

Spectral radius of graphs with given size and odd girth

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Abstract

Let $\mathcal{G}(m, k)$ be the set of graphs with size m and odd girth (the length of a shortest odd cycle) k . In this paper, we determine the graph maximizing the spectral radius among $\mathcal{G}(m, k)$ when m is odd. As byproducts, we show that, there is a number $\eta(m, k) > \sqrt{m - k + 3}$ such that every non-bipartite graph G with size m and spectral radius $\rho \geq \eta(m, k)$ must contain an odd cycle of length less than k unless m is odd and $G \cong SK_{k,m}$, which is the graph obtained by subdividing an edge $k - 2$ times of the complete bipartite graph $K_{2, \frac{m-k+2}{2}}$. This result implies the main results of Zhai and Shu [Discrete Math. 345 (2022)] and settles a conjecture of Li and Peng [The Electronic J. Combin. 29 (4) (2022)] as well.

Mathematics Subject Classifications: 05C50

1 Introduction

Let \mathcal{H} be a set of some fixed graphs. A graph is said to be \mathcal{H} -free if it does not contain subgraphs isomorphic to any members of \mathcal{H} . In 1907, Mantel [13] presented the following famous result, which aroused the study of the so-called Turán-Type extremal problem in graph theory.

Theorem 1 (Mantel [13]). *Let G be an n -vertex graph. If G is triangle-free, then $m(G) \leq m(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$, equality holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Problems involving triangles play an important role in the development of both extremal and spectral extremal graph theory. A classic Mantel's theorem, implies that every n -vertex graph of size $m > \lfloor \frac{n^2}{4} \rfloor$ must contain a triangle. Since then, much attentions have

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been paid to Turán-Type Extremal Problem and Mantel's Theorem has many interesting applications. Further related results may be found dotted throughout the literature in the nice survey [4] for example. For a graph G , let $\rho(G)$ be the spectral radius of G . In 1970, Nosal [21] proved that if a graph G is triangle-free with m edges, then $\rho(G) \leq \sqrt{m}$. This a classic result in spectral graph theory for triangle-free graphs, and it can be viewed as the spectral version of Mantel's Theorem, also usually called the spectral Mantel theorem. More precisely, Nikiforov [18] characterized the extremal triangle-free graphs attaining the upper bound. In what follows, we state this spectral result in a complete form.

Theorem 2 ([21, 18]). *Let G be a graph with m edges. If G is triangle-free, then $\rho(G) \leq \sqrt{m}$, equality holds if and only if G is a complete bipartite graph.*

During the years this striking and elegant result has attracted greatly significant attention (see, e.g., [15], [16], [11], [24], [20], [9] and the survey [19] for some highlights). In 2021, Lin, Ning and Wu [11] obtained a new spectral condition for triangles by majorization theory. Consequently, they enhanced Theorem 2 for non-bipartite graphs.

Theorem 3 (Lin, Ning and Wu [11]). *Let G be a non-bipartite graph with size m . If $\rho(G) \geq \sqrt{m-1}$, then G contains a triangle unless G is a C_5 .*

Theorem 3 can be viewed as a stability result on Theorem 2, since it excluded the complete bipartite graphs in our consideration. This type of result is also regarded as the second extremal graph problem in the literature. Moreover, by a well-known inequality $\rho(G) \geq \frac{2m(G)}{n}$ originated from Collatz and Sinogowitz [1], one have $\rho(G) > \sqrt{m-1}$, which provided by $m(G) > \lfloor \frac{n^2}{4} \rfloor$. Therefore, Theorem 3 is slightly stronger than Mantel's theorem. Note that the extremal graph in Theorem 3 is attained only for $m = 5$ and $G = C_5$. For $k \geq 1$, let $S_{2k-1}(K_{2, \frac{m-2k+1}{2}})$ be the graph obtained from $K_{2, \frac{m-2k+1}{2}}$ by putting $2k-1$ new vertices on an edge. Clearly, $S_{2k-1}(K_{2, \frac{m-2k+1}{2}})$ is non-bipartite and it also is $\{C_3, C_5, \dots, C_{2k+1}\}$ -free. In 2022, Zhai and Shu [25] proved a further improvement on Theorem 3. Let $\rho^*(m)$ be the largest root of $x^5 - mx^3 + (2m-5)x - (m-3) = 0$.

Theorem 4 (Zhai and Shu [25]). *Let G be a non-bipartite graph of size m . If $\rho(G) \geq \rho^*(m)$, then G contains a triangle unless m is odd and $G \cong S_1(K_{2, \frac{m-1}{2}})$.*

Notice that for $m \geq 6$, we have $\sqrt{m-2} < \rho^*(m) < \sqrt{m-1}$. Theorem 4 implies both Mantel's theorem and Lin, Ning and Wu's result as well. In 2022, Wang [22, Theorem 5] improved slightly Theorem 4 by determining the m -edge graphs G for every m , if G is a triangle-free and non-bipartite graph with $\rho(G) \geq \sqrt{m-2}$.

Very recently, by applying Cauchy's interlacing theorem of all eigenvalues, Li and Peng [10] found some forbidden induced subgraphs and presented an alternative proof of Theorem 4. Note that the unique extremal graph in Theorem 4 contains many copies of C_5 . Moreover, Li and Peng [10] considered the further stability result on Theorem 4 by forbidding both C_3 and C_5 as below. Let $\gamma(m)$ denote the largest root of $x^7 - mx^5 + (4m-14)x^3 - (3m-14)x - m + 5$.

Theorem 5 (Li and Peng [10]). *Let G be a graph with m edges. If G is $\{C_3, C_5\}$ -free and G is non-bipartite, then $\rho(G) \leq \gamma(m)$, equality holds if and only if m is odd and $G \cong S_3(K_{2, \frac{m-3}{2}})$.*

Moreover, Li and Peng [10, Conjecture 4.1] proposed the following conjecture.

Conjecture 6 (Li and Peng [10]). $\rho(G) \leq \rho(S_{2k-1}(K_{2, \frac{m-2k+1}{2}}))$, equality holds if and only if m is odd and $G \cong S_{2k-1}(K_{2, \frac{m-2k+1}{2}})$.

It is worth mentioning that the analogous result of Conjecture 6 for $\{C_3, C_5, \dots, C_{2k+1}\}$ -free non-bipartite graphs with given order n was previously proposed in [11]. The problem was proved by Lin and Guo [12] and also independently proved by Li, Sun and Yu [7, Theorem 1.6] using a different method.

In this paper, we aim to generalize all the above results and confirm Conjecture 6. Unlike the techniques in [10] where the authors developed the key ideas from [11], in present paper, we will expand the ideas mainly from [25]. The odd girth of a graph is defined as the length of a shortest odd cycle. Recall that $\mathcal{G}(m, k)$ is the set of graphs with size m and odd girth k . Let $C_k(a, b)$ be the graph obtained from a cycle $C_k = v_1v_2 \cdots v_kv_1$ by replacing the edge v_1v_k with a complete bipartite graph $K_{a,b}$ (see Fig.1). On the other hand, $C_k(a, b)$ is just a local blow up of the cycle C_k , and it also can be seen as a subdivision of $K_{a+1, b+1}$ by subdividing an edge with a path of length $k-3$. In particular, we denote $SK_{k,m} = C_k(1, \frac{m-k+2}{2})$, where $m \geq k \geq 5$ is an odd integer; see Fig.1. Clearly, $S_1(K_{2, \frac{m-1}{2}})$ in Theorem 4 is just $SK_{5,m}$, and $S_3(K_{2, \frac{m-3}{2}})$ in Theorem 5 is just $SK_{7,m}$.

