Enumeration of Sets of Mutually Orthogonal Latin Rectangles

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Submitted: Dec 22, 2022; Accepted: Jan 2, 2024; Published: Mar 8, 2024

Abstract

We study sets of mutually orthogonal Latin rectangles (MOLR), and a natural variation of the concept of self-orthogonal Latin squares which is applicable on larger sets of mutually orthogonal Latin squares and MOLR, namely that each Latin rectangle in a set of MOLR is isotopic to each other rectangle in the set. We call such a set of MOLR *co-isotopic*.

In the course of doing this, we perform a complete enumeration of sets of t mutually orthogonal $k \times n$ Latin rectangles for $k \leqslant n \leqslant 7$, for all $t < n$ up to isotopism, and up to paratopism. Additionally, for larger n we enumerate co-isotopic sets of MOLR, as well as sets of MOLR where the autotopism group acts transitively on the rectangles, and we call such sets of MOLR *transitive*.

We build the sets of MOLR row by row, and in this process we also keep track of which of the MOLR are co-isotopic and/or transitive in each step of the construction process. We use the prefix *stepwise* to refer to sets of MOLR with this property at each step of their construction.

Sets of MOLR are connected to other discrete objects, notably finite geometries and certain regular hypergraphs. Here we observe that all projective planes of order at most 9 except the Hughes plane can be constructed from a stepwise transitive MOLR.

Mathematics Subject Classifications: 05A99, 05B15, 05B25, 05B30

1 Introduction

Two Latin squares $L_A = (a_{ij})$ and $L_B = (b_{ij})$ of order n are said to be *orthogonal* if $|\{(a_{ij}, b_{ij}) : 1 \leqslant i, j \leqslant n\}| = n^2$, that is, if when superimposing L_A and L_B, we see each of the possible n^2 ordered pairs of symbols exactly once. Pairs of orthogonal Latin squares show up in many different areas of combinatorics, both applied and pure. On

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the applied side, Latin squares are well known as the source of statistical designs for experiments. For more involved experiments one may instead switch to a design based on a pair of orthogonal Latin squares, or more generally one can use a set of pairwise orthogonal Latin squares, referred to as *mutually orthogonal Latin squares* (MOLS). Here it is also possible to use mutually orthogonal $k \times n$ Latin rectangles (MOLR) when that suits the requirements of the statistical study. A less applied application of orthogonal Latin squares comes from the study of finite geometries. Here it is well known that a finite projective or affine plane of order n exists if and only if a set of $(n-1)$ pairwise orthogonal $n \times n$ Latin squares exists. Motivated by connections like these, the first aim of this paper is to perform a complete enumeration of distinct, in the suitable sense, sets of t mutually orthogonal $k \times n$ Latin rectangles, for as large values of the different parameters as possible. Our second aim is to push this enumeration further for a few special classes of such sets of MOLR. Our computational results regarding MOLR here extend those by Harvey and Winterer in [18]. Earlier work on MOLS is surveyed in the Handbook of Combinatorial Designs [12], and more recently, Egan and Wanless [14] enumerated MOLS of order up to 9.

A particular type of orthogonal Latin squares which has seen use as a design is self-orthogonal Latin squares. A Latin square L is said to be *self-orthogonal* if L and its transpose L^T are orthogonal. The existence of self-orthogonal Latin squares (SOLS) of order n for a number of values of n, not including $n = 10$, was established through many different constructions of infinite families of SOLS, e.g. in the seminal work by Mendelsohn [29]. In [19], Hedayat presented the first example of a self-orthogonal Latin square of order 10, and in [6], Brayton et al. showed that $n \times n$ self-orthogonal Latin squares exist for each $n \neq 2, 3, 6$. Self-orthogonal Latin squares of orders up to and including $n = 9$ were completely enumerated by Burger et al. in [11], and by the same authors for order 10 in [10].

Being self-orthogonal is a special case of the concept of being *conjugate orthogonal*, a concept introduced by Stein in [37]. Here a conjugate is defined by a permutation $\sigma \in S_3$ which interchanges the roles of rows, columns, and symbols in the square. The transpose corresponds to the σ which interchanges the roles of rows and columns. As Stein showed, for every σ except the identity there are Latin squares which are orthogonal to their σ-conjugate, and Phelps [33] later investigated the possible orders of σ-conjugate Latin squares, settling the question with a handful of exceptions. These exceptions were later settled by Bennett, Wu and Zhu in [3].

Regarding conjugate orthogonality, one may sometimes find Latin squares which have several pairwise orthogonal conjugates, and it is possible to find examples where all six conjugates are pairwise orthogonal, as in Belyavskaya and Popovich [1] and Bennett [2]. This construction can of course not produce larger sets than six, and applying it to $k \times n$ Latin rectangles where $k < n$ gives a maximum of four pairwise orthogonal rectangles. However, if transposition is viewed as one of many possible 'equivalence transformations' τ, $\tau(L) = L^T$, then replacing τ with some other form of 'equivalence' gives rise to a similar concept, where potentially a set of mutually orthogonal objects can be larger.

The main focus of the current paper is to enumerate sets of mutually orthogonal Latin rectangles. For small orders, we enumerate all MOLR up to certain notions of equivalence, which we define in Section 2. For larger orders, where full enumeration of all MOLR is not feasible, we have proceeded using several natural subclasses of sets of MOLR. The first such class consists of those MOLR where each constituent Latin rectangle is isotopic to every other Latin rectangle in the set. We call such a set of MOLR *co-isotopic*. The next class is the *transitive* sets of MOLR, where we call a set of MOLR transitive if the autotopism group of the set of MOLR acts transitively on the set of rectangles. So, every transitive set of MOLR is co-isotopic, but the reverse is not always true. Finally we consider *stepwise transitive* and *stepwise co-isotopic* sets of MOLR, which are even more restricted classes, where we require that the set of MOLR can be constructed by adding one row at a time in such way that each set of MOLR along the way is transitive or co-isotopic, respectively. These conditions are very restrictive, but it turns out that all but one of the finite projective and affine planes of small orders have a corresponding stepwise transitive set of $(n-1)$ MOLS.

We will also discuss how, for the orders we have reached, our results lead to a complete enumeration of certain finite geometries, in particular, resolvable projective, affine and hyperbolic planes.

The work in the present paper builds on and extends results from a previous paper of the current authors [21], where the focus was only on triples of MOLR. One distant goal is to approach the well-known long-standing open question of how large a maximum set of MOLS of order 10 is.

The paper is structured as follows. In Section 2 we give basic notation and formal definitions regarding t-tuples of MOLR. In Section 3 we state the questions guiding our investigation, describe briefly the algorithm used to find all sets of MOLR and give some practical information regarding the computer calculations. In Section 4, we give further background on finite geometries. In Section 5 we introduce a new reshaping transformation which maps a set of t MOLR of size $k \times n$ to a set of $(k-1)$ MOLR of size $(t+1) \times n$ and discuss some of the properties of this transformation.

In Section 6 we present the data our computer search resulted in, which can be downloaded from [26], together with further analysis and results. In particular, in Subsection 6.1 we give the total number of non-isotopic sets of MOLR for orders up to $n = 7$. In Subsection 6.2 we discuss our enumeration of subclasses of sets of larger order, $n \geqslant 8$. Then in Subsection 6.3 we discuss the autotopism groups of the sets of MOLR.

Finally, Section 7 concludes the main text with some open problems and observations. Here we also discuss the long-standing open problem of whether triples of mutually orthogonal Latin squares of order 10 exist. In earlier works, Franklin [16] constructed triples of MOLR, and Wanless [40] constructed 4-tuples of MOLR. We investigate the number of possible rows in stepwise transitive such examples. This is followed by a number of appendices containing more detailed data on autotopism group sizes and all non-isotopic stepwise transitive 8-MOLR with $n = 9$.

2 Basic notation and definitions for t-MOLR

A *Latin square* of order n is an $n \times n$ matrix with cells filled by n symbols, such that each row and each column contains each symbol exactly once. For $k \leqslant n$, a matrix with k rows and n columns whose cells are filled by n symbols such that each row contains each symbol exactly once and each column contains each symbol at most once is called a $k \times n$ *Latin rectangle*, and we shall refer to $k \times n$ as its *size*. In the following we use as symbol set $\{0, 1, \ldots, n-1\}$.

With *mutually orthogonal Latin squares* defined as above, we can extend the orthogonality condition to Latin rectangles. We say that two $k \times n$ Latin rectangles $A = (a_{i,j})$ and $B = (b_{i,j})$ are *orthogonal* if each ordered pair $(a_{i,j}, b_{i,j})$ appears at most once. Also, a set of pairwise orthogonal Latin rectangles is called a set of *mutually orthogonal Latin rectangles* (MOLR). A set of t pairwise orthogonal Latin rectangles is called a t-MOLR for short.

We say that a t-MOLR is *normalized* if it satisfies the following conditions:

(S1) (Ordering among columns) The symbols in the first row of each rectangle appear in the order $0, 1, \ldots, n-1$.

(S2) (Ordering among rectangles) The second row of the i:th rectangle is lexicographically larger than the second row of the $(i+1)$:th rectangle. In other words, if $a_1, a_2, \ldots, a_t$ are symbols in the position $(2, 1)$ in the t-MOLR, seen as an ordered t-tuple, then it holds that $a_1 > a_2 > \cdots > a_t$.

(S3) (Ordering among rows) The second row in the first rectangle is lexicographically larger than the third row, the third row is larger than the fourth row, and so on.

Note that (S2) relies on the rectangles having the same first row and being mutually orthogonal, so that all the entries in $(2, 1)$ are in fact distinct.

We use normalization to reduce the search space in our computer runs, but note that two different normalized t-MOLR may still be isotopic. For some further details, see our previous paper [21]. As our computations proceed by adding consecutive rows to t-MOLR, we will have use for the following term: An *extension* of size $k \times n$ is a t-MOLR which results from a t-MOLR of size $(k-1) \times n$ by adding one more row to each rectangle.

We will use several different notions of equivalence for t-MOLR. We recommend Section 2 of Egan and Wanless [14] for an in-depth discussion of the different symmetry and equivalence concepts for MOLS and we will follow that terminology. Let $(A_1, A_2, \ldots, A_t)$ be a t-MOLR of size $k \times n$, and let S_n denote the symmetric group on n elements. The following group of isotopisms acts on the set of t-MOLR: $G_{n,k,t} = S_t \times S_k \times S_n \times [S_n \times S_n \times \ldots \times S_n]$, where the initial S_t corresponds to permutations of the rectangles, the S_k corresponds to permutations of the rows, the next S_n corresponds to permutations of the columns, and each of the last t copies of S_n, in square brackets, corresponds to permutations of the symbols in the single rectangles. Two t-MOLR A and A' of size $k \times n$ are said to be *isotopic* if there exists a $g \in G_{n,k,t}$ such that $g(A) = A'$. The *autotopism group* of a t-MOLR A is defined as $\mathrm{Aut}(A) := \{g \in G_{n,k,t} \mid g(A) = A\}$.

The autotopism group of a t-MOLR of size $k \times n$ is also a subgroup of a larger group, the *autoparatopism* group, which can be described either using orthogonal arrays or, as we will do here, via a hypergraph representation of the MOLR, which will also be used in one of our proofs.

2.1 The standard hypergraph representation

Let $A = (L^1, L^2, \ldots, L^t)$ be a list of $k \times n$ MOLR. The *standard labelled hypergraph representation* $\mathcal{H}(A)$ *of* A is constructed as follows.

The vertex set V for $\mathcal{H}(A)$ consists of $t+2$ vertex classes, $V_1, V_2, \ldots V_{t+2}$. V_1 has one labelled vertex per row in the rectangles, and hence has size k. V_2 has one labelled vertex per column in the rectangles, and so has size n. For $i \geqslant 3$, V_i has n vertices and for each symbol from L^{i-2} there is one vertex labelled with that symbol.

The set of hyperedges of $\mathcal{H}(A)$ is as follows: For each $i \in V_1$ and $j \in V_2$ there is a hyperedge consisting of $(i, j, L^1_{i,j}, L^2_{i,j}, \ldots, L^t_{i,j})$. Thus each hyperedge contains exactly one vertex from each vertex class, and each hyperedge corresponds to a position in the rectangles and the tuple of symbols used there by the rectangles.

The hypergraph $H(A)$ is $(t+2)$-uniform, and $(t+2)$-partite, with at most one class of vertices, V_1, smaller than n. This hypergraph is also linear, i.e. any pair of vertices belongs to at most one edge, since two edges intersecting in two vertices would either correspond to having more than one symbol in a given position (i, j) in some rectangle, or a violation of orthogonality, or a repeated symbol in a row or column of some rectangle.

Conversely, let H be a linear $(t+2)$-uniform $(t+2)$-partite hypergraph with $t+1$ vertex classes $V_2, \ldots V_{t+2}$ of size n and one vertex class V_1 of size $k \leqslant n$, and assume that the vertices in each class V_i are labelled with a set of $|V_i|$ distinct symbols. Given a choice of an ordered pair of vertex classes, which must include V_1 as the first class if $k < n$, we can construct a t-MOLR A from H by the same description as above, letting the first class be the row indices and the second class be the column indices. For that t-MOLR A we have $H = \mathcal{H}(A)$ so we have an equivalence between t-MOLR and certain labelled hypergraphs. This is in fact just a different way of writing down the equivalence between t-MOLR and certain orthogonal arrays, with each hyperedge corresponding to a row in the orthogonal array.

