# Poset Structure Concerning Cylindric Diagrams

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**Abstract**

Cylindric diagrams admit structures of infinite $d$-complete posets with natural ordering. The purpose of this paper is to provide a realization of a cylindric diagram as a subset of an affine root system of type $A$ via colored hook lengths, and to present several characterizations of its poset structure. Furthermore, the set of order ideals of a cylindric diagram is described as a weak Bruhat interval of the affine Weyl group.

**Mathematics Subject Classifications:** 05E10, 06A11, 17B22, 20F55

## Introduction

A periodic (Young) diagram is a Young diagram consisting of infinitely many cells in $\mathbb{Z}^2$ which is invariant under parallel translations generated by a certain vector $\omega \in \mathbb{Z}^2$ called the period (see Figure 1). The image of a periodic diagram under the natural projection onto the cylinder $\mathbb{Z}^2/\mathbb{Z}\omega$ is called a cylindric diagram. Diagrams given as a set-difference of two cylindric diagrams are called cylindric skew diagrams.

We note that cylindric skew diagrams have been known to parameterize a certain class of irreducible modules over the Cherednik algebras (double affine Hecke algebras) ([12, 13]) and the (degenerate) affine Hecke algebras ([1, 6]) of type $A$, where standard tableaux on those diagrams also appear.

Let $\omega = (m, -\ell) \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$ and let $\theta$ be a cylindric diagram in $\mathbb{Z}^2/\mathbb{Z}\omega$. The lattice $\mathbb{Z}^2$ admits a partial order $\leqslant$ defined by

$$(a, b) \leqslant (c, d) \iff a \geqq c \text{ and } b \geqq d,$$

which induces a poset structure on $\mathbb{Z}^2/\mathbb{Z}\omega$ and also on $\theta$. Together with the content map $\mathbf{c} : \theta \to \mathbb{Z}/\kappa\mathbb{Z}$, where $\mathbf{c}(a, b) = b - a \mod \kappa$ and $\kappa = \ell + m$, the cylindric digram $\theta$ is a locally finite $\mathbb{Z}/\kappa\mathbb{Z}$-colored $d$-complete poset in the sense of [9, 10].

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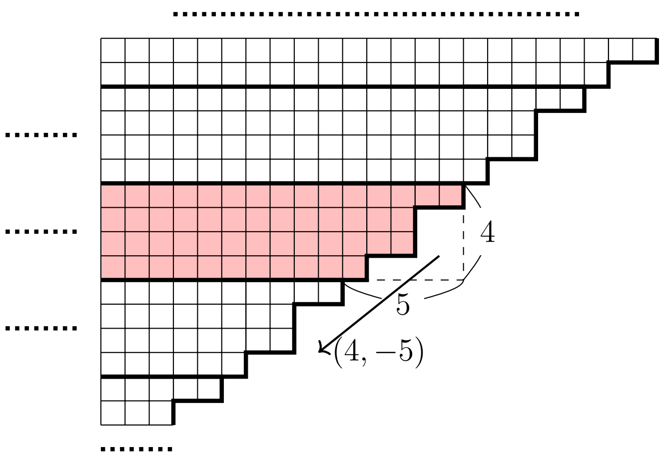



Figure 1: A periodic diagram of period $\omega = (4, -5)$.

The purpose of the present paper is to investigate the poset $(\theta, \leqslant)$ as well as the poset $(\mathcal{J}(\theta), \subset)$, where $\mathcal{J}(\theta)$ denotes the set of cylindric skew diagrams (or proper order ideals) included in $\theta$.

We briefly review a description in the classical case. Let $\lambda \subset \mathbb{Z}^2$ be a finite Young diagram. The associated Grassmannian permutation $w_\lambda$ is an element of the Weyl group of the root system $R$ of type $A_n$ where $n = \sharp\{\mathbf{c}(x) \mid x \in \lambda\}$. It is known that the poset $(\lambda, \leqslant)$ is dually isomorphic to the poset $(R(w_\lambda^{-1}), \leqslant^{\text{or}})$, where $R(w_\lambda^{-1}) := R_+ \cap w_\lambda^{-1} R_-$ and $\leqslant^{\text{or}}$ is the ordinary order (or the standard order) defined by

$$\alpha \leqslant^{\text{or}} \beta \iff \beta - \alpha \in \sum_{i \in [1,n]} \mathbb{Z}_{\geqq 0} \alpha_i$$

for $\alpha, \beta \in R(w_\lambda)$ with $\Pi$ being the set of simple roots ([7]).

Let $\theta$ be a cylindric diagram in $\mathbb{Z}^2/\mathbb{Z}\omega$. We would like to describe the poset $(\theta, \leqslant)$ in terms of the root system of type $A^{(1)}_{\kappa-1}$ with $\kappa = \ell + m$.

A key ingredient in our approach is the *colored hook length* ([2, 4]), given by

$$\mathbf{h}(x) = \sum_{y \in H(x)} \alpha_{\mathbf{c}(y)} \quad (x \in \theta),$$

where $H(x)$ denotes the hook at $x$ and $\alpha_i$ are simple roots. (See Section 2.1 for precise definitions.) We will show that the map $\mathbf{h}$ embeds the cylindric diagram $\theta$ into the set $R_+$ of positive (real) roots, and that the image $\mathbf{h}(\theta)$ is given by the inversion set $R(w_\theta)$ associated with a semi-infinite word $w_\theta$, which can be thought as an analogue of the Grassmannian permutation. Moreover, we show that the image $\mathbf{h}(\theta)$ is also characterized as the subset of $R_+$ consisting of those elements satisfying

$$\langle \zeta_\theta, \alpha^\vee \rangle = -1,$$

where $\zeta_\theta$ is a predominant integral weight determined by $\theta$ (see Section 2.2 and 2.3 for details).

Unlike the classical case, the ordinary order in $R(w_\theta)$ does not lead a poset isomorphism, and we need to introduce a modified order $\trianglelefteq$ in $R(w_\theta)$ by

$$\alpha \leqslant^{\mathrm{or}} \beta \iff \beta - \alpha \in \sum_{\gamma \in \Pi_\theta} \mathbb{Z}_{\geqq 0}\gamma,$$

to obtain a poset isomorphism $(\theta, \leqslant) \cong (R(w_\theta), \trianglelefteq)$, where $\Pi_\theta$ is a certain subset of the affine root system (see Section 3.1).

Another description of the poset $\theta$ is given by a linear extension or (reverse) standard tableau $\mathfrak{t}$ on $\theta$, which is by definition a bijective order preserving map $\theta \to \mathbb{Z}_{\geqq 1}$. A linear extension $\mathfrak{t} : \theta \to \mathbb{Z}_{\geqq 1}$ brings a poset structure to $\mathbb{Z}_{\geqq 1}$ and the resulting poset is an infinite analogue of the heap, which is originally introduced by Stembridge [7]. In summary, we have the following:

**Theorem** (Theorem 47 and Proposition 50)**.** *The followings are poset isomorphisms:*

$$(\mathbb{Z}_{\geqq 1}, \leqslant^{\mathrm{hp}}_{\mathfrak{t}}) \xleftarrow{\mathfrak{t}} (\theta, \leqslant) \xrightarrow{\mathbf{h}} (R(w_\theta), \trianglelefteq).$$

Another goal of this paper is to describe the poset structure $\mathcal{J}(\theta)$. For a finite Young diagram $\lambda$, it is known that the set $\mathcal{J}(\lambda)$ of order ideals of $\lambda$ is isomorphic to the interval $[e, w_\lambda] = \{u \in W \mid e \preceq u \preceq w_\lambda\}$ with weak right Bruhat order ([4, Proposition I]). For a cylindric diagram $\theta$, we define a "semi-infinite Bruhat interval" $[e, w_\theta)$, and we have the following:

**Theorem** (Theorem 58)**.** *The map*

$$\Phi : (\mathcal{J}(\theta), \subset) \to ([e, w_\theta), \preceq)$$

*given by* $\Phi(\xi) = w_\xi$ *is a poset isomorphism.*

# 1 Cylindric diagrams

## 1.1 Cylindric diagrams as posets

Let $(P, \leqslant)$ be a poset. For $x, y \in P$, define an *interval* $[x, y]$ by

$$[x, y] = \{z \in P \mid x \leqslant z \leqslant y\}.$$

We say that $y$ *covers* $x$ if $[x, y] = \{x, y\}$.

**Definition 1.** Let $(P, \leqslant)$ be a poset. A subset $J$ of $P$ is called an *order filter* (resp. *order ideal*) if the following condition holds:

$$x \in J,\ x \leqslant y \implies y \in J \qquad (\text{resp. } x \in J,\ x \geqslant y \implies y \in J).$$

An order filter (resp. order ideal) $J$ is said to be *proper* if $J \neq P$, and it is said to be *non-trivial* if $J \neq P$ nor $J \neq \emptyset$.

For $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$, we let $\mathbb{Z}\omega$ denote the subgroup of (the additive group) $\mathbb{Z}^2$ generated by $\omega$, and define the cylinder $\mathcal{C}_\omega$ by

$$\mathcal{C}_\omega = \mathbb{Z}^2/\mathbb{Z}\omega.$$

Let $\pi : \mathbb{Z}^2 \to \mathcal{C}_\omega$ be the natural projection. The cylinder $\mathcal{C}_\omega$ inherits a $\mathbb{Z}^2$-module structure via $\pi$.

Define a poset structure on $\mathbb{Z}^2$ by

$$(a, b) \leqslant (a', b') \iff a \geqq a' \text{ and } b \geqq b' \text{ as integers.}$$

For $x, y \in \mathcal{C}_\omega$, write $x \leqslant y$ if there exists $\tilde{x}, \tilde{y} \in \mathbb{Z}^2$ such that $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$ and $\tilde{x} \leqslant \tilde{y}$. It is not difficult to see the following:

**Lemma 2.** *Let $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$. Then the relation $\leqslant$ on $\mathcal{C}_\omega$ is a partial order, and the projection $\pi$ is order preserving.*

In the rest of this section, we fix $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$.

**Definition 3.** (1) A non-trivial order filter of $\mathcal{C}_\omega$ is called a *cylindric diagram*.
(2) A non-trivial order filter $\Theta$ of $\mathbb{Z}^2$ is called a *periodic diagram of period* $\omega$ if $\Theta + \omega = \Theta$.

**Lemma 4.** (1) *For a cylindric diagram $\theta$ in $\mathcal{C}_\omega$, the inverse image $\pi^{-1}(\theta)$ is a periodic diagram of period $\omega$.*
(2) *For a periodic diagram $\Theta$ of period $\omega$, the image $\pi(\Theta)$ is a cylindric diagram in $\mathcal{C}_\omega$.*

Figure 1 indicates a periodic diagram of period $\omega = (4, -5)$. The set consisting of colored cells is a fundamental domain with respect to the action of $\mathbb{Z}\omega$, and it is in one to one correspondence with the associated cylindric diagram.

**Definition 5.** Let $m, \ell \in \mathbb{Z}_{\geqq 1}$. A non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of (possibly negative) integers is called a *generalized partition of length $m$*. For $\omega = (m, -\ell) \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$, we denote by $\mathcal{P}_\omega$ the set of generalized partitions of length $m$ satisfying

$$\lambda_1 - \lambda_m \leqq \ell.$$

For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_\omega$, we define

$$\begin{aligned} \boldsymbol{\lambda} &= \{(a, b) \in \mathbb{Z}^2 \mid 1 \leqq a \leqq m,\ b \leqq \lambda_a\}, \\ \hat{\lambda} &= \boldsymbol{\lambda} + \mathbb{Z}\omega, \\ \mathring{\lambda} &= \pi(\hat{\lambda}). \end{aligned}$$

Note that $\boldsymbol{\lambda} = \hat{\lambda} \cap ([1, m] \times \mathbb{Z})$ and $\boldsymbol{\lambda}$ is a fundamental domain of $\hat{\lambda}$ with respect to the action of $\mathbb{Z}(m, -\ell)$.

If $\lambda \in \mathcal{P}_\omega$ then $\hat{\lambda}$ is a periodic diagram of period $\omega$ and $\mathring{\lambda}$ is a cylindric diagram. Moreover, any periodic (resp. cylindric) diagram of period $\omega$ is of the form $\hat{\lambda}$ (resp. $\mathring{\lambda}$) for some $\lambda \in \mathcal{P}_\omega$.

