Cobiased graphs: Single-element extensions and elementary quotients of graphic matroids

Daniel Slilaty[a] Thomas Zaslavsky[b]

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Abstract

Zaslavsky (1991) introduced a graphical structure called a biased graph and used it to characterize all single-element coextensions and elementary lifts of graphic matroids. We introduce a new, dual graphical structure that we call a cobiased graph and use it to characterize single-element extensions and elementary quotients of graphic matroids.

Mathematics Subject Classifications: 05B35, 05C22

1 Introduction

An *elementary lift* of a matroid M is a matroid of the form $N\backslash e_0$ in which $N/e_0 = M$ and e_0 is neither a loop nor coloop of N. Single-element coextensions and elementary lifts of graphic matroids were characterized in terms of graphic structures by Zaslavsky [10, 11]. Aside from the work in [10, 11], single-element coextensions and elementary lifts of graphic matroids have been objects of consistent interest in matroid theory and related fields. Guenin's investigation [2] into integral polyhedra related to binary elementary lifts of graphic matroids is notable.

An *elementary quotient* of a matroid M is a matroid of the form N/e_0 in which $N\backslash e_0 = M$ and e_0 is neither a loop nor coloop of N. Elementary quotients of graphic matroids have also been of consistent interest, in particular binary elementary quotients. Guenin's result in [2] applies not only to binary elementary lifts of graphic matroids but also binary elementary quotients. Guenin, Pivotto, and Wollan [3] explored the relationships between binary elementary lifts and quotients of graphic matroids. Seymour's original proof of the decomposition theorem for regular matroids [9] uses binary single-element extensions and

[a]Department of Mathematics and Statistics, Wright State University, Dayton, OH, USA (daniel.slilaty@wright.edu).
[b]Department of Mathematics and Statistics, Binghamton University, Binghamton, NY, USA (zaslav@math.binghamton.edu).

elementary quotients of graphic matroids. In the field of error-correcting codes, Hakimi and Bredeson [4] constructed binary codes using circuit spaces of binary single-element and multiple-element extensions and quotients of graphic matroids. Jungnickel and Vanstone [5] do the same with q-ary codes.

Despite all of the interest in single-element extensions and elementary quotients of graphic matroids, there has been no general description of them. Recski characterized connected single-element extensions and elementary quotients of graphic matroids [7, 8] that are representable over a given field and made some generalizations. In this paper we will fully characterize all single-element extensions and elementary quotients of graphic matroids in terms of graphical structures.

2 Cobiased Graphs

Let G be a graph and $X \subset V(G)$ or $X \subset G$. The *coboundary* of X is denoted by $\delta(X)$ and is the set of links (i.e., non-loop edges) of G with exactly one endpoint in X. Consider a tripartition $\{X_1, X_2, X_3\}$ of the vertices of one connected component of G into nonempty subsets such that each induced subgraph $G[X_i]$ is connected and there is at least one edge of G connecting each pair of these three subgraphs. The union of the three bonds (i.e., minimal edge cuts) $B_1 = \delta(X_1)$, $B_2 = \delta(X_2)$, and $B_3 = \delta(X_3)$ is called a *tribond* (see Figure 1).

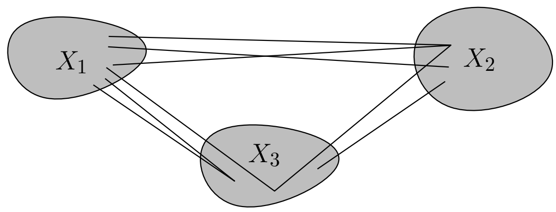

Figure 1: A tribond.

A *linear class* of bonds of G is a subset $\mathcal{L}$ of the set of all bonds in G satisfying the property that every tribond contains zero, one, or three bonds from $\mathcal{L}$; that is, a tribond cannot contain exactly two bonds from $\mathcal{L}$. We call the pair $(G, \mathcal{L})$ in which G is a graph and $\mathcal{L}$ a linear class of bonds a *cobiased graph*. Bonds in $\mathcal{L}$ are called *cobalanced* and bonds not in $\mathcal{L}$ are called *un-cobalanced*. (The name comes from the fact that cobalance of bonds is dual to balance of cycles in [10] *et seq.*) A linear class of bonds $\mathcal{L}$ is *trivial* when all bonds are cobalanced; that is, $\mathcal{L}$ is the set of all bonds of G.

3 Join and Complete Join Matroids of Cobiased Graphs

In this section we will describe two matroids associated with a cobiased graph $(G, \mathcal{L})$. These matroids are denoted by $J_0(G, \mathcal{L})$ and $J(G, \mathcal{L})$ and are called respectively the *complete join* and *join* matroids of $(G, \mathcal{L})$. The term "join" is used to echo the considerable amount of literature on T-joins of graphs that are dependent sets of binary elementary quotients of graphic matroids. The matroids $J_0(G,\mathcal{L})$ and $J(G,\mathcal{L})$ are defined in Section 3.1 in terms of their cocircuits using Crapo's Theorem [1, p. 62] on single-element extensions of matroids. Crapo's Theorem also immediately implies that $J_0(G,\mathcal{L})$ and $J(G,\mathcal{L})$ characterize respectively single-element extensions and elementary quotients of graphic matroids. From there we will determine the bases, independent sets, rank functions, and circuits of these two matroids. We will also provide a graphical description of deletions and contractions.

3.1 Cocircuits

A pair of bonds B_1, B_2 in a graph G is a *modular pair* when the number of connected components of $G - (B_1 \cup B_2)$ is two more than the number of connected components of G. Thus B_1, B_2 form a modular pair of bonds when: $B_1 \cup B_2$ forms a tribond, $B_1 \cup B_2$ form the configuration in Figure 1 but with no edges between X_1 and X_2 (see Figure 2, left), or B_1 and B_2 are in two distinct connected components of G (see Figure 2, right). When B_1, B_2 is a modular pair of bonds but $B_1 \cup B_2$ is not a tribond (i.e., $B_1 \cup B_2$ is one of the structures from Figure 2) we will call $B_1 \cup B_2$ a *dibond*. Note that a modular pair of bonds share an edge if and only if their union is a tribond.