2.2 Isotopism and paratopism

The isotopisms for a $k \times n$ t-MOLR A correspond to the transformations which freely permute the labels of the vertices within each vertex class of $\mathcal{H}(A)$, and also permute the indices of the vertex classes $V_3, V_4, \ldots, V_{t+2}$.

An isotopism is a special case of a more general type of transformation known as a *paratopism*. Two $k \times n$ t-MOLR A and B are *paratopic* if $\mathcal{H}(A)$ can be obtained from $\mathcal{H}(B)$ by permuting labels in the same way as for an isotopism, and freely permuting the indices of all vertex classes of equal size. The corresponding transformation of the MOLR is called a *paratopism* on the set of MOLR. Note that if $k < n$ then a paratopism between two MOLR cannot interchange the vertex class V_1 with any other class, since it is strictly smaller than the other classes, but for $k = n$ it may interchange V_1 with another class as well.

For $k = n$, if the paratopisms are restricted to only interchanging V_1 with V_2, while permuting the other classes freely we get the *trisotopisms*, and two t-MOLS related by a trisotopism are said to be *trisotopic*. Interchanging V_1 and V_2 corresponds to transposing the squares in the t-MOLS, so if two t-MOLS A and B are trisotopic they are either isotopic or A is isotopic to the t-MOLS obtained by transposing each square in B.

3 Generation of t-MOLR

The basic method for our generation routine is quite simple. We start with the set of all $1 \times n$ t-MOLR and find all possible extensions to $2 \times n$ t-MOLR, followed by an isotopy reduction where only one representative for each isotopy class is kept. As part of the isotopy reduction, the autotopism group of each representative is also determined and its size stored. The extension step is then repeated until the desired number of rows is reached. The algorithms and methods used to generate the t-MOLR are rather straightforward extensions of those used in our previous paper [21]. They were implemented in C++ and run in a parallelized version on the Kebnekaise and Abisko supercomputers at High Performance Computing Centre North (HPC2N). The total run time for all the data in the paper was a few hundred core-years.

In order to safeguard the correctness of our computational results, several steps were taken. We wrote separate implementations in Mathematica of both the main algorithm and another much simpler method with which we performed an independent generation of the data for smaller sizes, in order to help verify the correctness of the C++ implementation. All the data has been compared with the known classifications of MOLS and MOLR in the literature, primarily McKay, Meynert and Myrvold [27] and Egan and Wanless [14], and agrees with them.

The method described above was applied directly in order to first generate all t-MOLR of a given size when this was possible, and after each generation step we also classified the t-MOLR which belonged to one of the following two classes.

Definition 1. (a) A t-MOLR $A = (A_1, A_2, \ldots, A_t)$ is *co-isotopic* if all pairs of rectangles $A_i, A_j \in A$ are isotopic.

(b) A t-MOLR $A = (A_1, A_2, \ldots, A_t)$ is *transitive* if $\mathrm{Aut}(A)$ acts transitively on the set of rectangles in A. That is, for all pairs $A_i, A_j \in A$ there exists $\phi \in \mathrm{Aut}(A)$ which maps A_i to A_j.

When checking whether a t-MOLR is co-isotopic or transitive, we require the entire t-MOLR. This becomes a problem when generating t-MOLR by adding rows one by one. In order to find all co-isotopic t-MOLR we have to first generate all t-MOLR and then test them for co-isotopism and transitivity. We are therefore also interested in the two following recursively defined classes of t-MOLR which allow for more efficient generation.

Definition 2. (a) A $k \times n$ t-MOLR A is *stepwise co-isotopic* if A is co-isotopic and either $k = 1$, or $k \geqslant 2$ and A is the extension of a stepwise co-isotopic $(k-1) \times n$ t-MOLR.

(b) A $k \times n$ t-MOLR A is *stepwise transitive* if A is transitive and either $k = 1$, or $k \geqslant 2$ and A is the extension of a stepwise transitive $(k-1) \times n$ t-MOLR.

As we shall demonstrate in Section 6.2, the sets of $(n-1)$-MOLS corresponding to a classical projective plane over a finite field of order n are, in fact, always stepwise transitive.

For both of the classes in Definition 2 we can generate the $k \times n$ t-MOLR by starting out with the corresponding class of $(k-1) \times n$ t-MOLR, finding all their non-isotopic extensions and then discarding those that are not co-isotopic or transitive, respectively. These restrictions typically lead to a far smaller set of t-MOLR to extend to the next value of k, and thanks to this we were able to perform complete enumeration of these classes for larger values of n than in the general case.

In addition to producing the objects themselves, we also calculated the size of the autotopism group of each object. With some exceptions due to size restrictions, all the data we generated is available for download at [26]. Further details about the organization of the data are given there. Examples from this paper are also available as text files labelled "DATA" at https://doi.org/10.37236/9049.

We also include the size of the paratopism classes of MOLR in our tables. In order to obtain these classes, we used the package Nauty (see McKay and Piperno [28]) to reduce the set of isotopism classes to paratopism classes. In order to do this, the MOLR were encoded as graphs, using the method described by Egan and Wanless in [14].

4 Finite geometries

As mentioned in the introduction, sets of $n-1$ MOLS of size n have a well known connection to finite projective and affine planes. Here we will recollect some facts from finite geometry and demonstrate how a general t-MOLR can be translated into the finite geometry setting.

4.1 Basic notions for finite geometries

Definition 3. A pair $\mathcal{P} = (V, L)$, where V is a finite set of points and L is a set of subsets of V, which are called lines, is a *finite plane* if the following conditions are satisfied.

1. Each line has at least 2 points.
2. Any pair of points is contained in exactly one line.
3. There exists a point p and a line ℓ, where ℓ does not contain p.
4. There exists a set of 4 points such that no 3 of them lie on the same line.

Additionally, the plane may satisfy one of the following parallelity properties:

(P1) Every pair of lines has non-empty intersection.

(P2) Given a point p and a line ℓ, which does not contain p, there exists exactly one line through p which does not intersect ℓ.

(P3) Given a point p and a line ℓ, which does not contain p, there exist at least two lines through p which do not intersect ℓ.

A plane which satisfies $(P1)$ is called a *finite projective plane*, a plane which satisfies $(P2)$ is called a *finite affine plane*, and a plane which satisfies $(P3)$ is called a *finite hyperbolic plane*.

If each line in a finite projective plane contains exactly $n+1$ points, the finite projective plane is said to be of *order* n, and correspondingly, if each line in a finite affine plane contains exactly n points, the finite affine plane is said to be of order n.

A collection P of non-intersecting lines from $\mathcal{P}$ which form a partition of V is called a *parallel class*. A partition of the lines of $\mathcal{P}$ into parallel classes is called a *resolution* of $\mathcal{P}$, and a geometry which has at least one resolution is called a *resolvable* geometry.

A set of $n-1$ MOLS of order n is equivalent to a finite projective plane of order n, as pointed out by Bose [4]. The existence of a set of $n-1$ MOLS, for $n = p^r$, for a prime p, was demonstrated in a somewhat forgotten paper from 1896 by Moore [30]. See Ehrhardt [15] for a historical discussion of Moore's paper and the wider context of 19th-century design theory.

To relate our results on Latin rectangles and squares to the context of finite geometries, we will first recall the explicit correspondence between finite projective geometries on the one hand, and complete sets of mutually orthogonal Latin squares on the other hand. We will begin with noting that a finite projective plane of order n has n^2+n+1 points and n^2+n+1 lines, each containing $n+1$ points, and that each point in a finite projective plane of order n belongs to $n+1$ lines.

To get the correspondence between a finite projective plane of order n and a set of $n-1$ MOLS of order n, we first select a line of the projective plane, which we call the *line at infinity*, L_∞. Next, we select two points x_∞ and y_∞ on L_∞, and label the remaining $n-1$ points on L_∞ by $\ell_1, \ell_2, \dots, \ell_{n-1}$. The n remaining lines containing x_∞ are labelled by $X_1, X_2, \dots, X_n$, and the n remaining lines containing y_∞ are labelled $Y_1, Y_2, \dots, Y_n$.

The points $\ell_1, \ell_2, \dots, \ell_{n-1}$ will correspond to the $n-1$ Latin squares $L_1, L_2, \dots L_{n-1}$ in the set of MOLS, and the point $p_{i,j}$ at the intersection between X_i and Y_j will correspond to the cell (i,j) in the Latin squares. The symbol that goes in cell (i,j) in L_k is given by fixing a labelling using symbols $s_1, s_2, \dots, s_n$ of the n lines (excluding L_∞) that contain ℓ_k, and checking which of the symbols was assigned to the unique line passing through $p_{i,j}$.

Note that the ample room for choices of labelings gives different sets of MOLS, and that the construction can be reversed, that is, any $(n-1)$-MOLS gives a finite projective plane.

When $\mathcal{P}_n$ is the classical Galois plane over the finite field $GF(n)$, sometimes denoted by $PG(2,n)$, one can give a simple explicit form for a set of MOLS derived from that plane in the following way. Let $x \in GF(n)$ be a generator for the multiplicative group of $GF(n)$ and set $a_0 = 0$ and $a_i = x^{i-1}$ for $i = 1, \ldots, n-1$. Now, for $1 \leqslant k \leqslant n-1$ define an $n \times n$ Latin square L_k by setting position (i, j) equal to $a_i + a_k \times a_j$. As shown, e.g., in Dénes and Keedwell's Latin squares book [22], this defines an $(n-1)$-MOLS, which in turn also defines the classical Galois plane over $GF(n)$.

When n is a prime power, finite projective planes can always be constructed using the Galois field $GF(n)$, but there are projective planes that do not arise in this way. For example, as early as 1907, Veblen and MacLagan-Wedderburn [39] constructed 3 projective planes of order 9, not isomorphic to the standard projective plane of order 9 arising from $GF(9)$. Considerably later, Lam, Kolesova and Thiel [23] showed by an exhaustive computer search that these 4 projective planes of order 9 are, in fact, the only ones.

For some other n, notably $n = 6$ and $n = 14$, the Bruck-Ryser theorem [9] excludes the possibility of a projective plane. For $n = 6$, non-existence was already known, since Tarry [38] had proven that Euler's 36 officers problem (which asked for a set of just two MOLS of order 6) had no solution, leaving $n = 10$ the smallest open case. A delightful account of the search for a projective plane of order 10 can be found in Lam [24], and the non-existence was settled by Lam, Thiel and Swiercz in [25]. Combining this computational non-existence result with a result of Shrikhande [36], one gets that there does not exist a set of 7 or more MOLS of order 10. Since there are examples of pairs of orthogonal Latin squares of order 10, the maximum order of a t-MOLS for $n = 10$ lies in the interval $2 \leqslant t \leqslant 6$.

Given a projective plane $\mathcal{P}$ of order n we can construct an affine plane of the same order by deleting a line and all the points on it from $\mathcal{P}$. It is also well known that any affine plane can be obtained from a projective plane in this way. So, the existence of an affine or a projective plane of a given order are equivalent, and in turn equivalent to the existence of a set of $n-1$ MOLS of order n. Finite hyperbolic geometries have not been studied in as great detail as the projective or affine ones. The first axiomatization for finite hyperbolic planes was given by Graves in [17] and a few early constructions and structural theorems were given by Di Paola, Henderson, and Ostrom in [13, 20, 32], respectively. Sandler [35] also noted that one will obtain a finite hyperbolic plane from a finite projective plane by deleting three lines that do not intersect in a single point, together with all points on these lines, or equivalently by deleting two non-parallel lines from a finite affine plane.

Another well studied class of finite geometries are *nets*. They were introduced by Bruck [7, 8] who also showed that they, analogous to the situations for projective and affine planes, are equivalent to t-MOLS. A net $\mathcal{P}(A)$ can be constructed in the following way. Let $A = (A^1, A^2, \ldots, A^t)$ be a set of mutually orthogonal Latin squares of order n, and let the set of pairs $V = \{(i,j) \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$ be the point set. For each square A^s and symbol r we let the set of points such that $A^s_{i,j} = r$ be a line in L. We also add one line to L for each row of the squares and one for each column, and set $\mathcal{P}(A) = (V, L)$. We now note that each line in $\mathcal{P}(A)$ has n points, each point lies in $t+2$ lines, two lines defined by different squares intersect in exactly one point (since the squares are orthogonal), the set of lines defined by a single square are pairwise disjoint, and any pair of points lies in at most one line (again by orthogonality). Additionally, the set of lines defined by a single square form a partition of V, i.e., a parallel class, and the parallel classes defined by the different squares form a partition of L, so $\mathcal{P}(A)$ is a resolvable geometry. The net $\mathcal{P}(A)$ is of order n and degree t. Note that if $t = n-1$, then the net is an affine plane, and in general, nets are a particular class of *partial geometries* as defined by Bose [5].

4.2 Finite geometries from Latin rectangles

The classical finite geometric constructions which we have surveyed here are all connected to Latin squares. However, many of these constructions can fruitfully be extended to Latin rectangles and MOLR as well and we will briefly discuss how this can be done.

If we apply the same construction as for nets to a $k \times n$ t-MOLR we get a weaker geometry which we call a *partial net*. When $k < n$ we lose the property that every pair of lines from different parallel classes intersect. Of particular interest here are t-MOLR which are maximal with respect to either t or k, since these give rise to geometries which cannot be embedded in a larger partial net with the same value of n.