For a poset $P$ and its order filter $J$, we denote the set-difference $P \setminus J$ also by $P/J$. It is easy to see the following:

**Proposition 6.** *For a subset $\xi$ of $\mathcal{C}_\omega$, the following conditions are equivalent* :
(i) *$\xi$ is a proper order ideal of a cylindric diagram in $\mathcal{C}_\omega$.*
(ii) *$\xi$ is a set-difference $\theta/\eta$ of two cylindric diagrams $\theta, \eta$ in $\mathcal{C}_\omega$ with $\theta \supset \eta$.*
(iii) *$\xi$ is an intersection of a proper order ideal and a proper order filter of $\mathcal{C}_\omega$.*
(iv) *$\xi$ is a finite subset of $\mathcal{C}_\omega$ and satisfies the following condition:*

$$x, y \in \xi \implies [x, y] \subset \xi.$$

(v) *$\xi$ is a finite subset of $\mathcal{C}_\omega$ and satisfies the following condition:*

$$x, x + (1, 1) \in \xi \implies x + (0, 1), x + (1, 0) \in \xi \quad (\textit{the skew property})$$

**Definition 7.** A subset $\xi$ of $\mathcal{C}_\omega$ is called a *cylindric skew diagram* if it satisfies one of the conditions (i)–(v) in Proposition 6.

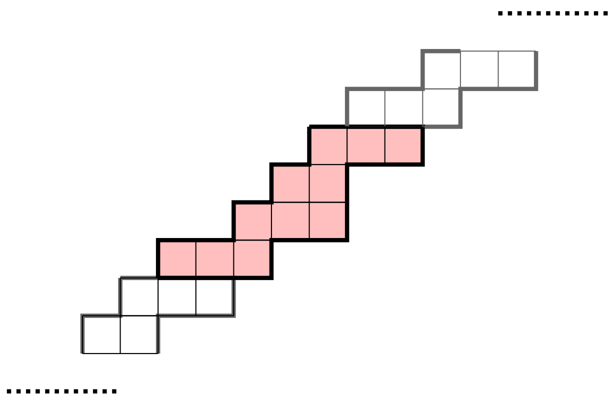


Figure 2: A cylindric skew diagram.

We denote the set of proper order ideals of $\theta$ by $\mathcal{J}(\theta)$ and regard it as a poset with the inclusion relation. Note that any $\xi \in \mathcal{J}(\theta)$ is a finite set and thus $\mathcal{J}(\theta) = \bigsqcup_{n=0}^{\infty} \mathcal{J}_n(\theta)$, where we put

$$\mathcal{J}_n(\theta) = \{\xi \in \mathcal{J}(\theta) \mid |\xi| = n\}.$$

### 1.2 Standard tableaux

In the rest of present section, fix a cylindric diagram $\theta$ in $\mathcal{C}_\omega$.

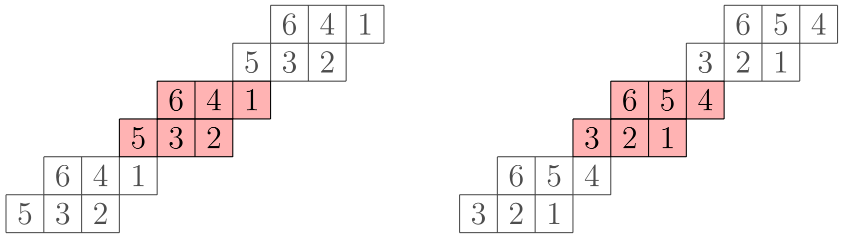
**Definition 8.** (1) For a cylindric diagram $\theta$, a *standard tableau* (or *linear extension*) of $\theta$ is a bijection $\mathfrak{t} : \theta \to \mathbb{Z}_{\geqq 1}$ satisfying

$$x < y \implies \mathfrak{t}(x) < \mathfrak{t}(y).$$

We denote by $\mathrm{ST}(\theta)$ the set of standard tableaux of $\theta$.
(2) For a finite poset $P$ with $|P| = n$, a standard tableau of $P$ is a bijection $\mathfrak{t} : P \to [1, n]$ satisfying

$$x < y \implies \mathfrak{t}(x) < \mathfrak{t}(y).$$

We denote by $\mathrm{ST}(P)$ the set of standard tableaux of $P$.



Standard tableau NOT standard tableau

Figure 3:

*Remark* 9. Our standard tableaux are usually referred to as reverse standard tableaux.

Let $\mathsf{t} \in \mathrm{ST}(\theta)$. It is easy to see that the subset $\mathsf{t}^{-1}([1,n])$ of $\theta$ is a proper order ideal, and moreover the restriction $\mathsf{t}|_{\mathsf{t}^{-1}([1,n])}$ is a standard tableau on $\mathsf{t}^{-1}([1,n])$. Conversely, for $\xi \in \mathcal{J}_n(\theta)$, any standard tableau on $\xi$ can be extended to a standard tableau on $\theta$. In summary, we have the following:

**Lemma 10.** *Let $n \in \mathbb{Z}_{\geqq 0}$. The correspondence $\mathsf{t} \mapsto \mathsf{t}^{-1}([1,n])$ gives a surjective map*

$$\mathrm{ST}(\theta) \to \mathcal{J}_n(\theta).$$

*Moreover, for each $\mathsf{t} \in \mathrm{ST}(\theta)$, the restriction $\mathsf{t} \mapsto \mathsf{t}|_{\mathsf{t}^{-1}([1,n])}$ gives a surjective map*

$$\mathrm{ST}(\theta) \to \mathrm{ST}(\mathsf{t}^{-1}([1,n])).$$

### 1.3 Content map and bottom set

Let $\Theta$ be a periodic diagram of period $\omega$. Define the *content map*

$$\mathbf{c} : \Theta \to \mathbb{Z}$$

by $\mathbf{c}(a,b) = b - a$. Put $\kappa = |\mathbf{c}(\omega)|$. Let $\theta = \pi(\Theta)$. Since $\mathbf{c}(x+\omega) = \mathbf{c}(x) - \kappa$, the content map $\mathbf{c}$ induces the map

$$\theta \to \mathbb{Z}/\kappa\mathbb{Z},$$

which we denote by the same symbol $\mathbf{c}$. It is easy to show the following:

**Proposition 11.** *For $x, y \in \theta$, the followings hold:*
(1) *If $\mathbf{c}(x) - \mathbf{c}(y) \equiv 0, \pm 1 \mod \kappa$, then $x$ and $y$ are comparable.*
(2) *If $x$ is covered by $y$, then $\mathbf{c}(x) - \mathbf{c}(y) \equiv \pm 1 \mod \kappa$.*

*Remark* 12. By Proposition 6 and Proposition 11, cylindric diagrams are infinite (locally finite) "$\mathbb{Z}/\kappa\mathbb{Z}$-colored $d$-complete posets" in the sense of [9, 10].

Let $i \in \mathbb{Z}/\kappa\mathbb{Z}$. By Proposition 11 (1), the inverse image $\mathbf{c}^{-1}(i)$ is non-empty totally ordered subset of $\theta$. Let $b_i$ denote the minimum element in $\mathbf{c}^{-1}(i)$.

**Definition 13.** Define the *bottom set* $\Gamma$ of $\theta$ by

$$\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}.$$

Figure 4 indicates the periodic diagram $\hat{\lambda}$ with $\lambda = (5,4,4,2) \in \mathcal{P}_{(4,-5)}$. The number in each cell is the content with modulo 9. Yellowed cells forms the bottom set of $\mathring{\lambda} = \pi(\hat{\lambda})$.

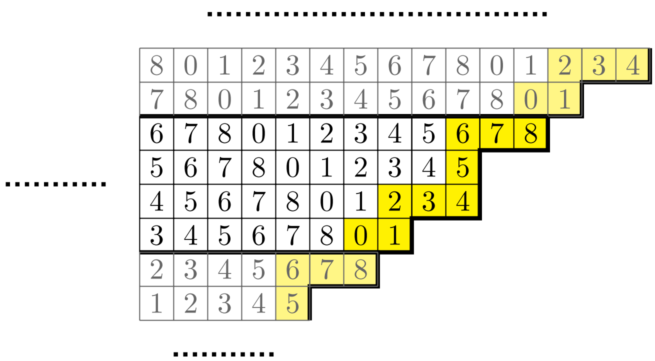



Figure 4:

## 1.4 Root systems and affine Weyl groups of type $A^{(1)}_{\kappa-1}$

Let $\kappa \in \mathbb{Z}_{\geqq 2}$. In the rest, we often identify $\mathbb{Z}/\kappa\mathbb{Z}$ with $\{0, 1, \ldots, \kappa-1\}$. Let $\mathfrak{h}$ be a $(\kappa+1)$-dimensional vector space and choose elements $\alpha_i^\vee$ $(i \in \mathbb{Z}/\kappa\mathbb{Z})$ and $d$ of $\mathfrak{h}$ so that

$$\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_{\kappa-1}^\vee, d\}$$

forms a basis for $\mathfrak{h}$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Define elements $\alpha_j$ $(j \in \mathbb{Z}/\kappa\mathbb{Z})$ and $\varpi_0$ of $\mathfrak{h}^*$ by

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}, \quad \langle \varpi_0, \alpha_i^\vee \rangle = \delta_{i0} \quad (i, j \in \mathbb{Z}/\kappa\mathbb{Z}),$$
$$\langle \alpha_j, d \rangle = \delta_{j0}, \quad \langle \varpi_0, d \rangle = 0,$$

where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Z}$ is the natural pairing and the integer $a_{ij}$ is defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i - j = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

for $\kappa \geqq 3$ and

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -2 & \text{if } i \neq j \end{cases}$$

for $\kappa = 2$. Then $\{\alpha_0, \alpha_1, \ldots, \alpha_{\kappa-1}, \varpi_0\}$ forms a basis for $\mathfrak{h}^*$. Define $\varpi_i \in \mathfrak{h}^*$ $(i = 1, 2, \ldots, \kappa-1)$ by

$$\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \langle \varpi_i, d \rangle = 0 \quad (j \in \mathbb{Z}/\kappa\mathbb{Z}).$$

The weights $\varpi_0, \varpi_1, \ldots, \varpi_{\kappa-1}$ are called fundamental weights. Put $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{\kappa-1}$ (resp. $\delta^\vee = \alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_{\kappa-1}^\vee$), which is called the null root (resp. the null coroot).

For $i \in \mathbb{Z}/\kappa\mathbb{Z}$, define the simple reflection $s_i \in GL(\mathfrak{h}^*)$ by

$$s_i(\zeta) = \zeta - \langle \zeta, \alpha_i^\vee \rangle \alpha_i \quad (\zeta \in \mathfrak{h}^*).$$

Define the *affine Weyl group* $W$ of type $A_{\kappa-1}^{(1)}$ as the subgroup of $GL(\mathfrak{h}^*)$ generated by simple reflections:

$$W = \langle s_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\rangle.$$

The following is well-known:

**Proposition 14.** *The group $W$ has the following fundamental relations*:

$$s_i^2 = 1, \tag{1.1}$$
$$s_i s_j = s_j s_i \quad (i - j \neq 0, \pm 1), \tag{1.2}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{1.3}$$

For $w \in W$, we define the *length* $\ell(w)$ of $w$ as the smallest $r$ for which an expression (or a word)

$$w = s_{i_1} s_{i_2} \cdots s_{i_r} \in W \quad (i_j \in \mathbb{Z}/\kappa\mathbb{Z})$$

exists. An expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is said to be reduced if $\ell(w) = r$.

Define the action of $W$ on $\mathfrak{h}$ by

$$s_i(h) = h - \langle \alpha_i, h\rangle \alpha_i^\vee \quad (h \in \mathfrak{h}).$$

We put

$$\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_{\kappa-1}\}, \quad \Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_{\kappa-1}^\vee\},$$
$$Q = \left\{ \sum_{i \in \mathbb{Z}/\kappa\mathbb{Z}} c_i \alpha_i \;\middle|\; c_i \in \mathbb{Z} \right\}, \quad Q_+ = \left\{ \sum_{i \in \mathbb{Z}/\kappa\mathbb{Z}} c_i \alpha_i \;\middle|\; c_i \in \mathbb{Z}_{\geqq 0} \right\}.$$

The set $\Pi$ (resp. $\Pi^\vee$) is called the set of simple roots (resp. the set of simple coroots), and $Q$ is called the root lattice. Put

$$R = W\Pi \subset \mathfrak{h}^*, \quad R^\vee = W\Pi^\vee \subset \mathfrak{h}.$$

Then $R$ (resp. $R^\vee$) is the set of real roots (resp. coroots) and $R \sqcup \mathbb{Z}\delta$ is the affine root system. Define the set $R_+$ of positive (real) roots and the set $R_-$ of negative (real) roots by

$$R_+ = R \cap Q_+ = \left\{ \sum_{i=0}^{\kappa-1} c_i \alpha_i \in R \mid c_i \in \mathbb{Z}_{\geqq 0} \right\}, \quad R_- = \left\{ \sum_{i=0}^{\kappa-1} c_i \alpha_i \in R \mid c_i \in \mathbb{Z}_{\leqq 0} \right\}.$$

For $\beta = \sum_{i=0}^{\kappa-1} k_i \alpha_i \in R$, define $\beta^\vee = \sum_{i=0}^{\kappa-1} k_i \alpha_i^\vee \in R^\vee$. Then the correspondence $\beta \mapsto \beta^\vee$ gives a bijection $R \to R^\vee$. Define the set of positive (resp. negative) coroots $R_+^\vee$ (resp. $R_-^\vee$) as the image of $R_+$ (resp. $R_-$) by this bijection.