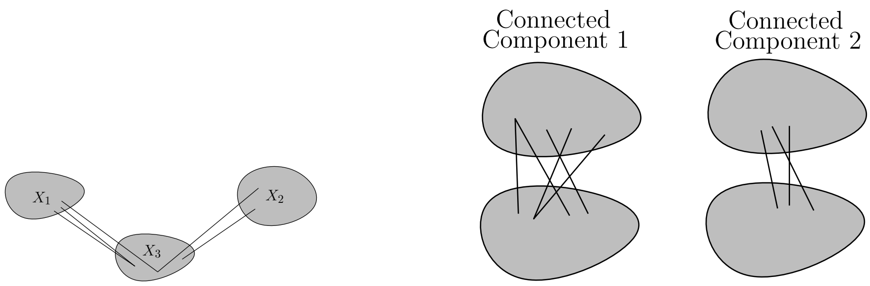

Figure 2: Every dibond is of exactly one of two possible types.

The matroid of Theorem 1 is $J_0(G,\mathcal{L})$, the complete join matroid of $(G,\mathcal{L})$. If $\mathcal{L}$ is trivial, then the single-element extension associated with $\mathcal{L}$ is just $M(G)$ along with a new element that is either a loop or coloop.

Theorem 1. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then there is a matroid with element set $E(G) \cup e_0$ in which e_0 is neither a loop nor a coloop and whose cocircuits consist of the following:*

(1) bonds in $\mathcal{L}$,

(2) sets of the form $B \cup e_0$ in which B is a bond not in $\mathcal{L}$, and

(3) tribonds and dibonds which do not contain a bond from $\mathcal{L}$.

Conversely, if N is a single-element extension of the graphic matroid $M(G)$ with new element e_0 which is neither a loop nor a coloop of N, then there is a non-trivial linear class of bonds $\mathcal{L}$ of G such that the cocircuits of N consist of the sets above.

Proof. This follows the cocircuit version of Crapo's Theorem [1, p.62] on single-element extensions of matroids as long as modularity of bonds as we have defined them graphically is exactly how modular pairs of cocircuits behave in the graphic matroid $M(G)$. This will complete our proof.

Let B_1 and B_2 be distinct bonds in G and let $H_1 = E - B_1$ and $H_2 = E - B_2$. Now B_1, B_2 is a modular pair of cocircuits $M(G)$ if and only if H_1, H_2 is a modular pair of hyperplanes in $M(G)$, if and only if $r(H_1) + r(H_2) = r(H_1 \cup H_2) + r(H_1 \cap H_2)$, if and only if $r(H_1 \cap H_2) = r(M(G)) - 2$, if and only if the flat $H_1 \cap H_2$ has 2 more connected components than does G, which is how we defined a modular pair of bonds. $\square$

The *join matroid* of a cobiased graph $(G, \mathcal{L})$ is defined as $J(G, \mathcal{L}) = J_0(G, \mathcal{L})/e_0$. If $\mathcal{L}$ is trivial, then $J(G, \mathcal{L}) = M(G)$. Theorem 2 is an immediate corollary of Theorem 1.

Theorem 2. *If $\mathcal{L}$ is a non-trivial linear class of bonds in G, then the cocircuits of $J(G, \mathcal{L})$ consist of the following:*

(1) bonds in $\mathcal{L}$ and

(2) tribonds and dibonds which do not contain a bond from $\mathcal{L}$.

Furthermore, if N is an elementary quotient of a graphic matroid $M(G)$, then $N = J(G, \mathcal{L})$ for some non-trivial linear class $\mathcal{L}$.

3.2 Bases and independent sets

Consider a partition $\pi = \{X_1, \ldots, X_k\}$ of $V(G)$ into nonempty parts such that each induced subgraph $G[X_i]$ is connected. Denote the set of such partitions for G by Lattice(G). The partial ordering is the usual refinement partial ordering; that is, given two such partitions $\pi_1 = \{X_1, \ldots, X_k\}$ and $\pi_2 = \{Y_1, \ldots, Y_l\}$ in Lattice(G), we have $\pi_2 \leqslant \pi_1$ when for each part $Y_i \in \pi_2$ there is a part $X_j \in \pi_1$ such that $Y_i \subseteq X_j$. It is well known that with this partial ordering Lattice(G) is indeed a lattice. If H is a subgraph of G or subset of $E(G)$, then H naturally induces a partition $\pi_H \in$ Lattice(G) corresponding to the connected components of $H \cup V(G)$. If $H \subseteq G$ with $V(H) = V(G)$, then Lattice(H) is a join subsemilattice of Lattice(G). (For the proof, let $\pi \in$ Lattice(G). Think of the edges of $\cup G[X_i]$ as a relation on $V(G)$ and extend it to an equivalence relation $\equiv_\pi$. If $\tau \in$ Lattice(G), extend $\equiv_\pi \cup \equiv_\tau$ to an equivalence relation $\equiv$; then the join $\pi \vee \tau$ is the partition that corresponds to $\equiv$. Supposing that $\pi, \tau \in$ Lattice(H), this formula for the join is the same whether viewed in H or in G.) In particular Lattice(G) is a join subsemilattice of Lattice(K_n), which is the usual partition lattice of the set $V(K_n)$.

Given $\pi = \{X_1, \ldots, X_k\} \in$ Lattice(G), denote the set of edges in $G[X_1] \cup \cdots \cup G[X_k]$ by Interior(π), or Interior$_G(\pi)$ when necessary. The collection of such edge sets is of course exactly the set of flats of the graphic matroid $M(G)$. The set of edges of G which are not in $\mathrm{Interior}(\pi)$ is denoted by $\mathrm{Exterior}(\pi)$ or $\mathrm{Exterior}_G(\pi)$. Note that $\mathrm{Exterior}(\pi)$ is a union of bonds.

Given a cobiased graph $(G, \mathcal{L})$, we call $\pi \in \mathrm{Lattice}(G)$ *cobalanced* with respect to $\mathcal{L}$ when every bond in $\mathrm{Exterior}(\pi)$ is cobalanced; otherwise the partition is *un-cobalanced.* The maximal element of $\mathrm{Lattice}(G)$ is π_G, that is, the partition of $V(G)$ given by the connected components of G itself. The coatoms of $\mathrm{Lattice}(G)$, that is, the elements of $\mathrm{Lattice}(G)$ which are covered by π_G, are those partitions π for which $\mathrm{Exterior}(\pi)$ is a bond.