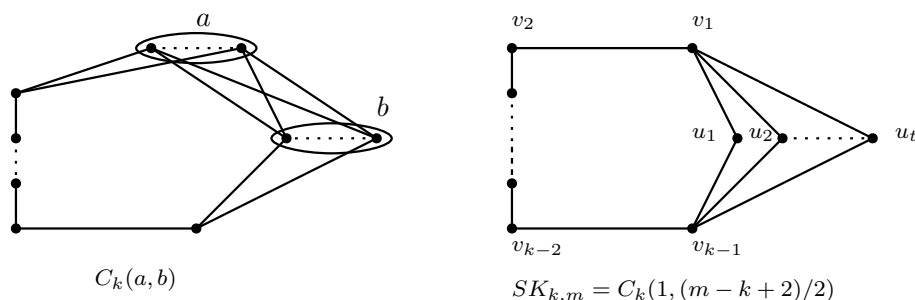


Figure 1: The graph $C_k(a, b)$ and $SK_{k,m}$.

Now we present the main result in this paper.

Theorem 7. *For two odd integers $m \geq k \geq 5$, if $G \in \mathcal{G}(m, k)$, then*

$$\rho(G) \leq \rho(SK_{k,m}),$$

with equality if and only if $G \cong SK_{k,m}$.

We actually get the following stronger result.

Theorem 8. *Let G be a non-bipartite graph with size m , and $k \geq 5$ an odd integer. If $\rho(G) \geq \frac{m-k+2}{\sqrt{m-k+1}} > \sqrt{m-k+3}$, then G contains an odd cycle of length less than k unless $k = 5$ and $G = C_5(2, 2)$, or m is odd and $G \cong SK_{k,m}$.*

Remark 9. On the one hand, from the proofs in Section 3, we actually obtain all non-bipartite graphs of size m satisfying $\rho > \sqrt{m-k+3}$ for any odd integer $k \geq 5$. On the other hand, though Theorem 8 removes the assumption of m being odd, it does not mean that the extremal graph maximizing ρ is obtained among $\mathcal{G}(m, k)$ when m is even. If $m = k + 5$ and $k = 5$, Theorem 8 indicates that the extremal graph is exactly $C_5(2, 2)$. For other cases, Theorem 8 only implies $\rho(G) < \frac{m-k+2}{\sqrt{m-k+1}}$ for any $G \in \mathcal{G}(m, k)$ when m is even. The extremal graph is still unknown in general.

By the knowledge of equitable partition [6, Page 198], $\rho(SK_{k,m})$ is the largest root of $p(x)$, where $p(x) = \det(xI - B)$ and B is the $k \times k$ matrix given as

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ \frac{m-k+2}{2} & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ \frac{m-k+2}{2} & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Denote by $\eta(m, k)$ the largest root of $p(x)$, where m is not necessary odd. Similar to Lemma 12, it is easy to see $\eta(m, k) \geq \frac{m-k+2}{\sqrt{m-k+1}} > \sqrt{m-k+3}$. Note that $\rho(C_5(2, 2)) = 2.9032 < 2.9191 = \eta(10, 5)$, Theorem 8 implies the following result immediately.

Corollary 10. *Let G be a non-bipartite graph with size m , and $k \geq 5$ an odd integer. If $\rho(G) \geq \eta(m, k)$, then G contains an odd cycle of length less than k unless $G \cong SK_{k,m}$.*

Clearly, Theorem 4 is the special case for $k = 5$ of Corollary 10, and Theorem 5 is the special case for $k = 7$. Furthermore, Corollary 10 completely solves Conjecture 6.

2 Preliminaries

At the beginning of this section, we shall give some terminologies and notations. For two disjoint subsets $S, T \subseteq V(G)$, $e(S)$ is the number of edges with both endpoints in S and $e(S, T)$ is the number of edges with one endpoint in S and the other in T . Given a vertex $u \in V(G)$, $N(u)$ is the neighborhood of u in G , and for $k \geq 2$, $N^k(u)$ is the set of vertices of distance k to u . Denote by $d_G(u)$ the degree of u and $d_S(u)$ the number of its neighbors in $S \subseteq V(G)$. Let $G[S]$ be the subgraph of G induced by S . Let $\phi(G)$ be the characteristic polynomial of $A(G)$. The spectral radius $\rho(G)$ is the largest root of $\phi(G)$. Its corresponding unit eigenvector is called the *Perron vector* of G . From the famous Perron-Frobenius theorem, Perron vector is a positive vector for a connected graph G .

An *internal path* of G is a sequence of vertices v_1, v_2, \dots, v_s with $s \geq 2$ such that:

- (i) the vertices in the sequence are distinct (except possibly $v_1 = v_s$),
- (ii) v_i is adjacent to v_{i+1} ($i = 1, 2, \dots, s - 1$),

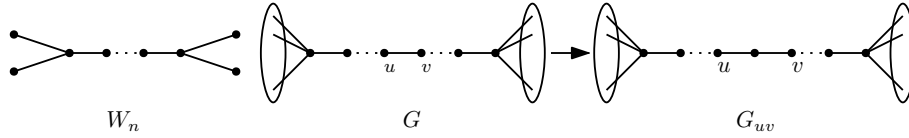


Figure 2: The graph W_n and the subdivision operation.

(iii) the vertex degrees satisfy $d(v_1) \geq 3$, $d(v_2) = \dots = d(v_{s-1}) = 2$ (unless $s = 2$) and $d(v_s) \geq 3$.

Lemma 11 ([5]). *Suppose that $G \neq W_n$ (see Fig.2) and uv is an edge on an internal path of G . Let G_{uv} be the graph obtained from G by the subdivision of the edge uv . Then $\rho(G_{uv}) < \rho(G)$.*

Now we give upper and lower bounds for the spectral radius of $SK_{k,m}$.

Lemma 12. *For every odd integers $m > k \geq 5$, we have*

$$\sqrt{m - k + 3} < \frac{m - k + 2}{\sqrt{m - k + 1}} \leq \rho(SK_{k,m}) < \sqrt{m - k + 4}.$$

Proof. For convenience, denote by $\rho = \rho(SK_{k,m})$ and $t = \frac{m-k+2}{2} \geq 2$. It suffices to show $\frac{2t}{\sqrt{2t-1}} \leq \rho < \sqrt{2t+2}$. By Lemma 11, we have $\rho < \rho(K_{2,t+1}) = \sqrt{2t+2}$.

For any positive number x , denote by $\alpha_x = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta_x = \frac{x-\sqrt{x^2-4}}{2}$. Therefore, one can easily verify that $\rho \geq \frac{2t}{\sqrt{2t-1}}$ if and only if $\frac{\rho}{2t\beta_\rho} \geq 1$. In what follows, we show $\frac{\rho}{2t\beta_\rho} \geq 1$.