For $i, j \in \mathbb{Z}$ with $i < j$, we define

$$\alpha_{ij} = \sum_{i \leqq k \leqq j-1} \alpha_{\bar{k}},$$

where $\bar{k} = k \mod \kappa\mathbb{Z} \in \mathbb{Z}/\kappa\mathbb{Z}$. The followings are well-known:

$$R_+ = \{\alpha_{ij} \mid i < j,\ j - i \notin \kappa\mathbb{Z}\} \tag{1.4}$$
$$= \{\alpha_{ij} + k\delta \mid 1 \leqq i < j \leqq \kappa,\ k \geqq 0\} \sqcup \{-\alpha_{ij} + k\delta \mid 1 \leqq i < j \leqq \kappa,\ k \geqq 1\}, \tag{1.5}$$
$$R_- = -R_+, \quad R = R_+ \sqcup R_-.$$

From the description of $R$ above, the following two lemmas follow easily and they will be used later:

**Lemma 15.** *If $\alpha \in R$, then $\alpha + k\delta \in R$ for all $k \in \mathbb{Z}$.*

**Lemma 16.** *Let $\alpha \in R \sqcup \mathbb{Z}\delta$ and $\beta \in R$. Then $\langle \alpha, \beta^\vee \rangle = 2$ if and only if $\alpha \equiv \beta \mod \delta$.*

## 2 Hooks in cylindric diagrams

### 2.1 Colored hook length

In this section, we will introduce colored hook length, which is a key ingredient in this paper.

Fix $\kappa, m, \ell \in \mathbb{Z}_{\geqq 1}$ with $\kappa = m + \ell$ and let $\theta$ be a cylindric diagram in $\mathcal{C}_{(m,-\ell)}$.

In the rest of this paper, we use the following notations:

$$\alpha(x) = \alpha_{\mathbf{c}(x)}, \quad s(x) = s_{\mathbf{c}(x)} \quad \text{for } x \in \theta.$$

**Definition 17.** For $x \in \theta$, put

$$\mathrm{Arm}(x) = \{x + (0,k) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\},$$
$$\mathrm{Leg}(x) = \{x + (k,0) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\},$$

and define

$$\mathbf{h}(x) = \alpha(x) + \sum_{y \in \mathrm{Arm}(x)} \alpha(y) + \sum_{y \in \mathrm{Leg}(x)} \alpha(y).$$

We call $\mathbf{h}(x)$ the *colored hook length* at $x$.

For $x \in \mathcal{C}_{(m,-\ell)} \setminus \theta$, we set $\mathbf{h}(x) = 0$ for convenience. It is easy to see that for $x \in \theta$

$$\mathbf{h}(x - (0,\ell)) = \mathbf{h}(x - (m,0)) = \mathbf{h}(x) + \delta$$

and

$$\mathbf{h}(x) = \alpha_{ij} \text{ for some integers } i < j. \tag{2.1}$$

**Example 18.** (See Figure 5.) Let $\omega = (4,-5)$. Then $\lambda = (5,3,3,1) \in \mathcal{P}_\omega$. For a cell $x = \pi(2,-4) \in \mathring{\lambda}$, we have $\mathbf{c}(x) = 3 + 9\mathbb{Z} \in \mathbb{Z}/9\mathbb{Z}$. The colored hook length at $x$ is

$$\begin{aligned}\mathbf{h}(x) &= \alpha_{-6} + (\alpha_{-5} + \alpha_{-4} + \alpha_{-3} + \alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1) \\ &\quad + (\alpha_{-7} + \alpha_{-8} + \alpha_{-9} + \alpha_{-10} + \alpha_{-11} + \alpha_{-12}) \\ &= \alpha_3 + (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_0 + \alpha_1) + (\alpha_2 + \alpha_1 + \alpha_0 + \alpha_8 + \alpha_7 + \alpha_6) \\ &= \delta + \alpha_0 + \alpha_1 + \alpha_6 + \alpha_7 + \alpha_8,\end{aligned}$$

which can be expressed as $\mathbf{h}(x) = \alpha_{-12,2}$.

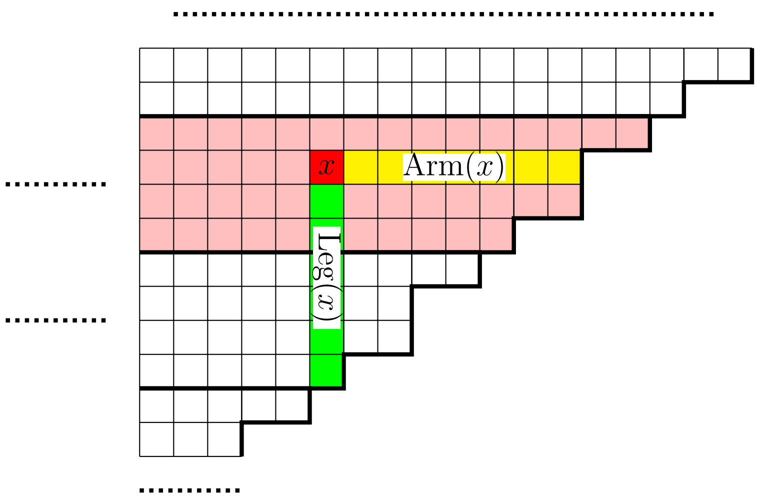



Figure 5: The sets $\mathrm{Arm}(x)$ and $\mathrm{Leg}(x)$ for $x$ in the cylindric diagram.

*Remark* 19. (1) For $x \in \theta$, the "multiset" $\mathrm{H}(x) := \{x\} \sqcup \mathrm{Arm}(x) \sqcup \mathrm{Leg}(x)$ is a cylindric analogue of the hook at $x$.
(2) A conjectural hook formula concerning the number of standard tableaux on cylindric skew diagrams is proposed in [11], where the hook length at $x \in \theta$ is given by $|H(x)| = |\mathrm{Arm}(x)| + |\mathrm{Leg}(x)| + 1$.

For $\alpha \in Q_+$, define

$$N(\alpha) = \max\{k \in \mathbb{Z} \mid \alpha - k\delta \in Q_+\}.$$

**Lemma 20.** *For $x \in \theta$, it holds that*

$$N(\mathbf{h}(x)) = \max\{k \in \mathbb{Z} \mid x + k(0,\ell) \in \theta\}.$$

*Proof.* We put $N(x) = \max\{k \in \mathbb{Z} \mid x + k(0,\ell) \in \theta\}$ and will show $N(\mathbf{h}(x)) = N(x)$.

Let $k \in \mathbb{Z}_{\geqq 0}$. Suppose that $x + k(0,\ell) \in \theta$. Then, $\mathbf{h}(x) - k\delta = \mathbf{h}(x + k(0,\ell)) \in Q_+$ and thus $N(\mathbf{h}(x)) \geqq N(x)$.

Suppose that $x - k(0,\ell) \notin \theta$. Noting that $x - k(0,\ell) = x + k(m,0)$, we have

$$|\mathrm{Arm}(x)| \leqq k\ell - 1, \quad |\mathrm{Leg}(x)| \leqq km - 1.$$

Thus we have $|\{x\} \cup \mathrm{Arm}(x) \cup \mathrm{Leg}(x)| \leqq k(\ell + m) - 1$ and hence $\mathbf{h}(x) - k\delta \notin Q_+$. This means $N(\mathbf{h}(x)) \leqq N(x)$. $\square$

Let $\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}$ be the bottom set of $\theta$, where $b_i$ is the minimum element of $\mathbf{c}^{-1}(i)$ as before.

For $\alpha = \sum_{i \in \mathbb{Z}/\kappa\mathbb{Z}} c_i \alpha_i \in Q_+$, define its *support* by

$$\mathrm{Supp}(\alpha) = \{b_i \mid c_i > 0 \ (i \in \mathbb{Z}/\kappa\mathbb{Z})\} \subset \Gamma.$$

For example, we have $\mathrm{Supp}(\delta) = \Gamma$. Let $x \in \theta$ with $N(\mathbf{h}(x)) = 0$. Then $\mathrm{Supp}(\mathbf{h}(x))$ is a non-empty, proper and connected subset of $\Gamma$.

**Lemma 21.** *Let $x \in \theta$. Then $\mathbf{h}(x) \in R_+$.*

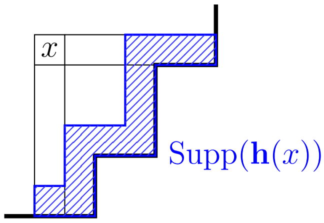
*Proof.* By (1.4) and (2.1), it is enough to show that $\mathbf{h}(x) \notin \mathbb{Z}\delta$.

Put $k = N(\mathbf{h}(x))$ and $x_0 = x + k(0, \ell)$. Then $x_0 \in \theta$ by Lemma 20 and $N(\mathbf{h}(x_0)) = 0$. Since $\emptyset \neq \mathrm{Supp}(\mathbf{h}(x_0)) \subsetneqq \Gamma$, we have $\mathbf{h}(x_0) \notin \mathbb{Z}\delta$ and thus $\mathbf{h}(x) = \mathbf{h}(x_0) + k\delta \notin \mathbb{Z}\delta$. □

Let $\Gamma_{\max}$ (resp. $\Gamma_{\min}$) denote the set of maximal (resp. minimal) elements in $\Gamma$. Note that $|\Gamma_{\max}| = |\Gamma_{\min}|$. One can easily see the following lemma. (See the figure below.)

**Lemma 22.** *Let $\alpha \in R_+$ with $N(\alpha) = 0$. Then $\alpha = \mathbf{h}(x)$ for some $x \in \theta$ if and only if*

$$|\mathrm{Supp}(\alpha) \cap \Gamma_{\max}| + 1 = |\mathrm{Supp}(\alpha) \cap \Gamma_{\min}|.$$




## 2.2 Predominant weights and hooks

**Definition 23.** We define $\zeta_\theta \in \mathfrak{h}^*$ by

$$\zeta_\theta = \sum_{i=0}^{\kappa-1} a_i \varpi_i, \tag{2.2}$$

$$\text{where } a_i = \begin{cases} 1 & \text{if } b_i \in \Gamma_{\max} \\ -1 & \text{if } b_i \in \Gamma_{\min} \\ 0 & \text{otherwise.} \end{cases}$$

Note that maximal and minimal elements are lined up alternatively in $\Gamma$. This implies that the weight $\zeta_\theta$ is predominant, namely, $\langle \zeta_\theta, \alpha^\vee \rangle \geqq -1$ for all $\alpha^\vee \in R_+^\vee$. Define

$$D(\zeta_\theta) = \{\alpha \in R_+ \mid \langle \zeta_\theta, \alpha^\vee \rangle = -1\}.$$

**Theorem 24.** *The correspondence $x \mapsto \mathbf{h}(x)$ gives a bijection*

$$\mathbf{h} : \theta \to D(\zeta_\theta).$$

*Proof.* First we will show that $\mathbf{h}(\theta) = D(\zeta_\theta)$. Let $\alpha \in R_+$ and put $\bar{\alpha} = \alpha - N(\alpha)\delta$. It follows from Lemma 20,

$$\alpha \in \mathbf{h}(\theta) \Leftrightarrow \bar{\alpha} \in \mathbf{h}(\theta).$$

On the other hand, as $\langle \zeta_\theta, \delta^\vee \rangle = 0$, it holds that

$$\alpha \in D(\zeta_\theta) \Leftrightarrow \bar{\alpha} \in D(\zeta_\theta).$$

Now we have $\mathbf{h}(\theta) = D(\zeta_\theta)$ by Lemma 22.

We will show the injectivity. Suppose that $\mathbf{h}(x) = \mathbf{h}(y)$. Then $N(\mathbf{h}(x)) = N(\mathbf{h}(y))$. Put $x_0 = x + N(\mathbf{h}(x))(0, \ell)$, $y_0 = y + N(\mathbf{h}(y))(0, \ell)$. Then we have $N(\mathbf{h}(x_0)) = N(\mathbf{h}(y_0)) = 0$ and thus $\mathbf{h}(x_0) = \mathbf{h}(y_0)$. Now we have $\mathrm{Supp}(\mathbf{h}(x_0)) = \mathrm{Supp}(\mathbf{h}(y_0))$ and this imples $x_0 = y_0$ and hence $x = y$. □

## 2.3 Weyl group elements and their inversion sets

The following proposition gives an alternative expression for $\mathbf{h}(x)$.