Theorem 3. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the bases of $J_0(G, \mathcal{L})$ consist of the following:*

(1) edge sets of maximal forests of G and

(2) sets of the form $F \cup e_0$ in which F is a maximal forest with one edge deleted such that the bond $\mathrm{Exterior}(\pi_F)$ *is un-cobalanced.*

When one edge is deleted from a maximal forest, one component tree is broken into two trees. The bond between those two trees is $\mathrm{Exterior}(\pi_F)$.

Proof. For a general matroid M, B is a basis if and only if B is a minimal set which intersects every cocircuit (see, for example, [6, p.77]). Now since edge sets of cycles are dependent in $M(G)$, they are also dependent in its single-element extension $J_0(G, \mathcal{L})$. So if B is a base of $J_0(G, \mathcal{L})$, then $B \backslash e_0$ is the edge set of a forest in G. Theorem 1 now implies the following: $e_0 \notin B$ if and only if B is a maximal forest and $e_0 \in B$ if and only if $B \backslash e_0$ is obtained from a maximal forest by the deletion of one edge e such that the bond $\mathrm{Exterior}(\pi_F)$ is un-cobalanced. □

Theorem 4. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the bases of $J(G, \mathcal{L})$ consist of the forests F for which* $\mathrm{Exterior}(\pi_F)$ *is an un-cobalanced bond.*

Proof. Because $\mathcal{L}$ is non-trivial, e_0 is not a loop or coloop of $J_0(G, \mathcal{L})$. Thus the bases of $J(G, \mathcal{L})$ are obtained from the bases of $J_0(G, \mathcal{L})$ that contain e_0 by removing e_0. The result now follows from Theorem 3. □

Theorem 5. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the independent sets of $J_0(G, \mathcal{L})$ consist of the following:*

(1) Edge sets of forests.

(2) Edge sets of the form $F \cup e_0$ in which F is a forest and π_F is un-cobalanced.

Proof. This follows from Theorem 3 because (1) and (2) describe exactly the subsets of the bases of $J_0(G, \mathcal{L})$. □

Theorem 6. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the independent sets of $J(G, \mathcal{L})$ consist of the edge sets of forests F such that π_F is un-cobalanced.*

Proof. This follows from Theorem 4 because these are exactly the subsets of the bases of $J(G, \mathcal{L})$. □

3.3 Rank

If H is a subgraph of G or subset of $E(G)$, then $|\pi_H|$ is the number of connected components of $H \cup V(G)$. Thus if $X \subseteq E(G)$, then $r_{M(G)}(X) = |V(G)| - |\pi_X|$.

Theorem 7. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G and $X \subseteq E(G)$, then*

(1) $r_{J_0(G,\mathcal{L})}(X) = |V(G)| - |\pi_X|$,

(2) $r_{J_0(G,\mathcal{L})}(X \cup e_0) = |V(G)| - |\pi_X|$ *when* π_X *is cobalanced, and*

(3) $r_{J_0(G,\mathcal{L})}(X \cup e_0) = |V(G)| - |\pi_X| + 1$ *when* π_X *is un-cobalanced.*

Proof. Part (1) follows from the fact that the rank of X in $M(G)$ and its single-element extension $J_0(G, \mathcal{L})$ must be the same. Theorem 5 implies that there is a circuit containing e_0 in the set $X \cup e_0$ if and only if π_X is cobalanced. This implies Parts (2) and (3). □

Theorem 8. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G and $X \subseteq E(G)$, then*

(1) $r_{J(G,\mathcal{L})}(X) = |V(G)| - |\pi_X| - 1$ *when* π_X *is cobalanced,*

(2) $r_{J(G,\mathcal{L})}(X) = |V(G)| - |\pi_X|$ *when* π_X *is un-cobalanced.*

Proof. This follows from Theorem 7 and the fact that $J(G, \mathcal{L}) = J_0(G, \mathcal{L})/e_0$. □

3.4 Circuits

If J is a forest in $(G, \mathcal{L})$ for which π_J is cobalanced and J is minimal with respect to this property, then J is called an *$\mathcal{L}$-join* of the cobiased graph $(G, \mathcal{L})$. Being an $\mathcal{L}$-join means that deleting any edge of J creates a bond that is not in $\mathcal{L}$.

Theorem 9. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the circuits of $J_0(G, \mathcal{L})$ consist of the following:*

(1) edges sets of the form $J \cup e_0$ in which J is an $\mathcal{L}$-join and

(2) edges sets of cycles.

Proof. Edge sets of cycles are circuits of $M(G)$ and therefore are also circuits in $J_0(G, \mathcal{L})$. Hence, any other circuit consists of the edge set of some forest along with e_0. Suppose that F is the edge set of a forest for which $F \cup e_0$ is a circuit. Theorem 5 implies that $F \cup e_0$ is independent when π_F is un-cobalanced and dependent when π_F is cobalanced. Thus $F \cup e_0$ is a circuit when π_F is cobalanced and F is minimal with respect to this property. Thus F is an $\mathcal{L}$-join. □

Theorem 10. *If $\mathcal{L}$ is a non-trivial linear class of bonds of G, then the circuits of $J(G, \mathcal{L})$ consist of the following:*

(1) edge sets of $\mathcal{L}$-joins and

(2) edge sets of cycles that do not contain $\mathcal{L}$-joins.

Proof. This follows from Theorem 9 and the fact that $J(G, \mathcal{L}) = J_0(G, \mathcal{L})/e_0$. □

3.5 Deletions and contractions

Let G be a graph and e a link in G. The bonds of G/e are the bonds of G which do not contain e. Define $(G, \mathcal{L})/e = (G/e, \mathcal{L}/e)$ in which $\mathcal{L}/e$ is the set of all bonds $B \in \mathcal{L}$ which do not contain e. Now $(G/e, \mathcal{L}/e)$ is a cobiased graph because any tribond of G/e is a tribond of G.

