficient to prove that $L(x) > xT(x)$ for every $x \geqslant 3$. Upon computation, we can get

$$L(x) - xT(x) = \frac{m+2}{6}x^2 + \frac{m-4}{3}x + \frac{4-m}{2} > 0.$$

Consequently, we have $\lambda(L_m) < \lambda(T_m)$, as desired. □

The next lemma provides a refinement on (5) for every $m \geqslant 62$.

Lemma 11. *Let m be even and $m \geqslant 62$. Then*

$$\sqrt{m-2} < \beta(m) < \sqrt{m-1.85}.$$

Proof. Firstly, we have $Z(\sqrt{m-2}) = -1 < 0$, which yields $\sqrt{m-2} < \beta(m)$. Secondly, one can check that $Z(\sqrt{m-1.85}) > 0$ for every $m \geqslant 62$, and $Z'(x) = 3x^2 - 2x - (m-2) > 0$ for $x \geqslant \sqrt{m-1.85}$. Therefore, we have $Z(x) > Z(\sqrt{m-1.85}) > 0$ for every $x > \sqrt{m-1.85}$, which yields $\beta(m) < \sqrt{m-1.85}$, as required. □

The following lemma is referred to as the eigenvalue interlacing theorem, also known as Cauchy interlacing theorem, which states that the eigenvalues of a principal submatrix of a Hermitian matrix interlace those of the underlying matrix; see, e.g., [53, pp. 52–53] or [54, pp. 269–271]. The eigenvalue interlacing theorem is a powerful tool to extremal combinatorics and plays a significant role in two recent breakthroughs [15, 16].

Lemma 12 (Eigenvalue Interlacing Theorem)**.** *Let H be an $n \times n$ Hermitian matrix partitioned as*

$$H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where A is an $m \times m$ principal submatrix of H for some $m \leqslant n$. Then for every $1 \leqslant i \leqslant m$,

$$\lambda_{n-m+i}(H) \leqslant \lambda_i(A) \leqslant \lambda_i(H).$$

Recall that $t(G)$ denotes the number of triangles in G. It is well-known that the value of (i,j)-entry of $A^k(G)$ is equal to the number of walks of length k in G starting from vertex v_i to v_j. Since each triangle of G contributes 6 closed walks of length 3, we can count the number of triangles and obtain

$$t(G) = \frac{1}{6}\sum_{i=1}^{n} A^3(i,i) = \frac{1}{6}\mathrm{Tr}(A^3) = \frac{1}{6}\sum_{i=1}^{n} \lambda_i^3. \tag{9}$$

The forthcoming lemma could be regarded as a triangle spectral counting lemma in terms of both the eigenvalues and the size of a graph. This could be viewed as a useful variant of (9) by using $\sum_{i=1}^n \lambda_i^2 = \mathrm{tr}(A^2) = \sum_{i=1}^n d_i = 2m$.

Lemma 13 (see [38])**.** *Let G be a graph on n vertices with m edges. If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are all eigenvalues of G, then*

$$t(G) = \frac{1}{6}\sum_{i=2}^{n}(\lambda_1 + \lambda_i)\lambda_i^2 + \frac{1}{3}(\lambda_1^2 - m)\lambda_1.$$

For convenience, we introduce a function $f(x)$, which will be frequently used in Section 3 to find the induced substructures that are forbidden in the extremal graph.

Lemma 14. *Let $f(x)$ be a function given as*

$$f(x) := (\sqrt{m-2.5} + x)x^2.$$

If $a \leqslant x \leqslant b \leqslant 0$, then

$$f(x) \geqslant \min\{f(a), f(b)\}.$$

Proof. The function $f(x)$ is increasing when $x \in (-\infty, -\frac{2}{3}\sqrt{m-2.5})$, and decreasing when $x \in [-\frac{2}{3}\sqrt{m-2.5}, 0]$. Thus the desired statement holds immediately. □

The following lemma [47] is also needed in this paper, it provides an operation on a connected graph and increases the adjacency spectral radius strictly.

Lemma 15 (Wu–Xiao–Hong [47], 2005)**.** *Let G be a connected graph and $(x_1, \ldots, x_n)^T$ be a Perron vector of G, where x_i corresponds to v_i. Assume that $v_i, v_j \in V(G)$ are vertices such that $x_i \geqslant x_j$, and $S \subseteq N_G(v_j) \setminus N_G(v_i)$ is a non-empty set. Denote $G^* = G - \{v_j v : v \in S\} + \{v_i v : v \in S\}$. Then $\lambda(G) < \lambda(G^*)$.*

2.2 Proof overview

As promised, we will interpret the key ideas and steps of the proof of Theorem 7. First of all, we would like to make a comparison of the proofs of Theorem 3 and Theorem 4. The proof of Theorem 3 in [23] is short and succinct. It relies on the base case in Conjecture 2, which states that if G is a triangle-free graph with $m \geqslant 2$ edges, then

$$\lambda_1^2(G) + \lambda_2^2(G) \leqslant m, \tag{10}$$

where the equality holds if and only if G is one of some specific bipartite graphs; see [23, 37]. Combining the condition in Theorem 3, we know that if G is a triangle-free non-bipartite graph such that $\lambda_1(G) \geqslant \sqrt{m-1}$, then $\lambda_2(G) < 1$. Such a bound on the second largest eigenvalue provides great convenience to characterize the local structure of G. For instance, combining $\lambda_2(G) < 1$ with the Cauchy interlacing theorem, we obtain that C_5 is a shortest odd cycle of G. However, it is not sufficient to use (10) for the proof of Theorem 4. Indeed, if G satisfies further that $\lambda(G) \geqslant \beta(m)$, then we get $\lambda_2(G) < 2$ only, since $\beta(m) \to \sqrt{m-2}$ as m tends to infinity. Nevertheless, this bound is invalid for our purpose to describe the local structure of G. The original proof of Zhai and Shu [50] for Theorem 4 avoids the use of (10) and applies the Perron components. Thus it needs to make more careful structure analysis of the desired extremal graph.

To overcome the aforementioned obstacle, we will get rid of the use of (10), and then exploit the information of all eigenvalues of graphs, instead of the second largest eigenvalue merely. Our proof of Theorem 7 grows out from the original proof [23] of Theorem 3, which provided a method to find forbidden induced substructures. We will frequently use Cauchy interlacing theorem and the triangle counting result in Lemma 13.

The main steps of our proof can be outlined as below. It introduces the main ideas of the approach of this paper for treating the problem involving triangles.

- ♠ Assume that G is a spectral extremal graph with even size, that is, G is a non-bipartite triangle-free graph and attains the maximum spectral radius. First of all, we will show that G is connected and it does not contain the odd cycle C_{2k+1} as an induced subgraph for every $k \geqslant 3$. Consequently, C_5 is a shortest odd cycle in G.
- ♡ Let S be the set of vertices of a copy of C_5 in G. By using Lemma 12 and Lemma 13, we will find more forbidden substructures in the desired extremal graph; see, e.g., the graphs H_1, H_2, H_3 in Lemma 17. In this step, we will characterize and refine the local structure on the vertices around the cycle S.
- ♣ Using the information on the local structure of G, we will show that $V(G) \setminus S$ has at most one vertex with distance two to S; see Claim 22. Moreover, there are at most three vertices of $V(G) \setminus S$ with exactly one neighbor on S, and all these vertices are adjacent to a same vertex of S.
- ♢ Combining with the three steps above, we will determine the structure of G and show some possible graphs with large spectral radius. By comparing the polynomials of graphs, we will prove that G is isomorphic to Y_m, T_m or L_m.

