Schnyder woods and Alon-Tarsi number of planar graphs

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Abstract

Thomassen in 1994 published a famous proof of the fact that the choosability of a planar graph is at most 5. Zhu in 2019 generalized this result by showing that the same bound holds for Alon-Tarsi numbers of planar graphs. We present an alternative proof of that fact, derived from the results on decompositions of planar graphs into trees known as Schnyder woods. It turns out that Thomassen's technique and our proof based on Schnyder woods have a lot in common. We discuss and explain the prominent role that counterclockwise 3-orientations play in proofs based on both these approaches.

Mathematics Subject Classifications: 05C10, 05C15, 05C31, 68R10

1 Introduction

Thomassen proved in [12] that the choice number of a planar graph is at most 5. This result is best possible as there exist planar graphs with choice number 5. The first such examples have been constructed by Voigt [13]. The proof technique introduced by Thomassen in [12] has been used in a number of follow-up papers. In particular, it has been generalised to derive analogous results for more restrictive variants of graph colorings. E.g. Schauz in [10] proved that online choice number (or paintability) of a planar graph is at most 5 as well. We are going to discuss yet another generalization by Zhu [14].

Theorem 1 (Zhu [14])**.** *The Alon-Tarsi number of a planar graph is at most 5.*

The main contribution of the current paper is a derivation of the above theorem from the results on the decompositions of planar graphs into trees. Such decompositions, called realizers, have been designed by Schnyder in [11] for the purpose of constructing succinct

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straight-line drawings of planar graphs. Our proof of Theorem 1 is conceptually (and technically) simpler than the one by Zhu. However, it turns out that both arguments are related. We show that in some sense the structure that is central in our derivation is also implicitly used in the works of Thomassen and Zhu.

In Section 2 we prepare the tools to be used in the proof of Theorem 1. We recall (and extend) basic results on the Alon-Tarsi polynomial method and on Schnyder decompositions. Then, in Section 3 we present our proof. Finally, in Section 4 we discuss its relation to the original proof of Thomassen for the choice number.

2 Preliminaries

2.1 Alon-Tarsi method

For a polynomial Q, let $\alpha(Q)$ be the minimum k such that there is a monomial m that occurs in Q with a nonzero coefficient, for which $\deg(Q) = \deg(m)$ and the maximum degree of any single variable in m is at most k. Note that, for a nonzero polynomials P and R, we have $\alpha(P) \leqslant \alpha(P \cdot R)$. For graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, *graph polynomial* f_G is defined as

$$f_G(x_1, x_2, \ldots, x_n) = \prod_{i<j \wedge \{v_i, v_j\} \in E} (x_i - x_j).$$

This polynomial has been discovered by a number of researchers. Most likely it was first used in 1891 by Petersen [9]. It appeared independently in the works of Matiyasevich [8]. A great deal of its todays popularity is owed to the paper of Alon and Tarsi [1].

For a vertex coloring c of graph G, we can see that $f_G(c(v_1), c(v_2), \ldots, c(v_n)) \neq 0$ if and only if the coloring is proper (where the colors are interpreted as elements of some ring, say $\mathbb{Z}$). Numerous applications of the Alon and Tarsi method inspired Jensen and Toft [6] to define the Alon-Tarsi number of a graph.

Definition 2 (Jensen and Toft [6])**.** *The Alon-Tarsi number of a graph G*, denoted by $AT(G)$, is defined as

$$AT(G) = \alpha(f_G) + 1.$$

The authors of both seminal papers [1, 8] observed that for any graph G

$$ch(G) \leqslant AT(G).$$

Alon and Tarsi proved the above proposition by applying Combinatorial Nullstelensatz to the graph polynomial f_G. Schauz [10] observed that this line of argument can be extended to the case of on-line choosability obtaining

$$ch_{OL}(G) \leqslant AT(G).$$

The following simple observation allows us to use triangular graphs when proving upper bounds for the Alon-Tarsi number for planar graphs.