For any odd integer $r \geq 5$, let $G_r = SK_{r,2t+r-2}$ and $\rho_r = \rho(G_r)$. We have proved that $\rho_r < \sqrt{2t+2}$. For each G_r , label G_r like $SK_{k,m}$ in Fig.1, that is, v_1 and v_{r-1} are the only vertices of degree greater than 2, $v_1v_2 \dots v_{r-1}$ is the path, and $N(v_1) \setminus \{v_2\} = N(v_{r-1}) \setminus \{v_{r-2}\} = \{u_1, \dots, u_t\}$. Let f be the positive eigenvector of G_r such that $A(G_r)f = \rho_r f$. By the symmetry of $G_r = SK_{r,2t+r-2}$, we may assume $f(u_1) = f(u_2) = \dots = f(u_t) = 2$. By $A(G_r)f = \rho_r f$, we have

$$\begin{cases} 2\rho_r = f(v_1) + f(v_{r-1}) = 2f(v_1), \\ \rho_r f(v_1) = 2t + f(v_2), \\ \rho_r f(v_i) = f(v_{i-1}) + f(v_{i+1}), \text{ for } 2 \leq i \leq \ell, \end{cases}$$

where $\ell = \frac{r-1}{2}$. By immediate calculations, we have

$$f(v_i) = \frac{\rho_r - 2t\beta_{\rho_r}}{\alpha_{\rho_r} - \beta_{\rho_r}} \alpha_{\rho_r}^i + \frac{2t\alpha_{\rho_r} - \rho_r}{\alpha_{\rho_r} - \beta_{\rho_r}} \beta_{\rho_r}^i$$

for $1 \leq i \leq \ell$. Since $f(v_\ell) > 0$, we have $\frac{\rho_r - 2t\beta_{\rho_r}}{\alpha_{\rho_r} - \beta_{\rho_r}} \alpha_{\rho_r}^\ell + \frac{2t\alpha_{\rho_r} - \rho_r}{\alpha_{\rho_r} - \beta_{\rho_r}} \beta_{\rho_r}^\ell > 0$, which gives that

$$\frac{\rho_r}{2t\beta_{\rho_r}} > \frac{1 - (\beta_{\rho_r}/\alpha_{\rho_r})^{\ell-1}}{1 - (\beta_{\rho_r}/\alpha_{\rho_r})^\ell} \quad (1)$$

Denote by $\varphi(x) = \frac{1-x^{\ell-1}}{1-x^\ell}$. For $0 < x < 1$, we have

$$\begin{aligned} (1-x^\ell)^2\varphi'(x) &= -(\ell-1)x^{\ell-2}(1-x^\ell) - (1-x^{\ell-1})(-\ell x^{\ell-1}) \\ &= x^{\ell-2}[\ell x(1-x^{\ell-1}) - (\ell-1)(1-x^\ell)] \\ &= x^{\ell-2}[\ell x - \ell + 1 - x^\ell] \\ &= x^{\ell-2}[\ell(x-1) + (1-x^\ell)] \\ &= x^{\ell-2}[(x-1)(\ell - x^{\ell-1} - x^{\ell-2} - \dots - x - 1)] \\ &< 0 \end{aligned}$$

Thus, $\varphi(x)$ is a decreasing function when $0 < x < 1$. Now we denote $\phi(x) = \frac{\beta_x}{\alpha_x} = \frac{x-\sqrt{x^2-4}}{x+\sqrt{x^2-4}}$. Notice that

$$\begin{aligned} (x + \sqrt{x^2 - 4})^2\phi'(x) &= (x + \sqrt{x^2 - 4})(1 - \frac{x}{\sqrt{x^2-4}}) - (1 + \frac{x}{\sqrt{x^2-4}})(x - \sqrt{x^2 - 4}) \\ &= -\frac{8}{\sqrt{x^2-4}} < 0. \end{aligned}$$

Thus $\phi(x)$ is a decreasing function. Let C_4^+ be the graph obtained from the cycle C_4 by attaching a new vertex to a vertex of the cycle C_4 . Note that $\rho(C_4^+) > 2.1$, and G_r contains C_4^+ as a subgraph because $t \geq 2$. We have $\rho_r > 2.1$. Therefore, we get $\beta_{\rho_r}/\alpha_{\rho_r} = \phi(\rho_r) < \phi(2.1) < 0.6$. This means that

$$\varphi(\beta_{\rho_r}/\alpha_{\rho_r}) = \frac{1 - (\beta_{\rho_r}/\alpha_{\rho_r})^{\ell-1}}{1 - (\beta_{\rho_r}/\alpha_{\rho_r})^\ell} > \varphi(0.6) = \frac{1 - 0.6^{\ell-1}}{1 - 0.6^\ell}. \quad (2)$$

Now suppose to the contrary that $\frac{\rho}{2t\beta_\rho} < 1$. There exists $\epsilon > 0$ such that $\frac{\rho}{2t\beta_\rho} < 1 - \epsilon$. Since $\lim_{\ell \rightarrow \infty} \frac{1-0.6^{\ell-1}}{1-0.6^\ell} = 1$, there exists $N > 0$ such that $\frac{1-0.6^{n-1}}{1-0.6^n} > 1 - \epsilon$ for any $n \geq N$. Take $r = \max\{k, 2N + 1\}$, and $\ell = \frac{r-1}{2}$. Keep in mind that the function

$$g(x) = \frac{x}{2t\beta_x} = \frac{x}{t(x - \sqrt{x^2 - 4})}$$

increases along with x increasing when $x > 2$, and $\rho \geq \rho_r > 2$ due to Lemma 11 and the fact $t \geq 2$. Combining (1) and (2) we have

$$\frac{\rho}{2t\beta_\rho} \geq \frac{\rho_r}{2t\beta_{\rho_r}} > \frac{1 - 0.6^{\ell-1}}{1 - 0.6^\ell} > 1 - \epsilon,$$

which is a contradiction. □

Lemma 13 ([23]). *Let u, v be two distinct vertices in a connected graph G , and $G' = G - \{vv_i \mid 1 \leq i \leq s\} + \{uv_i \mid 1 \leq i \leq s\}$, where $\{v_i \mid i = 1, 2, \dots, s\} \subseteq N_G(v) \setminus N_G(u)$. Assume that x is the Perron vector of G . If $x_u \geq x_v$, then $\rho(G) < \rho(G')$.*

Lemma 14 ([2]). *Let s, t, u, v be the four distinct vertices of a connected graph G and let $st, uv \in E(G)$, while $sv, tu \notin E(G)$. If $(x_s - x_u)(x_v - x_t) \geq 0$, where x is the Perron vector of G , then $\rho(G - st - uv + sv + tu) \geq \rho(G)$ with equality if and only if $x_s = x_u$ and $x_t = x_v$.*

Let H be a subgraph of G . For $u, v \in V(H)$, denote by $d_H(u, v)$ the distance of u, v in H , that is, the length of a shortest path from u to v in H . We close this part by the following result, which is immediate by simple observations and its proof is omitted.

Lemma 15. *Let C be an odd cycle of length $l \geq 5$ in graph G . For two vertices $x, y \in V(C)$ and $u \in V(G) \setminus V(C)$ with $x \sim u$ and $y \sim u$, if $d_C(x, y) \geq 2$, then G contains an odd cycle containing u with length at most l . Furthermore, if $d_C(x, y) \geq 3$, then such cycle has length at most $l - 1$.*

3 Proofs.

In this part, we give the proofs of Theorems 7 and 8. The techniques in our proofs are motivated by that of Zhai and Shu [25]. Avoiding fussy repetition and tedious calculations, we always assume that $k \geq 7$. Since there is nothing to prove when $m = k$, we always assume $m > k$. Let G^* be the extremal graph with the maximum spectral radius among all graphs in $\mathcal{G}(m, k)$. By Lemma 12, the lower bound of G^* is given by

$$\rho^2(G^*) \geq \rho^2(SK_{k,m}) \geq \frac{(m - k + 2)^2}{m - k + 1} > m - k + 3. \quad (3)$$

Our goal is to determine the structure of G^* . Let C_k be a shortest odd cycle of G^* . Keep in mind that G^* has no odd cycle with length less than k . We always assume that $\mathbf{x} = (x_1, x_2, \dots, x_{|V(G^*)|})^T$ is the Perron vector of G^* , and u^* is a vertex with the largest component in \mathbf{x} , i.e., $x_{u^*} = \max\{x_i \mid i \in V(G^*)\}$. Denoted by $A = N(u^*)$, $B = V(G^*) \setminus (A \cup u^*)$ and $\rho^* = \rho(G^*)$. Note that $e(A) = 0$ because G^* contains no triangle. This fact yields an upper bound of $e(B)$.

Lemma 16. $e(B) \leq k - 4$.