3 Some forbidden induced subgraphs

In this section, we always assume that G is a non-bipartite triangle-free graph with even size m and G attains the maximal spectral radius. Since L_m is triangle-free and non-bipartite, we get by Lemma 8 that

$$\lambda(G) \geqslant \lambda(L_m) > \sqrt{m-2.5}. \tag{11}$$

On the other hand, we obtain from Theorem 4 and Lemma 11 that

$$\lambda(G) < \beta(m) < \sqrt{m-1.85}. \tag{12}$$

Our aim in this section is to determine some forbidden induced substructures of the extremal graph G. In this process, we need to exclude 16 induced substructures for our purpose. One of the main research directions in the proof is to show that G has at least one triangle, i.e., $t(G) > 0$, whenever the substructure forms an induced copy in G. Throwing away some tedious calculations, the main tools used in our proof attribute to Cauchy Interlacing Theorem (Lemma 12) and the triangle counting result (Lemma 13).

Lemma 16. *For any odd integer $s \geqslant 7$, an extremal graph G does not contain C_s as an induced cycle. Consequently, C_5 is a shortest odd cycle in G.*

Proof. Since G is non-bipartite, let s be the length of a shortest odd cycle in G. Since G is triangle-free, we have $s \geqslant 5$. Moreover, a shortest odd cycle $C_s \subseteq G$ must be an induced odd cycle. It is well-known that the eigenvalues of C_s are given as $\left\{2\cos\frac{2\pi k}{s} : k = 0, 1, \ldots, s-1\right\}$. In particular, we have

$$\text{Eigenvalues}(C_7) = \{2, 1.246, 1.246, -0.445, -0.445, -1.801, -1.801\}.$$

Since C_s is an induced copy in G, we know that $A(C_s)$ is a principal submatrix of $A(G)$. Lemma 12 implies that for every $i \in \{1, 2, \ldots, s\}$,

$$\lambda_{n-s+i}(G) \leqslant \lambda_i(C_s) \leqslant \lambda_i(G).$$

where λ_i means the i-th largest eigenvalue. We next show that $s = 5$. For convenience, we write $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ for eigenvalues of G in the non-increasing order.

Suppose on the contrary that C_7 is an induced odd cycle of G, then $\lambda_2 \geqslant \lambda_2(C_7) = 2\cos\frac{2\pi}{7} \approx 1.246$ and $\lambda_3 \geqslant \lambda_3(C_7) = 2\cos\frac{12\pi}{7} \approx 1.246$. Recall in Lemma 14 that

$$f(x) = (\sqrt{m-2.5} + x)x^2.$$

Evidently, we get

$$f(\lambda_2) \geqslant f(1.246) \geqslant 1.552\sqrt{m-2.5} + 1.934$$

and

$$f(\lambda_3) \geqslant f(1.246) \geqslant 1.552\sqrt{m-2.5} + 1.934.$$

Our goal is to get a contradiction by applying Lemma 13 and showing $t(G) > 0$. It is not sufficient to obtain $t(G) > 0$ by using the positive eigenvalues of C_7 only. Next, we are going to exploit the negative eigenvalues of C_7. For $i \in \{4, 5, 6, 7\}$, we know that $\lambda_i(C_7) < 0$. The Cauchy interlacing theorem yields $\lambda_{n-3} \leqslant \lambda_4(C_7) = -0.445$, $\lambda_{n-2} \leqslant \lambda_5(C_7) = -0.445$, $\lambda_{n-1} \leqslant \lambda_6(C_7) = -1.801$ and $\lambda_n \leqslant \lambda_7(C_7) = -1.801$. To apply Lemma 14, we need to find the lower bounds on λ_i for each $i \in \{n-3, n-2, n-1, n\}$. We know from (11) that $\lambda_1 \geqslant \lambda(L_m) > \sqrt{m-2.5}$, and then $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 2.5 + 6.744) = m - 4.244$, which implies $-\sqrt{m-4.244} < \lambda_n \leqslant -1.801$. By Lemma 14, we get

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-4.244}), f(-1.801)\} > 0.8\sqrt{m-2.5}.$$

Similarly, we have $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < m - 1.001$. Combining with $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m-1.001)/2} < \lambda_{n-1} \leqslant -1.801$. By Lemma 14, we obtain

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m-1.001)/2}), f(-1.801)\} > 3.243\sqrt{m-2.5} - 5.841.$$

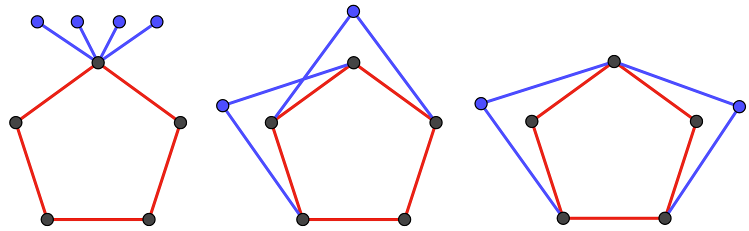
Using (11) and (12), we have $\sqrt{m-2.5} < \lambda_1 < \sqrt{m-1.85}$. By Lemma 13, we get

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+f(\lambda_3)+f(\lambda_n)+f(\lambda_{n-1})) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(7.147\sqrt{m-2.5} - 5\sqrt{m-1.85} - 1.973) > 0. \end{aligned}$$

This is a contradiction. By the monotonicity of $\cos x$, we can prove that C_s can not be an induced subgraph of G for each odd integer $s \geqslant 7$. Thus we get $s = 5$. $\square$

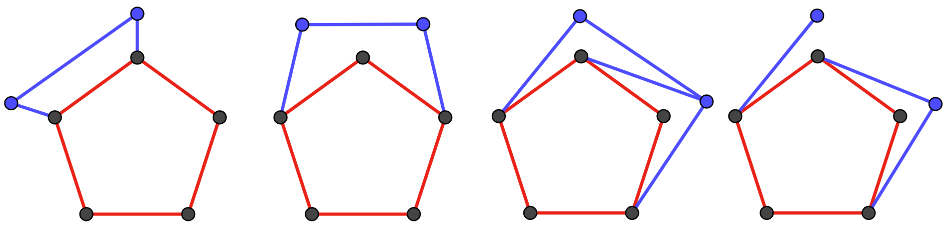
Using a similar method as in the proof of Lemma 16, we can prove the following lemmas, whose proofs are postponed to the Appendix. To avoid unnecessary calculations, we did not attempt to get the best bound on m, and we consider the case $m \geqslant 4.7 \times 10^5$.