The situation for deletions is only slightly more complicated. If B is a bond in $G\backslash e$, then either B or $B \cup e$ is a bond in G. We define $(G, \mathcal{L})\backslash e = (G\backslash e, \mathcal{L}\backslash e)$ in which $\mathcal{L}\backslash e$ is the set of all bonds B in $G\backslash e$ for which either B or $B \cup e \in \mathcal{L}$. Now $(G\backslash e, \mathcal{L}\backslash e)$ is a cobiased graph because if T is a tribond of $G\backslash e$, then either T or $T \cup e$ is a tribond of G.

Theorem 11. *If $(G, \mathcal{L})$ is a cobiased graph and e is a link in G, then*

(1) $[J_0(G, \mathcal{L})]\backslash e = J_0(G\backslash e, \mathcal{L}\backslash e)$,

(2) $[J(G, \mathcal{L})]\backslash e = J(G\backslash e, \mathcal{L}\backslash e)$,

(3) $[J_0(G, \mathcal{L})]/e = J_0(G/e, \mathcal{L}/e)$, *and*

(4) $[J(G, \mathcal{L})]/e = J(G/e, \mathcal{L}/e)$.

Proof. (2) We compare dependent sets. Let $D \subseteq E(G)$. If D is a dependent set of $[J(G, \mathcal{L})]\backslash e$, then $e \notin D$ and either D contains a cycle or D is a forest such that $\pi_D \in$ Lattice(G) is a cobalanced partition. If D contains a cycle, then it is a dependent set in $J(G\backslash e, \mathcal{L}\backslash e)$. If D is a forest, then because $e \notin D$, the partition of $V(G)$ associated with D is the same for G and $G\backslash e$. Thus π_D is still a cobalanced partition in Lattice$(G\backslash e)$, so D is a dependent set of $J(G\backslash e, \mathcal{L}\backslash e)$. Conversely, if D is a dependent set of $J(G\backslash e, \mathcal{L}\backslash e)$, then either D contains a cycle, in which case it is dependent in $[J(G, \mathcal{L})]\backslash e$, or D is a forest of $G\backslash e$ such that $\pi_D \in$ Lattice$(G\backslash e)$ is cobalanced. Again, the partition of $V(G)$ associated with D is the same in G as in $G\backslash e$, so $\pi_D \in$ Lattice(G) is cobalanced, from which it follows that D is dependent in $[J(G, \mathcal{L})]\backslash e$.

(1) The proof is similar to that of Part (2) with only the added detail of noting the presence of e_0 in dependent sets without cycles.

(3 and 4) These follow a similar strategy to the proofs of (1) and (2) but by comparing cocircuits. The details are left to the reader. □

3.6 Vertex union

An operation that preserves graphic matroids is the union of two graphs at a single vertex. This operation has the same property for cobiased graphs and join matroids, as we see in Theorem 12. That fact is hinted at within discussions in [8] but is not fully developed.

Theorem 12. *If $G_1 = H \cup K$ in which $H \cap K$ is a single vertex and $G_2 = H \cup K$ in which $H \cap K$ is empty, then*

(1) $\mathcal{L}$ is a linear class of bonds in G_1 if and only if $\mathcal{L}$ is a linear class of bonds in G_2 and

(2) $J_0(G_1,\mathcal{L}) = J_0(G_2,\mathcal{L})$ *and* $J(G_1,\mathcal{L}) = J(G_2,\mathcal{L})$.

Proof. The set of bonds in G_1 is exactly the set of bonds in G_2; furthermore, the tribonds of G_1 are the same as the tribonds of G_2 because no tribond can have edges in more than one block of a graph. This proves (1). Part (2) follows from these same facts and comparing cocircuits. $\square$

3.7 Two simple examples

Example 13. Pick vertices a, b in G. Define a bond as cobalanced when it does not separate a and b. Let $\mathcal{L}_{a,b}$ denote this set of cobalanced bonds.

Proposition 14. $(G, \mathcal{L}_{a,b})$ *is a cobiased graph.*

Proof. Consider a tribond corresponding to a tripartition $\{X, Y, Z\}$ of $V(G)$. If a, b are in the same part, all three bonds are cobalanced. If $a \in X$ and $b \in Y$, then the bond $\delta(Z)$ is cobalanced and the other two bonds are not. That is, an even number of the three are un-cobalanced. It follows that $\mathcal{L}_{a,b}$ is a linear class of bonds. $\square$

The linear class $\mathcal{L}_{a,b}$ has a special property: In every tribond the number of un-cobalanced bonds is even. This looks like a dual of the characteristic property of biased graphs derived from edge signs (gains in the 2-element group), that in every theta graph the number of unbalanced cycles is even (called "additive bias" in [10]). By analogy, let us call such a linear class of bonds *additive*. Signed graphs are especially simple (and important) gain graphs. That raises the following two questions: Is there similar importance for additive cobias? Is there a simple general construction of additively cobiased graphs, dual in some sense to the construction of additively biased graphs from signed graphs? Example 24 shows that $\mathbb{Z}_2$ does in fact relate to the class $\mathcal{L}_{a,b}$, which is a step in the direction of answers.

Example 15. More generally let $W \subseteq V(G)$ have even cardinality, define a bond to be *cobalanced* if it separates W into two even subsets, and let $\mathcal{L}_W^+$ be the linear class of all such bonds. Then $\mathcal{L}_W^+$ is also additive.

Both examples are forms of quotient labeled cobias; see Example 24.

4 Gain graphs and linear classes of bonds

An *oriented edge* in a graph is an edge e along with a chosen direction along that edge. If e is an oriented edge, then the reverse orientation is denoted by $-e$ when using additive notation and by e^{-1} when using multiplicative notation. (We will be using additive notation except in Sections 4.2 and 4.3.) The set of all possible oriented edges in G is denoted by $\vec{E}(G)$. An *oriented bond* $\vec{B}$ is a bond $B = \delta(X)$ along with a choice of orientation for the edges of B, either all away from X or all towards X. The reverse orientation of $\vec{B}$ is denoted by $-\vec{B}$ when using additive notation and $\vec{B}^{-1}$ when using multiplicative notation.