**Proposition 25.** *For any $x \in \theta$ and $\mathfrak{t} \in \mathrm{ST}(\theta)$, it holds that*

$$\mathbf{h}(x) = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n-1))\alpha(\mathfrak{t}^{-1}(n)), \tag{2.3}$$

*where $n = \mathfrak{t}(x)$.*

The proof of Proposition 25 will be given in the next section. In the rest of this section, we will see some consequences of the proposition.

Let $\mathfrak{t} \in \mathrm{ST}(\theta)$. For $n \in \mathbb{Z}_{\geqq 1}$, we define an element $w_{\theta,\mathfrak{t}}[n]$ of $W$ by

$$w_{\theta,\mathfrak{t}}[n] = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n)), \tag{2.4}$$

and we set $w_{\theta,\mathfrak{t}}[0] = e$.

**Example 26.** Let $\lambda = (5,4)$ and $\omega = (2,-3)$. For $\mathfrak{t} \in \mathrm{ST}(\mathring{\lambda})$ displayed in figure 6, we have $w_{\mathring{\lambda},\mathfrak{t}}[6] = s_4 s_2 s_1 s_3 s_0 s_2$.

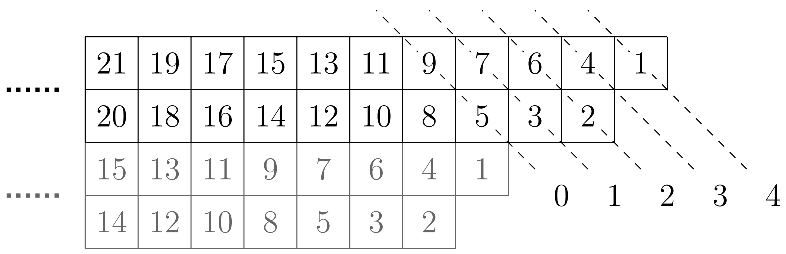



Figure 6:

**Proposition 27.** *The expression* (2.4) *is reduced.*

*Proof.* Put $p_k = \mathfrak{t}^{-1}(k)$ for $k \geqq 1$. By Proposition 25 and Theorem 24, we have

$$\mathbf{h}(p_k) = s(p_1)s(p_2) \cdots s(p_{k-1})\alpha(p_k) = w_{\theta,\mathfrak{t}}[k-1]\alpha(p_k) \in R_+$$

for all $k \in [1,n]$. This implies that

$$\ell(w_{\theta,\mathfrak{t}}[k-1]s(p_k)) > \ell(w_{\theta,\mathfrak{t}}[k-1]) \quad (k \in [1,n]).$$

Therefore we have $\ell(w_{\theta,\mathfrak{t}}[n]) = n$ and thus the expression (2.4) is reduced. $\square$

For $w \in W$, the set

$$R(w) = R_+ \cap wR_-$$

is called the *inversion set* of $w$. It is known for any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ that $\ell(w) = |R(w)|$ and

$$R(w) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, s_{i_1}s_{i_2}\alpha_{i_3}, \ldots, s_{i_1}s_{i_2}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}\}.$$

By (2.3), (2.4) and Proposition 27, we obtain the following proposition:

**Proposition 28.** *Let $\mathfrak{t} \in \mathrm{ST}(\theta)$ and $n \in \mathbb{Z}_{\geqq 1}$. Then it holds that*

$$R(w_{\theta,\mathfrak{t}}[n]) = \{\mathbf{h}(x) \mid x \in \mathfrak{t}^{-1}([1,n])\}.$$

*In particular, it holds that $R(w_{\theta,\mathfrak{t}}[n]) \subset D(\zeta_\theta)$.*

Define

$$R(w_{\theta,\mathfrak{t}}) = \bigcup_{n \geqq 1} R(w_{\theta,\mathfrak{t}}[n]).$$

Then

$$R(w_{\theta,\mathfrak{t}}) = \{\mathbf{h}(x) \mid x \in \theta\} = D(\zeta_\theta).$$

In particular, $R(w_{\theta,\mathfrak{t}})$ is independent of $\mathfrak{t}$ and we will denote it just by $R(w_\theta)$ in the rest.

*Remark* 29. The set $R(w_\theta)$ can be thought as the "inversion set" associated with the semi-infinite word

$$w_{\theta,\mathfrak{t}} := s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots\cdots .$$

**Definition 30.** Let $\zeta \in P$ be an integral weight.
(1) An element $w$ of $W$ is said to be $\zeta$-*pluscule* if

$$\langle \zeta, \alpha^\vee \rangle = -1 \quad \text{for all } \alpha \in R(w).$$

(2) An element $w$ of $W$ is said to be $\zeta$-*minuscule* if

$$\langle \zeta, \alpha^\vee \rangle = 1 \quad \text{for all } \alpha \in R(w^{-1}).$$

**Definition 31.** An element $w \in W$ is said to be *fully commutative* if any reduced expression of $w$ can be obtained from any other by using only the relations (1.2).

*Remark* 32. (1) An element $w \in W$ is $\zeta$-pluscule if and only if $w$ is $(w^{-1}\zeta)$-minuscule.
(2) It is known that if $w$ is $\zeta$-minuscule for some integral weight $\zeta$ then $w$ is fully commutative ([8]).

By Proposition 28, we have the following:

**Proposition 33.** *Let $\mathfrak{t} \in \mathrm{ST}(\theta)$ and $n \in \mathbb{Z}_{\geqq 1}$. Then $w_{\theta,\mathfrak{t}}[n]$ is $\zeta_\theta$-pluscule and fully commutative.*

### 2.4 Proof of Proposition 25

For $\mathfrak{t} \in \mathrm{ST}(\theta)$ and $x \in \theta$, we put

$$\gamma_{\mathfrak{t}}(x) = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n-1))\alpha(\mathfrak{t}^{-1}(n)), \tag{2.5}$$

where $n = \mathfrak{t}(x)$.

For $x \in \theta$, put $x^S = x + (1,0)$, $x^E = x + (0,1)$, $x^{SE} = x + (1,1) \in \mathcal{C}_\omega$. We will use the following lemma later:

**Lemma 34.** *Let $x \in \theta$.*
(1) *If $x \notin \Gamma$, then $x^S, x^E, x^{SE} \in \theta$ and*

$$\gamma_{\mathfrak{t}}(x) = \gamma_{\mathfrak{t}}(x^S) + \gamma_{\mathfrak{t}}(x^E) - \gamma_{\mathfrak{t}}(x^{SE}). \tag{2.6}$$

(2) *If $x \in \Gamma$, then $x^{SE} \notin \theta$ and*

$$\gamma_{\mathfrak{t}}(x) = \begin{cases} \alpha(x) + \gamma_{\mathfrak{t}}(x^S) + \gamma_{\mathfrak{t}}(x^E) & \text{if } x^S, x^E \in \theta & (2.7)\\ \alpha(x) + \gamma_{\mathfrak{t}}(x^S) & \text{if } x^S \in \theta, x^E \notin \theta & (2.8)\\ \alpha(x) + \gamma_{\mathfrak{t}}(x^E) & \text{if } x^E \in \theta, x^S \notin \theta & (2.9)\\ \alpha(x) & \text{if } x^S, x^E \notin \theta. & (2.10) \end{cases}$$

*Proof.* We put $p_k = \mathfrak{t}^{-1}(k)$ $(k \in \mathbb{Z}_{\geqq 1})$.
(1) Let $x = p_j$, $x^{SE} = p_i$, $x^E = p_{k_1}$ and $x^S = p_{k_2}$. Then $j > k_1, k_2 > i$ and we may assume that $k_2 < k_1$. Put $\mathbf{c}(x) = r$. Then $\mathbf{c}(x^E) = r - 1$, $\mathbf{c}(x^S) = r + 1$. We have

$$\gamma_{\mathfrak{t}}(x) = w_1 s(p_i) w_2 s(p_{k_1}) w_3 s(p_{k_2}) w_4 \alpha(p_j) = w_1 s_r w_2 s_{r+1} w_3 s_{r-1} w_4 \alpha_r,$$

where $w_1 = s(p_1) \cdots s(p_{i-1})$, $w_2 = s(p_{i+1}) \cdots s(p_{k_1-1})$, $w_3 = s(p_{k_1+1}) \cdots s(p_{k_2-1})$ and $w_4 = s(p_{k_2+1}) \cdots s(p_{j-1})$.

Note that $\mathbf{c}(p_d) - r \neq 0, \pm 1$ for all $d \in [i+1, j-1] \setminus \{k_1, k_2\}$. Actually, if $\mathbf{c}(p_d) - r = 0, \pm 1$ then $p_d$ is comparable with $p_j$ and $p_i$, and hence $p_j > p_d > p_i$. But such $d$ must be $k_1$ or $k_2$. Now we have

$$\begin{aligned} \gamma_{\mathfrak{t}}(x) &= w_1 s_r w_2 s_{r+1} w_3 s_{r-1} w_4 \alpha_r = w_1 s_r w_2 s_{r+1} w_3 s_{r-1} \alpha_r \\ &= w_1 s_r w_2 s_{r+1} w_3 (\alpha_{r-1} + \alpha_r) = \gamma_{\mathfrak{t}}(x^S) + w_1 s_r w_2 s_{r+1} w_3 \alpha_r \\ &= \gamma_{\mathfrak{t}}(x^E) + w_1 s_r w_2 s_{r+1} \alpha_r = \gamma_{\mathfrak{t}}(x^E) + w_1 s_r w_2 (\alpha_r + \alpha_{r+1}) \\ &= \gamma_{\mathfrak{t}}(x^S) + \gamma_{\mathfrak{t}}(x^E) + w_1 s_r w_2 \alpha_r = \gamma_{\mathfrak{t}}(x^S) + \gamma_{\mathfrak{t}}(x^E) + w_1 s_r \alpha_r \\ &= \gamma_{\mathfrak{t}}(x^S) + \gamma_{\mathfrak{t}}(x^E) - \gamma_{\mathfrak{t}}(x^{SE}). \end{aligned}$$

(2) Suppose that $x^S, x^E \notin \theta$, or equivalently, suppose that $x$ is minimal element in $\Gamma$. Let $x = p_j$. Then $p_d$ $(d \in [1, j-1])$ is not comparable with $p_j$. Hence

$$\gamma_{\mathfrak{t}}(x) = s(p_1) \cdots s(p_{j-1}) \alpha(p_j) = \alpha(p_j)$$

The other cases are reduced to the case where $x$ is minimal in $\Gamma$, via a similar argument as in the proof of the statement (1),

□

*Proposition 25.* Let $x \in \theta$. Put $x^S = x + (1, 0)$, $x^E = x + (0, 1)$, $x^{SE} = x + (1, 1)$. It is easy to see the following:

$$\mathbf{h}(x) = \begin{cases} \mathbf{h}(x^S) + \mathbf{h}(x^E) - \mathbf{h}(x^{SE}) & \text{if } x \notin \Gamma \\ \alpha(x) + \mathbf{h}(x^S) + \mathbf{h}(x^E) & \text{if } x \in \Gamma \text{ and } x^S, x^E \in \theta \\ \alpha(x) + \mathbf{h}(x^S) & \text{if } x \in \Gamma \text{ and } x^S \in \theta,\ x^E \notin \theta \\ \alpha(x) + \mathbf{h}(x^E) & \text{if } x \in \Gamma \text{ and } x^E \in \theta,\ x^S \notin \theta \\ \alpha(x) & \text{if } x \in \Gamma \text{ and } x^S, x^E \notin \theta \end{cases} \tag{2.11}$$

On the other hand, we have shown that $\gamma_t(x)$ satisfies the same recurrence relations in Lemma 34. $\square$

# 3 Poset structure of cylindric diagrams

## 3.1 Partial orders on the inversion set

Recall that $Q$ denote the root lattice: $Q = \bigoplus_{i \in \mathbb{Z}/\kappa\mathbb{Z}} \mathbb{Z}\alpha_i$.

**Definition 35.** Define the partial order $\leqslant^{\mathrm{or}}$ on $Q$ by

$$\alpha \leqslant^{\mathrm{or}} \beta \iff \beta - \alpha \in Q_+ = \bigoplus_{i \in \mathbb{Z}/\kappa\mathbb{Z}} \mathbb{Z}_{\geqq 0}\alpha_i$$

The order $\leqslant^{\mathrm{or}}$ is called the *ordinary order*.