Lemma 17. *G does not contain any graph of $\{H_1, H_2, H_3\}$ as an induced subgraph.*



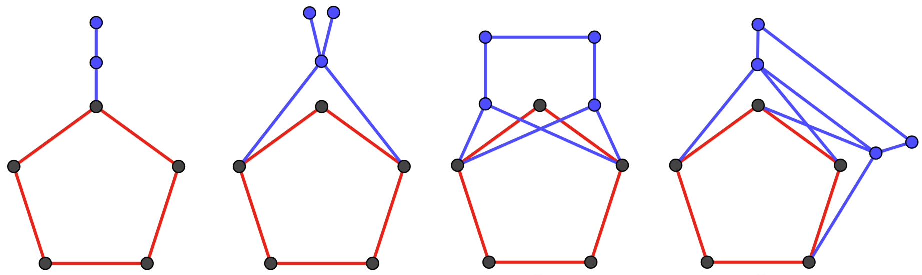
H_1 H_2 H_3

Lemma 18. *G does not contain any graph of $\{T_1, T_2, T_3, T_4\}$ as an induced subgraph.*



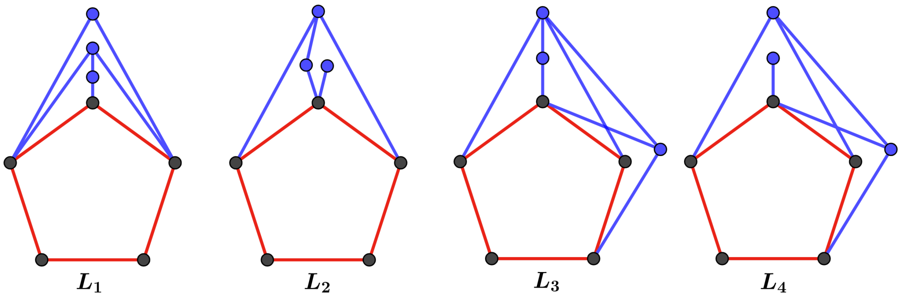
T_1 T_2 T_3 T_4

Lemma 19. *Any graph of $\{J_1, J_2, J_3, J_4\}$ can not be an induced subgraph of G.*



J_1 J_2 J_3 J_4

Lemma 20. *Any graph of* $\{L_1, L_2, L_3, L_4\}$ *can not be an induced subgraph of* G.



4 Proof of the main theorem

It is the time to show the proof of Theorem 7.

Proof of Theorem 7. Suppose that G is a non-bipartite triangle-free graph with m edges ($m \geqslant 4.7 \times 10^5$ is even) such that G attains the maximum spectral radius. Thus we have $\lambda(G) \geqslant \lambda(L_m)$ since L_m is one of the triangle-free non-bipartite graphs. Our goal is to prove that $G = Y_m$ if $\frac{m}{3} \in \mathbb{N}$; $G = T_m$ if $\frac{m-1}{3} \in \mathbb{N}$, and $G = L_m$ if $\frac{m-2}{3} \in \mathbb{N}$. First of all, we can see that G must be connected. Otherwise, we can choose G_1 and G_2 as two different components, where G_1 attains the spectral radius of G. By identifying two vertices from G_1 and G_2, respectively, we get a new graph with larger spectral radius, which is a contradiction. By Lemma 16, we can draw the following claim.

Claim 21. C_5 *is a shortest odd cycle in* G.

By Claim 21, we denote by $S = \{u_1, u_2, u_3, u_4, u_5\}$ the set of vertices of a copy of C_5, where $u_i u_{i+1} \in E(G)$ and $u_5 u_1 \in E(G)$. Let $N(S) := \big(\cup_{u \in S} N(u)\big) \setminus S$ be the union of neighborhoods of vertices of S, and let $d_S(v) = |N(v) \cap S|$ be the number of neighbors of v in the set S. Clearly, we have $d_S(v) \in \{0, 1, 2\}$ for every $v \in V(G) \setminus S$. Otherwise, if $d_S(v) \geqslant 3$, then one can find a triangle immediately, a contradiction.

Claim 22. $V(G) \setminus S$ *does not contain a vertex with distance* 3 *to* S, *and* $V(G) \setminus S$ *has at most one vertex with distance* 2 *to* S.

Proof. This claim is a consequence of Lemmas 17 and 19. Firstly, suppose on the contrary that $V(G) \setminus S$ contains a vertex which has distance 3 to S. Let w_1 be such a vertex and $P_4 = w_1 w_2 w_3 u_1$ be a shortest path of length 3. Then w_2 can not be adjacent to any vertex of S. Since G is triangle-free, we know that neither $w_3 u_2$ nor $w_3 u_5$ can be an edge, and at least one of $w_3 u_3$ and $w_3 u_4$ is not an edge. If $w_3 u_3 \notin E(G)$ and $w_3 u_4 \notin E(G)$, then $\{w_2, w_3\} \cup S$ induces a copy of J_1, contradicting with Lemma 19. If $w_3 u_3 \in E(G)$ and $w_3 u_4 \notin E(G)$, then $\{w_1, w_2, w_3\} \cup (S \setminus \{u_2\})$ forms an induced copy of J_1 since $w_1 w_3, w_1 u_i$ and $w_2 u_i$ are not edges of G, a contradiction. By symmetry, $w_3 u_3 \notin E(G)$ and $w_3 u_4 \in E(G)$ yield a contradiction similarly.

Secondly, suppose on the contrary that $V(G) \setminus S$ contains two vertices, say w_1, w_2, which have distance 2 to S. Let v_1 and v_2 be two vertices out of S such that $w_1 \sim v_1 \sim S$ and $w_2 \sim v_2 \sim S$. Since J_1 can not be an induced copy of G and G is triangle-free, we know that $d_S(v_1) = d_S(v_2) = 2$. If $v_1 = v_2$, then $\{w_1, w_2, v_1\} \cup S$ forms an induced copy of J_2 in G, we get a contradiction by Lemma 19. Thus, we get $v_1 \neq v_2$. Without loss of generality, we may assume that $N_S(v_1) = \{u_1, u_3\}$. By Lemma 17, G does not contain H_3 as an induced subgraph, we get $N_S(v_2) \neq \{u_3, u_5\}$ and $N_S(v_2) \neq \{u_1, u_4\}$. By symmetry, we have either $N_S(v_2) = \{u_2, u_4\}$ or $N_S(v_2) = \{u_1, u_3\}$. For the former case, since H_2 is not an induced subgraph of G by Lemma 17, we get $v_1v_2 \in E(G)$. If $w_1w_2 \in E(G)$, then G contains J_4 as an induced subgraph, which is a contradiction by Lemma 19. Thus $w_1w_2 \notin E(G)$. By Lemma 15, one can compare the Perron components of v_1 and v_2, and then move w_1 and w_2 together, namely, either making w_1 adjacent to v_2, or w_2 adjacent to v_1. In this process, the resulting graph remains triangle-free and non-bipartite as well. However, it has larger spectral radius than G, which contradicts with the maximality of the spectral radius of G. For the latter case, i.e., $N_S(v_1) = N_S(v_2) = \{u_1, u_3\}$. Since J_3 is not an induced copy in G, a similar argument shows $w_1w_2 \notin E(G)$, and then it also leads to a contradiction. □

By Claim 22, we shall partition the remaining proof in two cases, which are dependent on whether $V(G) \setminus S$ contains a vertex with distance 2 to the 5-cycle S.

Case 1. Every vertex of $V(G) \setminus S$ is adjacent a vertex of S.

In this case, we have $V(G) = S \cup N(S)$. For convenience, we denote $N(S) = V_1 \cup V_2$, where $V_i = \{v \in N(S) : d_S(v) = i\}$ for each $i = 1, 2$. At the first glance, different vertices of V_1 can be joined to different vertices of S. By Lemma 18, G does not contain T_1 and T_2 as induced subgraphs, we obtain that V_1 is an independent set in G. Using Lemma 15, we can move all vertices of V_1 together such that *all of them are adjacent to a same vertex of* S, and get a new graph with larger spectral radius. Note that this process can keep the resulting graph being triangle-free and non-bipartite since V_1 is edge-less and S is still a copy of C_5. By Lemma 17, H_1 can not be an induced subgraph of G, then $|V_1| \leqslant 3$.