Our more symmetric classes of t-MOLR correspond to geometries with certain symmetries. A transitive t-MOLR gives rise to a geometry with the property that the automorphism group of the geometry acts transitively on the set of parallel classes in a specific resolution of the geometry. A stepwise transitive set in turn gives us a geometry with the property that the geometry can be built by stepwise adding new points in a way which preserves the transitivity of the resolution.

5 A reshaping transformation

For the specific case of $(n-1)$-MOLS, several authors (see Egan and Wanless [14] for further details) have studied a notion of equivalence where two such MOLS are equivalent if they define the same projective plane. Using the relation between MOLS and projective planes described in Section 4, this means that two such equivalent MOLS are constructed by selecting different lines at infinity from the projective plane. For t-MOLR there is in general no extension to a projective plane, but we will define a similar transformation T which maps a $k \times n$ t-MOLR A to a $(t+1) \times n$ $(k-1)$-MOLR $T(A)$.

In preparation for defining this transformation, we will first define a hypergraph $G(A) = (W, E)$ with labelled vertices and hyperedges, from a $k \times n$ t-MOLR $A = \{L^1, L^2, \ldots, L^t\}$ where each rectangle uses the symbols $1, 2, \ldots, n$. The hypergraph $G(A)$ is similar, but not identical, to the standard hypergraph representation.

Let the vertex set W be the set of ordered pairs (i, j), where $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$ together with the singletons $0 \leqslant q \leqslant t$. A vertex (i, j) is given the label i and a singleton vertex q is labelled by the corresponding symbol q.

Next we will define the set E of labelled hyperedges. For each $q \geqslant 1$ and symbol s we add a hyperedge consisting of the singleton q and the pairs (i, j) such that $L^q_{ij} = s$, and label this edge by s. We think of the singletons q as an indexing of the rectangles, and the edges containing the singleton q keep track of symbol positions in L^q. Permuting symbols in the Latin rectangles $L^1, L^2, \ldots L^t$ thus only affects the edge labels of G. For each column index $1 \leqslant i \leqslant n$ we add a labelled hyperedge which consists of all pairs (i, j) with $i = 1, 2, \ldots, k$ and the singleton $q = 0$, labelled by j.

Note that for $k = n$, $t = n - 1$ the labelled hypergraph $G(A)$, with labels removed and interpreting edges as lines, is equivalent to the projective plane defined by A with the point in the line at infinity where the row-lines intersect removed, and the row-lines deleted.

Lemma 4. *The hypergraph $G(A) = (W, E)$ is linear, $(k+1)$-uniform, and $(k+1)$-partite, with k vertex classes $W_1, W_2, \ldots, W_k$ of size n corresponding to the rows of A, and one vertex class W_0 of size $t + 1$ corresponding to the added singletons.*

Proof. By construction, W_i, $1 \leqslant i \leqslant k$, has size n, and W_0 has size $t+1$. Each hyperedge contains exactly one vertex from each vertex class, so the hypergraph is $(k + 1)$-uniform and $(k + 1)$-partite.

In $G(A)$, two edges intersecting in a singleton q have no further common vertices since they either correspond to positions of different symbols in the same rectangle, or to distinct columns of a rectangle. Edges containing distinct singletons cannot intersect in two other vertices, since that would either mean that the same symbol appears twice in a column for the same rectangle, or that a pair of symbols appears twice in a pair of rectangles, hence violating orthogonality. So the hypergraph is linear. ☐

Lemma 4 says that $G(A)$ satisfies the properties required for being the standard hypergraph representation for some MOLR. Hence we can interpret $G(A)$, with a specified $(k+1)$-partition of its vertex set, as the standard hypergraph representation $\mathcal{H}(T(A))$ for some $(k-1)$-tuple $T(A)$ of $(t+1) \times n$ MOLR, by interpreting the vertex class W_0 as corresponding to V_1, the row indices for $T(A)$, and taking one of the other W_i, which we take to be W_1, as corresponding to V_2, the column indices in $T(A)$. The remaining $W_2, W_3, \ldots, W_k$ are interpreted as the indexing of the rectangles, $V_i = W_{i-1}$ for $i = 3, \ldots, k + 1$.

This implicitly defines a transformation T from a $k \times n$ t-MOLR A to a $(t + 1) \times n$ $(k - 1)$-MOLR $T(A)$, so if one of the MOLR exists, then so does the other. Note that $T(T(A))$ gives a MOLR with the same parameters as A. However, the transformation does not induce an isomorphism of the paratopism group for the set of $k \times n$ t-MOLR to that for $(t+1) \times n$ $(k-1)$-MOLR. For most parameters these two paratopism groups do not have the same size, and so cannot be isomorphic. This in turn means that we do not necessarily have the same number of paratopism classes for $k \times n$ t-MOLR and $(t+1) \times n$ $(k - 1)$-MOLR. For small n, with $t < n - 1$, it turns out from our computational results that the number of paratopism classes coincide, see Tables 2, 4, 6 and 8, but we expect this to fail for n slightly larger than those we consider here. A full investigation of how the transformation T interacts with paratopism is beyond the scope of the current article, but we note some of the properties of T in the following theorem.

Theorem 5. *The mapping T maps the set of $k \times n$ t-MOLR to the set of $(t+1) \times n$ $(k-1)$-MOLR, and $T(A)$ has the identity permutation as the first row in each rectangle.*

Two MOLR A and B have $T(A) = T(B)$ if and only if B can be obtained from A by only permuting the symbols of each rectangle.

Proof. It follows directly from the definition that T maps the set of $k \times n$ t-MOLR to the set of $(t+1) \times n$ $(k-1)$-MOLR. Since each edge in $G(A)$ incident to the vertex given by the singleton 0 consists of the pairs (i, j) with a fixed second coordinate, the MOLR $T(A)$ has the identity permutation as the first row of each rectangle.

Suppose B can be obtained from A by only permuting symbols in the rectangles. Then $G(A)$ and $G(B)$ with removed edge labels will leave the same unlabelled hypergraph, and thus represent the same MOLR, up to permutation of symbols in each rectangle. As we have a fixed choice of index classes $W_0 = V_1$, $W_1 = V_2$, this in turn means that $T(A) = T(B)$.

Conversely, suppose $T(A) = T(B)$. Then the standard hypergraph representations of $T(A)$ and $T(B)$ of course coincide, $\mathcal{H}(T(A)) = \mathcal{H}(T(B)) = \mathcal{H}$. Removing the edge labels from $G(A)$ and $G(B)$ leaves the same unlabelled hypergraph $\mathcal{H}$ if $G(A)$ and $G(B)$ differed only in the order of the edge labels, corresponding to permutations of the symbols in some of the rectangles of A and B. □

6 Computational results and analysis

We now turn to the results and analysis of our computational work.

6.1 The number of t-MOLR

In Tables 1–7, we present data on the number of isotopism classes and paratopism classes of $k \times n$ t-MOLR.

	2×4	3×4	4×4
$t = 1$	2	2	2
$t = 2$	3	2	1
$t = 3$	2	1	1

Table 1: The number of isotopism classes of t-MOLR for $n = 4$.

	2×4	3×4	4×4
$t = 1$	2	2	2
$t = 2$	2	2	1
$t = 3$	2	1	1

Table 2: The number of paratopism classes of t-MOLR for $n = 4$.

	2×5	3×5	4×5	5×5
$t = 1$	2	3	3	2
$t = 2$	5	14	2	2
$t = 3$	4	1	1	1
$t = 4$	3	1	1	1

Table 3: The number of isotopism classes of t-MOLR for $n = 5$.

	2×5	3×5	4×5	5×5
$t = 1$	2	3	3	2
$t = 2$	3	9	1	1
$t = 3$	3	1	1	1
$t = 4$	2	1	1	1

Table 4: The number of paratopism classes of t-MOLR for $n = 5$.

	2×6	3×6	4×6	5×6	6×6
$t = 1$	4	16	56	40	22
$t = 2$	28	1526	2036	85	0
$t = 3$	103	2572	513	7	0
$t = 4$	92	118	12	8	0
$t = 5$	33	0	0	0	0

Table 5: The number of isotopism classes of t-MOLR for $n = 6$.

	2×6	3×6	4×6	5×6	6×6
$t = 1$	4	14	44	33	12
$t = 2$	14	575	745	44	0
$t = 3$	44	745	179	5	0
$t = 4$	33	44	5	5	0
$t = 5$	17	0	0	0	0

Table 6: The number of paratopism classes of t-MOLR for $n = 6$.

	2×7	3×7	4×7	5×7	6×7	7×7
$t=1$	4	56	1398	6941	3479	564
$t=2$	100	514 162	49 415 812	21 290 125	11 582	20
$t=3$	2858	65 883 453	323 112 477	55 545	16	4
$t=4$	17 609	35 469 948	68 659	204	7	3
$t=5$	10 626	22 982	19	5	5	1
$t=6$	1895	23	2	1	1	1

Table 7: The number of isotopism classes of t-MOLR for $n=7$.

	2×7	3×7	4×7	5×7	6×7	7×7
$t=1$	4	45	808	3712	1895	147
$t=2$	45	172 622	16 481 351	7 103 198	4013	7
$t=3$	808	16 481 351	80 797 488	14 121	12	1
$t=4$	3712	7 103 198	14 121	82	4	1
$t=5$	1895	4013	12	4	4	1
$t=6$	324	11	2	1	1	1

Table 8: The number of paratopism classes of t-MOLR for $n=7$.

In the data for $n \leqslant 7$ some patterns can be observed, somewhat interrupted by the exceptional behavior for $n=6$. If we consider fixed values of t and n and increase k we always see a unimodal sequence, and the peak of the sequence appears at a lower value of k when t is increased. The patterns conform well with the number of constraints on the symbols, as a function of t and k. If we instead keep n and k fixed and increase t we see a similar pattern, though here there is an exception for $n=6$, $k=5$, where there is a local minimum at $t=3$. These observations motivate the following questions.

Question 6. Is the number of t-MOLR for fixed n and t a unimodal sequence in k?

Question 7. For $n \geqslant 7$, is the number of t-MOLR for fixed n and k a unimodal sequence in t?

Additionally we see in the tables for paratopism classes that there is a symmetry between the entries below and above the diagonal, with exceptions for the cases with $k=n$ or $t=n-1$. For the case of 6×6 Latin squares (that is, $t=1$, $k=n=6$), this exception corresponds to a change from paratopism to trisotopism as the equivalence relation, 17 is the number of trisotopism classes and 12 the number of paratopism classes (see Egan and Wanless [14]). For $n=7$ we have three entries for $k=n$ which do not match the corresponding ones for $t=n-1$. For these small values of n this partial symmetry is explained by the reshaping transformation described in Section 5, but as discussed there we expect the symmetry to fail in general for large enough n.

We have also classified the small sets of MOLR according to some further properties. In Tables 9–11, for $n = 4, 5, 6$, we give the number of t-MOLR that are A) co-isotopic, B) transitive, C) stepwise co-isotopic and D) stepwise transitive. In a sense, these four classes are gradually more regular, and the data in the tables gives the total numbers from each such class in the form A, B, C, D in each cell. In Table 12, the data is presented in the form A, B, and in Table 13, the data is in the form C, D. Comparing the data in Tables 1-7 with the data in Tables 9-13, it is clear that when k or t is small compared to n, most t-MOLR have none of these stronger regularity properties, but whenever a t-MOLR exists, we also have a co-isotopic t-MOLR with the same parameters.

Whenever there exists a t-MOLR, we also find transitive t-MOLR for most parameters. The exception is $n = 7$, where there exist t-MOLS ($k = 7$, that is) with $t = 4, 5$ but no corresponding transitive t-MOLR, demonstrating that the autotopism group for the 6-MOLR does not have orbits of length 4 and 5. As a further example of observations from the data, for $n = 7, k = 4$ (see Table 12) there exist co-isotopic 5-MOLR, but no transitive 5-MOLR, and *a fortiori*, no stepwise transitive 5-MOLR.

	2×4	3×4	4×4
$t = 2$	2, 2, 2, 2	2, 2, 1, 1	1, 1, 1, 1
$t = 3$	1, 1, 1, 1	1, 1, 1, 1	1, 1, 1, 1

Table 9: The number of non-isotopic t-MOLR for $n = 4$ sorted by increasing regularity.

	2×5	3×5	4×5	5×5
$t = 2$	4, 3, 4, 3	11, 9, 7, 6	2, 1, 2, 1	2, 1, 2, 1
$t = 3$	3, 2, 3, 2	1, 0, 0, 0	1, 0, 0, 0	1, 0, 0, 0
$t = 4$	2, 2, 2, 2	1, 1, 1, 1	1, 1, 1, 1	1, 1, 1, 1

Table 10: The number of non-isotopic t-MOLR for $n = 5$ sorted by increasing regularity.

	2×6	3×6	4×6	5×6
$t = 2$	12, 11, 12, 11	280, 170, 158, 103	229, 160, 66, 50	43, 36, 13, 12
$t = 3$	16, 6, 16, 6	115, 29, 32, 4	62, 39, 4, 1	4, 3, 0, 0
$t = 4$	9, 8, 9, 8	19, 17, 15, 15	4, 3, 0, 0	4, 4, 0, 0
$t = 5$	2, 2, 2, 2	0, 0, 0, 0	0, 0, 0, 0	0, 0, 0, 0

Table 11: The number of non-isotopic t-MOLR for $n = 6$ sorted by increasing regularity.