Proof. Since $e(A) = 0$, we have $\sum_{i \in A} d_A(i)x_i = 0$. Note that

$$\rho^{*2}x_{u^*} = \rho^*(\rho^*x_{u^*}) = \rho^* \left(\sum_{v \in A} x_v \right) = \sum_{v \in A} \rho^*x_v = \sum_{v \in A} \sum_{u \in N(v)} x_u.$$

According to the assumption (3), we have

$$\begin{aligned} (m - k + 3)x_{u^*} &< d(u^*)x_{u^*} + \sum_{i \in A} d_A(i)x_i + \sum_{j \in B} d_A(j)x_j = d(u^*)x_{u^*} + \sum_{j \in B} d_A(j)x_j \\ &\leq (d(u^*) + e(A, B))x_{u^*} = (m - e(B))x_{u^*}. \end{aligned}$$

It leads to $e(B) < k - 3$, and thus $e(B) \leq k - 4$. □

Lemma 17. u^* must be on a shortest odd cycle of G^* .

Proof. Let C be a shortest odd cycle of G^* . If $u^* \in C$, there is nothing to prove. Otherwise, since $e(A) = 0$ and $e(B) \leq k - 4$, there exists at least two vertices $x, y \in A \cap V(C)$ with $d_C(x, y) \geq 2$. Hence we will get a cycle containing u^* with length at most k by Lemma 15. \square

Lemma 18. B induces a P_{k-3} with some isolates possibly.

Proof. According to Lemma 17, let $C = u^*u_0v_1v_2 \dots v_{k-3}u_a u^*$ be a shortest odd cycle of G^* , where $u_0, u_a \in A$. We claim that $v_1, v_2, \dots, v_{k-3} \in B$ since otherwise there would be a shorter odd cycle in G^* . Note that $e(B) \leq k - 4$. The result follows. \square

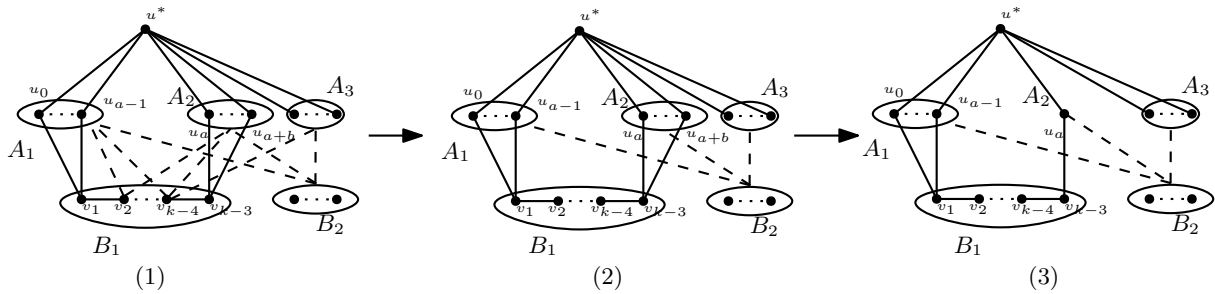


Figure 3: The structures yielded by Lemmas 18, 19 and 20 where the dashed lines between two parts means that the adjacency relations between the vertices in one part and those in the other part are still unknown.

Let $B_1 = V(P_{k-3}) = \{v_1, v_2, \dots, v_{k-3}\}$. Then $B_2 = B \setminus B_1$ is an independent set in G . Note that $N_A(v_1) \cap N_A(v_{k-3}) = \emptyset$ since otherwise G^* would contain a shorter odd cycle. So, we may denote $A_1 = N_A(v_1) = \{u_0, u_1, \dots, u_{a-1}\}$ for some $a \geq 1$ and $A_2 = N_A(v_{k-3}) = \{u_a, u_{a+1}, \dots, u_{a+b}\}$ for some $b \geq 0$. Set $A_3 = A \setminus (A_1 \cup A_2)$. Clearly, A_1, A_2 and A_3 are independent sets. Now the structure of G^* is as shown in Fig.3 (1).

Lemma 19. $d(v_i) = 2$ for each $i = 2, 3, \dots, k - 4$.

Proof. Suppose to the contrary that $v_i \sim u$ for some $u \in A$ and $2 \leq i \leq k - 4$. Consider the cycle $C = u^*u_0v_1 \dots v_{k-3}u_a u^*$. If $u \in \{u_0, u_a\}$, clearly, then there would be an odd cycle of length shorter than k . Indeed, if $v_i \sim u_0$, then either the cycle $u_0v_1v_2 \dots v_iv_0$ or the cycle $u^*u_0v_iv_{i+1} \dots v_{k-3}u_a u^*$ is an odd cycle with shorter length. This is a contradiction. If $u \in A \setminus \{u_0, u_a\}$, noticing $d_C(u^*, v_i) \geq 3$, $u^* \sim u$ and $v_i \sim u$, Lemma 15 indicates there would be a shorter odd cycle as well, a contradiction. \square

According to Lemma 19, the structure of G^* is as shown in Fig.3 (2). Without loss of generality, we may assume that $x_{v_1} \geq x_{v_{k-3}}$ in what follows.

Lemma 20. $|A_2| = 1$, i.e., $A_2 = \{u_a\}$.

Proof. Suppose to the contrary that $|A_2| \geq 2$, that is, $A_2 = \{u_a, \dots, u_{a+b}\}$ with $b \geq 1$. Let $G' = G^* - \{u_{a+i}v_{k-3} \mid i = 1, 2, \dots, b\} + \{u_{a+i}v_1 \mid i = 1, 2, \dots, b\}$. Clearly, $G' \in \mathcal{G}(m, k)$ and Lemma 13 implies $\rho(G') > \rho(G^*)$, a contradiction. \square

Lemma 21. *If $B_2 = \emptyset$, then $G^* \cong SK_{k,m}$.*

Proof. If $c = |A_3| = 0$, we have $G^* \cong SK_{k,m}$, it follows the result. In the following, we suppose that $c = |A_3| \geq 1$. At this time, G^* is isomorphic to $SK_{k,m-c}$ by attaching c pendent edges at its maximal degree vertex. Notice that $m = 2a + c + k - 2$. Let G' obtained from G^* by contracting the internal path P_{k-3} as an edge v_1v_2 , that is, G' is obtained from $SK_{5,m-c}$ by attaching c pendent edges to its maximal degree vertex. By Lemma 11, we have $\rho(G^*) < \rho(G')$. Next we will show $\rho(G') \leq \frac{m-k+2}{\sqrt{m-k+1}} = \frac{2a+c}{\sqrt{2a+c-1}}$, which yields a contradiction. By the knowledge of equitable partition [6, Page 198], $\rho(G')$ is the largest root of $f(x) = x^6 - (2a+c+3)x^4 + (4a+2c+ac+1)x^2 - 2ax - ac$. The derivative function of $f(x)$ is $f'(x) = 6x^5 - 4(2a+c+3)x^3 + 2(4a+2c+ac+1)x - 2a$. One can verify that $f'(x) > 0$ for $x > \sqrt{2a+c+1}$. Thus $f(x)$ is increase when $x \geq \sqrt{2a+c+1}$. By computation,

$$f(\sqrt{2a+c+1}) = (ac-1)(2a+c+1) - 2a\sqrt{2a+c+1} - ac.$$

If $a = 1, c \geq 3$, then $f(\sqrt{2a+c+1}) = f(\sqrt{c+3}) = (c+3)^2 - 5(c+3) - 2\sqrt{c+3} + 3$ and we can verify it is an increase function about c , and thus $f(\sqrt{c+3}) \geq f(\sqrt{3+3}) = 7 - \frac{1}{\sqrt{6}} > 0$. Thus, $\rho(G') < \sqrt{2a+c+1} < \frac{2a+c}{\sqrt{2a+c-1}}$.