We can fix a vertex $v \in N(S)$ and assume that $N_S(v) = \{u_1, u_3\}$. For each $w \in V(G) \setminus (S \cup \{v\})$, since G contains no triangles and no H_3 as an induced subgraph by Lemma 17, we know that $N_S(w) \neq \{u_3, u_5\}$ and $N_S(w) \neq \{u_4, u_1\}$. It is possible that $N_S(w) = \{u_1, u_3\}, \{u_2, u_4\}$ or $\{u_5, u_2\}$. Furthermore, if $N_S(w) = \{u_1, u_3\}$, then $wv \notin E(G)$ since G contains no triangle; if $N_S(w) = \{u_2, u_4\}$, then $wv \in E(G)$ since G contains no induced copy of H_2. We denote $N_{i,j} := \{w \in V(G) \setminus S : N_S(w) = \{u_i, u_j\}\}$. Note that G has no induced copy of H_3, then at least one of the sets $N_{2,4}$ and $N_{5,2}$ is empty.

Subcase 1.1. If both $N_{2,4} = \varnothing$ and $N_{5,2} = \varnothing$, then $V_2 = N_{1,3}$ and $V(G) = S \cup V_1 \cup N_{1,3}$. By Lemma 18, T_3 and T_4 can not be induced subgraphs of G. Hence, all vertices of V_1 are adjacent to the vertex u_1 or u_2 by symmetry. Next, we will show that $|V_1| \in \{1, 3\}$, and then we prove that V_1 and $N_{1,3}$ form a complete bipartite graph or an empty graph.

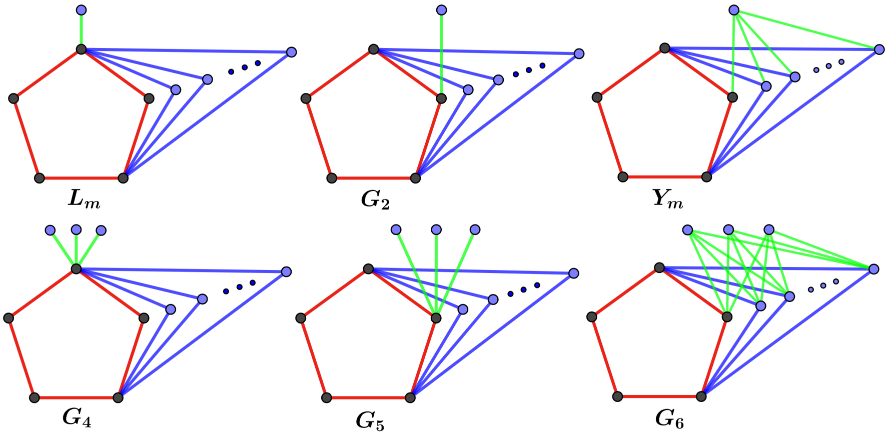


Figure 3: The structure of G when $|V_1| = 1$ or $|V_1| = 3$.

Suppose that all vertices V_1 are adjacent to u_1. Then there is no edge between V_1 and $N_{1,3}$ since G is triangle-free. Note that $m = 5+2|N_{1,3}|+|V_1|$ is even, we get $|V_1| \in \{1, 3\}$; see L_m and G_4 in Figure 3.

If $|V_1| = 1$, then G is the desired extremal graph L_m;

If $|V_1| = 3$, then by computation, we get $\lambda(G_4)$ is the largest root of

$$F_4(x) := x^6 - mx^4 + (\tfrac{7m}{2} - 14)x^2 + (6-m)x + 9 - \tfrac{3m}{2}.$$

Clearly, we can check that $L(x) < F_4(x)$ for each $x \geqslant 1$, and so $\lambda(G_4) < \lambda(L_m)$.

Suppose that all vertices of V_1 are adjacent to u_2. If there is no edge between V_1 and $N_{1,3}$, then $|V_1| \in \{1, 3\}$ and G is isomorphic to G_2 or G_5; see Figure 3. By computations or Lemma 15, we can get $\lambda(G) < \lambda(L_m)$; If there exists an edge between V_1 and $N_{1,3}$, then we claim that V_1 and $N_{1,3}$ form a complete bipartite subgraph by Lemma 20. Indeed, Lemma 20 asserts that G does not contain L_1 as an induced subgraph. In other words, if $v \in V_1$ is a vertex which is adjacent to one vertex of $N_{1,3}$, then v will be adjacent to all vertices of $N_{1,3}$. Note that G does not contain L_2 as an induced subgraph, which means that other vertices of V_1 are also adjacent to all vertices of $N_{1,3}$. Observe that $m = 5+2|N_{1,3}|+|V_1|(1+|N_{1,3}|)$ is even, which yields that $|V_1|$ is odd, and so $|V_1| \in \{1, 3\}$. Consequently, G is isomorphic to either Y_m or G_6; see Figure 3.

If $|V_1| = 1$, then $|N_{1,3}| = \frac{m-6}{3}$ and $G = Y_m$. By Lemma 9, we get $\lambda(L_m) < \lambda(Y_m)$. Thus, Y_m is the required extremal graph whenever $m = 3t$ for some even $t \in \mathbb{N}$.

If $|V_1| = 3$, then $|N_{1,3}| = \frac{m-8}{5}$ and $\lambda(G_6)$ is the largest root of

$$F_6(x) := x^4 - x^3 + (2-m)x^2 + (m-3)x + \tfrac{3m-9}{5}.$$

It is not hard to check that $\lambda(G_6) < \lambda(L_m)$. Indeed, by calculation, we know that the largest roots of $x^2F_6(x)$ and $L(x)$ are located in $(\sqrt{m-3}, \sqrt{m-2})$. Moreover, we denote

$$D(x) := L(x) - x^2F_6(x) = x^5 - 2x^4 + (3-m)x^3 + (\tfrac{19m}{10} - \tfrac{26}{5})x^2 + (4-m)x + 2 - \tfrac{m}{2}.$$

Clearly, we can verify that $D(\sqrt{m-3}) < 0$ and $D(\sqrt{m-2}) < 0$. Furthermore, one can prove that $\frac{\mathrm{d}}{\mathrm{d}x}D(x) > 0$ for each $x \geqslant \sqrt{m-3}$. Consequently, it leads to $D(x) < 0$ for every $x \in (\sqrt{m-3}, \sqrt{m-2})$, and so $L(x) < x^2 F_6(x)$, which yields $\lambda(G_6) < \lambda(L_m)$.

Subcase 1.2. Without loss of generality, we may assume that $N_{2,4} \neq \varnothing$ and $N_{5,2} = \varnothing$, then $N(S) = V_1 \cup N_{1,3} \cup N_{2,4}$. By Lemma 17, H_2 can not be an induced subgraph of G. Thus, $N_{1,3}$ and $N_{2,4}$ induce a complete bipartite subgraph in G. Now, we consider the vertices of V_1. Recall that all vertices of V_1 are adjacent to a same vertex of S. By Lemma 18, G does not contain T_3 and T_4 as induced subgraphs. Then the vertices of V_1 can not be adjacent to u_1, u_4 or u_5. By Lemma 20, we know that L_3 and L_4 can not be induced subgraph of G. Thus, all vertices of V_1 can not be adjacent to u_2 or u_3. To sum up, we get $V_1 = \varnothing$, and so $N(S) = N_{1,3} \cup N_{2,4}$. We denote $A = N_{1,3} \cup \{u_2, u_4\}$ and $B = N_{2,4} \cup \{u_3, u_1\}$. Let $|A| = a$ and $|B| = b$. Then we observe that G is isomorphic to the subdivision of the complete bipartite graph $K_{a,b}$ by subdividing the edge u_1u_4 of $K_{a,b}$. Note that $m = ab + 1$ and $a, b \geqslant 3$ are odd integers. Without loss of generality, we may assume that $a \geqslant b$.