Fact 3. *If G is a subgraph of H, then*

$$AT(G) \leqslant AT(H).$$

Proof. Graph polynomial f_G divides graph polynomial f_H. Therefore $\alpha(f_G) \leqslant \alpha(f_H)$, and $AT(G) \leqslant AT(H)$. □

The next lemma is also from the work of Alon and Tarsi [1]. A graph is called *even (odd)*, depending on the parity of the number of its edges. Recall that a directed graph F is *an Eulerian subgraph of an orientation D of graph G* if F is a subgraph of D and for every vertex in F, the in-degree in F is equal to the out-degree in F.

Lemma 4 (Alon and Tarsi [1])**.** *Consider a graph G and its orientation D. Let k be the maximum out-degree in this orientation. If the number of even Eulerian subgraphs of D is different from the number of odd Eulerian subgraphs of D then*

$$AT(G) - 1 \leqslant k.$$

A generalized version of the above lemma is proved in the next section.

The coloring number of a graph G, denoted by $col(G)$ is the minimum number k such that there exists an acyclic orientation of the edges of G for which out-degree of every vertex is at most $k-1$. Since the empty graph is always an (even) Eulerian subgraph of any oriented graph, by Lemma 4 we have

$$AT(G) \leqslant col(G).$$

2.2 Extension to augmented graphs

We define an extended version of the graph polynomial for graphs with augmented edges.

Definition 5. An *Augmented orientation* is a tuple (G, w, D) where G is a graph, D is an orientation of G, and $w : E \to \mathbb{N}$ assigns positive strengths to the edges of G. An edge with strength k will be called a *k-edge*. An edge with strength 2 will be called a *double edge*. *Augmented in-degree* (resp. *augmented out-degree*) is defined as the sum of strengths of the ingoing (resp. outgoing) edges.

Let (G, w, D) be an augmented orientation. We define a *graph polynomial* for an augmented graph as follows.

$$W_{G,w}(x_1, \dots, x_n) = \prod_{i<j \wedge e=\{v_i,v_j\} \in E} (x_i^{w(e)} - x_j^{w(e)}).$$

Remark 6. Observe that, in general, $W_{G,w}$ is different from the polynomial $f_{G'}$ of a multi-graph G', constructed by replacing every k-edge of (G, w, D) with k parallel edges. Note also that $W_{G,w} = f_G$ if every edge has strength 1.

Using identity

$$(x_a^{w(e)} - x_b^{w(e)}) = (x_a - x_b)\left(\sum_{i=0}^{w(e)-1} x_a^{w(e)-1-i} x_b^i\right),$$

polynomial $W_{G,w}$ can be rewritten into

$$\begin{aligned} W_{G,w}(x_1,\ldots,x_n) &= \left(\prod_{i<j\wedge e=\{v_i,v_j\}\in E} (x_i - x_j)\right)\cdot P(x_1,\ldots,x_n) \\ &= f_G(x_1,\ldots,x_n)\cdot P(x_1,\ldots,x_n), \end{aligned}$$

where P is a nonzero polynomial. From this we conclude the following fact

$$AT(G) - 1 = \alpha(f_G) \leqslant \alpha(W_{G,w}).$$

Definition 7 (Eulerian structure)**.** A graph F without isolated vertices is *an Eulerian structure in augmented orientation* (G, w, D) if F is a subgraph of G and for every vertex v its augmented in-degree is equal to its augmented out-degree in augmented orientation (F, w_F, D_F), where D_F is D restricted to F and w_F is w restricted to F.

The proof of the next lemma is a straightforward extension of the proof of Alon and Tarsi [1] to the case of graphs with augmented edges.