	2×7	3×7	4×7	5×7	6×7	7×7
$t = 2$	42, 29	14 464, 3549	65 156, 27 299	22 432, 18 836	409, 392	9, 6
$t = 3$	318, 15	49 370, 647	2985, 1578	111, 36	11, 6	4, 1
$t = 4$	691, 21	1622, 110	84, 67	67, 53	7, 3	3, 0
$t = 5$	176, 6	49, 42	2, 0	4, 2	5, 3	1, 0
$t = 6$	26, 5	7, 7	2, 2	1, 1	1, 1	1, 1

Table 12: The number of non-isotopic co-isotopic and transitive t-MOLR for $n = 7$.

	2×7	3×7	4×7	5×7	6×7	7×7
$t = 2$	42, 29	7423, 2175	14 960, 10 029	4163, 3923	91, 84	6, 4
$t = 3$	318, 15	13 975, 185	283, 160	8, 5	4, 1	4, 1
$t = 4$	691, 21	585, 48	12, 1	3, 0	3, 0	3, 0
$t = 5$	176, 6	48, 42	2, 0	1, 0	1, 0	1, 0
$t = 6$	26, 5	6, 4	2, 2	1, 1	1, 1	1, 1

Table 13: The number of non-isotopic stepwise co-isotopic and stepwise transitive t-MOLR for $n = 7$.

The numbers of stepwise co-isotopic and stepwise transitive t-MOLR are by definition smaller than (or equal to) the numbers of co-isotopic and transitive t-MOLR respectively, but we again find stepwise co-isotopic examples for most parameter values. For $n = 7$, $k = 4$, we have no stepwise transitive sets of MOLR, and additionally, there are no stepwise transitive 5-MOLR for $n = 7$, $k = 5, 6$.

Here we note that for each $n \leqslant 7$ all $(n-1)$-tuples of MOLS are stepwise transitive, if they exist. Since in each case the maximum set of MOLS for these n corresponds to a Galois projective plane (that is, constructed from the corresponding Galois field) this reflects the high degree of symmetry of these planes. For $n = 9$ there are several projective planes, some of which are not Galois projective planes, and we will investigate that case below.

6.2 Larger orders

For $n \geqslant 8$ we have not generated all t-MOLR, even though our programs are in principle able to do so. The problem here is that the number of t-MOLR becomes so large that several peta-byte would be required to store them on disc, and any kind of analysis of the whole set would become impractical. Instead, we have focused on two interesting subclasses, the stepwise co-isotopic and the stepwise transitive t-MOLR. These classes are restrictive enough to let us push the generation program a few more steps, and we have already seen that they contain a number of interesting examples.

In Table 14 we give the number of stepwise co-isotopic and stepwise transitive t-MOLR for $n = 8$ and in Table 15 we give the number of stepwise transitive t-MOLR for $n = 9$. For $n = 8$ it is clear that for small parameters k and t, the stepwise co-isotopic t-MOLR far outnumber the stepwise transitive ones. We also find stepwise co-isotopic t-MOLR for all parameters, but not stepwise transitive ones. This motivates the following question.

Question 8. For $n \geqslant 7$, is there a stepwise co-isotopic t-MOLR for every pair t, k that allows a t-MOLR?

For $n = 9$ we only have data for the stepwise transitive class, since the number of stepwise co-isotopic t-MOLR is too large. Here it is clear that the possible values of t are quite restricted. We note two interesting facts. First, there is a unique stepwise transitive 6-MOLS, which in turn has unique stepwise transitive restrictions to 8 and 7 rows. We present this object in Figure 1. Second, there are 5 stepwise transitive 8-MOLS, which are presented in Appendix G.

As mentioned earlier, it is known that there are exactly 4 projective planes of order 9. The Galois plane corresponds to the 8-MOLS with autotopism group of order 10 368, see Table 42 in Appendix F. The other four 8-MOLS can be divided into two pairs, such that both 8-MOLS in one pair correspond to the Hall plane, and those in the other pair correspond to the dual of the Hall plane. In Appendix G we include data on this pairing. This leaves the Hughes plane of order 9 as the smallest projective plane which cannot be defined by a stepwise transitive MOLS.

With this in mind one may ask about the situation for larger orders as well. Wanless [41] has found that 8 of the 22 projective planes of order 16 cannot be constructed via a co-isotopic MOLS, and hence they cannot be constructed from a stepwise transitive set of MOLS either. In the online catalogue provided by Royle (currently available via [34]), these are the planes labelled JOHN, BBS4, BBH2 and their duals, BBH1 (which is self-dual), and, finally, either MATH or its dual DMATH. In the case of MATH or DMATH, the test did not give enough information to discern which one of these two planes was constructible in this fashion.

On the other hand, we can prove the following result, where by 'the standard way', we refer to the construction given in Section 4. Theorem 5.2.5 in [22] shows, in our terminology, that the tuples coming from the Galois planes are co-isotopic. However, a closer inspection of their proof leads to this stronger statement.

Theorem 9. *The set M of $(n-1)$ MOLS $L_1, L_2, \ldots, L_{n-1}$ corresponding to a projective plane constructed in the standard way from the finite field $GF(n)$ is stepwise transitive.*

Proof. Let $a_0 = 0$ and $a_i = x^{i-1}$ for $i = 1, 2, \ldots, n-1$, where x is a generating element of $GF(n)$, and note that by definition column j of L_k has entries

$$a_0 + a_k a_j, a_1 + a_k a_j, \ldots, a_{n-1} + a_k a_j$$

in this order. Column 0 therefore coincides among all of the L_k. Now, let π be the column permutation that leaves column 0 in place, and shifts the sequence of columns

	2×8	3×8	4×8	5×8	6×8	7×8	8×8
$t=2$	186, 99	446 443, 45 429	4 432 284, 1 097 655	3 826 527, 2 569 679	242 732, 206 612	484, 305	70, 13
$t=3$	11 565, 66	9 144 025, 7627	178 502, 41 505	628, 75	111, 32	10, 6	7, 3
$t=4$	216 950, 152	1 648 723, 4284	3547, 712	58, 20	4, 0	3, 0	3, 0
$t=5$	509 622, 19	2652, 0	267, 0	2, 0	1, 0	1, 0	1, 0
$t=6$	91 013, 109	975, 908	155, 146	1, 0	1, 0	1, 0	1, 0
$t=7$	4538, 5	2, 2	2, 2	1, 1	1, 1	1, 1	1, 1

Table 14: The number of non-isotopic stepwise co-isotopic and stepwise transitive t-MOLR for $n=8$.

	2×9	3×9	4×9	5×9	6×9	7×9	8×9	9×9
$t=2$	126	1 418 577	560 524 587	20019499500	67480364637	5 872 237 985	14 940 988	28 955
$t=3$	202	72 836	1 746 912	0	0	0	0	0
$t=4$	1067	356 680	2 640 163	645 453	1816	31	7	5
$t=5$	17	0	0	0	0	0	0	0
$t=6$	543	21 620	244	33	16	1	1	1
$t=7$	39	1532	300	0	0	0	0	0
$t=8$	54	48	27	22	16	9	7	5

Table 15: The number of non-isotopic stepwise transitive t-MOLR for $n=9$.

0	1	2	3	4	5	6	7	8
8	7	6	5	3	2	4	1	0
7	0	8	4	1	6	5	3	2
6	2	0	1	8	7	3	5	4
5	8	4	6	2	3	7	0	1
4	3	5	7	0	1	8	2	6
3	4	1	0	7	8	2	6	5
2	6	7	8	5	0	1	4	3
1	5	3	2	6	4	0	8	7

0	1	2	3	4	5	6	7	8
7	8	5	4	2	6	3	0	1
1	7	6	0	3	8	2	4	5
2	0	3	6	5	4	1	8	7
3	4	8	7	1	0	5	6	2
6	5	1	2	8	7	0	3	4
8	2	4	5	6	3	7	1	0
4	3	0	1	7	2	8	5	6
5	6	7	8	0	1	4	2	3

0	1	2	3	4	5	6	7	8
6	5	8	0	7	3	1	4	2
8	4	7	6	2	1	3	5	0
4	7	5	8	1	2	0	6	3
7	3	0	2	6	8	4	1	5
5	6	4	1	3	0	2	8	7
2	8	6	7	0	4	5	3	1
1	0	3	5	8	6	7	2	4
3	2	1	4	5	7	8	0	6

0	1	2	3	4	5	6	7	8
3	0	7	6	8	1	5	2	4
6	5	4	8	7	2	0	1	3
1	6	8	5	0	3	7	4	2
2	7	1	0	5	4	8	3	6
7	2	3	4	6	8	1	0	5
5	3	0	1	2	6	4	8	7
8	4	5	2	1	7	3	6	0
4	8	6	7	3	0	2	5	1

0	1	2	3	4	5	6	7	8
2	6	4	7	1	8	0	3	5
3	8	0	5	6	7	4	2	1
5	4	7	2	3	0	8	1	6
8	5	3	1	7	6	2	4	0
1	0	6	8	2	4	7	5	3
4	7	8	6	5	1	3	0	2
6	2	1	4	0	3	5	8	7
7	3	5	0	8	2	1	6	4

0	1	2	3	4	5	6	7	8
1	4	0	8	6	7	2	5	3
2	6	5	7	8	3	1	0	4
8	3	1	4	7	6	5	2	0
4	2	6	5	0	1	3	8	7
3	8	7	0	5	2	4	6	1
7	5	3	2	1	0	8	4	6
5	7	8	6	3	4	0	1	2
6	0	4	1	2	8	7	3	5

Figure 1: The unique stepwise transitive 6-MOLS of order 9.

$1, 2, \ldots, n-1$ one step forward, cyclically. Then π maps the square L_i to the square L_{i+1}, for $i \neq n-1$, and maps L_{n-1} to L_1. So, π is an isotopism of the whole set of MOLS and M is transitive.

When restricting to the first s rows, the argument works unchanged, so the set M is also stepwise transitive. $\square$

A full characterization of the family of projective planes which correspond to stepwise transitive sets of MOLS would of course be interesting, but even simpler questions are left open.

Question 10. For large n, what proportion of the projective planes of order n correspond to co-isotopic or stepwise transitive sets of MOLS?

Here it seems likely that asymptotically the proportion is 0. It would be of interest to use the existing catalogues of finite projective planes, the currently most extensive being that of Moorhouse [31] which now contains several hundred thousand examples, to check how common these properties are among the known non-Galois examples.

6.3 Autotopism group sizes of *t*-MOLR

We have computed the order of the autotopism group for all sets of MOLR discussed so far in the paper. Detailed statistics of these orders are given in Appendices A to F. We will here discuss some of the symmetry properties of sets of MOLR in general, and some additional observations based on our data.

First let us note that the case of $2 \times n$ sets of MOLR is somewhat special. If we follow the construction for a partial net using a $2 \times n$ t-MOLR A, each of the lines which do not correspond to a row has two points, and no such line connects two vertices in the same row. This means that if we delete the two lines with n points, we have a bipartite graph $g(A)$, where the two rows give us the bipartition. Additionally, the edges coming from each rectangle add a perfect matching, as do the edges coming from the columns, so we have a $(t+1)$-regular bipartite graph with a natural edge colouring given by these matchings. The autotopism group of A now corresponds to automorphisms of this edge-coloured graph which map the matching given by the column-lines to itself. If we assume that A is normalized, we can also invert this construction and reconstruct the t-MOLR A. Now, for $t = n-1$ this defines a proper edge colouring of $K_{n,n}$, i.e., a Latin square, so here we obtain a mapping from $2 \times n$ $(n-1)$-MOLR to a Latin square $L(A)$, and an autotopism of A defines an autotopism of $L(A)$ which fixes one symbol, again corresponding to the column lines in the partial net.

Given a regularity property we may also look at how it interplays with restrictions of a set of MOLR. First let us note that any subset of rectangles from a co-isotopic t-MOLR is co-isotopic, so this case is trivial, and the same is true for a stepwise co-isotopic t-MOLR. For transitivity, the group structure comes into play. Given a transitive t-MOLR A with autotopism group G we get a subgroup G' which describes the action of G on the set of rectangles. If G' has an element g of order r we will obtain a transitive r-MOLR from A by taking the orbit of a single rectangle from A under g. Whenever we have a transitive t-MOLR with autotopism group of order t this implies the existence of p-MOLR for the same size $k \times n$ for every prime factor p of t. For general parameters we observe that when t or k is small, transitive t-MOLR with autotopism group of order exactly t are common.

Following this, we say that a transitive t-MOLR A is G-complete if there does not exist a $t' > t$ and a t'-MOLR B, with $A \subsetneq B$, such that A is the orbit of a rectangle in B under an element $g \in \text{Aut}(B)$, and otherwise we say that A is G-incomplete. As noted, there are many examples of t-MOLR with autotopism groups of size t, and hence we will also have many incomplete r-MOLR of the same size and r a divisor of t.

Observation 11. *In our data we found the following:*

1. *Among the stepwise transitive sets of MOLR for* $n = 8$ *there is one* 3*-MOLS with autotopism group of order* 48*, which is G-complete. We display that example in Figure* 2.