If $b = 3$, then $m = 3a + 1$ for some $a \in \mathbb{N}^*$. In this case, we get $G = T_m$. Invoking Lemma 10, we have $\lambda(L_m) < \lambda(T_m)$ and thus T_m is the desired extremal graph.

If $b \geqslant 5$, then $m = ab + 1$ and $\lambda(SK_{a,b})$ is the largest root of

$$F_{a,b}(x) := x^5 - mx^3 + (3m - 2 - 2a - 2b)x - 2m + 2a + 2b.$$

Recall in (8) that $\lambda(L_m)$ is the largest root of $L(x)$. We can verify that

$$L(x) - xF_{a,b}(x) = -(\tfrac{m}{2} + 5 - 2a - 2b)x^2 + (4 + m - 2a - 2b)x - \tfrac{m}{2} + 2.$$

Since $b \geqslant 5$ and $m = ab + 1$, we get $\frac{m}{2} + 5 - 2a - 2b = \frac{1}{2}((a-4)(b-4) - 5) > 0$. It follows that $L(x) \leqslant xF_{a,b}(x)$ for every $x \geqslant 3$. Thus, we get $\lambda(SK_{a,b}) < \lambda(L_m)$, as required.

Case 2. There is exactly one vertex of $V(G) \setminus S$ with distance 2 to the cycle S. Let w_2, w_1 be two vertices with $w_2 \sim w_1 \sim S$, and $N_S(w_1) = \{u_1, u_3\}$ by Lemma 19. We denote $V(G) \setminus (S \cup \{w_1, w_2\}) := V_1 \cup V_2$, where $V_i = \{v \in V(G) : v \notin S \cup \{w_1, w_2\}, d_S(v) = i\}$ for each $i = 1, 2$. Similar with the argument in Case 1, using Lemmas 18 and 15, one can move all vertices of V_1 such that all of them are adjacent to a same vertex of S. By Lemma 17, H_1 is not an induced subgraph of G. Then $|V_1| \leqslant 3$.

Let $v \in V_2$ be any vertex. We claim that $N_S(v) = \{u_1, u_3\}$. Indeed, Lemma 17 implies that $N_S(v) \neq \{u_3, u_5\}$ and $N_S(v) \neq \{u_1, u_4\}$. If $N_S(v) = \{u_2, u_4\}$, then $w_1v \in E(G)$ since G does not contain H_2 as an induced subgraph by Lemma 17 again. Consequently, $S \cup \{w_1, w_2, v\}$ forms an induced copy of L_4, which contradicts with Lemma 20. Thus, we get $N_S(v) \neq \{u_2, u_4\}$. Similarly, we can show $N_S(v) \neq \{u_2, u_5\}$. In conclusion, we obtain $N_S(v) = \{u_1, u_3\}$ for every $v \in V_2$. Since m is even, we get $|V_1| \in \{0, 2\}$.

First of all, suppose that $|V_1| = 0$. Then G is isomorphic to G_2, the graph obtained from a C_5 by blowing up the vertex u_2 exactly $\frac{m-4}{2}$ times and then hanging an edge to u_2; see Figure 3. By computations, we $\lambda(G_2) < \lambda(L_m)$, as desired.

Now, suppose that $|V_1| = 2$ and $V_1 := \{v_1, v_2\}$. By Lemma 18, we know that T_3 and T_4 are not induced subgraphs of G. Then the vertices of V_1 can not be adjacent to u_4

and u_5. By symmetry of u_1 and u_3, there are two possibilities, namely, all vertices of V_1 are adjacent to u_1 or u_2. If all vertices of V_1 are adjacent to u_1, then $v_1w_2 \notin E(G)$ and $v_2w_2 \notin E(G)$ since T_1 can not be an induced subgraph of G by Lemma 18. By comparing the Perron components of u_1 and w_1, one can move v_1, v_2 and w_2 together using Lemma 15. Thus, G is isomorphic to G_4 or G_5 in Figure 3. If all vertices of V_1 are adjacent to u_2, then $v_1w_2 \notin E(G)$ and $v_2w_2 \notin E(G)$ since J_3 is not an induced subgraph in G by Lemma 19. A similar argument shows that G is isomorphic to G_5 in Figure 3. By direct computations, we can obtain $\lambda(G_4) < \lambda(L_m)$ and $\lambda(G_5) < \lambda(L_m)$. This completes the proof. □

5 Concluding remarks

Although we have solved Question 5 for every $m \geqslant 4.7 \times 10^5$, our proof requires a lot of calculations of eigenvalues. As shown in Figure 2, there are three kinds extremal graphs depending on $m \,(\mathrm{mod}\ 3) \in \{0, 1, 2\}$. Thus, it seems unavoidable to make calculations and comparisons among the spectral radii of these three graphs. Unlike the odd case in Theorem 4, the bound $\beta(m)$ is sharp for all odd integers $m \in \mathbb{N}$. For the even case, Theorem 7 presents all extremal graph for $m \geqslant 4.7 \times 10^5$. We do not try our best to optimize the lower bound on m. In addition, for $m \in \{6, 8, 10\}$, Lemma 8 gives $\lambda(L_m) > \sqrt{m-2}$. Using a result in [46, Theorem 5], we can prove that Y_6, L_8 and T_{10} are extremal graphs when $m \in \{6, 8, 10\}$, respectively. In view of this evidence, it is possible to find a new proof of Question 5 to characterize the extremal graphs for every $m \geqslant 12$.

The blow-up of a graph G is a new graph obtained from G by replacing each vertex $v \in V(G)$ with an independent set I_v, and for two vertices $u, v \in V(G)$, we add all edges between I_u and I_v whenever $uv \in E(G)$. It was proved in [23, 37] that if G is a triangle-free graph with $m \geqslant 2$ edges, then $\lambda_1^2(G) + \lambda_2^2(G) \leqslant m$, where the equality holds if and only if G is a blow-up of a member of the family $\mathcal{G} = \{P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, P_5 \cup K_1\}$. This result confirmed the base case of a conjecture of Bollobás and Nikiforov [4]. Observe that all extremal graphs in this result are bipartite graphs. Therefore, it is possible to consider the maximum of $\lambda_1^2(G) + \lambda_2^2(G)$ in which G is triangle-free and non-bipartite.

The extremal problem was also studied for non-bipartite triangle-free graphs with given number of vertices. We write $SK_{s,t}$ for the graph obtained from the complete bipartite graph $K_{s,t}$ by subdividing an edge. In 2021, Lin, Ning and Wu [23] proved that if G is a non-bipartite triangle-free graph on n vertices, then

$$\lambda(G) \leqslant \lambda\big(SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}\big), \tag{13}$$

and equality holds if and only if $G = SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Comparing this result with Theorem 4, one can see that the extremal graphs with given order and size are extremely different although both of them are subdivisions of complete bipartite graphs. Roughly speaking, the former is nearly balanced, but the latter is exceedingly unbalanced.

Later, Li and Peng [20] extended (13) to the non-r-partite K_{r+1}-free graphs with n vertices. Notice that the extremal graph in (13) has many copies of C_5. There is another way to extend (13) by considering the non-bipartite graphs on n vertices without any copy of $\{C_3, C_5, \ldots, C_{2k+1}\}$ where $k \geqslant 2$. This was done by Lin and Guo [24] as well as Li, Sun and Yu [17] independently. Subsequently, the corresponding spectral problem for graphs with m edges was studied in [42, 19, 28]. However, the extremal graphs in this setting can be achieved only for odd m. Hence, we propose the following question for interested readers[3].