The restriction of the ordinary order defines a poset structure on $R(w_\theta)$.

Let $\theta$ be a cylindric diagram in $\mathcal{C}_\omega$ with $|\omega| = \kappa$. We have introduced a poset structure on $\theta$ and also have seen that the map $\mathbf{h}$ gives a bijection between $\theta$ and $R(w_\theta)$. Remark that this is not a poset isomorphism as seen in the following example:

**Example 36.** Let $\lambda = (4,2)$, $\omega = (2,-2)$ and consider the cylindric diagram $\mathring{\lambda}$ in $\mathcal{C}_\omega$. Then $x = \pi(1,2)$ and $y = \pi(2,1)$ are incomparable in $\mathring{\lambda}$. On the other hand, $\mathbf{h}(x) = \delta + \alpha_3$ and $\mathbf{h}(y) = \alpha_0 + \alpha_2 + \alpha_3$, and hence $\mathbf{h}(x) - \mathbf{h}(y) = \alpha_1 + \alpha_3$. This implies $\mathbf{h}(y) \leqslant^{\mathrm{or}} \mathbf{h}(x)$.

We will introduce a modified ordinary order $\trianglelefteq$, for which we will have $(\theta, \leqslant) \cong (R(w_\theta), \trianglelefteq)$.

Let $\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}$ be the bottom set of $\theta$, where $b_i$ is the element such that $\mathbf{c}(b_i) = i$. Let $\Gamma_{\max}$ (resp. $\Gamma_{\min}$) denote the set of maximal (resp. minimal) elements in $\Gamma$.

**Definition 37.** Define

$$\Pi_\theta = \Pi_\theta^0 \sqcup \Pi_\theta^{\mathrm{arm}} \sqcup \Pi_\theta^{\mathrm{leg}}.$$

Here,

$$\Pi_\theta^0 = \{\alpha(x) \mid x \in \Gamma \setminus (\Gamma_{\max} \sqcup \Gamma_{\min})\},$$
$$\Pi_\theta^{\mathrm{arm}} = \left\{ \alpha(x) + \sum_{y \in \mathrm{Arm}(x)} \alpha(y) \;\middle|\; x \in \Gamma_{\max} \right\},$$
$$\Pi_\theta^{\mathrm{leg}} = \left\{ \alpha(x) + \sum_{y \in \mathrm{Leg}(x)} \alpha(y) \;\middle|\; x \in \Gamma_{\max} \right\}.$$

Note that $\Pi_\theta \subset R_+ \sqcup \mathbb{Z}_{\geqq 0}\delta$.

**Example 38.** For the cylindric diagram described in Fig. 4, we have

$$\begin{aligned}
&\Pi_\theta^0 = \{\alpha_3, \alpha_5, \alpha_7\},\\
&\Pi_\theta^{\mathrm{arm}} = \{\alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_3 + \alpha_4, \alpha_0 + \alpha_1\},\\
&\Pi_\theta^{\mathrm{leg}} = \{\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2, \alpha_0 + \alpha_8\}.
\end{aligned}$$

**Example 39.** Let $\lambda = (n)$ and $\omega = (1, -n+1)$. Then, for the corresponding cylindric diagram $\mathring{\lambda}$, we have

$$\begin{aligned}
&\Pi_{\mathring{\lambda}}^0 = \{\alpha_1, \alpha_2, \dots, \alpha_{n-2}\},\\
&\Pi_{\mathring{\lambda}}^{\mathrm{arm}} = \{\delta\},\\
&\Pi_{\mathring{\lambda}}^{\mathrm{leg}} = \{\alpha_0 + \alpha_{n-1}\}.
\end{aligned}$$

**Definition 40.** Define the partial order $\trianglelefteq$ on $R(w_\theta)$ by

$$\alpha \trianglelefteq \beta \iff \beta - \alpha \in \sum_{\gamma \in \Pi_\theta} \mathbb{Z}_{\geqq 0}\gamma = \left\{ \sum_{\gamma \in \Pi_\theta} k_\gamma \gamma \;\middle|\; k_\gamma \in \mathbb{Z}_{\geqq 0} \ (\forall \gamma \in \Pi_\theta) \right\}. \tag{3.1}$$

**Proposition 41.** *Let $x, y \in \theta$. Then*

$$x \leqslant y \implies \mathbf{h}(x) \trianglelefteq \mathbf{h}(y).$$

*In other words, the bijection*

$$\mathbf{h} : (\theta, \leqslant) \to (R(w_\theta), \trianglelefteq)$$

*is order preserving.*

*Proof.* We assume that $y$ covers $x$, and will show that $\mathbf{h}(y) - \mathbf{h}(x) \in \Pi_\theta$ by induction on $y$ concerning the poset structure on $\theta$.

Put $y^S = y + (1,0)$, $y^E = y + (0,1)$, $y^{SE} = y + (1,1)$. Then $x = y^S$ or $x = y^E$.

When $y \in \Gamma$, it follows from (2.11) that $\mathbf{h}(y) - \mathbf{h}(x) \in \Pi_\theta$.

Suppose that $y \notin \Gamma$. Note that $y^{SE} \in \theta$. Since $y^E$ covers $y^{SE}$ and $y > y^E$, we have $\mathbf{h}(y^E) - \mathbf{h}(y^{SE}) \in \Pi_\theta$ by induction hypothesis. By the recursion relation (2.11), we have

$$\mathbf{h}(y) - \mathbf{h}(y^S) = \mathbf{h}(y^E) - \mathbf{h}(y^{SE}) \in \Pi_\theta.$$

Similar argument implies $\mathbf{h}(y) - \mathbf{h}(y^E) \in \Pi_\theta$. In both cases, we have $\mathbf{h}(y) - \mathbf{h}(x) \in \Pi_\theta$. Therefore, the statement is proved. □

It is easy to see that

$$\alpha \trianglelefteq \beta \implies \alpha \leqslant^{\mathrm{or}} \beta$$

for any $\alpha, \beta \in R(w_\theta)$. Thus we have the following:

**Corollary 42.** *Let $x, y \in \theta$. Then*

$$x < y \implies \mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y).$$

### 3.2 Poset isomorphism

Our next goal is to prove that the order preserving map

$$\mathbf{h} : (\theta, \leqslant) \to (R(w_\theta), \trianglelefteq)$$

is actually a poset isomorphism. We start with some preparations.

As before, we denote by $\mathrm{Supp}(\alpha)$ the support of $\alpha \in Q$. The following lemma is almost obvious from Definition 37.

**Lemma 43.** *Let $\alpha \in \Pi_\theta$. Then*

$$|\mathrm{Supp}(\alpha) \cap \Gamma_{\max}| = |\mathrm{Supp}(\alpha) \cap \Gamma_{\min}| = \begin{cases} 0 & (\alpha \in \Pi_\theta^0) \\ 1 & (\alpha \in \Pi_\theta^{\mathrm{arm}} \sqcup \Pi_\theta^{\mathrm{leg}}). \end{cases} \tag{3.2}$$

It is easy to see the next lemma:

**Lemma 44.** (1) *Let $\alpha \in R_+$. Then $N(\alpha) = \max\{N \in \mathbb{Z} \mid \alpha - N\delta \in R_+\}$.*
(2) *Let $x, y \in \theta$. If $x < y$ then $N(\mathbf{h}(x)) \leqq N(\mathbf{h}(y))$.*

*Proof.* (1) Follows from Lemma 15.
(2) Suppose $x < y$. By Corollary 42, we have $N(\mathbf{h}(x))\delta \leqslant^{\mathrm{or}} \mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y)$.

As $\mathbf{h}(y) - N(\mathbf{h}(x))\delta$ is in $R$ by Lemma 15, it must be a positive root. This means $N(\mathbf{h}(x)) \leqq N(\mathbf{h}(y))$. ☐

**Lemma 45.** *Let $x, y \in \theta$ such that $N(\mathbf{h}(x)) = N(\mathbf{h}(y)) = 0$. Then*

$$x < y \iff \mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y).$$

*In particular, if $x$ and $y$ are incomparable, then $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are also incomparable with respect to $\leqslant^{\mathrm{or}}$.*

*Proof.* By Corollary 42, we have

$$x < y \implies \mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y).$$

We shall prove the opposite implication. Suppose $\mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y)$. Then noting that $0 <^{\mathrm{or}} \mathbf{h}(x), \mathbf{h}(y) <^{\mathrm{or}} \delta$, we have $\mathrm{Supp}(\mathbf{h}(x)) \subset \mathrm{Supp}(\mathbf{h}(y)) \subsetneq \Gamma$. This implies $x < y$. ☐

**Lemma 46.** *Let $x, y \in \theta$. Suppose that $x$ and $y$ are incomparable in $\theta$. Then $N(\mathbf{h}(y)) - N(\mathbf{h}((x)) = 1, 0$ or $-1$. Moreover the followings hold:*
(1) *If $N(\mathbf{h}(y)) - N(\mathbf{h}(x)) = 1$, then*

$$\mathbf{h}(y) - \delta \leqslant^{\mathrm{or}} \mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y).$$

(2) *If $N(\mathbf{h}(y)) - N(\mathbf{h}(x)) = -1$, then*

$$\mathbf{h}(x) - \delta \leqslant^{\mathrm{or}} \mathbf{h}(y) \leqslant^{\mathrm{or}} \mathbf{h}(x).$$

(3) *If $N(\mathbf{h}(y)) - N(\mathbf{h}(x)) = 0$, then $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are incomparable with respect to $\leqslant^{\mathrm{or}}$.*

*Proof.* In this proof, we denote $N(\mathbf{h}(x))$ by $N(x)$ for $x \in \theta$. Put

$$x_k = x + (N(x) - k)(0, \ell), \quad y_k = y + (N(y) - k)(0, \ell)$$

for $k \in \mathbb{Z}_{\geqq 0}$. Then $N(x_k) = N(y_k) = k$. Putting $n = N(x)$, one can see that

$$\theta \setminus (\{z \in \theta \mid z \geqq x\} \sqcup \{z \in \theta \mid z \leqq x\}) = [x_{n-1} - (1,1), x_{n+1} + (1,1)].$$

As $x$ and $y$ are incomparable, $y$ belongs to this interval and hence

$$x_{n-1} < y < x_{n+1} \tag{3.3}$$

and $n - 1 \leqq N(y) \leqq n + 1$ by Lemma 44. Namely, we have $N(y) - N(x) = -1, 0$ or $1$.

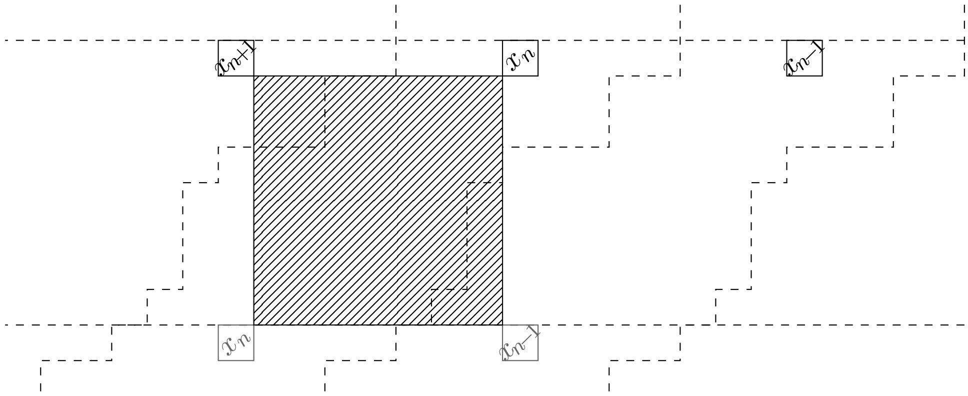


Figure 7: The cells in the shadow are incomparable with $x = x_n$.