2. *For* $n = 9$ *none of the stepwise transitive* 4*-MOLR are G-complete. The* 4*-MOLR with autotopism groups of orders* 5184 *and* 2592 *both correspond to the* 8*-MOLS with autotopism group of order* 10 368*. The two* 4*-MOLR with autotopism group of order* 64 *correspond to the two* 8*-MOLR with autotopism group of order* 384.

3. *For* $n = 9$ *the* 8*-MOLS with autotopism group of order* 31 104 *does not correspond to any G-incomplete stepwise transitive* 4*-MOLS.*

0	1	2	3	4	5	6	7
7	6	5	4	3	2	1	0
6	7	4	5	2	3	0	1
5	4	7	6	1	0	3	2
4	5	6	7	0	1	2	3
3	2	1	0	7	6	5	4
2	0	3	1	6	4	7	5
1	3	0	2	5	7	4	6

0	1	2	3	4	5	6	7
6	7	4	5	2	3	0	1
5	4	7	6	1	0	3	2
2	3	0	1	6	7	4	5
1	0	3	2	5	4	7	6
7	6	5	4	3	2	1	0
4	2	1	7	0	6	5	3
3	5	6	0	7	1	2	4

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
3	2	1	0	7	6	5	4
7	6	5	4	3	2	1	0
5	4	7	6	1	0	3	2
4	5	6	7	0	1	2	3
6	3	0	5	2	7	4	1
2	7	4	1	6	3	0	5

Figure 2: The G-complete 3-MOLS with $n = 8$

4. *For $n = 9$ the stepwise transitive set of 6 MOLS is G-complete.*

For stepwise transitive sets of MOLR, restrictions become far less well-behaved. Given a transitive t-MOLR A and an autotopism g which has order r on the set of rectangles, we know that we will obtain a transitive r-MOLR A' from A. However, assuming that A is stepwise transitive does not necessarily lead to stepwise transitivity for A'. In order for this to happen it must also be the case that each of the stepwise transitive sets of MOLR which are used to construct A have autotopisms with the same orbit as g, and this is not always the case. We see one such example at $n = 9$, where a stepwise transitive 6-MOLS exists, but no stepwise transitive triple.

7 Concluding remarks

In this paper we have focused on enumeration of t-MOLR up to $n = 9$, coming tantalisingly close to the, in this setting, special value $n = 10$. As we have mentioned, a significant theoretical and computational effort led to the result that there is no set of 9 mutually orthogonal Latin squares of order 10, and hence no projective plane of order 10. However, an even more basic question remains:

Question 12. Is there a triple of mutually orthogonal Latin squares of order 10?

A large number of pairwise orthogonal Latin squares of order 10 are known, and as mentioned above, the self-orthogonal Latin squares of order 10 have been completely enumerated. An exhaustive search by McKay, Meynert, and Myrvold [27] proved that no square of order 10 with non-trivial autoparatopism group is part of an orthogonal triple. The total number of Latin squares of order 10 with trivial autoparatopism group was however too large for a complete search for orthogonal triples.

Note that these results do not immediately exclude the existence of transitive, or even stepwise transitive, triples of MOLS or MOLR of order 10, since the autotopism group of a single square or rectangle in such a triple can be trivial. Our data provides several such examples for rectangles. However, as an anonymous reviewer pointed out, from a transitive triple of MOLS one can, by translation to an orthogonal array, construct a triple of MOLS where at least one square has non-trivial autoparatopisms. Thus the result of

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	1	0	2
8	9	3	5	7	1	0	2	6	4
7	6	4	0	3	2	9	8	5	1
6	7	5	4	1	8	2	0	9	3
5	4	6	1	9	0	8	3	2	7
4	5	0	7	2	6	1	9	3	8
3	0	1	2	8	9	4	6	7	5
2	3	8	9	6	7	5	4	1	0

0	1	2	3	4	5	6	7	8	9
8	9	6	5	0	7	2	4	3	1
4	7	0	6	2	3	1	8	9	5
5	3	8	9	1	4	0	2	7	6
2	4	3	1	8	0	9	5	6	7
1	0	4	2	3	6	7	9	5	8
9	2	7	0	6	8	5	1	4	3
6	8	9	7	5	2	3	0	1	4
3	5	1	4	7	9	8	6	0	2

0	1	2	3	4	5	6	7	8	9
4	5	3	2	6	9	1	8	7	0
9	6	7	0	8	4	2	5	1	3
1	9	0	4	5	3	8	6	2	7
3	0	8	6	2	1	7	9	5	4
7	2	5	9	1	8	3	0	4	6
8	4	1	5	9	7	0	3	6	2
2	3	6	8	7	0	5	4	9	1
6	8	4	7	0	2	9	1	3	5

Figure 3: A 3-MOLR of size 9×10, whose restriction to the first 8 rows is stepwise transitive.

McKay, Meynert, and Myrvold [27] does in fact rule out the existence of transitive triples of MOLS of order 10 as well.

There are other restricted versions of Question 12 which remain.

Question 13. Is there a co-isotopic 3-MOLS of order 10?

Question 14. Is there a stepwise co-isotopic 3-MOLS of order 10?

A negative answer to Question 13 would lead to another extension of McKay, Meynert, and Myrvold's result from [27]. For Question 14, a more specialised version of the type of search we have performed might be able to handle the case $t = 3$ for $n = 10$ as well.

In [14] Egan and Wanless tried to find an example of 3 Latin squares of order 10 that come as close to being mutually orthogonal as possible. They presented an example of 3 Latin squares such that the first is orthogonal to the other two, and the final two produce 91 different symbol pairs when superimposed. There have also been earlier examples of MOLR and pairwise almost orthogonal Latin squares for $n = 10$. In [16], Franklin constructed examples of triples of pairwise orthogonal 9×10 rectangles, in order to be used in the construction of designs, and in [40] Wanless constructed a set of 4 such rectangles.

Using our program we performed a partial search for stepwise transitive MOLR with $n = 10$. We found several examples of stepwise transitive triples of 8×10 MOLR, and some of these could be extended to triples of 9×10 MOLR, but not while preserving transitivity. In Figure 3 we give one such example. This example can be uniquely extended to 3 Latin squares, such that all positions which break orthogonality lie in the last row. Unfortunately, none of the examples we found could be extended to a triple of MOLS.

Acknowledgments

The authors would like to thank Ian Wanless for providing us with useful references and data on projective planes. We would also like to thank one anonymous reviewer for pointing out the connection to orthogonal arrays mentioned in the last section, and the other diligent reviewers for many helpful comments.

The computational work was performed on resources provided by the Swedish National Infrastructure for Computing (SNIC) at High Performance Computing Center North (HPC2N). This work was supported by the Swedish strategic research programme eSSENCE. This work was supported by The Swedish Research Council grant 2014-4897.

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A Sizes of autotopism groups of MOLR for $n = 4$

2×4	Σ	8	16
non-ISO	3	1	2
co-iso.	2	0	2
transitive	2	0	2
st. co-iso.	2	0	2
st. trans.	2	0	2

3×4	Σ	8	24
non-ISO	2	1	1
co-iso.	2	1	1
transitive	2	1	1
st. co-iso.	1	0	1
st. trans.	1	0	1

4×4	Σ	96
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

Table 16: 2-MOLR for $n = 4$.

2×4	Σ	16	48
non-ISO	2	1	1
co-iso.	1	0	1
transitive	1	0	1
st. co-iso.	1	0	1
st. trans.	1	0	1

3×4	Σ	72
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

4×4	Σ	288
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

Table 17: 3-MOLR for $n = 4$.

B Sizes of autotopism groups of MOLR for $n = 5$

2×5	Σ	2	4	10	20
non-ISO	5	1	2	1	1
co-iso.	4	0	2	1	1
transitive	3	0	2	0	1
st. co-iso.	4	0	2	1	1
st. trans.	3	0	2	0	1

3×5	Σ	1	2	4	10	20
non-ISO	14	1	7	3	2	1
co-iso.	11	1	4	3	2	1
transitive	9	0	4	3	1	1
st. co-iso.	7	0	1	3	2	1
st. trans.	6	0	1	3	1	1

4×5	Σ	20	40
non-ISO	2	1	1
co-iso.	2	1	1
transitive	1	0	1
st. co-iso.	2	1	1
st. trans.	1	0	1

5×5	Σ	100	200
non-ISO	2	1	1
co-iso.	2	1	1
transitive	1	0	1
st. co-iso.	2	1	1
st. trans.	1	0	1

Table 18: 2-MOLR for $n = 5$.

2×5	Σ	2	6	10
non-ISO	4	1	2	1
co-iso.	3	0	2	1
transitive	2	0	2	0
st. co-iso.	3	0	2	1
st. trans.	2	0	2	0

3×5	Σ	10
non-ISO	1	1
co-iso.	1	1
st. co-iso.	1	1

4×5	Σ	20
non-ISO	1	1
co-iso.	1	1
st. co-iso.	1	1

5×5	Σ	100
non-ISO	1	1
co-iso.	1	1

Table 19: 3-MOLR for $n = 5$.

2×5	Σ	6	24	40
non-ISO	3	1	1	1
co-iso.	2	0	1	1
transitive	2	0	1	1
st. co-iso.	2	0	1	1
st. trans.	2	0	1	1

3×5	Σ	40
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

4×5	Σ	80
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

5×5	Σ	400
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

Table 20: 4-MOLR for $n = 5$.

C Sizes of autotopism groups of MOLR for $n = 6$

2×6	Σ	1	2	4	6	8	12	24	72
non-ISO	28	1	7	7	1	3	6	2	1
co-iso.	12	0	3	2	0	3	1	2	1
transitive	11	0	2	2	0	3	1	2	1
st. co-iso.	12	0	3	2	0	3	1	2	1
st. trans.	11	0	2	2	0	3	1	2	1

3×6	Σ	1	2	3	4	6	8	12	24	216
non-ISO	1526	1155	252	18	59	19	8	11	3	1
co-iso.	280	89	117	1	40	13	8	8	3	1
transitive	170	0	100	0	40	10	8	8	3	1
st. co-iso.	158	43	63	1	26	9	8	4	3	1
st. trans.	103	0	52	0	26	9	8	4	3	1

4×6	Σ	1	2	3	4	6	8	12	16	24	48
non-ISO	2036	1425	425	30	78	35	16	21	1	3	2
co-iso.	229	36	112	5	31	11	16	12	1	3	2
transitive	160	0	92	0	29	9	15	9	1	3	2
st. co-iso.	66	7	27	1	11	3	8	6	0	2	1
st. trans.	50	0	22	0	10	3	6	6	0	2	1

5×6	Σ	1	2	3	4	6	8	12	16	24	48
non-ISO	85	5	25	2	26	4	11	7	2	1	2
co-iso.	43	0	10	0	13	3	6	7	2	0	2
transitive	36	0	6	0	11	2	6	7	2	0	2
st. co-iso.	13	0	2	0	0	1	4	2	2	0	2
st. trans.	12	0	1	0	0	1	4	2	2	0	2

Table 21: 2-MOLR for $n = 6$.

2×6	Σ	1	2	4	6	8	12	24	36	72
non-ISO	103	24	25	26	2	7	13	4	1	1
co-iso.	16	3	4	3	0	0	3	1	1	1
transitive	6	0	0	0	0	0	3	1	1	1
st. co-iso.	16	3	4	3	0	0	3	1	1	1
st. trans.	6	0	0	0	0	0	3	1	1	1

3×6	Σ	1	2	3	4	6	12	18	36
non-ISO	2572	1980	442	54	27	55	6	4	4
co-iso.	115	41	32	11	2	18	3	4	4
transitive	29	0	0	6	0	13	2	4	4
st. co-iso.	32	11	11	2	2	2	1	1	2
st. trans.	4	0	0	0	0	0	1	1	2

4×6	Σ	1	2	3	4	6	8	9	12	18	24	36
non-ISO	513	93	194	96	37	64	3	2	11	9	1	3
co-iso.	62	1	8	11	1	23	0	2	3	9	1	3
transitive	39	0	0	3	0	18	0	2	3	9	1	3
st. co-iso.	4	0	0	0	0	3	0	0	0	0	0	1
st. trans.	1	0	0	0	0	0	0	0	0	0	0	1

5×6	Σ	3	6	9	18
non-ISO	7	2	2	1	2
co-iso.	4	0	1	1	2
transitive	3	0	0	1	2

Table 22: 3-MOLR for $n = 6$.

2×6	Σ	1	2	3	4	8	12	16	18	24	36	48
non-ISO	92	14	18	1	28	8	10	2	1	4	4	2
co-iso.	9	0	1	0	2	2	0	2	0	0	0	2
transitive	8	0	0	0	2	2	0	2	0	0	0	2
st. co-iso.	9	0	1	0	2	2	0	2	0	0	0	2
st. trans.	8	0	0	0	2	2	0	2	0	0	0	2

3×6	Σ	1	2	3	4	6	8	12	16	18	24	36	48	144
non-ISO	118	16	38	5	22	10	6	9	2	4	1	3	1	1
co-iso.	19	0	0	0	5	1	6	2	2	0	1	0	1	1
transitive	17	0	0	0	5	0	6	1	2	0	1	0	1	1
st. co-iso.	15	0	0	0	3	0	6	1	2	0	1	0	1	1
st. trans.	15	0	0	0	3	0	6	1	2	0	1	0	1	1

4×6	Σ	3	6	9	12	18	24	72
non-ISO	12	2	2	2	1	3	1	1
co-iso.	4	0	0	1	1	0	1	1
transitive	3	0	0	0	1	0	1	1

5×6	Σ	6	9	18	24	36	72
non-ISO	8	1	2	1	1	1	2
co-iso.	4	0	0	0	1	1	2
transitive	4	0	0	0	1	1	2

Table 23: 4-MOLR for $n = 6$.