Question 23. For even m, what is the extremal graph attaining the maximum spectral radius over all non-bipartite $\{C_3, C_5, \ldots, C_{2k+1}\}$-free graphs with m edges?

We write $q(G)$ for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the *signless Laplacian matrix* $Q(G) = D(G) + A(G)$, where $D(G) = \mathrm{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix. A theorem of He, Jin and Zhang [11] implies that if G is a triangle-free graph on n vertices, then $q(G) \leqslant n$, with equality if and only if G is a complete bipartite graph (need not be balanced). This result can also be viewed as a spectral version of Mantel's theorem. It is worth mentioning that Liu, Miao and Xue [25] characterized the maximum signless Laplacian spectral radius among all non-bipartite triangle-free graphs with given order n and size m, respectively. Fortunately, the corresponding extremal graphs are independent of the parity of m. Soon after, they [30] also provided the extensions for graphs without any copy of $\{C_3, C_5, \ldots, C_{2k+1}\}$.

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A Proofs of Lemmas 17, 18, 19 and 20

In the appendix, we shall provide the detailed proof of some lemmas in Section 3.

Proof of Lemma 17. Suppose on the contrary that G contains H_i as an induced subgraph for some $i \in \{1, 2, 3\}$. To obtain a contradiction, we shall show that $t(G) > 0$ by using Lemma 13. The eigenvalues of graphs H_1, H_2 and H_3 can be seen in Table 1.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
H_1	2.578	1.373	0.618	0	0	0	-0.451	-1.618	-2.501
H_2	2.641	1	0.723	0.414	-0.589	-1.775	-2.414		
H_3	2.681	1	0.642	0	0	-2	-2.323		

Table 1: Eigenvalues of H_1, H_2 and H_3.

First of all, we consider the case that H_1 is an induced subgraph in G. The Cauchy interlacing theorem implies $\lambda_{n-9+i}(G) \leqslant \lambda_i(H_1) \leqslant \lambda_i(G)$ for every $i \in \{1, 2, \ldots, 9\}$. We denote $\lambda_i = \lambda_i(G)$ for short. Obviously, we have

$$f(\lambda_2) \geqslant f(1.371) \geqslant 1.879\sqrt{m-2.5} + 2.576$$

and

$$f(\lambda_3) \geqslant f(0.618) \geqslant 0.381\sqrt{m-2.5} + 0.236.$$

Moreover, for each $i \in \{4,5,6\}$, we know that $\lambda_i \geqslant 0$, which gives $f(\lambda_i) \geqslant 0$. Next, we shall consider the negative eigenvalues of G. The Cauchy interlacing theorem implies $\lambda_{n-2} \leqslant \lambda_7(H_1) = -0.451$ and $\lambda_{n-1} \leqslant \lambda_8(H_1) = -1.618$ and $\lambda_n \leqslant \lambda_9(H_1) = -2.501$. Moreover, we get from Lemma 8 that $\lambda_1 \geqslant \lambda(L_m) > \sqrt{m-2.5}$ and $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) \leqslant 2m - (m - 2.5 + 5.091) = m - 2.591$, which implies $-\sqrt{m-2.591} < \lambda_n \leqslant -2.501$. By Lemma 14, we have

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.591}), f(-2.501)\} > 0.04\sqrt{m-2.5}.$$

Since $\lambda_{n-1}^2 + \lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-2}^2) < m + 0.026$ and $\lambda_{n-1}^2 \leqslant \lambda_n^2$, we get $-\sqrt{(m+0.026)/2} < \lambda_{n-1} \leqslant -1.618$. By Lemma 14, we get

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m+0.026)/2}), f(-1.618)\} > 2.617\sqrt{m-2.5} - 4.235.$$

Moreover, we have $-\sqrt{(m+0.23)/3} < \lambda_{n-2} \leqslant -0.451$ and then

$$f(\lambda_{n-2}) \geqslant \min\{f(-\sqrt{(m+0.23)/3}), f(-0.451)\} > 0.203\sqrt{m-2.5} - 0.091.$$

Theorem 4 and Lemma 11 imply

$$\lambda_1 < \beta(m) < \sqrt{m-1.85}.$$

By Lemma 13, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.12\sqrt{m-2.5} - 5\sqrt{m-1.85} - 1.514) > 0, \end{aligned}$$

where the last inequality holds for $m \geqslant 188$. This is a contradiction.

Second, assume that H_2 is an induced subgraph of G. Then Cauchy interlacing theorem gives $\lambda_2 \geqslant 1, \lambda_3 \geqslant 0.723$ and $\lambda_4 \geqslant 0.414$. Similarly, we get

$$\begin{aligned} f(\lambda_2) &\geqslant f(1) = \sqrt{m-2.5} + 1, \\ f(\lambda_3) &\geqslant f(0.723) \geqslant 0.522\sqrt{m-2.5} + 0.377 \end{aligned}$$

and

$$f(\lambda_4) \geqslant f(0.414) \geqslant 0.171\sqrt{m-2.5} + 0.07.$$

The negative eigenvalues of H_2 imply that $\lambda_{n-2} \leqslant -0.589$, $\lambda_{n-1} \leqslant -1.775$ and $\lambda_n \leqslant -2.414$. As $\lambda_n^2 \leqslant 2m-(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_{n-2}^2+\lambda_{n-1}^2) < 2m-(m-2.5+5.191) = m-2.691$, we get $-\sqrt{m-2.691} \leqslant \lambda_n \leqslant -2.414$. Lemma 14 gives

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.691}), f(-2.414)\} > 0.09\sqrt{m-2.5}.$$

In addition, we have $-\sqrt{(m+0.459)/2} \leqslant \lambda_{n-1} \leqslant -1.775$ and

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m+0.459)/2}), f(-1.775)\} > 3.15\sqrt{m-2.5} - 5.592.$$

Moreover, we get $\sqrt{(m+0.805)/3} < \lambda_{n-2} \leqslant -0.589$ and

$$f(\lambda_{n-2}) \geqslant \min\{f(-\sqrt{(m+0.805)/3}), f(-0.589)\} > 0.346\sqrt{m-2.5} - 0.204.$$

Using Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.279\sqrt{m-2.5} - 5\sqrt{m-1.85} - 4.349) > 0, \end{aligned}$$

where the last inequality holds for $m \geqslant 258$, which is also a contradiction.

Finally, if H_3 is an induced subgraph of G, then we get $\lambda_2 \geqslant 2$ and $\lambda_3 \geqslant 0.642$. Thus

$$f(\lambda_2) \geqslant f(1) = \sqrt{m-2.5} + 1$$

and

$$f(\lambda_3) \geqslant f(0.642) \geqslant 0.412\sqrt{m-2.5} + 0.264.$$

Moreover, Cauchy interlacing theorem gives $\lambda_{n-1} \leqslant -2$ and $\lambda_n \leqslant -2.323$. Since $\lambda_n^2 \leqslant 2m-(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_{n-1}^2) < 2m-(m-2.5+5.412) = m-2.912$, we get $-\sqrt{m-2.912} < \lambda_n \leqslant -2.323$. Then

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.912}), f(-2.323)\} \geqslant 0.2\sqrt{m-2.5}.$$

Similarly, we have $-\sqrt{(m+1.087)/2} < \lambda_{n-1} \leqslant -2$ and

$$f(\lambda_{n-1}) \geqslant \min\{f(-\sqrt{(m+1.087)/2}), f(-2)\} \geqslant 4\sqrt{m-2.5} - 8.$$

Combining Lemma 13 with $\lambda_1 < \sqrt{m-1.85}$, we get

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3}\lambda_1 \\ &> \frac{1}{6}(5.612\sqrt{m-2.5} - 5\sqrt{m-1.85} - 6.736) > 0, \end{aligned}$$

where the last inequality holds for $m \geqslant 162$, which is a contradiction. □

Using the similar method as in the proofs of Lemmas 16 and 17, we can prove Lemmas 18, 19 and 20 as well. For simplicity, we next present a brief sketch only.