Let Γ be a group. A Γ-*gain graph* is a pair (G, φ) in which G is a graph and $\varphi\colon E(G) \to \Gamma$ is a mapping such that $\varphi(-e) = -\varphi(e)$ for an additive group and $\varphi(e^{-1}) = \varphi(e)^{-1}$ for a multiplicative group. (All our additive groups are abelian. Our multiplicative groups are not assumed to be abelian.) The function φ is called a Γ-*gain function*. Gain graphs give rise to biased graphs (see [10]). Similarly, they give rise to cobiased graphs, though not always.

4.1 Cobiased graphs from additive gain graphs

If Γ is an additive group and (G, φ) is a Γ-gain graph, then for each oriented bond $\vec{B}$, define $\varphi(\vec{B}) = \sum_{e \in \vec{B}} \varphi(e)$. Say that the bond B is *cobalanced* when $\varphi(\vec{B}) = -\varphi(-\vec{B}) = 0$. Let $\mathcal{L}_\varphi$ be the set of cobalanced bonds of (G, φ).

Proposition 16. *If Γ is an additive group and (G, φ) is a Γ-gain graph, then $(G, \mathcal{L}_\varphi)$ is a cobiased graph.*

Proof. Consider a tribond containing the three bonds $\delta(X_1)$, $\delta(X_2)$, $\delta(X_3)$ and assume without loss of generality that $\delta(X_1), \delta(X_2) \in \mathcal{L}_\varphi$. Let $\vec{B}_1$ and $\vec{B}_2$ be the oriented bonds obtained from $\delta(X_1)$ and $\delta(X_2)$ by orienting all edges away from X_1 and X_2. Let $\vec{B}_3$ be the oriented bond obtained from $\delta(X_3)$ by orienting all of its edges towards X_3. Now $\varphi(\vec{B}_3) = \varphi(\vec{B}_1) + \varphi(\vec{B}_2) = 0$, which implies that a tribond cannot have exactly two bonds in $\mathcal{L}_\varphi$, which is our result. □

4.2 Cobiased planar graphs using gains over arbitrary groups

Let Γ be a multiplicative group (not necessarily abelian), let G be a connected graph embedded in the plane, and let (G, φ) be a Γ-gain graph. Consider an oriented bond $\vec{B}$ in G. The oriented edges of $\vec{B}$ correspond to a closed walk in the topological dual graph G^*. Thus there is a well-defined cyclic ordering of the edges of $\vec{B}$ up to a choice of a starting edge and clockwise or counterclockwise direction. So, given such an oriented bond $\vec{B}$, let $e_1, \ldots, e_k$ be a cyclic ordering with e_1 as the starting edge. Define $\varphi(\vec{B}) = \varphi(e_1) \cdots \varphi(e_k)$. Note that any other choice of starting edge yields a product $\varphi(e_i) \cdots \varphi(e_k)\varphi(e_1) \cdots \varphi(e_{i-1})$, which is conjugate to $\varphi(e_1) \cdots \varphi(e_k)$ in the group Γ. Furthermore, a different choice of direction yields a product that is the inverse of the original. Therefore, $\varphi(\vec{B}) = 1$ for any one choice of starting edge and direction if and only if $\varphi(\vec{B}) = 1$ for all possible choices of starting edge and direction. Say that a bond B is *cobalanced* when $\varphi(\vec{B}) = 1$ for some choice of starting edge and direction and let $\mathcal{L}_\varphi$ be the set of cobalanced bonds given by φ.

Proposition 17. *If G is a graph embedded in the plane, Γ is a multiplicative group, and (G, φ) is a Γ-gain graph, then $(G, \mathcal{L}_\varphi)$ is a cobiased graph.*

Proof. This proof is similar to the one for Proposition 16, but with the added concern of picking starting edges and directions for each bond in a tribond to match with the others. □

4.3 An example not realizable by gains

The example here is essentially the topological dual of [10, Example 5.8]. Consider the labeled graph $G \cong K_{2,4}$ shown on the left in Figure 3 with all edges oriented in the downward direction. Let $\mathcal{L} = \{a_1a_2a_3a_4, a_1a_2b_3b_4, b_1b_2a_3a_4\}$. Up to reembedding of G, a tribond in G is of one of the two types shown on the right of Figure 3. Note that any such tribond contains at most one bond from $\mathcal{L}$. Thus $(G, \mathcal{L})$ is a cobiased graph.

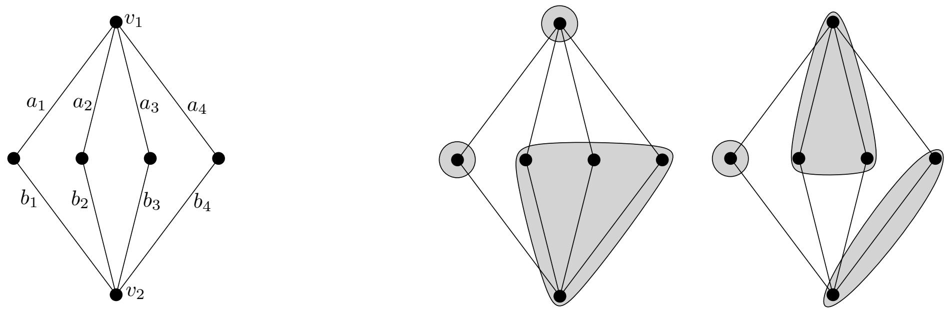

Figure 3: All edges are oriented in the downward direction. Every tribond is of one of the two types shown.

By way of contradiction, assume that there is a multiplicative gain function φ for which $\mathcal{L} = \mathcal{L}_\varphi$. For simplicity, let us denote $\varphi(a_i)$ and $\varphi(b_i)$ by a_i and b_i. Therefore $a_1a_2a_3a_4 = 1$, $a_1a_2b_3b_4 = 1$, and $b_1b_2a_3a_4 = 1$. Thus

$$1 = (a_1a_2b_3b_4)^{-1}a_1a_2a_3a_4(b_1b_2a_3a_4)^{-1} = (b_1b_2b_3b_4)^{-1},$$

which implies that $b_1b_2b_3b_4 \in \mathcal{L}$, a contradiction.