If $a \geq 2$ and $c \geq 2$, we have

$$\begin{aligned} f(\sqrt{2a+c+1}) &= (ac-1)(2a+c+1) - 2a\sqrt{2a+c+1} - ac \\ &= ac\left[\left(1 - \frac{1}{ac}\right)(2a+c+1) - \frac{2}{c}\sqrt{2a+c+1} - 1\right] \\ &\geq ac\left[\left(1 - \frac{1}{4}\right)(2a+c+1) - \sqrt{2a+c+1} - 1\right] \\ &= \frac{1}{4}ac[3(2a+c+1) - 4\sqrt{2a+c+1} - 4] \\ &> 0, \end{aligned}$$

due to $\sqrt{m-k+3} = \sqrt{2a+c+1} \geq \sqrt{7} > 2$. Hence, $\rho(G') < \sqrt{2a+c+1} < \frac{2a+c}{\sqrt{2a+c-1}}$.

If $a = 1$ and $c = 2$, we have $f\left(\frac{2a+c}{\sqrt{2a+c-1}}\right) = f\left(\frac{4}{\sqrt{3}}\right) = 4.6405 > 0$. It follows that $\rho(G') \leq \frac{4}{\sqrt{3}} = \frac{2a+c}{\sqrt{2a+c-1}}$.

If $a = 1$ and $c = 1$, then $\rho^* \leq \rho(G') = 2.115 < \frac{3}{\sqrt{2}} = \frac{2a+c}{\sqrt{2a+c-1}}$. □

According to Lemma 20, if $B_2 \neq \emptyset$, the structure of G^* is as shown in Fig.3 (3). Now we turn our eyes on the components of the Perron vector \mathbf{x} . Without loss of generality, we may assume that $x_{u_0} \geq x_{u_1} \geq \dots \geq x_{u_{a-1}}$ and $B_2 \neq \emptyset$ in the rest proof.

Lemma 22. *If $B_2 \neq \emptyset$, then for any $w \in B_2$, let $s = d_A(w)$, we have $s \geq 2$ and the value*

$$M(w) = a(x_{v_1} - x_{u^*}) + s(x_w - x_{u^*}) + x_{v_{k-3}} > 0. \tag{4}$$

Proof. If $s = 1$ and $u_i \sim w$, then $d_G(w) = d_A(w) = 1$ since B_2 is an independent set. The graph $G' = G^* - u_iw + u^*w \in \mathcal{G}(m, k)$ and $\rho(G') > \rho(G^*)$ due to Lemma 13, which is impossible. Hence $d_A(w) = s \geq 2$.

Let $N^2(u^*)$ be the set of vertices $v \in V(G)$ such that there exists $u \in V(G)$ satisfying $u^* \sim u \sim v$. Trivially, we have $u^* \in N^2(u^*)$. According to $A(G^*)\mathbf{x} = \rho^*\mathbf{x}$, we get

$$\begin{aligned} (\rho^*)^2 x_{u^*} &= (A^2 x)_{u^*} = \sum_{v \in N^2(u^*)} d_A(v) x_v \\ &= d_A(v_1) x_{v_1} + d_A(v_{k-3}) x_{v_{k-3}} + d_A(w) x_w + \sum_{v \in N^2(u^*) \setminus \{v_1, v_{k-3}, w\}} d_A(v) x_v \\ &\leq a x_{v_1} + x_{v_{k-3}} + s x_w + (m - (k - 4) - a - 1 - s) x_{u^*} \\ &= (m - k + 3) x_{u^*} + a(x_{v_1} - x_{u^*}) + s(x_w - x_{u^*}) + x_{v_{k-3}} \\ &= (m - k + 3) x_{u^*} + M(w). \end{aligned}$$

Since $(\rho^*)^2 > (m - k + 3)$, we get $M(w) > 0$. □

For components of \mathbf{x} corresponding to A , we get the following result.

Lemma 23. $x_{u_{a-1}} \geq x_{u_a}$ and if $A_3 \neq \emptyset$, then $x_{u_a} \geq x_w$ for any $w \in A_3$.

Proof. Suppose to the contrary that $x_{u_{a-1}} < x_{u_a}$. Recall that $x_{v_1} \geq x_{v_{k-3}}$. Let

$$G' = G^* - u_{a-1}v_1 - u_a v_{k-3} + u_{a-1}v_{k-3} + u_a v_1.$$

Clearly, $G' \in \mathcal{G}(m, k)$ and Lemma 14 implies $\rho(G') > \rho(G)$, a contradiction. Suppose to the contrary that $x_a \leq x_w$ for some $w \in A_3$. Let $G'' = G^* - u_a v_{k-3} + u_w v_{k-3}$. Similarly, $G'' \in \mathcal{G}(m, k)$ and $\rho(G'') > \rho(G)$, a contradiction. □

According to Lemma 23, we know that $x_{u_{a-1}} \geq x_{u_a}$ and $x_{u_a} \geq x_w$ for any $w \in A_3$. We denote $A_3 = \{u_{a+1}, u_{a+2}, \dots, u_{a+c}\}$ for some $c \geq 0$. By sorting the vertices of A_1 and A_3 , we may assume further that

$$x_{u_0} \geq \dots \geq x_{u_{a-1}} \geq x_{u_a} \geq x_{u_{a+1}} \geq \dots \geq x_{u_{a+c}}. \tag{5}$$

Lemma 24. If $B_2 \neq \emptyset$, then for $w \in B_2$, let $d(w) = s$, we have $N(w) = \{u_0, u_1, \dots, u_{s-1}\}$.

Proof. Suppose to the contrary that $u_i \sim w$ with $i \geq s$. Therefore, there exists $1 \leq j \leq s - 1$ such that $u_j \not\sim w$. Thus, $G' = G^* - u_i w + u_j w \in \mathcal{G}(m, k)$ with $\rho(G') > \rho^*$ due to (5) and Lemma 13, this is a contradiction. □

The following two lemmas (Lemmas 25 and 26) are key ingredients in the proof of our main result. To some extent, it characterized clearly the structure of the desired extremal graph. More precisely, we will show that if $B_2 \neq \emptyset$, then $A_3 = \emptyset$ and every vertex of B_2 is adjacent to every vertex of $A_1 \cup A_2$. We now begin the details in earnest.

Lemma 25. If $B_2 \neq \emptyset$, then $A_3 = \emptyset$.

Proof. In what follows, we will prove that $A_3 = \emptyset$ whenever $B_2 \neq \emptyset$. Otherwise, we assume to the contrary that $A_3 = \{u_{a+1}, u_{a+2}, \dots, u_{a+c}\}$, where $c = |A_3| \geq 1$. We will show that $\rho(G^*) < \rho(SK_{k,m})$, which leads to a contradiction. Since $B_2 \neq \emptyset$, choosing a vertex $w \in B_2$, we compute the values $\rho^*M(w)$ and $(\rho^*)^2M(w)$ by using the equation $A(G^*)\mathbf{x} = \rho^*\mathbf{x}$. Note that Lemma 24 gives

$$\rho^*x_w = \sum_{i=0}^{s-1} x_{u_i} = \rho^*x_{u^*} - \sum_{j=s}^{a+c} x_{u_j}.$$

Moreover, we have

$$\begin{cases} \rho^*x_{v_1} = \sum_{i=0}^{a-1} x_{u_i} + x_{v_2}, \\ \rho^*x_{v_{k-3}} = x_{v_{k-4}} + x_{u_a}. \end{cases}$$