Proof of Lemma 18. First of all, the eigenvalues of $T_1, \ldots, T_4$ can be given as below.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
T_1	2.377	1.273	0.801	0	−0.554	−1.651	−2.246
T_2	2.342	1	1	0.470	−1	−1.813	−2
T_3	2.641	1	0.723	0.414	−0.589	−1.775	−2.414
T_4	2.447	1.176	0.656	0	−0.264	−1.832	−2.183

Table 2: Eigenvalues of T_1, T_2, T_3 and T_4.

Suppose on the contrary that T_1 is an induced subgraph of G. Then Lemma 12 gives $\lambda_2 \geqslant 1.273$ and $\lambda_3 \geqslant 0.801$. Thus, we get

$$f(\lambda_2) \geqslant f(1.273) \geqslant 1.62\sqrt{m-2.5} + 2.062$$

and

$$f(\lambda_3) \geqslant f(0.801) \geqslant 0.641\sqrt{m-2.5} + 0.513.$$

Moreover, using the same technique in Lemma 17, the negative eigenvalues of T_1 implies $\lambda_n^2 \leqslant 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 2.5 + 5.299) = m - 2.799$, and so $-\sqrt{m-2.799} < \lambda_n \leqslant -2.246$. By Lemma 14, it follows that

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.799}), f(-2.246)\} > 0.14\sqrt{m-2.5}.$$

Similarly, we can get

$$f(\lambda_{n-1}) \geqslant f(-1.651) \geqslant 2.725\sqrt{m-2.5} - 4.5$$

and

$$f(\lambda_{n-2}) \geqslant f(-0.554) \geqslant 0.306\sqrt{m-2.5} - 0.17.$$

Using Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.432\sqrt{m-2.5} - 5\sqrt{m-1.85} - 2.095) > 0, \end{aligned}$$

which is a contradiction. Thus, G does not contain T_1 as an induced subgraph.

Suppose on the contrary that G contains T_2 as an induced subgraph. Then Lemma 12 implies $\lambda_2 \geqslant 1, \lambda_3 \geqslant 1$ and $\lambda_4 \geqslant 0.47$. Thus, we have

$$\begin{aligned} f(\lambda_2) &\geqslant f(1) = \sqrt{m-2.5} + 1, \\ f(\lambda_3) &\geqslant f(1) = \sqrt{m-2.5} + 1 \end{aligned}$$

and

$$f(\lambda_4) \geqslant f(0.47) \geqslant 0.22\sqrt{m-2.5} + 0.103.$$

Moreover, we have $-\sqrt{m-4.01} < \lambda_n \leqslant -2$. Then Lemma 14 leads to

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-4.01}), f(-2)\} \geqslant 0.7\sqrt{m-2.5}.$$

In addition, we have

$$f(\lambda_{n-1}) \geqslant f(-1.813) \geqslant 3.28\sqrt{m-2.5} - 5.959$$

and

$$f(\lambda_{n-2}) \geqslant f(-1) = \sqrt{m-2.5} - 1.$$

By Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, it follows that

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(7.2\sqrt{m-2.5} - 5\sqrt{m-1.85} - 4.856) > 0, \end{aligned}$$

a contradiction. So G does not contain T_2 as an induced subgraph.

Suppose on the contrary that T_3 is an induced subgraph of G. Using Lemma 12, we obtain $\lambda_2 \geqslant 1, \lambda_3 \geqslant 0.723$ and $\lambda_4 \geqslant 0.414$. Then

$$\begin{aligned} f(\lambda_2) &\geqslant f(1) = \sqrt{m-2.5} + 1, \\ f(\lambda_3) &\geqslant f(0.723) \geqslant 0.522\sqrt{m-2.5} + 0.377 \end{aligned}$$

and

$$f(\lambda_4) \geqslant f(0.414) \geqslant 0.171\sqrt{m-2.5} + 0.07.$$

Moreover, we can get $-\sqrt{m-2.695} \leqslant \lambda_n \leqslant -2.414$. By Lemma 14, we have

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.695}), f(-2.414)\} \geqslant 0.08\sqrt{m-2.5}.$$

Similarly, we obtain

$$f(\lambda_{n-1}) \geqslant f(-1.775) \geqslant 3.15\sqrt{m-2.5} - 5.592$$

and

$$f(\lambda_{n-2}) \geqslant f(-0.589) \geqslant 0.346\sqrt{m-2.5} - 0.204.$$

Consequently, Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$ gives

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.269\sqrt{m-2.5} - 5\sqrt{m-1.85} - 4.349) > 0, \end{aligned}$$

where the last inequality holds for $m \geqslant 276$. This leads to a contradiction. Hence T_3 can not be an induced subgraph of G.

Suppose on the contrary that T_4 is an induced subgraph of G. Applying Lemma 12, we obtain $\lambda_2 \geqslant 1.176$ and $\lambda_3 \geqslant 0.656$. Thus

$$f(\lambda_2) \geqslant f(1.176) \geqslant 1.382\sqrt{m-2.5} + 1.626$$

and

$$f(\lambda_3) \geqslant f(0.656) \geqslant 0.43\sqrt{m-2.5} + 0.282.$$

Moreover, we have $-\sqrt{m-2.741} \leqslant \lambda_n \leqslant -2.183$. Lemma 14 implies

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.741}), f(-2.183)\} \geqslant 0.1\sqrt{m-2.5}.$$

Similarly, one can get

$$f(\lambda_{n-1}) \geqslant f(-1.832) \geqslant 3.356\sqrt{m-2.5} - 6.148$$

and

$$f(\lambda_{n-2}) \geqslant f(-0.264) \geqslant 0.069\sqrt{m-2.5} - 0.018.$$

Finally, combining Lemma 13 with $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.337\sqrt{m-2.5} - 5\sqrt{m-1.85} - 4.258) > 0, \end{aligned}$$

a contradiction. Therefore T_4 can not be an induced subgraph of G. □

The proofs of Lemmas 19 and 20 can proceed in a similar way.

Proof of Lemma 19. The eigenvalues of graphs $J_1, \dots, J_4$ can be computed as follows.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
J_1	2.151	1.268	0.618	0.420	−0.895	−1.618	−1.944		
J_2	2.554	1.223	0.618	0.565	0	−0.942	−1.618	−2.401	
J_3	2.900	1.362	0.690	0.618	0.618	−0.273	−1.618	−1.618	−2.679
J_4	3.082	1.380	0.827	0.670	0.338	−0.406	−1.209	−1.726	−2.956

Table 3: Eigenvalues of J_1, J_2, J_3 and J_4.

If J_1 is an induced subgraph of G, then Lemma 12 implies $\lambda_2 \geqslant 1.268, \lambda_3 \geqslant 0.618$ and $\lambda_4 \geqslant 0.42$. Moreover, the negative eigenvalues of J_1 gives $\lambda_{n-2} \leqslant -0.895$ and $\lambda_{n-1} \leqslant -1.618$. Then $-\sqrt{m-3.085} \leqslant \lambda_n \leqslant -1.944$ and

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-3.085}), f(-1.944)\} > 0.25\sqrt{m-2.5}.$$

By Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+f(\lambda_3)+f(\lambda_4)+f(\lambda_{n-2})+f(\lambda_{n-1})+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.835\sqrt{m-2.5} - 5\sqrt{m-1.85} - 2.603) > 0, \end{aligned}$$

a contradiction. Thus J_1 can not be an induced subgraph of G.