(1) Suppose that $N(y) - N(x) = 1$. In this case,

$$\mathbf{h}(y) - \mathbf{h}(x) = \mathbf{h}(y_0) + \delta - \mathbf{h}(x_0).$$

By definition, $\mathbf{h}(x_0)$ and $\mathbf{h}(y_0)$ are positive roots. By Lemma 15, $\mathbf{h}(x_0) - \delta$ is also a root and it is not positive. Therefore $\delta - \mathbf{h}(x_0) \in R_+$ and $\mathbf{h}(y) - \mathbf{h}(x)$ is a sum of two positive roots. This implies $\mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y)$. Combining with (3.3), we have $x - \delta \leqslant^{\mathrm{or}} y \leqslant^{\mathrm{or}} x$.
(2) Follows from (1).
(3) Suppose that $N(y) - N(x) = 0$. Note that $x_0$ and $y_0$ are incomparable this case, and it follows from Lemma 45 that $\mathbf{h}(y_0)$ and $\mathbf{h}(x_0)$ are also incomparable with respect to $\leqslant^{\mathrm{or}}$. Now we have

$$\mathbf{h}(y) - \mathbf{h}(x) = \mathbf{h}(y_0) + N(y)\delta - \mathbf{h}(x_0) - N(x)\delta = \mathbf{h}(y_0) - \mathbf{h}(x_0).$$

and hence $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are incomparable with respect to $\leqslant^{\mathrm{or}}$. □

**Theorem 47.** *The map*

$$\mathbf{h} : (\theta, \leqslant) \to (R(w_\theta), \trianglelefteq)$$

*is a poset isomorphism.*

*Proof.* By Proposition 41, we have

$$x \leqslant y \implies \mathbf{h}(x) \trianglelefteq \mathbf{h}(y),$$

$$y \leqslant x \implies \mathbf{h}(y) \trianglelefteq \mathbf{h}(x).$$

Thus the statement follows if we prove that

$$x \text{ and } y \text{ are incomparable} \implies \mathbf{h}(x) \text{ and } \mathbf{h}(y) \text{ are incomparable with respect to } \trianglelefteq.$$

Suppose that $x$ and $y$ are incomparable. Then, putting $n = N(\mathbf{h}(x))$, we have $N(\mathbf{h}(y)) = n+1, n$ or $n-1$ by Lemma 46.

First we assume that $N(y) = n$. Then Lemma 46 implies that $\mathbf{h}(x)$ and $\mathbf{h}(y)$ must be incomparable.

Next, assume that $N(y) = n+1$. Then

$$\mathbf{h}(y) - \mathbf{h}(x) = \mathbf{h}(y_0) + \delta - \mathbf{h}(x_0),$$

where $x_0 = x + n(0,\ell)$ and $y_0 = y + (n+1)(0,\ell)$. By Lemma 22, we have

$$|\mathrm{Supp}(\mathbf{h}(y_0)) \cap \Gamma_{\max}| + 1 = |\mathrm{Supp}(\mathbf{h}(y_0)) \cap \Gamma_{\min}|,$$
$$|\mathrm{Supp}(\delta - \mathbf{h}(x_0)) \cap \Gamma_{\max}| = |\mathrm{Supp}(\delta - \mathbf{h}(x_0)) \cap \Gamma_{\min}| + 1.$$

They are not compatible with Lemma 43 and thus we have

$$\mathbf{h}(y_0) \notin \mathbb{Z}_{\geqq 0}\Pi_\theta, \quad \delta - \mathbf{h}(x_0) \notin \mathbb{Z}_{\geqq 0}\Pi_\theta \tag{3.4}$$

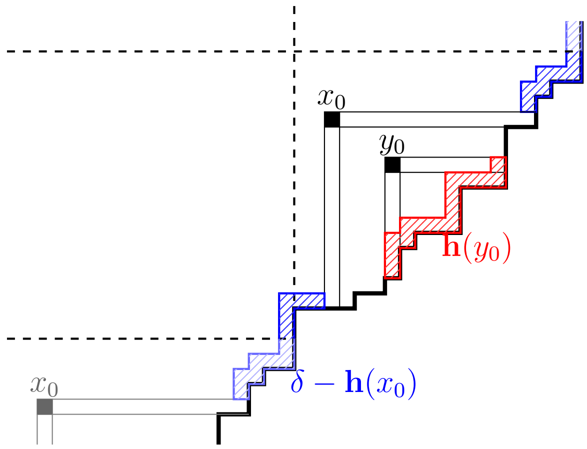



Figure 8:

We need to show that $\mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0)) \notin \mathbb{Z}_{\geqq 0}\Pi_\theta$. It follows from Lemma 46 that

$$0 \leqslant^{\mathrm{or}} \mathbf{h}(y_0) \leqslant^{\mathrm{or}} \mathbf{h}(x_0) \leqslant^{\mathrm{or}} \delta$$

and thus we have

$$0 \leqslant^{\mathrm{or}} \mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0)) \leqslant^{\mathrm{or}} \delta,$$

and

$$\mathrm{Supp}\,(\mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0))) = \mathrm{Supp}(\mathbf{h}(y_0)) \sqcup \mathrm{Supp}(\delta - \mathbf{h}(x_0)).$$

By (3.3), it holds that $y = y_{n+1} < x_{n+1}$. Thus $y_0 < x_0$ and moreover $x_0$ and $y_0$ are not located in the same row or column. Hence

$$x_0^{\mathrm{arm}}, x_0^{\mathrm{leg}} \notin \mathrm{Supp}\,(\mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0)))\,, \tag{3.5}$$

where $x_0^{\mathrm{arm}}$ (resp. $x_0^{\mathrm{leg}}$) is the minimal element in $\{x_0\} \cup \mathrm{Arm}(x_0)$ (resp. $\{x_0\} \cup \mathrm{Leg}(x_0)$).

Suppose that

$$\mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0)) = \sum_{i=1}^{r} \beta_i$$

with $\beta_1, \ldots, \beta_r \in \Pi_\theta$. Then $0 \leqslant^{\mathrm{or}} \beta_i \leqslant^{\mathrm{or}} \delta$ $(i = 1, \ldots, r)$, $0 \leqslant^{\mathrm{or}} \sum_{i=1}^{r} \beta_i \leqslant^{\mathrm{or}} \delta$ and

$$\mathrm{Supp}\left(\sum_{i=1}^{r} \beta_i\right) = \bigsqcup_{i=1}^{r} \mathrm{Supp}(\beta_i).$$

Note that each $\mathrm{Supp}(\beta_i)$ is an interval in $\theta$. Combining with (3.5), this implies that

$$\mathrm{Supp}(\beta_i) \subset \mathrm{Supp}\,(\mathbf{h}(y_0)) \quad \text{or} \quad \mathrm{Supp}(\beta_i) \subset \mathrm{Supp}\,(\delta - \mathbf{h}(x_0)))\,.$$

Thus there exist $i_1, \ldots, i_s$ for which we have $\mathbf{h}(y_0) = \beta_{i_1} + \cdots + \beta_{i_s}$, but this contradics (3.4). Therefore $\mathbf{h}(y) - \mathbf{h}(x) = \mathbf{h}(y_0) + (\delta - \mathbf{h}(x_0))$ cannot be a sum of elements in $\Pi_\theta$, and thus $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are incomparable with respect to $\trianglelefteq$.

The same argument implies that $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are incomparable also in the case where $N(\mathbf{h}(y)) = n - 1$. □

**Proposition 48.** *Let $\alpha, \beta \in R(w_\theta)$ with $\alpha \trianglelefteq \beta$. Then there exists a sequence*

$$\alpha = \gamma_1, \gamma_2, \ldots, \gamma_k = \beta$$

*in $R(w_\theta)$ such that $\gamma_{i+1} - \gamma_i \in \Pi_\theta$ $(i = 1, \ldots, k-1)$.*

*In other words, the partial order $\trianglelefteq$ on $R(w_\theta)$ coincides with the transitive closure of the relations*

$$\alpha \trianglelefteq\!\cdot\, \beta \ \textit{whenever}\ \beta - \alpha \in \Pi_\theta. \tag{3.6}$$

*Proof.* Let $\trianglelefteq^{\mathrm{tc}}$ denote the transitive closure of the relations above. It follows from the same argument in the proof of Proposition 41 that

$$x \leqslant y \implies \mathbf{h}(x) \trianglelefteq^{\mathrm{tc}} \mathbf{h}(y)$$

for any $x, y \in \theta$. It is clear that

$$\mathbf{h}(x) \trianglelefteq^{\mathrm{tc}} \mathbf{h}(y) \implies \mathbf{h}(x) \trianglelefteq \mathbf{h}(y).$$

Combining with Theorem 47, the statement follows. □

### 3.3 Heaps

Let $\theta$ be a cylindric diagram. Recall that standard tableaux on $\theta$ have been defined as order preserving bijection from $(\theta, \leqslant)$ to $(\mathbb{Z}_{\geqq 1}, \leqq)$. Through the bijection $\mathfrak{t}$, the set $\mathbb{Z}_{\geqq 1}$ inherits a partial order from $\theta$, which we will investigate in this section.

**Definition 49.** Let $\mathfrak{t} \in \mathrm{ST}(\theta)$. Define a partial order $\preceq_{\mathfrak{t}}$ on $\mathbb{Z}_{\geqq 1}$ as the transitive closure of the relations

$$a \preceq_{\mathfrak{t}} b \text{ whenever } a \leqq b \text{ and either } s_{i_a} s_{i_b} = s_{i_b} s_{i_a} \text{ or } i_a = i_b.$$

where $i_k = \mathbf{c}(\mathfrak{t}^{-1}(k))$ for $k \in \mathbb{Z}$. The poset $(\mathbb{Z}_{\geqq 1}, \preceq_{\mathfrak{t}})$ is called the *heap* of $w_{\theta,\mathfrak{t}}$.

**Proposition 50.** *Let $\theta$ be a cylindric diagram and $\mathfrak{t}$ a standard tableau on $\theta$. Then, the map $\mathfrak{t} : \theta \to \mathbb{Z}_{\geqq 1}$ gives a poset isomorphism*

$$(\theta, \leqslant) \cong (\mathbb{Z}_{\geqq 1}, \preceq_{\mathfrak{t}}).$$

*Proof.* Let $x, y \in \theta$. Suppose that $x < y$ is a covering relation in $\theta$. Then $y = x - (1, 0)$ or $y = x - (0, 1)$ and it is easy to see that $\mathfrak{t}(x) < \mathfrak{t}(y)$ and $s(x)s(y) \neq s(y)s(x)$. Hence $\mathfrak{t}(x) \preceq_{\mathfrak{t}} \mathfrak{t}(y)$.

Conversely, suppose that $\mathfrak{t}(x) \prec_{\mathfrak{t}} \mathfrak{t}(y)$ is a covering relation in $\mathbb{Z}_{\geqq 1}$. Then $s(x)s(y) \neq s(y)s(x)$ or $\mathbf{c}(x) = \mathbf{c}(y)$, and hence $\mathbf{c}(x) - \mathbf{c}(y) \neq 0, \pm 1$. By Proposition 11 (1), $x$ and $y$ are comparable. Since $\mathfrak{t}$ is order preserving, we must have $x < y$, and hence $\mathbf{h}$ is a poset isomorphism. ☐

The posets $(\mathbb{Z}_{\geqq 1}, \preceq_{\mathfrak{t}})$ are thought as semi-infinite analogue of heaps introduced by Stembridge [8]. Stembridge also introduced the heap order on the inversion sets. We treat a slightly modified version of heap order by Nakada [2].

**Definition 51.** Define a partial order $\leqslant^{\mathrm{hp}}$ on $R(w_\theta)$ as the transitive closure of the relations

$$\alpha \leqslant^{\mathrm{hp}} \beta \text{ whenever } \alpha \leqslant^{\mathrm{or}} \beta \text{ and } \langle \alpha, \beta^\vee \rangle \neq 0.$$

**Proposition 52.** *The map $\mathbf{h} : \theta \to R(w_\theta)$ gives a poset isomorphism*

$$(\theta, \leqslant) \cong (R(w_\theta), \leqslant^{\mathrm{hp}}).$$

*In other words, the partial order $\leqslant^{\mathrm{hp}}$ and $\trianglelefteq$ on $R(w_\theta)$ coincide.*

*Proof.* Let $x, y \in \theta$. Suppose that $x < y$ is a covering relation in $\theta$. Then $\mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y)$ and $\mathbf{h}(y) - \mathbf{h}(x) \in \Pi_\theta \subset R \sqcup \mathbb{Z}\delta$. We have

$$\langle \mathbf{h}(y) - \mathbf{h}(x), \mathbf{h}(y)^\vee \rangle = 2 - \langle \mathbf{h}(x), \mathbf{h}(y)^\vee \rangle.$$

If $\langle \mathbf{h}(y), \mathbf{h}(x)^\vee \rangle = 0$ then $\mathbf{h}(y) - \mathbf{h}(x) \equiv \mathbf{h}(y) \mod \mathbb{Z}\delta$ by Lemma 16, and thus $\mathbf{h}(x) = k\delta$ for some $k \in \mathbb{Z}$. This is a contradiction. Therefore $\langle \mathbf{h}(x), \mathbf{h}(y)^\vee \rangle \neq 0$, from which it follows that $\mathbf{h}(x) \leqslant^{\mathrm{hp}} \mathbf{h}(y)$.