2×6	Σ	1	2	4	6	8	12	16	20	24	36	48	72	240
non-ISO	33	1	5	6	1	7	1	2	1	3	2	1	2	1
co-iso.	2	0	0	0	0	0	0	0	1	0	0	0	0	1
transitive	2	0	0	0	0	0	0	0	1	0	0	0	0	1
st. co-iso.	2	0	0	0	0	0	0	0	1	0	0	0	0	1
st. trans.	2	0	0	0	0	0	0	0	1	0	0	0	0	1

Table 24: 5-MOLR for $n = 6$.

D Sizes of autotopism groups of MOLR for $n = 7$

2×7	Σ	1	2	4	14	24	28	48
non-ISO	100	21	55	18	2	1	1	2
co-iso.	42	3	16	18	2	0	1	2
transitive	29	0	8	18	0	0	1	2
st. co-iso.	42	3	16	18	2	0	1	2
st. trans.	29	0	8	18	0	0	1	2

3×7	Σ	1	2	3	4	6	7	14	21	24	28	42	72
non-ISO	514 162	508 132	5880	48	65	23	4	4	2	1	1	1	1
co-iso.	14 464	10 835	3524	6	65	23	1	4	2	1	1	1	1
transitive	3549	0	3455	0	65	23	0	2	0	1	1	1	1
st. co-iso.	7423	5017	2302	6	65	23	1	4	2	0	1	1	1
st. trans.	2175	0	2082	0	65	23	0	2	0	0	1	1	1

4×7	Σ	1	2	3	4	6	7	8	14	21	28	42
non-ISO	49 415 812	49 363 791	51 060	428	444	54	11	14	6	2	1	1
co-iso.	65 156	37 639	27 054	16	365	54	4	14	6	2	1	1
transitive	27 299	0	26 867	0	361	54	0	14	1	0	1	1
st. co-iso.	14 960	4249	10 418	16	205	54	0	11	3	2	1	1
st. trans.	10 029	0	9775	0	187	54	0	11	0	0	1	1

5×7	Σ	1	2	4	5	7	10	14	28
non-ISO	21 290 125	21 243 988	45 872	227	10	6	12	9	1
co-iso.	22 432	3508	18 672	227	1	6	8	9	1
transitive	18 836	0	18 599	227	0	0	5	4	1
st. co-iso.	4163	99	3935	121	0	0	5	2	1
st. trans.	3923	0	3799	118	0	0	5	0	1

6×7	Σ	1	2	3	4	5	6	7	10	12	42	84
non-ISO	11 582	10 912	492	20	24	102	11	1	12	4	3	1
co-iso.	409	2	345	0	24	8	9	1	12	4	3	1
transitive	392	0	342	0	24	0	8	0	12	4	1	1
st. co-iso.	91	0	61	0	4	2	7	0	12	2	2	1
st. trans.	84	0	60	0	3	0	7	0	12	1	0	1

7×7	Σ	2	3	6	12	21	42	294	588
non-ISO	20	5	5	3	1	1	2	2	1
co-iso.	9	1	0	2	1	0	2	2	1
transitive	6	0	0	2	1	0	2	0	1
st. co-iso.	6	0	0	1	1	0	1	2	1
st. trans.	4	0	0	1	1	0	1	0	1

Table 25: 2-MOLR for $n = 7$.

2×7	Σ	1	2	3	4	6	12	14	42
non-ISO	2858	2300	512	3	28	9	2	3	1
co-iso.	318	194	100	3	6	9	2	3	1
transitive	15	0	0	3	0	9	2	0	1
st. co-iso.	318	194	100	3	6	9	2	3	1
st. trans.	15	0	0	3	0	9	2	0	1

3×7	Σ	1	2	3	6	7	9	14	18	21	42	63
non-ISO	65 883 453	65 822 447	60 195	635	143	17	3	3	4	4	1	1
co-iso.	49 370	48 126	566	542	116	4	3	3	4	4	1	1
transitive	647	0	0	524	113	0	3	0	4	1	1	1
st. co-iso.	13 975	13 397	305	189	64	4	3	3	4	4	1	1
st. trans.	185	0	0	125	51	0	3	0	4	0	1	1

4×7	Σ	1	2	3	4	6	7	9	12	14	18	21	42	63
non-ISO	323 112 477	323 002 195	107 997	1975	120	116	43	10	3	8	2	6	1	1
co-iso.	2985	1232	147	1474	1	91	9	10	3	8	2	6	1	1
transitive	1578	0	0	1468	0	90	0	10	3	0	2	3	1	1
st. co-iso.	283	59	27	136	1	34	2	10	3	4	2	3	1	1
st. trans.	160	0	0	113	0	30	0	10	3	0	2	0	1	1

5×7	Σ	1	2	3	4	6	7	14	21	42
non-ISO	55 545	52 981	2500	32	5	2	15	8	1	1
co-iso.	111	31	21	32	0	2	15	8	1	1
transitive	36	0	0	32	0	2	0	0	1	1
st. co-iso.	8	0	0	2	0	2	0	3	0	1
st. trans.	5	0	0	2	0	2	0	0	0	1

6×7	Σ	1	2	3	4	6	12	42	126
non-ISO	16	1	4	1	1	3	2	3	1
co-iso.	11	0	1	0	1	3	2	3	1
transitive	6	0	0	0	0	3	2	0	1
st. co-iso.	4	0	0	0	0	0	0	3	1
st. trans.	1	0	0	0	0	0	0	0	1

7×7	Σ	294	882
non-ISO	4	3	1
co-iso.	4	3	1
transitive	1	0	1
st. co-iso.	4	3	1
st. trans.	1	0	1

Table 26: 3-MOLR for $n = 7$.

2×7	Σ	1	2	4	8	14	16	28
non-ISO	17 609	15 981	1545	64	12	2	4	1
co-iso.	691	489	164	19	12	2	4	1
transitive	21	0	0	5	12	0	4	0
st. co-iso.	691	489	164	19	12	2	4	1
st. trans.	21	0	0	5	12	0	4	0

3×7	Σ	1	2	3	4	6	7	12	14	21	28	42
non-ISO	35 469 948	35 420 362	48 685	626	226	21	12	1	7	6	1	1
co-iso.	1622	949	539	9	110	4	1	1	4	3	1	1
transitive	110	0	0	0	109	0	0	1	0	0	0	0
st. co-iso.	585	251	243	9	67	4	1	1	4	3	1	1
st. trans.	48	0	0	0	47	0	0	1	0	0	0	0

4×7	Σ	1	2	3	4	6	7	8	14	16	21	28	42	56	84
non-ISO	68 659	67 073	1354	74	110	2	20	12	5	1	3	2	1	1	1
co-iso.	84	3	5	0	52	0	1	12	3	1	2	2	1	1	1
transitive	67	0	0	0	51	0	0	12	0	1	0	1	0	1	1
st. co-iso.	12	0	1	0	2	0	0	2	2	1	2	1	1	0	0
st. trans.	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0

5×7	Σ	1	2	4	7	8	14	28	56
non-ISO	204	96	41	32	6	18	6	4	1
co-iso.	67	1	0	31	6	18	6	4	1
transitive	53	0	0	31	0	18	0	3	1
st. co-iso.	3	0	0	0	0	0	2	1	0

6×7	Σ	4	6	24	42	84
non-ISO	7	1	1	2	2	1
co-iso.	7	1	1	2	2	1
transitive	3	1	0	2	0	0
st. co-iso.	3	0	0	0	2	1

7×7	Σ	294	588
non-ISO	3	2	1
co-iso.	3	2	1
st. co-iso.	3	2	1

Table 27: 4-MOLR for $n = 7$.

2×7	Σ	1	2	4	5	8	10	14
non-ISO	10 626	9590	957	48	4	24	2	1
co-iso.	176	122	37	4	4	6	2	1
transitive	6	0	0	0	4	0	2	0
st. co-iso.	176	122	37	4	4	6	2	1
st. trans.	6	0	0	0	4	0	2	0

3×7	Σ	1	2	3	5	6	7	10	14	21
non-ISO	22 982	21 848	1039	39	30	7	1	12	1	5
co-iso.	49	3	1	0	30	0	0	12	1	2
transitive	42	0	0	0	30	0	0	12	0	0
st. co-iso.	48	2	1	0	30	0	0	12	1	2
st. trans.	42	0	0	0	30	0	0	12	0	0

4×7	Σ	1	2	4	8	14	21
non-ISO	19	2	4	8	3	1	1
co-iso.	2	0	0	0	0	1	1
st. co-iso.	2	0	0	0	0	1	1

5×7	Σ	4	14	20
non-ISO	5	2	1	2
co-iso.	4	1	1	2
transitive	2	0	0	2
st. co-iso.	1	0	1	0

6×7	Σ	20	24	42	120
non-ISO	5	1	1	1	2
co-iso.	5	1	1	1	2
transitive	3	1	0	0	2
st. co-iso.	1	0	0	1	0

7×7	Σ	294
non-ISO	1	1
co-iso.	1	1
st. co-iso.	1	1

Table 28: 5-MOLR for $n = 7$.

2×7	Σ	1	2	3	4	5	6	8	10	12	16	48	84
non-ISO	1895	1505	328	2	29	4	5	14	2	1	2	2	1
co-iso.	26	7	7	1	1	1	2	0	1	1	2	2	1
transitive	5	0	0	0	0	0	1	0	0	1	0	2	1
st. co-iso.	26	7	7	1	1	1	2	0	1	1	2	2	1
st. trans.	5	0	0	0	0	0	1	0	0	1	0	2	1

3×7	Σ	1	2	3	6	12	21	84	126
non-ISO	23	8	4	1	5	1	1	1	2
co-iso.	7	0	0	0	3	1	0	1	2
transitive	7	0	0	0	3	1	0	1	2
st. co-iso.	6	0	0	0	2	1	0	1	2
st. trans.	4	0	0	0	0	1	0	1	2

4×7	Σ	84	126
non-ISO	2	1	1
co-iso.	2	1	1
transitive	2	1	1
st. co-iso.	2	1	1
st. trans.	2	1	1

5×7	Σ	84
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

6×7	Σ	252
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

7×7	Σ	1764
non-ISO	1	1
co-iso.	1	1
transitive	1	1
st. co-iso.	1	1
st. trans.	1	1

Table 29: 6-MOLR for $n = 7$.

E Sizes of autotopism groups of MOLR for $n = 8$

2×8	Σ	1	2	4	8	12	16	30	32	60	64	128
st. co-iso.	186	52	65	43	10	2	2	1	6	1	2	2
st. trans.	99	0	35	39	10	2	2	0	6	1	2	2

3×8	Σ	1	2	3	4	6	8	12	16	30	32	48	60	64	192
st. co-iso.	446 443	394 387	50 556	11	1311	28	101	22	12	2	8	1	1	1	2
st. trans.	45 429	0	43 990	0	1265	27	99	22	12	1	8	1	1	1	2

4×8	Σ	1	2	3	4	6	8	12	16	24	32	48	64	192	768
st. co-iso.	4 432 284	3 196 674	1 222 057	168	11 703	343	1002	50	194	11	54	11	14	1	2
st. trans.	1 097 655	0	1 086 038	0	10 123	317	857	50	181	9	52	11	14	1	2

5×8	Σ	1	2	4	8	16	32
st. co-iso.	3 826 527	884 803	2 929 324	11 629	702	65	4
st. trans.	2 569 679	0	2 558 291	10 693	629	62	4

6×8	Σ	1	2	3	4	6	8	12	16	32	96
st. co-iso.	242 732	7954	227 619	2	6289	46	611	23	181	6	1
st. trans.	206 612	0	200 153	0	5745	35	503	22	147	6	1

7×8	Σ	1	2	3	4	6	8	12	16	24	32	48	56	96
st. co-iso.	484	28	175	5	109	72	48	29	9	2	2	2	1	2
st. trans.	305	0	92	0	82	64	24	29	7	2	2	1	0	2

8×8	Σ	4	8	16	32	64	448
st. co-iso.	70	9	19	22	8	11	1
st. trans.	13	0	2	4	4	3	0

Table 30: 2-MOLR for $n = 8$.

2×8	Σ	1	2	3	4	6	8	12	16	24	32	48	96	384
co-iso.	11 565	10 583	803	24	83	22	20	14	6	2	4	2	1	1
transitive	66	0	0	24	0	22	0	14	0	2	0	2	1	1
st. co-iso.	11 565	10 583	803	24	83	22	20	14	6	2	4	2	1	1
st. trans.	66	0	0	24	0	22	0	14	0	2	0	2	1	1

3×8	Σ	1	2	3	4	6	8	12	16	18	24	36	48	144	576
st. co-iso.	9 144 025	9 121 524	13 878	7463	479	564	27	64	8	4	3	3	5	1	2
st. trans.	7627	0	0	7067	0	485	0	57	0	4	3	3	5	1	2

4×8	Σ	1	2	3	4	6	8	9	12	16	18	24	32	36	48	64	72	96	144	192	2304
st. co-iso.	178 502	91 562	40 127	41 876	2464	1665	397	38	177	94	26	39	8	3	10	5	2	3	1	3	2
st. trans.	41 505	0	0	39 650	0	1566	0	38	168	0	26	36	0	3	7	0	2	3	1	3	2

5×8	Σ	1	2	3	4	6	8	12	16	24	48
st. co-iso.	628	34	142	311	59	34	26	5	5	10	2
st. trans.	75	0	0	38	0	25	0	3	0	7	2

6×8	Σ	2	3	4	6	8	12	16	24	48
st. co-iso.	111	18	8	42	8	8	12	8	2	5
st. trans.	32	0	8	0	8	0	12	0	2	2

7×8	Σ	8	9	24	56	168
st. co-iso.	10	2	3	2	1	2
st. trans.	6	0	3	1	0	2

8×8	Σ	48	64	192	448	1344
st. co-iso.	7	1	2	1	1	2
st. trans.	3	1	0	0	0	2

Table 31: 3-MOLR for $n = 8$.