If J_2 is an induced subgraph of G, then Lemma 12 implies $\lambda_2 \geqslant 1.223, \lambda_3 \geqslant 0.618$ and $\lambda_4 \geqslant 0.565$. In addition, the negative eigenvalues of J_2 gives $\lambda_{n-2} \leqslant -0.942$ and $\lambda_{n-1} \leqslant -1.618$. Then $-\sqrt{m-3.202} \leqslant \lambda_n \leqslant -2.401$ and

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-3.202}), f(-2.401)\} > 0.3\sqrt{m-2.5}.$$

Applying Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we have

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+f(\lambda_3)+f(\lambda_4)+f(\lambda_{n-2})+f(\lambda_{n-1})+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(6.002\sqrt{m-2.5} - 5\sqrt{m-1.85} - 2.826) > 0, \end{aligned}$$

a contradiction. So J_2 can not be an induced subgraph of G.

If J_3 is an induced subgraph of G, then Lemma 12 implies $\lambda_2 \geqslant 1.362, \lambda_3 \geqslant 0.690$, $\lambda_4 \geqslant 0.618$ and $\lambda_5 \geqslant 0.618$. Additionally, the negative eigenvalues of J_3 gives $\lambda_{n-3} \leqslant -0.273$, $\lambda_{n-2} \leqslant -1.618$ and $\lambda_{n-1} \leqslant -1.618$. Then $-\sqrt{m-5.907} \leqslant \lambda_n \leqslant -2.679$ and

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-5.907}), f(-2.679)\} > 1.5\sqrt{m-2.5}.$$

Using Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we get

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+\cdots+f(\lambda_5)+f(\lambda_{n-3})+\cdots+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(9.907\sqrt{m-2.5} - 5\sqrt{m-1.85} - 5.164) > 0, \end{aligned}$$

a contradiction. Therefore J_3 can not be an induced subgraph of G.

If J_4 is an induced subgraph of G, then Lemma 12 implies $\lambda_2 \geqslant 1.380, \lambda_3 \geqslant 0.827$, $\lambda_4 \geqslant 0.670$ and $\lambda_5 \geqslant 0.338$. Furthermore, the negative eigenvalues of J_4 gives $\lambda_{n-3} \leqslant -0.406$, $\lambda_{n-2} \leqslant -1.209$ and $\lambda_{n-1} \leqslant -1.726$. Then $-\sqrt{m-5.259} \leqslant \lambda_n \leqslant -2.956$ and

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-5.259}), f(-2.956)\} > 1.3\sqrt{m-2.5}.$$

Due to Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, we obtain

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+\cdots+f(\lambda_5)+f(\lambda_{n-3})+\cdots+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(9.059\sqrt{m-2.5} - 5\sqrt{m-1.85} - 3.442) > 0, \end{aligned}$$

a contradiction. Henceforth J_4 can not be an induced subgraph of G. □

Proof of Lemma 20. By computation, we can obtain the eigenvalues of $L_1, \dots, L_4$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
L_1	2.950	1.156	0.618	0.522	0	−0.790	−1.618	−2.838
L_2	2.753	1.204	0.641	0.618	−0.253	−0.700	−1.618	−2.645
L_3	3.141	1.139	0.763	0	0	−0.277	−1.745	−3.021
L_4	2.964	1	0.764	0.513	0	−0.710	−1.722	−2.809

Table 4: Eigenvalues of L_1, L_2, L_3 and L_4.

Suppose on the contrary that G contains L_1 as an induced subgraph. Then Lemma 12 yields $\lambda_2 \geqslant 1.156$, $\lambda_3 \geqslant 0.618$ and $\lambda_4 \geqslant 0.522$. The negative eigenvalues of L_1 implies $\lambda_{n-2} \leqslant -0.790$ and $\lambda_{n-1} \leqslant -1.618$. Then $-\sqrt{m-2.734} \leqslant \lambda_n \leqslant -2.838$. By Lemma 14, we have

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.734}), f(-2.838)\} > 0.1\sqrt{m-2.5}.$$

According to Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, it follows that

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+f(\lambda_3)+f(\lambda_4)+f(\lambda_{n-2})+f(\lambda_{n-1})+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.334\sqrt{m-2.5} - 5\sqrt{m-1.85} - 2.805) > 0, \end{aligned}$$

which is a contradiction. Thus G does not contain L_1 as an induced subgraph.

Suppose on the contrary that G contains L_2 as an induced subgraph. Lemma 12 yields $\lambda_2 \geqslant 1.204$, $\lambda_3 \geqslant 0.641$ and $\lambda_4 \geqslant 0.618$. The negative eigenvalues of L_2 implies $\lambda_{n-3} \leqslant -0.253$, $\lambda_{n-2} \leqslant -0.7$ and $\lambda_{n-1} \leqslant -1.618$. Then $-\sqrt{m-2.918} \leqslant \lambda_n \leqslant -2.645$. Lemma 14 gives

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.918}), f(-2.645)\} > 0.2\sqrt{m-2.5}.$$

By Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, it follows that

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2)+f(\lambda_3)+f(\lambda_4)+f(\lambda_{n-3})+\cdots+f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.618\sqrt{m-2.5} - 5\sqrt{m-1.85} - 2.35) > 0, \end{aligned}$$

which is a contradiction. Therefore G does not contain L_2 as an induced subgraph.

Suppose on the contrary that G contains L_3 as an induced subgraph. Lemma 12 yields $\lambda_2 \geqslant 1.139$ and $\lambda_3 \geqslant 0.763$. The negative eigenvalues of L_3 implies $\lambda_{n-2} \leqslant -0.277$ and $\lambda_{n-1} \leqslant -1.745$. Then $-\sqrt{m-2.504} \leqslant \lambda_n \leqslant -3.021$. Lemma 14 gives

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.504}), f(-3.021)\} > 0.001\sqrt{m-2.5}.$$

By Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, it follows that for $m \geqslant 4.7 \times 10^5$,

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.005\sqrt{m-2.5} - 5\sqrt{m-1.85} - 3.412) > 0, \end{aligned}$$

which is a contradiction. Thus, G does not contain L_3 as an induced subgraph.

Suppose on the contrary that G contains L_4 as an induced subgraph. Lemma 12 yields $\lambda_2 \geqslant 1$, $\lambda_3 \geqslant 0.764$ and $\lambda_4 \geqslant 0.513$. The negative eigenvalues of L_4 implies $\lambda_{n-2} \leqslant -0.710$ and $\lambda_{n-1} \leqslant -1.722$. Then $-\sqrt{m-2.818} \leqslant \lambda_n \leqslant -2.809$. Lemma 14 gives

$$f(\lambda_n) \geqslant \min\{f(-\sqrt{m-2.818}), f(-2.809)\} > 0.15\sqrt{m-2.5}.$$

By Lemma 13 and $\lambda_1 < \beta(m) < \sqrt{m-1.85}$, it follows that

$$\begin{aligned} t(G) &> \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2.5}{3}\lambda_1 \\ &> \frac{1}{6}(5.468\sqrt{m-2.5} - 5\sqrt{m-1.85} - 3.883) > 0, \end{aligned}$$

which is a contradiction. Hence G does not contain L_4 as an induced subgraph. □