Next, suppose that $\mathbf{h}(x) \leqslant^{\mathrm{hp}} \mathbf{h}(y)$ is a covering relation. Put $x_0 = x + N(\mathbf{h}(x))(0, \ell)$ and $y_0 = x + N(\mathbf{h}(y))(0, \ell)$. Then $\mathbf{h}(x_0) = \mathbf{h}(x) - N(\mathbf{h}(x))\delta,\ \mathbf{h}(y_0) = \mathbf{h}(y) - N(\mathbf{h}(y))\delta$ and

$$\langle \mathbf{h}(x_0), \mathbf{h}(y_0)^\vee \rangle = \langle \mathbf{h}(x), \mathbf{h}(y)^\vee \rangle \neq 0 \tag{3.7}$$

by assumption.

We assume that $x$ and $y$ are incomparable. Then as $\mathbf{h}(x) \leqslant^{\mathrm{or}} \mathbf{h}(y)$, we have $N(\mathbf{h}(y)) = N(\mathbf{h}(x)) + 1$ and

$$\mathbf{h}(y_0) \leqslant^{\mathrm{or}} \mathbf{h}(x_0) \leqslant^{\mathrm{or}} \delta \tag{3.8}$$

by Lemma 46. Moreover, by (3.5) in the proof of Theorem 47, we have

$$y_0 \notin \mathrm{Arm}(x_0) \cup \mathrm{Leg}(x_0). \tag{3.9}$$

(See also Figure 8.)

Recall that positive roots $\mathbf{h}(x_0)$ and $\mathbf{h}(y_0)$ can be expressed as $\mathbf{h}(x_0) = \alpha_{ij}$ and $\mathbf{h}(y_0) = \alpha_{kl}$ for some $i, j, k, l \in \mathbb{Z}$ with $i < j,\ k < l$. By (3.8) and (3.9), the indices $i, j, k$ and $l$ can be chosen in such a way that they satisfy $j - i \leqq \kappa - 1$ and $i < k < l < j$. Thus we have

$$\begin{aligned}
\langle \mathbf{h}(x_0), \mathbf{h}(y_0)^\vee \rangle &= \langle \alpha_{ij}, \alpha_{kl}^\vee \rangle = \langle \alpha_{k-1\,l+1}, \alpha_{kl}^\vee \rangle \\
&= \langle \alpha_{k-1}, \alpha_{kl}^\vee \rangle + \langle \alpha_k, \alpha_{kl}^\vee \rangle + \sum_{d=k+1}^{l-1} \langle \alpha_d, \alpha_{kl}^\vee \rangle + \langle \alpha_{l-1}, \alpha_{kl}^\vee \rangle + \langle \alpha_l, \alpha_{kl}^\vee \rangle \\
&= -1 + 1 + 0 + 1 - 1 = 0
\end{aligned}$$

This contradicts (3.7). Therefore $x$ and $y$ are comparable, and thus $x < y$ as $\mathbf{h}(x) < \mathbf{h}(y)$. □

# 4 Poset structure of the set of order ideals

## 4.1 Standard tableaux on cylindric skew diagrams

For a poset $P$, let $\mathcal{J}(P)$ denote the set of proper order ideals and regard $\mathcal{J}(P)$ as a poset with the inclusion relation.

Let $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$ and fix a cylindric diagram $\theta$ in $\mathcal{C}_\omega$. In this section, we will investigate the poset structure of the set $\mathcal{J}(\theta)$ of order ideals of $\theta$, in other words, cylindric skew diagrams included in $\theta$.

Recall that any cylindric skew diagram $\xi \in \mathcal{J}(\theta)$ is a finite set and $\mathcal{J}(\theta) = \bigsqcup_{n=0}^{\infty} \mathcal{J}_n(\theta)$, where

$$\mathcal{J}_n(\theta) = \{\xi \in \mathcal{J}(\theta) \mid |\xi| = n\}.$$

For $\xi \in \mathcal{J}_n(\theta)$ and $\mathfrak{t} \in \mathrm{ST}(\xi)$, define a word $w_{\xi,\mathfrak{t}}$ by

$$w_{\xi,\mathfrak{t}} = s(\mathfrak{t}^{-1}(1)) s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n)). \tag{4.1}$$

We sometimes regard $w_{\xi,\mathfrak{t}}$ as a Weyl group element.

**Proposition 53.** *The word $w_{\xi,\mathfrak{t}}$ is reduced. As an element of Weyl group, $w_{\xi,\mathfrak{t}}$ is fully commutative and independent of $\mathfrak{t}$.*

*Proof.* It follows from Lemma 10 that the standard tableau $\mathfrak{t}$ on $\xi$ can be extended to a standard tableau $\tilde{\mathfrak{t}}$ on $\theta$, for which we have $w_{\theta,\tilde{\mathfrak{t}}}[n] = w_{\xi,\mathfrak{t}}$. By Proposition 27 and Proposition 33, the right hand side of (4.1) is a reduced expression and $w_{\xi,\mathfrak{t}}$ is a fully commutative element of $W$. It follows from Proposition 28 that

$$R(w_{\xi,\mathfrak{t}}) = \{\mathbf{h}(x) \mid x \in \xi\}.$$

Hence the set $R(w_{\xi,\mathfrak{t}})$ is independent of $\mathfrak{t}$ and so is $w_{\xi,\mathfrak{t}}$. □

We denote by $w_\xi$ the Weyl group element determined by the word $w_{\xi,\mathfrak{t}}$ for a/any standard tableau $\mathfrak{t} \in \mathrm{ST}(\xi)$.

**Lemma 54** (See [8, Theorem 3.2])**.** *The map*

$$\mathfrak{t} \mapsto w_{\xi,\mathfrak{t}} = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n))$$

*gives a bijection from $\mathrm{ST}(\xi)$ to the set of reduced expressions for $w_\xi$.*

*Proof.* First, we prove that the correspondence is injective. For $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathrm{ST}(\xi)$, consider two words $w_{\xi,\mathfrak{t}_1} = s(p_1)s(p_2)\cdots s(p_n)$ and $w_{\xi,\mathfrak{t}_2} = s(q_1)s(q_2)\cdots s(q_n)$, where $p_k = \mathfrak{t}_1^{-1}(k)$ and $q_k = \mathfrak{t}_2^{-1}(k)$. Assume that $w_{\xi,\mathfrak{t}_1} = w_{\xi,\mathfrak{t}_2}$ as words. Then $\mathbf{c}(p_1) = \mathbf{c}(q_1)$ and it holds that $p_1$ and $q_1$ are minimal elements of $\xi$. Hence we have $p_1 = q_1$. Inductively, we have $p_k = q_k$ for any $k \in [1, n]$ by similar argument.

Next, we prove that the map is surjective. Take $\mathfrak{t} \in \mathrm{ST}(\xi)$ and put $p_j = \mathfrak{t}^{-1}(j)$ ($j \in [1, n]$). Then $w_{\xi,\mathfrak{t}} = s(p_1)s(p_2)\cdots s(p_n)$, which is a reduced expression of $w_\xi$.

Suppose that $s(p_k)s(p_{k+1}) = s(p_{k+1})s(p_k)$. Then $\mathbf{c}(p_k) - \mathbf{c}(p_{k+1}) \neq \pm 1$, and thus $p_k$ is not covered by $p_{k+1}$. This means that $p_k$ and $p_{k+1}$ are incomparable. Define the map $\mathfrak{t}^{(k)} : \xi \to [1, n]$ by

$$\mathfrak{t}^{(k)}(p_j) = \begin{cases} k+1 & \text{if } j = k, \\ k & \text{if } j = k+1, \\ j & \text{otherwise.} \end{cases}$$

Then $\mathfrak{t}^{(k)} \in \mathrm{ST}(\xi)$ and $w_{\xi,\mathfrak{t}^{(k)}} = s(p_1)s(p_2)\cdots s(p_{k+1})s(p_k)\cdots s(p_n)$. Now full commutativity of $w_\xi$ implies the surjectivity. □

### 4.2 Bruhat intervals

For $v, w \in W$, we write $v \lessdot w$ if $\ell(w) = \ell(v) + 1$ and $w = vs_i$ for some simple reflection $s_i$. Write $v \prec w$ if there is a sequence $v = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_n = w$. It is clear that the relation $\preceq$ is a partial order of $W$, and it is called the *weak right Bruhat order.*

For $w \in W$, we define

$$[e, w] = \{x \in W \mid e \preceq x \preceq w\}.$$

Note that when $\ell(w) = n$, we have

$$[e, w] = \left\{ s_{i_1} s_{i_2} \cdots s_{i_k} \in W \;\middle|\; \begin{array}{l} 0 \leqq k \leqq n \text{ and there exist } i_{k+1}, \ldots, i_n \text{ such that} \\ s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_n} \text{ is a reduced expression for } w \end{array} \right\}. \tag{4.2}$$

Let $\theta$ be a cylindric diagram. For $\mathfrak{t} \in \mathrm{ST}(\theta)$, we define

$$[e, w_{\theta,\mathfrak{t}}) = \bigcup_{n=1}^{\infty} [e, w_{\theta,\mathfrak{t}}[n]].$$

We will see that the "semi-infinite Bruhat interval" $[e, w_{\theta,\mathfrak{t}})$ is actually independent of $\mathfrak{t} \in \mathrm{ST}(\theta)$.

**Lemma 55.** *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two standard tableaux on $\theta$. Then for each $n \geqq 1$, there exist $r \geqq n$ and $\mathfrak{s} \in \mathrm{ST}(\theta)$ for which it holds that $w_{\theta,\mathfrak{s}}[r] = w_{\theta,\mathfrak{t}_1}[r]$ as elements of $W$ and $w_{\theta,\mathfrak{s}}[n] = w_{\theta,\mathfrak{t}_2}[n]$ as words.*

*Proof.* Choose $r \geqq n$ such that $\mathfrak{t}_2^{-1}[1, n] \subset \mathfrak{t}_1^{-1}[1, r]$. Put $\xi_1 = \mathfrak{t}_1^{-1}[1, r]$ and $\xi_2 = \mathfrak{t}_2^{-1}[1, n]$. Note that $\xi_1 \setminus \xi_2$ is an order ideal of the cylindric diagram $\theta \setminus \xi_2$. Take $\mathfrak{t} \in \mathrm{ST}(\theta \setminus \xi_2)$ such that $\mathfrak{t}^{-1}[1, r-n] = \xi_1 \setminus \xi_2$ (Lemma 10). Define a map $\mathfrak{s} : \theta \to \mathbb{Z}_{\geqq 1}$ by

$$\mathfrak{s}(p) = \begin{cases} \mathfrak{t}(p) + n & (p \in \theta \setminus \xi_2) \\ \mathfrak{t}_2(p) & (p \in \xi_2) \end{cases}$$

Then we have $\mathfrak{s} \in \mathrm{ST}(\theta)$, which satisfies the desired conditions by Proposition 53. □

**Proposition 56.** *Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two standard tableaux of $\theta$. Then*

$$[e, w_{\theta,\mathfrak{t}_1}) = [e, w_{\theta,\mathfrak{t}_2}) \textit{ as subsets of } W.$$

*Proof.* Let $n \geqq 1$. By Lemma 55, there exist $r \geqq n$ and $\mathfrak{s} \in \mathrm{ST}(\theta)$ such that $w_{\theta,\mathfrak{s}}[r] = w_{\theta,\mathfrak{t}_1}[r]$ and $w_{\theta,\mathfrak{s}}[n] = w_{\theta,\mathfrak{t}_2}[n]$. Now we have

$$[e, w_{\theta,\mathfrak{t}_2}[n]] = [e, w_{\theta,\mathfrak{s}}[n]] \subset [e, w_{\theta,\mathfrak{s}}[r]] \subset [e, w_{\theta,\mathfrak{t}_1}[r]].$$

Hence we obtain

$$[e, w_{\theta,\mathfrak{t}_2}) = \bigcup_{n=1}^{\infty} [e, w_{\theta,\mathfrak{t}_2}[n]] \subset [e, w_{\theta,\mathfrak{t}_1}).$$

Similarly, we obtain $[e, w_{\theta,\mathfrak{t}_1}) \subset [e, w_{\theta,\mathfrak{t}_2})$, and hence $[e, w_{\theta,\mathfrak{t}_1}) = [e, w_{\theta,\mathfrak{t}_2})$. □

We denote $[e, w_{\theta,\mathfrak{t}})$ just by $[e, w_\theta)$ in the rest. We have

$$[e, w_\theta) = \bigcup_{\xi \in \mathcal{J}(\theta)} [e, w_\xi]$$

by the following lemma:

**Lemma 57.** *Let $v \in W$. Then $v \in [e, w_\theta)$ if and only if $v = w_\xi$ for some $\xi \in \mathcal{J}(\theta)$.*

*Proof.* Let $v \in [e, w_\theta)$. Then $v \in [e, w_{\theta,\mathfrak{t}}[n]]$ for some $\mathfrak{t} \in \mathrm{ST}(\theta)$ and $n$. By Lemma 54, there exist $\mathfrak{t}' \in \mathrm{ST}(\theta)$ and $k$ such that $v = w_{\theta,\mathfrak{t}'}[k]$. Putting $\xi = \mathfrak{t}'^{-1}[1, k]$, we have $v = w_\xi$.