2×8	Σ	1	2	3	4	6	8	12	16	24	32	48	64	96	128	384
st. co-iso.	216 950	212 259	4241	20	313	8	52	2	31	1	9	6	3	1	3	1
st. trans.	152	0	0	0	98	0	17	0	20	1	4	5	3	0	3	1

3×8	Σ	1	2	3	4	6	8	12	16	24	32	48
st. co-iso.	1 648 723	1 596 362	45 732	46	6203	20	279	5	58	6	8	4
st. trans.	4284	0	0	0	4028	0	196	2	46	2	8	2

4×8	Σ	1	2	3	4	6	8	12	16	24	32	48	64	96
st. co-iso.	3547	296	1420	14	1273	5	340	5	106	12	45	1	26	4
st. trans.	712	0	0	0	510	0	124	4	32	6	15	0	21	0

5×8	Σ	2	4	8	16	24	32	48	64
st. co-iso.	58	2	10	25	9	4	4	2	2
st. trans.	20	0	3	5	3	1	4	2	2

6×8	Σ	16	24	48
st. co-iso.	4	1	1	2

7×8	Σ	56	168
st. co-iso.	3	1	2

8×8	Σ	448	1344
st. co-iso.	3	1	2

Table 32: 4-MOLR for $n = 8$.

2×8	Σ	1	2	4	5	8	10	16	20	32	64	128
st. co-iso.	509 622	505 439	3896	185	9	52	8	21	2	5	3	2
st. trans.	19	0	0	0	9	0	8	0	2	0	0	0

3×8	Σ	1	2	3	4	6	8	12	16	24	48
st. co-iso.	2652	2112	419	2	75	13	15	2	9	2	3

4×8	Σ	1	2	3	4	6	8	12	16	24	32	48	64	96	192
st. co-iso.	267	129	30	12	30	1	23	3	17	2	11	1	4	2	2

5×8	Σ	8	16
st. co-iso.	2	1	1

6×8	Σ	16
st. co-iso.	1	1

7×8	Σ	56
st. co-iso.	1	1

8×8	Σ	448
st. co-iso.	1	1

Table 33: 5-MOLR for $n = 8$.

2×8	Σ	1	2	3	4	5	6	8	10	12	16	24	32	48	64	96	128	384
st. co-iso.	91 013	89 017	1597	59	138	4	71	35	8	37	22	3	7	7	2	2	2	2
st. trans.	109	0	0	0	0	0	60	0	0	36	0	3	0	6	0	2	0	2

3×8	Σ	1	2	3	4	6	8	12	18	24	36	48	96	288
st. co-iso.	975	3	10	18	7	797	3	120	2	5	5	1	3	1
st. trans.	908	0	0	0	0	775	0	119	0	4	5	1	3	1

4×8	Σ	3	6	9	12	18	24	32	36	48	96	144
st. co-iso.	155	2	122	2	14	8	1	1	1	1	2	1
st. trans.	146	0	122	0	13	8	0	0	1	1	0	1

5×8	Σ	24
st. co-iso.	1	1

6×8	Σ	48
st. co-iso.	1	1

7×8	Σ	168
st. co-iso.	1	1

8×8	Σ	1344
st. co-iso.	1	1

Table 34: 6-MOLR for $n = 8$.

2×8	Σ	1	2	3	4	5	6	7	8	10	32	42	48	84	128	192	2688
st. co-iso.	4538	4439	67	9	3	2	3	1	2	4	1	2	1	1	1	1	1
st. trans.	5	0	0	0	0	0	0	1	0	0	0	2	0	1	0	0	1

3×8	Σ	21	168
st. co-iso.	2	1	1
st. trans.	2	1	1

4×8	Σ	168	672
st. co-iso.	2	1	1
st. trans.	2	1	1

5×8	Σ	168
st. co-iso.	1	1
st. trans.	1	1

6×8	Σ	336
st. co-iso.	1	1
st. trans.	1	1

7×8	Σ	1176
st. co-iso.	1	1
st. trans.	1	1

8×8	Σ	9408
st. co-iso.	1	1
st. trans.	1	1

Table 35: 7-MOLR for $n = 8$.

F Sizes of autotopism groups of MOLR for $n = 9$

2×9	Σ	4	6	12	16	24	36	72	80	648
st. trans.	126	95	3	13	4	3	3	2	2	1

3×9	Σ	2	4	6	8	12	18	24	36	40	54	72	108	1944
st. trans.	1 418 577	1 413 987	3905	511	8	115	19	8	13	3	2	3	2	1

4×9	Σ	2	4	6	8	12	16	18	24	36	54	72	80	108	144
st. trans.	560 524 587	560 437 428	82 404	3520	958	220	15	19	12	5	2	1	1	1	1

5×9	Σ	2	4	6	8	12	16	18	24	36	54	72	108	144
st. trans.	20019499500	20019088250	400 369	9228	1223	370	6	29	9	11	2	1	1	1

6×9	Σ	2	4	6	8	12	18	24	36	54	72	108	324	648
st. trans.	67480364637	67479927127	423 557	12 685	521	577	89	32	28	3	6	9	2	1

7×9	Σ	2	4	6	8	12	18	24	36	54	108	216
st. trans.	1 577 270 689	1 577 056 397	208 362	4900	384	595	13	18	13	2	4	1

8×9	Σ	2	4	6	8	12	16	18	24	36	54	72	96	108	864
st. trans.	14 940 988	14 917 761	20 900	1879	114	268	8	22	13	13	2	1	2	4	1

9×9	Σ	2	4	6	8	12	16	18	24	36	54	72	96	108	144	162	324	972	7776
st. trans.	28 955	25 570	2802	307	110	99	3	13	11	17	2	3	2	6	1	1	5	2	1

Table 36: 2-MOLR for $n = 9$.

2×9	Σ	3	6	12	18	24	36	54	108
st. trans.	202	86	102	4	3	2	3	1	1

3×9	Σ	3	6	9	12	18	36	54	162	324
st. trans.	72 836	71 109	1618	40	6	56	1	4	1	1

4×9	Σ	3	6	9	12	18	36	54
st. trans.	1 746 912	1 742 486	4209	149	37	21	4	6

Table 37: 3-MOLR for $n = 9$.

2×9	Σ	4	8	12	16	24	144
st. trans.	1017	881	121	5	2	7	1

3×9	Σ	4	8	12	16	24	36	48	72	108	432
st. trans.	356 680	355 023	1511	112	6	18	4	1	3	1	1

4×9	Σ	4	8	12	16	24	32	36	72	144	288
st. trans.	2 640 163	2 635 762	4131	95	147	1	17	2	5	2	1

5×9	Σ	4	8	12	16	32	36	72	144	288
st. trans.	645 453	641 633	3467	30	305	11	1	3	2	1

6×9	Σ	4	8	12	16	24	36	48	72	216	432
st. trans.	1816	1662	124	15	9	1	1	1	1	1	1

7×9	Σ	4	8	16	72	144
st. trans.	31	3	12	14	1	1

8×9	Σ	8	64	288	576
st. trans.	7	2	3	1	1

9×9	Σ	64	576	2592	5184
st. trans.	5	2	1	1	1

Table 38: 4-MOLR for $n = 9$.

2×9	Σ	5	10
st. trans.	17	7	10

Table 39: 5-MOLR for $n = 9$.

2×9	Σ	6	12	24	36	54	72	108	216
st. trans.	543	422	104	7	4	1	2	2	1

3×9	Σ	6	12	18	24	36	54	72	108	324	648
st. trans.	21 620	20 528	946	69	12	48	5	3	3	5	1

4×9	Σ	6	12	18	24	36	54	72	108
st. trans.	244	157	41	24	2	8	3	3	6

5×9	Σ	6	12	18	36	54	108
st. trans.	33	10	11	1	4	3	4

6×9	Σ	6	12	24	36	72	108	216	324	648
st. trans.	16	1	1	1	2	2	2	2	3	2

7×9	Σ	12
st. trans.	1	1

8×9	Σ	48
st. trans.	1	1

9×9	Σ	432
st. trans.	1	1

Table 40: 6-MOLR for $n = 9$.

2×9	Σ	7	14
st. trans.	39	37	2

3×9	Σ	7	21
st. trans.	1532	1522	10

4×9	Σ	7	21
st. trans.	300	298	2

Table 41: 7-MOLR for $n = 9$.

2×9	Σ	8	16	32	48	56	96	864
st. trans.	54	39	9	2	1	1	1	1

3×9	Σ	8	16	48	96	144	288	432	2592
st. trans.	48	22	17	3	2	1	1	1	1

4×9	Σ	16	24	32	48	64	144	288	432	576
st. trans.	27	11	2	2	3	4	1	1	1	2

5×9	Σ	16	24	48	64	144	288	432	576
st. trans.	22	8	2	3	4	1	1	1	2

6×9	Σ	16	48	96	144	288	432	864	2592
st. trans.	16	6	3	2	1	1	1	1	1

7×9	Σ	16	48	96	288	864
st. trans.	9	2	2	3	1	1

8×9	Σ	48	384	1152	3456
st. trans.	7	2	3	1	1

9×9	Σ	384	3456	10 368	31 104
st. trans.	5	2	1	1	1

Table 42: 8-MOLR for $n = 9$.

G The stepwise transitive 8-MOLR for $n = 9$

0	1	2	3	4	5	6	7	8
8	7	6	2	1	0	3	4	5
7	6	8	0	2	1	5	3	4
6	8	7	1	0	2	4	5	3
5	4	3	6	7	8	2	1	0
4	3	5	8	6	7	0	2	1
3	5	4	7	8	6	1	0	2
2	0	1	4	5	3	7	8	6
1	2	0	5	3	4	8	6	7

0	1	2	3	4	5	6	7	8
7	6	8	0	2	1	5	3	4
5	4	3	6	7	8	2	1	0
2	0	1	4	5	3	7	8	6
3	5	4	7	8	6	1	0	2
1	2	0	5	3	4	8	6	7
8	7	6	2	1	0	3	4	5
4	3	5	8	6	7	0	2	1
6	8	7	1	0	2	4	5	3

0	1	2	3	4	5	6	7	8
6	8	7	1	0	2	4	5	3
2	0	1	4	5	3	7	8	6
3	5	4	7	8	6	1	0	2
4	3	5	8	6	7	0	2	1
7	6	8	0	2	1	5	3	4
1	2	0	5	3	4	8	6	7
8	7	6	2	1	0	3	4	5
5	4	3	6	7	8	2	1	0

0	1	2	3	4	5	6	7	8
5	4	3	6	7	8	2	1	0
3	5	4	7	8	6	1	0	2
4	3	5	8	6	7	0	2	1
8	7	6	2	1	0	3	4	5
6	8	7	1	0	2	4	5	3
7	6	8	0	2	1	5	3	4
1	2	0	5	3	4	8	6	7
2	0	1	4	5	3	7	8	6

0	1	2	3	4	5	6	7	8
4	3	5	8	6	7	0	2	1
1	2	0	5	3	4	8	6	7
7	6	8	0	2	1	5	3	4
6	8	7	1	0	2	4	5	3
3	5	4	7	8	6	1	0	2
2	0	1	4	5	3	7	8	6
5	4	3	6	7	8	2	1	0
8	7	6	2	1	0	3	4	5

0	1	2	3	4	5	6	7	8
3	5	4	7	8	6	1	0	2
8	7	6	2	1	0	3	4	5
1	2	0	5	3	4	8	6	7
7	6	8	0	2	1	5	3	4
2	0	1	4	5	3	7	8	6
5	4	3	6	7	8	2	1	0
6	8	7	1	0	2	4	5	3
4	3	5	8	6	7	0	2	1

0	1	2	3	4	5	6	7	8
2	0	1	4	5	3	7	8	6
4	3	5	8	6	7	0	2	1
8	7	6	2	1	0	3	4	5
1	2	0	5	3	4	8	6	7
5	4	3	6	7	8	2	1	0
6	8	7	1	0	2	4	5	3
7	6	8	0	2	1	5	3	4
3	5	4	7	8	6	1	0	2

0	1	2	3	4	5	6	7	8
1	2	0	5	3	4	8	6	7
6	8	7	1	0	2	4	5	3
5	4	3	6	7	8	2	1	0
2	0	1	4	5	3	7	8	6
8	7	6	2	1	0	3	4	5
4	3	5	8	6	7	0	2	1
3	5	4	7	8	6	1	0	2
7	6	8	0	2	1	5	3	4

Figure 4: The stepwise transitive 8-MOLS of size 9×9 with $|\mathrm{Aut}| = 10\,368$, corresponding to the Galois plane.