4.4 Cycle shifting

Let (G, φ) be a Γ-gain graph in which Γ is an additive group. Let C be a cycle in G with oriented edges $e_1, \ldots, e_k$ in cyclic order in one direction around C. For any $a \in \Gamma$, let $\psi_{C,a}$ be the Γ-gain function on G for which $\varphi(e_i) = a$ and $\varphi(e) = 0$ for all e such that $\pm e \notin \{e_1, \ldots, e_k\}$. Now for any oriented bond $\vec{B}$, $\varphi(\vec{B}) = (\varphi + \psi_{C,a})(\vec{B})$. Thus $\mathcal{L}_\varphi = \mathcal{L}_{\varphi+\psi_{C,a}}$. We call this operation *shifting*. We say that two Γ-gain functions φ_1 and φ_2 (or two Γ-gain graphs (G, φ_1) and (G, φ_2)) are *shifting equivalent* when φ_1 is obtained from φ_2 via a sequence of shifts. Note that this relation is an equivalence relation.

Theorem 18. *If Γ is an additive group and (G, φ) and (G, ψ) are Γ-gain graphs, then (G, φ) and (G, ψ) are shifting equivalent if and only if $\varphi(\vec{B}) = \psi(\vec{B})$ for all oriented bonds $\vec{B}$ in G.*

In order to prove Theorem 18, we need a concept of normalizing gains. So if (G, φ) is a Γ-gain graph and T is a maximal forest in G, then we say that φ is *T-normalized* when φ is zero on each edge of G not in T.

Proposition 19. *Let Γ be an additive group, (G, φ) a Γ-gain graph, and T a maximal forest of G. Then there is a unique T-normalized Γ-gain function φ_T that is shifting equivalent to φ.*

Proof. If e is an edge outside T, let $C(e)$ be the fundamental cycle in $T \cup e$. Now $\varphi_T = \varphi - \sum_{e \notin T} \psi_{C(e),\varphi(e)}$ is a Γ-gain function that is shifting equivalent to φ and is zero outside T.

We prove that φ_T is uniquely determined. Let ψ be any T-normalized Γ-gain function that, like φ_T, is shifting equivalent to φ. If e is in the tree T_1 of T, then $T_1 - e$ consists of two trees connected by a bond B in which e is the only edge of T, thus the only edge in B for which ψ may be nonzero. Orienting e and B compatibly, we have

$$\psi(e) = \psi(\vec{B}) = \varphi(\vec{B}). \tag{1}$$

In particular, $\varphi_T(e) = \varphi(\vec{B}) = \psi(e)$ for e in T. Thus, $\psi = \varphi_T$. $\square$

Proof of Theorem 18. If φ and ψ are shifting equivalent, then $\varphi(\vec{B}) = \psi(\vec{B})$ for all oriented bonds $\vec{B}$ in G from the definition of shifting. Conversely, assume that $\varphi(\vec{B}) = \psi(\vec{B})$ for all oriented bonds $\vec{B}$ in G. Let T be a maximal forest in G. Then $\psi_T = \psi = \varphi = \varphi_T$ on bonds, which implies $\psi_T(e) = \varphi_T(e)$ for all edges by equation (1). It follows that ψ is shifting equivalent to φ. $\square$

4.5 Cobiased graphs from vertex labelings

Let G be a graph and let Γ be an additive group. Consider a vertex labeling $\pi \colon V(G) \to \Gamma$. If H is a subgraph of G, we write $\pi(H) = \sum_{v \in V(H)} \pi(v)$. Call π a Γ-*quotient labeling* when for each connected component H of G, $\pi(H) = 0$. Now if $B = \delta(X)$ is a bond of G and $\vec{B}$ is an orientation of B towards X, then define $\pi(\vec{B}) = \pi(X)$ and say that B is cobalanced when $\pi(\vec{B}) = 0$. Let $\mathcal{L}_\pi$ be the set of cobalanced bonds relative to π.

Such Γ-quotient labelings were used by Recski [7, 8], for Γ equal to the additive group of a field, to characterize vector representations of single-element extensions and elementary quotients of graphic matroids over fields as well as defining some more general extension and elementary-quotient constructions for graphic matroids.

Proposition 20. *Let Γ be an additive group and π a Γ-quotient labeling of a graph G. Then $(G, \mathcal{L}_\pi)$ is a cobiased graph.*

Proof. Consider a tribond containing the three bonds $\delta(X_1)$, $\delta(X_2)$, $\delta(X_3)$ and assume without loss of generality that $\delta(X_1), \delta(X_2) \in \mathcal{L}_\pi$. Let $\vec{B}_1$ and $\vec{B}_2$ be the oriented bonds obtained from $\delta(X_1)$ and $\delta(X_2)$ by orienting all edges away from X_1 and X_2. Let $\vec{B}_3$ be the oriented bond obtained from $\delta(X_3)$ by orienting all of its edges towards X_3. Now

$$\pi(\vec{B}_3) = \pi(X_3) = -\pi(X_1 \cup X_2) = -\pi(X_1) - \pi(X_2) = \pi(\vec{B}_1) + \pi(\vec{B}_2) = 0,$$

which implies that a tribond cannot have exactly two bonds in $\mathcal{L}_\pi$, which implies our result. $\square$

It might seem that quotient labelings are different than gains; however, they are actually equivalent constructions. Theorem 21 describes how to get gains from a quotient labeling and Theorem 23 describes how to get a quotient labeling from gains.

Given a Γ-quotient labeling π of G and a maximal forest T in G, define a T-normalized Γ-gain function $\varphi_{\pi,T}$ as follows. For each oriented edge e not in T, let $\varphi_{\pi,T}(e) = 0$. For an edge e in T, let $B = \delta(X)$ be the bond $\mathrm{Exterior}(\pi_{T\backslash e})$ and say that $\vec{B}$ is the orientation of B directed towards X. Orient e towards X as well. Now set $\varphi_{\pi,T}(e) = \pi(X)$.

Theorem 21. *Let Γ be an additive group and π a Γ-quotient labeling of graph G. If T is a maximal forest of G, then for every oriented bond $\vec{B}$ in G, $\varphi_{\pi,T}(\vec{B}) = \pi(\vec{B})$.*

Proposition 22 is necessary for the proof of Theorem 21.