Let $\xi \in \mathcal{J}(\theta)$. Then there exist $\mathfrak{t} \in \mathrm{ST}(\theta)$ and $n$ such that $w_\xi = w_{\theta,\mathfrak{t}}[n]$. Therefore $w_\xi \in [e, w_\theta)$. □

The following theorem can be seen as a semi-infinite version of the results established in [8] (see also [3, 5]).

**Theorem 58.** *Let $\theta$ be a cylindric Young diagram in $\mathcal{C}_\omega$.*
(1) *The map*
$$\Phi : (\mathcal{J}(\theta), \subset) \to ([e, w_\theta), \preceq)$$
*given by $\Phi(\xi) = w_\xi$ is a poset isomorphism.*
(2) *The map*
$$\Psi : ([e, w_\theta), \preceq) \to (\mathcal{J}(R(w_\theta)), \subset)$$
*given by $\Psi(w) = R(w)$ is a poset isomorphism.*

*Proof.* We will show (1) and (2) togather. Note that the poset isomorphism $\mathbf{h} : \theta \to R(w_\theta)$ induces a poset isomorphism $\mathcal{J}(\theta) \to \mathcal{J}(R(w_\theta))$, under which $\xi \in \mathcal{J}(\theta)$ corresponds to
$$\{\mathbf{h}(x) \mid x \in \xi\} = R(w_\xi) = \Psi \circ \Phi(\xi).$$
Hence $\Psi \circ \Phi$ is bijective and thus $\Phi$ is injective. As $\Phi$ is surjective by Lemma 57, $\Phi$ is bijective. Thus $\Psi$ is also bijective.

We will show that $\Phi$ and $\Psi$ are order preserving.

Suppose that $\xi'$ covers $\xi$, or equivalently that $\xi' = \xi \sqcup \{x\}$ for a maximal element $x$ of $\xi'$. Then there exists $\mathfrak{t} \in \mathrm{ST}(\xi')$ satisfying $\mathfrak{t}^{-1}(n) = x$, for which we have
$$w_{\xi'} = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n-1))s(\mathfrak{t}^{-1}(n)) = w_\xi s(x),$$
This implies that $w_{\xi'}$ covers $w_\xi$. Hence $\Phi$ is order preserving.

It is easy to see that $v \preceq w$ implies $R(v) \subset R(w)$. Hence $\Psi$ is order preserving.

As we know that $(\Psi \circ \Phi)^{-1}$ is order preserving, it holds that $\Phi^{-1}$ and $\Psi^{-1}$ are also order preserving. □

**Proposition 59.** *Let $\theta$ be a cylindric diagram. Then*
$$[e, w_\theta) = \{w \in W \mid w \textit{ is } \zeta_\theta\textit{-pluscule}\}$$

*Proof.* It follows from Proposition 33 that any element of $[e, w_\theta)$ is $\zeta_\theta$-pluscule.

Let $w \in W$ be $\zeta_\theta$-pluscule and $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ its reduced expression. We will show that $w \in [e, w_\theta)$ by induction on $n = \ell(w)$. By induction hypothesis, $v := s_{i_1} s_{i_2} \cdots s_{i_{n-1}}$ belongs to $[e, w_\theta)$, and thus $v = w_\xi$ for some $\xi \in \mathcal{J}(\theta)$.

Let $x$ be the minimum element of $\mathbf{c}^{-1}(i_n) \cap (\theta \setminus \xi)$ and put $\xi' = \xi \sqcup \{x\}$. Take $\mathfrak{t} \in \mathrm{ST}(\xi')$ such that $\mathfrak{t}(n) = x$. Then $w = s(\mathfrak{t}^{-1}(1))s(\mathfrak{t}^{-1}(2)) \cdots s(\mathfrak{t}^{-1}(n))$. Since $w$ is $\zeta_\theta$-pluscule, if $i_n = i_k$ then there exist $j_+, j_- \in [k, n]$ such that $j_+ = i_n + 1$ and $j_- = i_n - 1$ by [7, Proposition 2.3]. This implies that the subset $\xi'$ satisfies the condition (v) in Proposition 6. Therefore $\xi'$ is a cylindric skew diagram in $\theta$ and $w = w_{\xi \sqcup \{x\}}$. Therefore $w \in [e, w_\theta)$. □

### 4.3 Skew diagrams and classical case

Let $\theta$ be a cylindric diagram in $\mathcal{C}_\omega$. Let $\xi \in \mathcal{J}_n(\theta)$ and take $\mathfrak{t} \in \mathrm{ST}(\theta)$ such that $\xi = \mathfrak{t}^{-1}[1,n]$. Then we have $w_{\theta,\mathfrak{t}}[n] = w_\xi$ and $\mathbf{h}(\xi) = R(w_\xi)$. Thus the next theorem follows easily from Theorem 47:

**Theorem 60.** *Let $\xi \in \mathcal{J}_n(\theta)$.*
(1) *The map $\mathbf{h} : (\xi, \leqslant) \to (R(w_\xi), \trianglelefteq)$ is a poset isomorphism.*
(2) *For $\mathfrak{t} \in \mathrm{ST}(\xi)$, the map $\mathfrak{t} : (\xi, \leqslant) \to ([1,n], \leqslant_{\mathfrak{t}}^{\mathrm{hp}})$ is a poset isomorphism.*

Note that $\mathcal{J}(\xi) = \{\eta \in \mathcal{J}(\theta) \mid \eta \subset \xi\}$. Theorem 58 implies the following:

**Theorem 61.** *Let $\xi \in \mathcal{J}(\theta)$.*
(1) *The map $\Phi : (\mathcal{J}(\xi), \subset) \to ([e, w_\xi], \preceq)$ given by $\Phi(\eta) = w_\eta$ is a poset isomorphism.*
(2) *The map $\Psi : ([e, w_\xi], \preceq) \to (\mathcal{J}(R(w_\xi)), \subset)$ given by $\Psi(w) = R(w)$ is a poset isomorphism.*

In the rest, we will see that description for non-cylindric diagrams can be deduced from the results above. Let $m \in \mathbb{Z}_{\geqq 1}$ and let $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m)$ be partitions such that $\lambda_i \geqq \mu_i \geqq 0$ ($i \in [1,m]$). Under the notation in Section 1.1, the associated classical skew Young diagram is represented as the subset $\boldsymbol{\lambda}/\boldsymbol{\mu}$ of $\mathbb{Z}^2$:

$$\boldsymbol{\lambda}/\boldsymbol{\mu} = \left\{(a,b) \in \mathbb{Z}^2 \mid a \in [1,m],\ b \in [\mu_a + 1, \lambda_a]\right\}.$$

Note that the classical normal Young diagram associated with $\lambda$ is a special skew diagram $\boldsymbol{\lambda}/\boldsymbol{\phi}$ with $\phi = (0, 0, \ldots, 0)$.

To connect classical diagrams and cylindric diagrams, we take $\ell \in \mathbb{Z}_{\geqslant 1}$ such that

$$\ell \geqq \lambda_1 - \mu_m.$$

Then the partitions $\lambda, \mu$ are $\ell$-restricted, and moreover it is easy to see that the skew diagram $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is isomorphic to the cylindric skew diagram $\mathring{\lambda}/\mathring{\mu} = \pi(\boldsymbol{\lambda}/\boldsymbol{\mu})$ as a poset. Under this identification $\boldsymbol{\lambda}/\boldsymbol{\mu} = \mathring{\lambda}/\mathring{\mu}$, Theorem 60 and Lemma 45 for the order ideal $\mathring{\lambda}/\mathring{\mu}$ of the cylindric diagram $\mathring{\lambda}$ imply the followings:

$$([1,n], \leqslant_{\mathfrak{t}}^{\mathrm{hp}}) \cong (\boldsymbol{\lambda}/\boldsymbol{\mu}, \leqslant) \cong (R(w_{\boldsymbol{\lambda}/\boldsymbol{\mu}}), \trianglelefteq) = (R(w_{\boldsymbol{\lambda}/\boldsymbol{\mu}}), \leqslant^{\mathrm{or}})$$

for each $\mathfrak{t} \in \mathrm{ST}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \mathrm{ST}(\mathring{\lambda}/\mathring{\mu})$, and it follows from Theorem 61 that

$$(\mathcal{J}(\boldsymbol{\lambda}/\boldsymbol{\mu}), \subset) \cong ([e, w_{\boldsymbol{\lambda}/\boldsymbol{\mu}}], \preceq) \cong (\mathcal{J}(R(w_{\boldsymbol{\lambda}/\boldsymbol{\mu}}), \subset).$$

Remark that by redefining the content as

$$\mathbf{c}(a,b) = b - a + m - \mu_m,$$

we have $\mathbf{c}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \subset [1, \kappa - 1]$, and

$$w_{\boldsymbol{\lambda}/\boldsymbol{\mu}} \in \bar{W},\ R(w_{\boldsymbol{\lambda}/\boldsymbol{\mu}}) \subset \bar{R},$$

where $\bar{W}$ and $\bar{R}$ denote the Weyl group and the root system of type $A_{\kappa-1}$ respectively.

We will see the relation between the results above and preceding works. Let $n \in \mathbb{Z}_{\geqq 1}$ and $\lambda$ be a partition of $n$. Fix $\mathfrak{t} \in \mathrm{ST}(\boldsymbol{\lambda}/\boldsymbol{\phi})$ and put

$$w_\lambda := w_{\boldsymbol{\lambda}/\boldsymbol{\phi}} = s(\mathfrak{t}^{-1}(n))s(\mathfrak{t}^{-1}(n-1)) \cdots s(\mathfrak{t}^{-1}(1)).$$

The element $w_\lambda$ is independent of $\mathfrak{t}$ and it is called the Grassmannian permutation associated with $\lambda$.

It has been shown in [7, 3] that the map

$$\mathbf{coh} : \boldsymbol{\lambda}/\boldsymbol{\phi} \to R(w_\lambda^{-1})$$

given by

$$\mathbf{coh}(x) = s(\mathfrak{t}^{-1}(n))s(\mathfrak{t}^{-1}(n-1)) \cdots s(\mathfrak{t}^{-1}(k+1))\alpha(\mathfrak{t}^{-1}(k)), \tag{4.3}$$

where $k = \mathfrak{t}^{-1}(x)$, leads an *dual isomorphism* of posets:

$$\mathbf{coh} : (\boldsymbol{\lambda}/\boldsymbol{\phi}, \leqslant) \to (R(w_\lambda^{-1}), \leqslant^{\mathrm{or}}), \tag{4.4}$$

where $\leqslant^{\mathrm{or}}$ is the ordinary order as before.

On the other hand, as a classical version of Theorem 60, we have a poset isomorphism

$$\mathbf{h} : (\boldsymbol{\lambda}/\boldsymbol{\phi}, \leqslant) \to (R(w_\lambda), \trianglelefteq). \tag{4.5}$$

Now define the map $\iota : R \to R$ by $\iota(\alpha) = -w_\lambda^{-1}\alpha$. Then it follows immediately from the expression (2.3) and (4.3) that $\iota \circ \mathbf{h}(x) = \mathbf{coh}(x)$ for all $x \in \boldsymbol{\lambda}/\boldsymbol{\phi}$. Therefore we have the following:

**Proposition 62.** *The restriction of $\iota$ gives a dual poset isomorphism*

$$\iota : (R(w_\lambda), \trianglelefteq) \to (R(w_\lambda^{-1}), \leqslant^{\mathrm{or}})$$

*and moreover $\iota \circ \mathbf{h} = \mathbf{coh}$. In other words, the following diagram of poset isomorphisms commutes :*

$$\begin{array}{ccc} (\boldsymbol{\lambda}/\boldsymbol{\phi}, \leqslant) & \xrightarrow{\ \mathbf{h}\ } & (R(w_\lambda), \trianglelefteq) \\ {\scriptstyle \mathbf{coh}}\downarrow & \swarrow{\scriptstyle \iota} & \\ (R(w_\lambda^{-1}), \leqslant^{\mathrm{or}})^{\mathrm{op}} & & \end{array} \tag{4.6}$$

*where $(R(w_\lambda^{-1}), \leqslant^{\mathrm{or}})^{\mathrm{op}}$ denotes the poset obtained from $(R(w_\lambda^{-1}), \leqslant^{\mathrm{or}})$ by reversing the order.*

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