0	1	2	3	4	5	6	7	8
8	7	6	2	1	0	3	4	5
7	6	8	0	2	1	5	3	4
6	8	7	1	0	2	4	5	3
5	4	3	6	7	8	2	1	0
4	3	5	8	6	7	0	2	1
3	5	4	7	8	6	1	0	2
2	0	1	4	5	3	7	8	6
1	2	0	5	3	4	8	6	7

0	1	2	3	4	5	6	7	8
7	6	8	0	2	1	5	3	4
5	4	3	6	7	8	2	1	0
2	0	1	4	5	3	7	8	6
3	5	4	7	8	6	1	0	2
1	2	0	5	3	4	8	6	7
8	7	6	2	1	0	3	4	5
4	3	5	8	6	7	0	2	1
6	8	7	1	0	2	4	5	3

0	1	2	3	4	5	6	7	8
6	8	7	1	0	2	4	5	3
1	2	0	5	3	4	8	6	7
5	4	3	6	7	8	2	1	0
4	3	5	8	6	7	0	2	1
8	7	6	2	1	0	3	4	5
2	0	1	4	5	3	7	8	6
7	6	8	0	2	1	5	3	4
3	5	4	7	8	6	1	0	2

0	1	2	3	4	5	6	7	8
5	4	3	6	7	8	2	1	0
3	5	4	7	8	6	1	0	2
4	3	5	8	6	7	0	2	1
8	7	6	2	1	0	3	4	5
6	8	7	1	0	2	4	5	3
7	6	8	0	2	1	5	3	4
1	2	0	5	3	4	8	6	7
2	0	1	4	5	3	7	8	6

0	1	2	3	4	5	6	7	8
4	3	5	8	6	7	0	2	1
2	0	1	4	5	3	7	8	6
8	7	6	2	1	0	3	4	5
6	8	7	1	0	2	4	5	3
5	4	3	6	7	8	2	1	0
1	2	0	5	3	4	8	6	7
3	5	4	7	8	6	1	0	2
7	6	8	0	2	1	5	3	4

0	1	2	3	4	5	6	7	8
3	5	4	7	8	6	1	0	2
8	7	6	2	1	0	3	4	5
1	2	0	5	3	4	8	6	7
7	6	8	0	2	1	5	3	4
2	0	1	4	5	3	7	8	6
5	4	3	6	7	8	2	1	0
6	8	7	1	0	2	4	5	3
4	3	5	8	6	7	0	2	1

0	1	2	3	4	5	6	7	8
2	0	1	4	5	3	7	8	6
6	8	7	1	0	2	4	5	3
3	5	4	7	8	6	1	0	2
1	2	0	5	3	4	8	6	7
7	6	8	0	2	1	5	3	4
4	3	5	8	6	7	0	2	1
5	4	3	6	7	8	2	1	0
8	7	6	2	1	0	3	4	5

0	1	2	3	4	5	6	7	8
1	2	0	5	3	4	8	6	7
4	3	5	8	6	7	0	2	1
7	6	8	0	2	1	5	3	4
2	0	1	4	5	3	7	8	6
3	5	4	7	8	6	1	0	2
6	8	7	1	0	2	4	5	3
8	7	6	2	1	0	3	4	5
5	4	3	6	7	8	2	1	0

Figure 5: The stepwise transitive 8-MOLS of size 9×9 with $|\mathrm{Aut}| = 31\,104$, corresponding to the dual of the Hall plane.

0	1	2	3	4	5	6	7	8
8	7	6	5	3	4	2	1	0
7	8	1	6	2	0	3	5	4
6	0	8	7	1	2	4	3	5
5	4	3	2	6	7	8	0	1
4	3	5	1	7	8	0	6	2
3	5	4	0	8	6	1	2	7
2	6	7	8	0	1	5	4	3
1	2	0	4	5	3	7	8	6

0	1	2	3	4	5	6	7	8
7	6	8	4	5	3	1	0	2
3	5	4	0	6	7	2	8	1
5	4	3	2	7	8	0	1	6
1	8	6	7	2	0	4	3	5
6	7	0	8	1	2	5	4	3
8	2	7	6	0	1	3	5	4
4	3	5	1	8	6	7	2	0
2	0	1	5	3	4	8	6	7

0	1	2	3	4	5	6	7	8
6	5	4	0	2	1	3	8	7
4	7	6	5	8	3	1	2	0
1	2	0	8	5	6	7	4	3
8	3	7	1	0	2	5	6	4
2	0	1	7	6	4	8	3	5
5	6	8	4	3	7	2	0	1
7	8	3	2	1	0	4	5	6
3	4	5	6	7	8	0	1	2

0	1	2	3	4	5	6	7	8
5	4	7	2	1	0	8	3	6
6	3	8	1	0	2	7	4	5
8	7	4	5	6	3	2	0	1
7	6	5	4	3	8	0	1	2
3	8	6	0	2	1	4	5	7
2	0	1	8	7	4	5	6	3
1	2	0	6	5	7	3	8	4
4	5	3	7	8	6	1	2	0

0	1	2	3	4	5	6	7	8
4	8	5	1	0	2	7	6	3
1	2	0	7	5	8	4	3	6
3	6	7	0	2	1	5	8	4
2	0	1	6	8	4	3	5	7
7	5	8	4	3	6	1	2	0
6	7	3	2	1	0	8	4	5
8	4	6	5	7	3	0	1	2
5	3	4	8	6	7	2	0	1

0	1	2	3	4	5	6	7	8
3	2	1	6	8	7	0	5	4
8	6	7	4	3	1	5	0	2
2	3	5	1	0	4	8	6	7
4	5	0	8	7	6	1	2	3
1	4	3	2	5	0	7	8	6
7	8	6	5	2	3	4	1	0
5	0	4	7	6	8	2	3	1
6	7	8	0	1	2	3	4	5

0	1	2	3	4	5	6	7	8
2	3	0	8	7	6	5	4	1
5	0	3	2	1	4	8	6	7
4	5	1	6	8	7	3	2	0
3	2	4	0	5	1	7	8	6
8	6	7	5	0	3	2	1	4
1	4	5	7	6	8	0	3	2
6	7	8	4	3	2	1	0	5
7	8	6	1	2	0	4	5	3

0	1	2	3	4	5	6	7	8
1	0	3	7	6	8	4	2	5
2	4	5	8	7	6	0	1	3
7	8	6	4	3	0	1	5	2
6	7	8	5	1	3	2	4	0
5	2	4	6	8	7	3	0	1
4	3	0	1	5	2	7	8	6
3	5	1	0	2	4	8	6	7
8	6	7	2	0	1	5	3	4

Figure 6: The stepwise transitive 8-MOLS of size 9×9 with $|\mathrm{Aut}| = 384$, corresponding to the Hall plane.

0	1	2	3	4	5	6	7	8
8	7	6	5	3	2	4	1	0
7	8	4	6	5	3	2	0	1
6	4	8	1	0	7	3	5	2
5	0	1	8	2	4	7	6	3
4	2	5	7	1	0	8	3	6
3	5	0	2	8	6	1	4	7
2	6	3	0	7	1	5	8	4
1	3	7	4	6	8	0	2	5

0	1	2	3	4	5	6	7	8
7	8	5	6	2	1	3	0	4
6	5	1	7	8	4	0	3	2
2	7	3	4	5	0	8	1	6
3	2	8	1	7	6	5	4	0
8	3	0	2	6	7	4	5	1
1	4	6	5	0	8	7	2	3
4	0	7	8	1	3	2	6	5
5	6	4	0	3	2	1	8	7

0	1	2	3	4	5	6	7	8
6	0	8	7	5	4	2	3	1
1	2	6	4	3	7	8	5	0
7	3	1	8	6	2	4	0	5
8	4	7	5	1	0	3	2	6
5	7	4	6	2	8	0	1	3
2	6	3	0	7	1	5	8	4
3	5	0	2	8	6	1	4	7
4	8	5	1	0	3	7	6	2

0	1	2	3	4	5	6	7	8
5	6	0	8	7	3	1	4	2
3	4	8	2	1	0	7	6	5
1	0	7	6	3	8	5	2	4
4	8	3	0	6	7	2	5	1
2	5	6	4	0	1	3	8	7
6	7	5	1	2	4	8	3	0
8	3	4	7	5	2	0	1	6
7	2	1	5	8	6	4	0	3

0	1	2	3	4	5	6	7	8
4	5	7	1	0	6	8	2	3
2	3	0	5	7	8	1	4	6
3	6	5	0	1	4	2	8	7
6	7	4	2	8	3	0	1	5
1	0	3	8	5	2	7	6	4
7	2	8	4	6	0	3	5	1
5	8	1	6	3	7	4	0	2
8	4	6	7	2	1	5	3	0

0	1	2	3	4	5	6	7	8
3	2	1	4	8	7	0	5	6
8	7	5	0	6	2	3	1	4
5	8	4	7	2	6	1	3	0
2	3	0	6	5	1	4	8	7
7	6	8	1	3	4	5	0	2
4	0	7	8	1	3	2	6	5
1	4	6	5	0	8	7	2	3
6	5	3	2	7	0	8	4	1

0	1	2	3	4	5	6	7	8
2	3	4	0	1	8	7	6	5
4	0	3	1	2	6	5	8	7
8	5	6	2	7	1	0	4	3
1	6	5	7	0	2	8	3	4
6	4	7	5	8	3	1	2	0
5	8	1	6	3	7	4	0	2
7	2	8	4	6	0	3	5	1
3	7	0	8	5	4	2	1	6

0	1	2	3	4	5	6	7	8
1	4	3	2	6	0	5	8	7
5	6	7	8	0	1	4	2	3
4	2	0	5	8	3	7	6	1
7	5	6	4	3	8	1	0	2
3	8	1	0	7	6	2	4	5
8	3	4	7	5	2	0	1	6
6	7	5	1	2	4	8	3	0
2	0	8	6	1	7	3	5	4

Figure 7: The stepwise transitive 8-MOLS of size 9×9 with $|\mathrm{Aut}| = 384$, corresponding to the dual of the Hall plane.

0	1	2	3	4	5	6	7	8
8	7	6	2	1	0	3	4	5
7	6	8	0	2	1	5	3	4
6	8	7	1	0	2	4	5	3
5	4	3	6	7	8	2	1	0
4	3	5	8	6	7	0	2	1
3	5	4	7	8	6	1	0	2
2	0	1	4	5	3	7	8	6
1	2	0	5	3	4	8	6	7

0	1	2	3	4	5	6	7	8
7	8	5	1	2	4	0	6	3
2	0	1	6	3	8	4	5	7
3	4	6	5	7	0	8	1	2
6	3	4	8	5	2	7	0	1
5	7	8	0	1	3	2	4	6
1	2	0	4	6	7	3	8	5
8	5	7	2	0	6	1	3	4
4	6	3	7	8	1	5	2	0

0	1	2	3	4	5	6	7	8
6	3	4	8	5	2	7	0	1
1	2	0	4	6	7	3	8	5
5	7	8	0	1	3	2	4	6
7	8	5	1	2	4	0	6	3
3	4	6	5	7	0	8	1	2
2	0	1	6	3	8	4	5	7
4	6	3	7	8	1	5	2	0
8	5	7	2	0	6	1	3	4

0	1	2	3	4	5	6	7	8
5	4	3	6	7	8	2	1	0
3	5	4	7	8	6	1	0	2
4	3	5	8	6	7	0	2	1
8	7	6	2	1	0	3	4	5
6	8	7	1	0	2	4	5	3
7	6	8	0	2	1	5	3	4
1	2	0	5	3	4	8	6	7
2	0	1	4	5	3	7	8	6

0	1	2	3	4	5	6	7	8
4	5	8	0	3	6	1	2	7
6	7	3	5	1	2	8	4	0
2	0	1	7	8	4	3	6	5
3	6	7	4	0	1	5	8	2
1	2	0	6	5	8	7	3	4
8	4	5	2	7	3	0	1	6
7	3	6	8	2	0	4	5	1
5	8	4	1	6	7	2	0	3

0	1	2	3	4	5	6	7	8
3	6	7	4	0	1	5	8	2
8	4	5	2	7	3	0	1	6
1	2	0	6	5	8	7	3	4
4	5	8	0	3	6	1	2	7
2	0	1	7	8	4	3	6	5
6	7	3	5	1	2	8	4	0
5	8	4	1	6	7	2	0	3
7	3	6	8	2	0	4	5	1

0	1	2	3	4	5	6	7	8
2	0	1	5	6	7	8	3	4
5	3	7	8	0	4	2	6	1
8	6	4	2	3	1	5	0	7
1	2	0	7	8	3	4	5	6
7	5	3	4	2	6	1	8	0
4	8	6	1	5	0	7	2	3
6	4	8	0	7	2	3	1	5
3	7	5	6	1	8	0	4	2

0	1	2	3	4	5	6	7	8
1	2	0	7	8	3	4	5	6
4	8	6	1	5	0	7	2	3
7	5	3	4	2	6	1	8	0
2	0	1	5	6	7	8	3	4
8	6	4	2	3	1	5	0	7
5	3	7	8	0	4	2	6	1
3	7	5	6	1	8	0	4	2
6	4	8	0	7	2	3	1	5

Figure 8: The stepwise transitive 8-MOLS of size 9×9 with $|\mathrm{Aut}| = 3456$, corresponding to the Hall plane.