By immediate calculations, we have

$$\begin{aligned} \rho^*M(w) &= a\rho^*x_{v_1} + s\rho^*x_w - (a+s)\rho^*x_{u^*} + \rho^*x_{v_{k-3}} \\ &= ax_{v_2} + x_{v_{k-4}} - (a-1)x_{u_a} - a \sum_{i=a+1}^{a+c} x_{u_i} - s \sum_{j=s}^{a+c} x_{u_j}, \end{aligned} \quad (6)$$

and

$$(\rho^*)^2M(w) = a(x_{v_1} + x_{v_3}) + (x_{v_{k-5}} + x_{v_{k-3}}) - (a-1)\rho^*x_{u_a} - a \sum_{i=a+1}^{a+c} \rho^*x_{u_i} - s \sum_{j=s}^{a+c} \rho^*x_{u_j}. \quad (7)$$

Note that $\rho^*x_{u_a} \geq x_{u^*} + x_{v_{k-3}}$ and $\rho^*x_{u_j} \geq x_{u^*}$ for $0 \leq j \leq a+c$. We get from (7) that

$$\begin{aligned} (\rho^*)^2M(w) &\leq ax_{v_1} + ax_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - (a-1)(x_{u^*} + x_{v_{k-3}}) \\ &\quad - acx_{u^*} - s(a+c-s+1)x_{u^*} \\ &= ax_{v_1} + ax_{v_3} + x_{v_{k-5}} - (a-2)x_{v_{k-3}} - f(s)x_{u^*}, \end{aligned} \quad (8)$$

where $f(s)$ is defined as

$$f(s) = -s^2 + (a+c+1)s + ac + a - 1.$$

Note that Lemma 22 gives $M(w) > 0$, and thus $\rho^*M(w) > 0$ and $(\rho^*)^2M(w) > 0$.

Case 1. $a \geq 2$. In this case, from (8), we have

$$0 < (\rho^*)^2M(w) \leq ax_{v_1} + ax_{v_3} + x_{v_{k-5}} - (a-2)x_{v_{k-3}} - f(s)x_{u^*} \leq ((2a+1) - f(s))x_{u^*}. \quad (9)$$

It leads to $f(s) < 2a+1$. Since $2 \leq s \leq a+c+1$, we have either $3a+ac+2c-3 = f(2) \leq f(s) < 2a+1$ or $ac+a-1 = f(a+c+1) \leq f(s) < 2a+1$. Thus we get $c < 1 + 2/a \leq 2$ and so $c = 1$.

Since $c = 1$, we have $f(s) = -s^2 + (a+2)s + 2a - 1$. Recall that $2 \leq s \leq a+c+1$. By $f(s) < 2a+1$, that is, $s[-s+(a+2)] < 2$, it can be deduced that $s = a+2$. It yields

that $w \sim u$ for any $w \in B_2$ and $u \in A$. Obviously, $x_{u_0} = \cdots = x_{u_{a-1}}$ and $x_w = x_{u^*}$ for any $w \in B_2$ because A_1 and $B_2 \cup \{x_{u^*}\}$ are orbits of G^* acting by $\text{Aut}(G^*)$. Equation (7) turns to be

$$(\rho^*)^2 M(w) = a(x_{v_1} + x_{v_3}) + (x_{v_{k-5}} + x_{v_{k-3}}) - (a-1)\rho^* x_{u_a} - a\rho^* x_{u_{a+1}}.$$

Since $\rho^* x_{u_a} = (b_2 + 1)x_{u^*} + x_{v_{k-3}}$ and $\rho^* x_{u_{a+1}} = (b_2 + 1)x_{u^*}$ where $b_2 = |B_2|$, we have

$$\begin{aligned} (\rho^*)^2 M(w) &= a(x_{v_1} + x_{v_3}) + x_{v_{k-5}} - (a-2)x_{v_{k-3}} - (2a-1)(b_2+1)x_{u^*} \\ &\leq ((2a+1) - (2a-1)(b_2+1))x_{u^*}. \end{aligned}$$

Therefore, $(2a+1) - (2a-1)(b_2+1) > 0$, and thus $b_2 < \frac{2}{2a-1} \leq \frac{2}{3}$. It leads to $b_2 = 0$, which contradicts the assumption of $B_2 \neq \emptyset$.

Case 2. $a = 1$. In this case, from (8), we have

$$0 < (\rho^*)^2 M(w) \leq x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - f(s)x_{u^*} \leq (4 - f(s))x_{u^*}. \quad (10)$$

It leads to $f(s) = -s^2 + (c+2)s + c < 4$. Since $2 \leq s \leq c+2$, we have either $3c = f(2) \leq f(s) < 4$ or $c = f(c+2) \leq f(s) < 4$. Hence $c = 1, 2$ or 3 .

Subcase 2.1. $c = 1$.

Since $2 \leq s \leq a+c+1 = 3$, we have $s = 2$ or 3 . Set $B_{2,1} = \{w \in B_2 \mid d(w) = 2\}$, $B_{2,2} = \{w \in B_2 \mid d(w) = 3\}$, $b_1 = |B_{2,1}|$ and $b_2 = |B_{2,2}|$. Similarly, $x_w = x_{u^*}$ for any $w \in B_{2,2}$. If $b_1 \geq 1$, for $w \in B_{2,1}$, (7) turns to be

$$(\rho^*)^2 M(w) = x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - 3\rho^* x_{u_2}.$$

Since $\rho^* x_{u_2} = (1 + b_2)x_{u^*}$, we have

$$\begin{aligned} (\rho^*)^2 M(w) &= x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - 3(b_2+1)x_{u^*} \\ &\leq (4 - 3(b_2+1))x_{u^*}. \end{aligned}$$

Therefore, $3(b_2+1) < 4$, and thus $b_2 < 1/3$. It leads to $b_2 = 0$. In this case, as similar to the proof of Lemma 21, we construct a new graph G' by contracting the path $v_1 v_2 \cdots v_{k-3}$ as an edge $v_1 v_2$ from G^* . The value $\rho(G')$ is just the root of the function

$$f(x) = x^5 - x^4 - (2b_1 + 4)x^3 + (2b_1 + 3)x^2 + (2b_1 + 1)x - 2b_1.$$

By calculations, we have $f(x) > 0$ whenever $x > \sqrt{2b_1 + 4}$. It means that $\rho^* \leq \rho(G') < \sqrt{2b_1 + 4} = \sqrt{m-k+3}$, a contradiction. Thus, $b_1 = 0$ and $b_2 \geq 1$. For $w \in B_{2,2}$, (7) turns to be

$$(\rho^*)^2 M(w) = x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - \rho^* x_{u_2}.$$

Since $\rho^* x_{u_2} = (1 + b_2)x_{u^*}$, we have

$$\begin{aligned} (\rho^*)^2 M(w) &= x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - (b_2+1)x_{u^*} \\ &\leq (4 - (b_2+1))x_{u^*}. \end{aligned}$$

Therefore, $(b_2 + 1) < 4$, and thus $b_2 < 3$. Let G_1 and G_2 be the graphs shown in Fig.4. According to Lemma 11, if $b_2 = 1$, then $\rho^* < \rho(G_1) = 2.632 < \sqrt{m - k + 3} = \sqrt{7}$; if $b_2 = 2$, then $\rho^* < \rho(G_2) = 3.133 < \sqrt{m - k + 3} = \sqrt{10}$. They are both impossible since $\rho^* > \sqrt{m - k + 3}$.

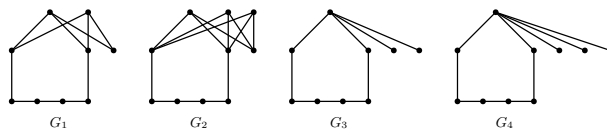


Figure 4: The graphs G_1 and G_2 in the proof of Lemma 25.

Subcase 2.2. $c \in \{2, 3\}$.