Proposition 22. *Let Γ be an additive group and π a Γ-quotient labeling of graph G. If T and T' are maximal forests in G, then $\varphi_{\pi,T}$ and $\varphi_{\pi,T'}$ are shifting equivalent.*

Proof. One shift operation can be performed in each connected component of G. Thus the result is true if and only if it is true for connected graphs, so we may assume that G is connected. Consider the following well-known operation on spanning trees, which we will call *edge exchange* in this proof. If T is a spanning tree of G and $e \notin T$, then for any edge $f \neq e$ on the unique cycle in $T \cup e$, $(T\backslash f) \cup e$ is a spanning tree of G. It is well known that if G is a connected graph and T and T' are spanning trees of G, then there is a sequence of spanning trees $T_1, \ldots, T_k \subseteq (T \cup T')$ such that $T = T_1$, $T' = T_k$, and T_{i+1} is obtained from T_i by an edge exchange. So to complete the proof it suffices to show that $\varphi_{\pi,T_{i+1}}$ is obtained from φ_{π,T_i} by a single shift operation.

Say that $T_{i+1} = (T_i\backslash f) \cup e$ and let $g \notin \{e, f\}$ be any other edge in T_i. Thus $T_i\backslash\{f, g\}$ has exactly three connected components S_1, S_2, S_3 as shown in Figure 4. Orient edges f and g as indicated. There are two cases for the placement of e relative to f and g as shown in the figure.

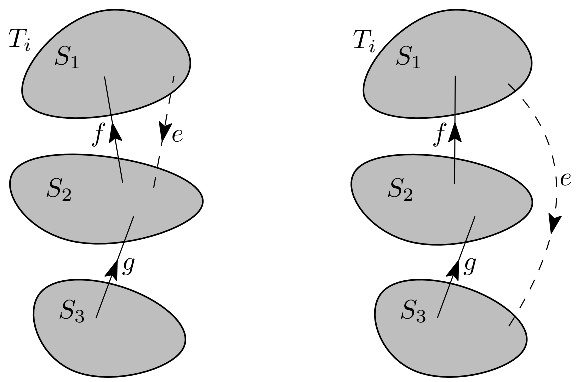

Figure 4: Figure for the proof of Proposition 22.

Let C be the unique cycle in $T_i \cup e$ oriented in the opposite direction to e and f. Note that g is in C in the right configuration of Figure 4 but not in the left configuration. Now let $a = \varphi_{\pi,T_i}(f)$. We prove that $\varphi_{\pi,T_i} + \psi_{C,a} = \varphi_{\pi,T_{i+1}}$, which will satisfy our requirement. First, by definition, $(\varphi_{\pi,T_i} + \psi_{C,a})(f) = 0 = \varphi_{\pi,T_{i+1}}(f)$ and $(\varphi_{\pi,T_i} + \psi_{C,a})(e) = -a = \varphi_{\pi,T_{i+1}}(e)$. Second, for the configuration on the left of Figure 4,

$$\varphi_{\pi,T_i}(g) = (\varphi_{\pi,T_i} + \psi_{C,a})(g) = -\pi(S_3) = \varphi_{\pi,T_{i+1}}(g).$$

Finally, for the configuration on the right,

$$(\varphi_{\pi,T_i} + \psi_{C,a})(g) = \varphi_{\pi,T_i}(g) - a = -\pi(S_1) - \pi(S_3) = -\pi(S_1 \cup S_3) = \varphi_{\pi,T_{i+1}}(g).$$

Since g was chosen arbitrarily we have proven that $\varphi_{\pi,T_i} + \psi_{C,a} = \varphi_{\pi,T_{i+1}}$. □

Proof of Theorem 21. Let B be a bond of G. If B intersects T in one edge (i.e., $B = \text{Exterior}(\pi_{T\backslash e})$) then $\pi(\vec{B}) = \varphi_{\pi,T}(\vec{B})$ by the definition of $\varphi_{\pi,T}$. If $|B \cap T| \geqslant 2$, then let T' be any maximal forest for which $|B \cap T'| = 1$. By Proposition 22 $\varphi_{\pi,T'}$ and $\varphi_{\pi,T}$ are shifting equivalent, so $\pi(\vec{B}) = \varphi_{\pi,T'}(\vec{B}) = \varphi_{\pi,T}(\vec{B})$, as required. □

Conversely, assume that (G, φ) is a Γ-gain graph. Assume that G is completely split apart along cut vertices so that each connected component of G is a block. This operation, of course, does not change the join matroids (Theorem 12). To simplify the discussion, assume that G is loopless and has no isolated vertices. (Loops in G are always loops in the join matroids and isolated vertices have no effect on the matroids.) Therefore, for each vertex v, $\delta(v)$ is a bond of G. Let $\vec{B}_v$ be the bond $\delta(v)$ oriented towards v. Define $\pi_\varphi(v) = \varphi(\vec{B}_v)$.

Theorem 23. *If (G, φ) is a Γ-gain graph in which each connected component is a block, then π_φ is a quotient labeling and for each oriented bond $\vec{B}$ in G, $\varphi(\vec{B}) = \mathcal{L}_{\pi_\varphi}(\vec{B})$.*

Proof. For each connected component H of G,

$$\sum_{v \in V(H)} \pi_\varphi(v) = \sum_{v \in V(H)} \varphi(\vec{B}_v) = \sum_{e \in \vec{E}(H)} \varphi(e) = 0.$$

Thus π_φ is a quotient labeling. In a similar fashion, if $\vec{B}$ is the orientation of bond $\delta(X)$ directed towards v, then

$$\sum_{v \in X} \pi_\varphi(v) = \sum_{v \in X} \varphi(\vec{B}_v) = \sum_{e \in \vec{B}} \varphi(e) = \varphi(\vec{B}),$$

as required. □

Example 24 (Nonseparating bonds)**.** We generalize the examples of Section 3.7 to an arbitrary subset W of $V(G)$ by defining $\mathcal{L}_W$ as the set of bonds $\delta(X)$ that do not separate W; i.e., $W \subseteq X$ or $W \subseteq V(G) \backslash X$. It is easy to verify that this set is a linear class, but it is not additive (in the sense of Section 3.7) if $|W| > 2$ since it is possible for every part of the tripartition of a tribond to contain a vertex of W.