Lemma 8. *Consider augmented orientation* (G, w, D)*. Let* k *be the maximum augmented out-degree in this augmented orientation. If the number of even Eulerian structures in* (G, w, D) *is different than the number of odd Eulerian structures in* (G, w, D) *then*

$$AT(G) - 1 = \alpha(f_G) \leqslant \alpha(W_{G,w}) \leqslant k.$$

Proof. We define function S from the set of orientations of G to the set of monomials that can potentially occur in $W_{G,w}$. Given an orientation, from every term $(x_i^{w(e)} - x_j^{w(e)})$ of the product defining $W_{G,w}$, we choose $x_i^{w(e)}$ if the orientation directs edge $e = v_iv_j$ away from v_i; otherwise, we choose $-x_j^{w(e)}$. Then, the value of S on the orientation is the product of the chosen terms.

We can see that

$$W_{G,w} = \sum_{R\text{ - orientation of }G} S(R).$$

The degree of x_i in $S(R)$ is exactly the augmented out-degree of v_i in (augmented) orientation (G, w, R). Consider two orientations R_1, R_2 such that $|S(R_1)| = |S(R_2)|$ (i.e. the monomials produced from R_1 and R_2 differ at most by sign). Let A be the set of edges that are oriented differently in R_1 than in R_2. We can notice that A induces Eulerian structure in both (G, w, R_1) and (G, w, R_2) (because augmented out-degree of every vertex is the same in R_1 and R_2). Moreover, the sign of $S(R_1)$ is equal to the sign of $S(R_2)$ if and only if the number of edges in A is even. Therefore, for every Eulerian structure

B in (G, w, R_1), when we change the orientation of the edges belonging to B, we get an orientation R_3 for which $|S(R_1)| = |S(R_3)|$. Let $CO(R), CE(R)$ be respectively, the number of Eulerian structures in (G, w, R) with odd and even number of edges.

The above discussion allows us to conclude that, whenever $CO(D) \neq CE(D)$, we have

$$\sum_{R:|S(R)|=|S(D)|} S(R) \neq 0.$$

Since the maximum augmented out-degree of (G, w, D) is at most k, we obtain that $\alpha(S(R)) \leqslant k$. That implies

$$AT(G) - 1 = \alpha(f_G) \leqslant \alpha(W_{G,w}) \leqslant k.$$

□

2.3 Schnyder woods

In this section, we recall the definitions and basic properties of Schnyder labellings. We are going to use them in the main proof. Considered theorems are trivial for graphs with fewer than 3 vertices. In this section, we always assume that plane graphs under consideration have at least 3 vertices.

Definition 9. A *triangular graph* is a plane graph whose faces are triangles.

Clearly, every planar graph (with at least 3 vertices) is a subgraph of some triangular graph on the same vertex set. By monotonicity of the Alon-Tarsi number (Fact 3) it is enough to prove the upper bound for triangular graphs. Observe also that triangular graphs are 2-connected.

2.3.1 3-orientations and realizers

Definition 10 (Schnyder [11])**.** A *realizer* of a triangular graph G is a tuple of three sets of oriented edges (T_r, T_g, T_b) such that, after ignoring edge orientations, these three sets form a partition of the interior edges of G and such that for each interior vertex v of G it holds:

1. v has out-degree one in each of T_r, T_g, T_b.
2. The counterclockwise order of the edges incident to v is the following: outgoing edge of T_r, incoming edges of T_b, outgoing edge of T_g, incoming edges of T_r, outgoing edge of T_b, incoming edges of T_g.

Note that in the above definition, for any interior vertex v there might be no incoming edges in any of the sets.

Schnyder proved in [11] that every triangular graph has a realizer. In addition to partitioning the interior edges into three sets, the realizer also orients the interior edges of G. In this orientation, every interior vertex has out-degree exactly 3. Such orientations of the interior edges of a triangular graph are called *internal* 3-*orientations*. It is easy to check that, by Euler's formula, in every such orientation, no interior edge that is adjacent to an exterior vertex can be directed away from the exterior vertex. De Fraysseix and Ossona de Mendez observed in [3] that an internal 3-orientation uniquely determines a realizer. Therefore, there is a bijection between the set of realizers and the set of internal 3-orientations of a graph G. Both of these perspectives are going to be useful for our needs. (Note that we are ignoring edges of the outer triangle.) We use the following observation from [11].