In this case, it is easy to verify that $f(s) = -s^2 + (c + 2)s + c < 4$ implies $s = c + 2$. Also, $x_w = x_{u^*}$ for any $w \in B_2$. Now (7) turns to be

$$(\rho^*)^2 M(w) = x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - \rho^* \sum_{i=2}^{c+1} x_{u_i}.$$

Since $2 \leq i \leq c + 1$ and $\rho^* x_{u_i} = (b + 1)x_{u^*}$ where $b = |B_2|$, we have

$$\begin{aligned} (\rho^*)^2 M(w) &= x_{v_1} + x_{v_3} + x_{v_{k-5}} + x_{v_{k-3}} - c(b + 1)x_{u^*} \\ &\leq (4 - c(b + 1))x_{u^*}. \end{aligned}$$

Therefore, $4 - c(b + 1) > 0$, and thus $b = 0$, which contradicts the assumption $B_2 \neq \emptyset$.

The proof is completed. \square

Lemma 26. *If $B_2 \neq \emptyset$, then $N_G(w) = A_1 \cup A_2$ for every $w \in B_2$.*

Proof. Denote $s = d(w)$, we have $2 \leq s \leq a + 1$ by Lemma 22 and Lemma 25. If $a = 1$, then $s = 2$, the result $N_G(w) = A_1 \cup A_2$ follows immediately. Now, we consider $a \geq 2$, it only needs to show that $s = a + 1$. According to $A(G^*)\mathbf{x} = \rho^*\mathbf{x}$, we have

$$\rho^* x_{u^*} = \sum_{i=0}^{a-1} x_{u_i} + x_{u_a} \quad \text{and} \quad \rho^* x_{v_1} = \sum_{i=0}^{a-1} x_{u_i} + x_{v_2}.$$

Thus $x_{u_a} - x_{v_2} = \rho(x_{u^*} - x_{v_1}) \geq 0$ and then $x_{u_a} \geq x_{v_2}$. Moreover,

$$\rho^* x_{u_a} = x_{u^*} + x_{v_{k-3}} \quad \text{and} \quad \rho^* x_{v_{k-4}} = x_{v_{k-5}} + x_{v_{k-3}}.$$

It follows that $\rho^*(x_{u_a} - x_{v_{k-4}}) = x_{u^*} - x_{v_{k-5}} \geq 0$, i.e. $x_{u_a} \geq x_{v_{k-4}}$. Therefore, from (6), we have

$$\begin{aligned} \rho^* M(w) &= ax_{v_2} + x_{v_{k-4}} - (a - 1)x_{u_a} - s \sum_{j=s}^a x_{u_j} \\ &\leq ax_{v_2} + x_{v_{k-4}} - (a - 1)x_{u_a} - s(a - s + 1)x_{u_a} \\ &\leq (a + 1 - (a - 1) - s(a - s + 1))x_{u_a}. \end{aligned}$$

Note that $a + 1 - (a - 1) - s(a - s + 1) = s^2 - (a + 1)s + 2 \leq 0$ whenever $2 \leq s \leq a$, and $\rho^*M(w) > 0$. We have $s = a + 1$. □

Recall the definition of $C_k(a, b)$ (see Fig.1). Lemma 25 and Lemma 26 indicate that in Fig.3, if $B_2 \neq \emptyset$, then $A_3 = \emptyset$ and every vertex of B_2 is adjacent to every vertex of $A_1 \cup A_2$. In other words, A_1 and $B_2 \cup \{u^*\}$ form a complete bipartite, and $v_1v_2 \cdots v_{k-3}u_a$ is a path of length $k - 3$. Thus $G^* \in C_k(a, b)$ for some $a \geq 1$ and $b = |B_2| + 1 \geq 1$. In what follows, we consider the spectral radius of $C_k(a, b)$.

Lemma 27. *If $a, b \geq 2$, then $\rho(C_k(a, b)) \leq \sqrt{ab + a + b}$ unless $a = 2$ and $b = 2$.*

Proof. Without loss of generality, assume $a \geq b \geq 2$. According to Lemma 11, we have $\rho(C_k(a, b)) \leq \rho(C_7(a, b)) < \rho(C_5(a, b))$. By the knowledge of equitable partition [6, Page 198], $\rho(C_7(a, b))$ and $\rho(C_5(a, b))$ are respectively the largest roots of $f(x)$ and $g(x)$, where

$$\begin{cases} f(x) = x^7 - (ab + a + b + 4)x^5 + (5ab + 3a + 3b + 3)x^3 - (5ab + a + b)x - 2ab, \\ g(x) = x^5 - (ab + a + b + 2)x^3 + (3ab + a + b)x - 2ab. \end{cases}$$

Note that the derivative function of $g(x)$ is $g'(x) = 5x^4 - 3(ab + a + b + 2)x^2 + 3ab + a + b$. It is clear that $g'(x) > 0$ whenever $x \geq \sqrt{ab + a + b}$. We have $g(x)$ is increase whenever $x \geq \sqrt{ab + a + b}$, and thus $g(x) \geq g(\sqrt{ab + a + b})$ whenever $x \geq \sqrt{ab + a + b}$. By immediate calculation, we have

$$g(\sqrt{ab + a + b}) = \sqrt{ab + a + b} \cdot (ab - a - b) - 2ab.$$

If $b \geq 4$, then $g(\sqrt{ab + a + b}) \geq b(4a - 2a) - 2ab = 0$. It means $g(x) \geq 0$ whenever $x \geq \sqrt{ab + a + b}$, and thus $\rho(C_5(a, b)) \leq \sqrt{ab + a + b}$. Hence $\rho(C_k(a, b)) < \sqrt{ab + a + b}$.

If $b = 3$, then $g(\sqrt{ab + a + b}) = \sqrt{4a + 3} \cdot (2a - 3) - 6a$. If $a \geq 5$, then

$$\begin{aligned} g(\sqrt{ab + a + b}) &= \sqrt{4a + 3}(2a - 3) - 6a \\ &\geq \sqrt{23} \cdot (2a - 3) - 6a = (2\sqrt{23} - 6)a - 3\sqrt{23} \\ &\geq (2\sqrt{23} - 6) \cdot 5 - 3\sqrt{23} > 0. \end{aligned}$$

It means $\rho(C_k(a, b)) < \rho(C_5(a, b)) < \sqrt{ab + a + b}$. If $a = 4$, then $f(x) = x^7 - 23x^5 + 84x^3 - 67x - 24$, whose largest root is $4.327 < \sqrt{ab + a + b} = \sqrt{19}$. Hence $\rho(C_k(a, b)) \leq \rho(C_7(a, b)) < \sqrt{ab + a + b}$. If $a = 3$, then $f(x) = x^7 - 19x^5 + 66x^3 - 51x - 18$, whose largest root is $3.846 < \sqrt{ab + a + b} = \sqrt{15}$. Hence $\rho(C_k(a, b)) \leq \rho(C_7(a, b)) < \sqrt{ab + a + b}$.

If $b = 2$, then $g(\sqrt{ab + a + b}) = \sqrt{3a + 2} \cdot (a - 2) - 4a$. If $a \geq 9$, then

$$\begin{aligned} g(\sqrt{ab + a + b}) &= \sqrt{3a + 2} \cdot (a - 2) - 4a \\ &\geq \sqrt{29} \cdot (a - 2) - 4a = (\sqrt{29} - 4)a - 2\sqrt{29} \\ &\geq (\sqrt{29} - 4) \cdot 9 - 2\sqrt{29} > 0. \end{aligned}$$

It means $\rho(C_k(a, b)) < \rho(C_5(a, b)) < \sqrt{ab + a + b}$. If $3 \leq a \leq 8$, as similar to the case of $b = 3$, the largest root of $f(x)$ is less than $\sqrt{ab + a + b}$ (see Tab.1), and thus $\rho(C_k(a, b)) \leq \rho(C_7(a, b)) < \sqrt{ab + a + b}$.