This linear class exemplifies Γ-quotient labeling with the group $\Gamma = \mathbf{Z}_{|W|}$. The label of a vertex is 0 is $\pi(v) = 0$ if $v \notin W$ and $\pi(v) = 1$ if $v \in W$. Since $\pi(\delta(X)) \equiv |W \cap X|$ mod $|W|$ for any $X \subseteq V(G)$, only nonseparating bonds are cobalanced relative to π.

We interpret Example 15 in a similar way. Using the same group $\mathbb{Z}_2$, assign vertex values as in the previous example.

4.6 Deletions and contractions for gains

Let φ be a Γ-gain function. The set $\mathcal{L}_\varphi/e$ is defined in Section 3.5 as the set of bonds in $\mathcal{L}_\varphi$ that do not contain e and it is shown that $\mathcal{L}_\varphi/e$ is a linear class of bonds of G/e. Let φ/e be the Γ-gain function defined on G/e by restriction of φ to $E(G/e) = E(G)\backslash e$. Proposition 25 is immediate.

Proposition 25. *If φ is a Γ-gain function and e is an edge of G, then $\mathcal{L}_\varphi/e = \mathcal{L}_{\varphi/e}$.*

The set $\mathcal{L}_\varphi\backslash e$ is defined in Section 3.5 as the set of bonds B in $G\backslash e$ for which either B or $B\cup e$ is a bond in $\mathcal{L}_\varphi$. As long as e is not an isthmus of G, there is a Γ-gain function ψ on G that is shifting equivalent to φ and for which $\psi(e) = 0$. (See Proposition 19.) Define $\psi\backslash e$ to be the Γ-gain function on $G\backslash e$ defined by restriction of ψ to $E(G\backslash e) = E(G)\backslash e$.

Proposition 26. *If φ is a Γ-gain function and e is a non-isthmus edge of G, then there is ψ that is shifting equivalent to φ such that $\psi(e) = 0$ and $\mathcal{L}_\varphi\backslash e = \mathcal{L}_\psi\backslash e = \mathcal{L}_{\psi\backslash e}$.*

Proof. The existence of ψ is implied by Proposition 19. Now, if $B \in \mathcal{L}_{\psi\backslash e}$, then $(\psi\backslash e)(\vec{B}) = 0$. Since B is a bond of $G\backslash e$, there is a bond $B_e \in \{B, B\cup e\}$ of G. Since $\psi(e) = 0$, we get $\varphi(\vec{B}_e) = \psi(\vec{B}_e) = 0$. This implies that $B_e \in \mathcal{L}_\varphi$, which implies that $B \in \mathcal{L}_\varphi\backslash e$. Conversely, if $B \in \mathcal{L}_\varphi\backslash e$, there is a bond $B_e \in \{B, B\cup e\}$ of $\mathcal{L}_\varphi$ for which $\psi(\vec{B}_e) = \varphi(\vec{B}_e) = 0$. This implies that $(\psi\backslash e)(\vec{B}) = 0$, which makes $B \in \mathcal{L}_{\psi\backslash e}$. □

4.7 Deletions and contractions for quotient labelings

Let π be a Γ-quotient labeling of G and let e be a link in G with endpoints u and v. Let w be the vertex obtained by the contraction of e in G. Define π/e to be the labeling on $V(G/e)$ given by $(\pi/e)(x) = \pi(x)$ when $x \in V(G)\cap V(G/e)$ and $(\pi/e)(w) = \pi(u)+\pi(v)$.

Proposition 27. *If π is a Γ-quotient labeling of G and e is a link in G, then π/e is a Γ-quotient labeling of G/e and $\mathcal{L}_\pi/e = \mathcal{L}_{\pi/e}$.*

Proof. A bond B of G/e is in $\mathcal{L}_\pi/e$ if and only if B is a bond of G not containing e and $B \in \mathcal{L}_\pi$ if and only if both endpoints of e are in X or both endpoints of e are not in X where $B = \delta(X)$ if and only if B is a bond of G/e in $\mathcal{L}_{\pi/e}$. □

For Proposition 28, we define the vertex labeling $\pi\backslash e$ on $G\backslash e$ by $\pi\backslash e = \pi$.

Proposition 28. *If π is a Γ-quotient labeling of G and e is a link in G, then $\pi\backslash e$ is a Γ-quotient labeling of $G\backslash e$ if and only if e is not an un-cobalanced isthmus of $(G, \mathcal{L}_\pi)$. Furthermore, if e is not an un-cobalanced isthmus of $(G, \mathcal{L}_\pi)$, then $\mathcal{L}_\pi\backslash e = \mathcal{L}_{\pi\backslash e}$.*

Proof. The first statement is obvious. Now if e is not an un-cobalanced isthmus, consider a bond B in $G\backslash e$ and let $B_e \in \{B, B\cup e\}$ be the corresponding bond in G. Now $B \in \mathcal{L}_\pi\backslash e$ if and only $B_e \in \mathcal{L}_\pi$ if and only if $B \in \mathcal{L}_{\pi\backslash e}$. □

4.8 Gains and quotient labelings using fields

Let $\mathbb{F}$ be any field. Denote the additive group of $\mathbb{F}$ by $\mathbb{F}^+$. Scalar multiplication of $\mathbb{F}^+$-gain functions and quotient labelings using a nonzero element of $\mathbb{F}$ does not affect the resulting linear class of bonds. The proof of Proposition 29 is evident. Readers who are familiar with partial fields will note that this operation generalizes immediately to partial fields.

Proposition 29. *If (G, φ) is an $\mathbb{F}^+$-gain graph, π is an $\mathbb{F}^+$-quotient labeling, and a is a nonzero element of $\mathbb{F}$, then*

(1) $\mathcal{L}_\varphi = \mathcal{L}_{a\varphi}$ and

(2) $a\pi$ is a quotient labeling of G with $\mathcal{L}_{a\pi} = \mathcal{L}_\pi$.

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