The proof is completed. □

(b, a)	$f(x)$	largest root	$\sqrt{ab + a + b}$
(3, 4)	$x^7 - 23x^5 + 84x^3 - 67x - 24$	4.3266	4.3589
(3, 3)	$x^7 - 19x^5 + 66x^3 - 51x - 18$	3.8461	3.8730
(2, 8)	$x^7 - 30x^5 + 113x^3 - 90x - 32$	5.0753	5.0990
(2, 7)	$x^7 - 27x^5 + 100x^3 - 79x - 28$	4.7720	4.7958
(2, 6)	$x^7 - 24x^5 + 87x^3 - 68x - 24$	4.4488	4.4721
(2, 5)	$x^7 - 21x^5 + 74x^3 - 57x - 20$	4.1001	4.1231
(2, 4)	$x^7 - 18x^5 + 61x^3 - 46x - 16$	3.7230	3.7417
(2, 3)	$x^7 - 15x^5 + 48x^3 - 35x - 12$	3.3065	3.3166

Table 1: The function $f(x)$ and its largest root used in the proof of Lemma 27.

By immediate calculations, we have $\rho(C_5(2, 2)) = 2.9032 > 7/\sqrt{6} = \frac{m(C_5(2,2))-5+2}{\sqrt{m(C_5(2,2))-5+1}}$. For $k \geq 7$, we get the following result.

Lemma 28. $\rho(C_k(2, 2)) < \frac{7}{\sqrt{6}} = \frac{m(C_k(2,2))-k+2}{\sqrt{m(C_k(2,2))-k+1}}$ for $k \geq 7$.

Proof. By immediate calculations, we have $\rho(C_k(2, 2)) \leq \rho(C_7(2, 2)) = 2.84 < \frac{7}{\sqrt{6}}$. \square

Proof of Theorem 7. According to Lemma 20, G^* is of the form (3) shown in Fig.3. If $B_2 = \emptyset$, then Lemma 21 implies that $G^* \cong SK_{k,m}$. If $B_2 \neq \emptyset$, then Lemmas 25 and 26 mean that $G^* = C_k(a, b)$ for some a, b . Without loss of generality, assume that $a \geq b$. It only need to show $b = 1$. Suppose to the contrary that $b \geq 2$. Lemma 27 indicates that $a = b = 2$. However, in this case $m = 8 + k - 3$ is even, a contradiction. \square

Proof of Theorem 8. It suffices to show that any $\{C_3, C_5, \dots, C_{k-2}\}$ -free non-bipartite graph G with m edges different from $C_5(2, 2)$ or $SK_{k,m}$ has spectral radius $\rho < \frac{m-k+2}{\sqrt{m-k+1}}$. Since G is non-bipartite, we may assume that G has odd girth r with $r \geq k$, and G has maximum spectral radius among $\mathcal{G}(m, r)$. Suppose to the contrary that $\rho \geq \frac{m-k+2}{\sqrt{m-k+1}}$. Therefore, we have $\rho \geq \frac{m-k+2}{\sqrt{m-k+1}} \geq \frac{m-r+2}{\sqrt{m-r+1}} > \sqrt{m-r+3}$. Noticing all the lemmas do not using the assumption of m being odd, from Lemmas 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 and 27 by replacing k with r in them, we get $G = C_r(a, b)$ with $a \geq b$, and either $a = 2$ and $b = 2$, or $b = 1$. If $r = k$, then Lemma 28 implies $G \cong C_5(2, 2)$ or $SK_{k,m} = C_k(a, 1)$, a contradiction. If $r > k \geq 5$, then Lemma 28 implies that $G \cong SK_{r,m} = C_r(a, 1)$. However, Lemma 12 indicates $\rho < \sqrt{m-r+4} \leq \sqrt{m-(k+2)+4} < \sqrt{m-k+3} < \frac{m-k+2}{\sqrt{m-k+1}}$, a contradiction. \square

4 Open problems for cycles

Theorem 7 determines the extremal graphs for non-bipartite graphs without any short odd cycle of $\{C_3, C_5, \dots, C_{2k+1}\}$. In this section, we will conclude some recent development on this topic and propose some open problems for readers. Let $K_k \vee I_t$ be the graph consisting of a clique on k vertices and an independent set on t vertices in which each

vertex of the clique is adjacent to each vertex of the independent set. A well-known conjecture in extremal spectral graph theory involving consecutive cycles states that

Conjecture 29 (Zhai–Lin–Shu [24]). Let k be fixed and m be large enough. If G is a graph with m edges and

$$\lambda(G) \geq \frac{k - 1 + \sqrt{4m - k^2 + 1}}{2},$$

then G contains a cycle of length t for every $t \leq 2k + 2$, unless $G = K_k \vee I_{\frac{1}{k}}(m - \binom{k}{2})$.

One may consider naturally the problem of finding the maximum spectral radius among all $\{C_3, C_4, \dots, C_{2k+2}\}$ -free graphs with m edges. We remark here that this problem is an easy consequence of Nikiforov in [17], which implies that the star graph on m edges attains the maximum spectral radius. Indeed, Nikiforov [17] proved that for $m \geq 10$, every C_4 -free graph G satisfies $\rho(G) \leq \sqrt{m}$, equality holds if and only if G is uniquely a star with m edges. The above problem is a direct corollary by noting that the star graph contains no copies of C_t for every $3 \leq t \leq 2k + 2$.

The following Conjecture 30 seems weaker than Conjecture 29 at first glance. While they are equivalent since the bound in right hand side is monotonically increasing on $k \in [2, +\infty)$. So it is reasonable to attribute this conjecture to Zhai, Lin and Shu.

Conjecture 30 (Zhai–Lin–Shu). Let k be fixed and G be a graph of sufficiently large size m without isolated vertices. If G is C_{2k+1} -free or C_{2k+2} -free, then

$$\lambda(G) \leq \frac{k - 1 + \sqrt{4m - k^2 + 1}}{2},$$

equality holds if and only if $G = K_k \vee I_{\frac{1}{k}}(m - \binom{k}{2})$.

In 2021, Zhai, Lin and Shu [24] proved this conjecture in the case of $k = 2$ and odd m , and later Gao, Lou and Huang [14] proved the case of $k = 2$ and even m . These cases were also provided by Li, Sun and Wei [8]. Conjecture 30 remains open for $k \geq 3$.

Let C_t^Δ denote the graph on $t + 1$ vertices obtained from C_t and C_3 by identifying an edge. It was proved in [24] that the complete bipartite graphs attain the maximum spectral radius among $\{C_3^\Delta, C_4^\Delta\}$ -free graphs with m edges. In [20], Nikiforov enhanced that the same result still holds for C_3^Δ -free graphs. Very recently, Li, Sun and Wei [8] determined the extremal graph for C_4^Δ -free or C_5^Δ -free when the size m is odd, and soon after, Fang, You and Huang [3] determined the extremal graph for even m . Observe that $C_{2k+1} \subseteq C_{2k}^\Delta$ and $C_{2k+2} \subseteq C_{2k+1}^\Delta$. Motivated by Conjecture 30, Yongtao Li tells privately us that it is also interesting to consider the following conjecture.

Conjecture 31 (Yongtao Li). Let $k \geq 3$ and G be a graph of sufficiently large size m without isolated vertices. If G is C_{2k}^Δ -free or C_{2k+1}^Δ -free, then

$$\lambda(G) \leq \frac{k - 1 + \sqrt{4m - k^2 + 1}}{2},$$

equality holds if and only if $G = K_k \vee I_{\frac{1}{k}}(m - \binom{k}{2})$.

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