Proposition 11 (Theorem 4.5 in [11])**.** *Let G be a triangular graph with realizer (T_r, T_g, T_b). Then T_r, T_g, T_b are trees and each of T_r, T_g, T_b spans all interior vertices of G.*

Every large enough triangular graph admits a number of different realizers. Natural operation of inverting the edges of a directed triangle in an internal 3-orientation allows to transform one internal 3-orientation into another. Starting from this notion, Brehm [2] studied the graph of orientations and discovered that it is naturally organized into a structure of a distributive lattice. The top and the bottom elements of that lattice are the unique orientations without respectively clockwise and counter-clockwise cycles. The counterclockwise orientation described in the next proposition plays a central role in our proof.

Proposition 12 (Theorem 1.3.3 and Lemma 1.7.7 in [2])**.** *For every triangular graph, there exists exactly one internal 3-orientation in which all directed cycles are oriented counterclockwise.*

Proof. Theorem 1.3.3 and Lemma 1.7.7 in [2] imply the proposition for 4-connected triangular graphs. It can be extended to all triangular graphs in a standard way. We describe it below.

Suppose that a triangular graph is not 4-connected. Then it contains a separating triangle. Consider a subgraph enclosed by such a triangle, together with the triangle. Let us call it *inner graph*. After removing internal vertices and edges of the inner graph we obtain the *outer graph*. Observe that both these graphs are triangular. It is easy to check that every internal 3-orientation of the inner graph can be combined with every internal 3-orientation of the outer graph to construct an internal 3-orientation of the whole graph. As we already noted, the edges adjacent to external vertices of a 3-orientation have to be directed towards these vertices. That implies that, for every 3-orientation of the whole graph, every directed cycle is either contained completely in the inner graph or does not use any of its internal edges.

With this property in mind, we can construct a counterclockwise orientation of a triangular graph by a series of the following steps. Pick a maximal 4-connected subgraph of the graph, fix the unique counterclockwise internal 3-orientation for this subgraph and then remove all its internal vertices and edges from the graph. Note that in the fixed orientation of the subgraph no edge was directed away from the external vertex of this subgraph. The remaining graph is still triangular and the procedure can be repeated as long as the graph is not a triangle. By the property discussed above, the orientation of the whole graph constructed in this way is indeed counterclockwise. Theorem 1.3.3 in [2] asserts that it is unique (it can be also deduced from the above argument). □

2.3.2 Warm-up example

We illustrate how the basic results on the realizers of planar graphs can be used to derive an alternative proof of the following strengthening of one of the results from [7]. (The actual bound from [7] was for the Alon-Tarsi number.)

Theorem 13 (Kim, Kim and Zhu [7]). *For every planar graph G there exists a forest F such that*

$$col(G - F) \leqslant 3.$$

We are going to use the following version of Theorem 4.6 from [11].

Proposition 14. *Consider a triangular graph $G = (V, E)$ with realizer (T_r, T_g, T_b). Then, the oriented graph $(V, T_r \cup T_b)$ is acyclic.*

Proof of Theorem 13. By the monotonicity of the coloring number, it is enough to prove the theorem for triangulations. Let $G = (V, E)$ be a triangulation and let (T_r, T_g, T_b) be a realizer of G (note that realizers exist by the results of Schnyder [11]). By Proposition 14, oriented graph $H = (V, T_r \cup T_g)$ is acyclic. Moreover, by the definition of a realizer, every interior vertex has out-degree 2 in H and every exterior vertex has in-degree 0. Therefore, we can add to H the edges of the outercycle of G and orient them in such a way that the resulting graph H' is still acyclic. Then, the out-degrees of all vertices H' are still at most 2. This implies that $col(H')$ is at most 3. Note that after dropping the orientation, the edges of H' are precisely the edges of $G - T_b$. Let F be the set of edges T_b with dropped orientations. The orientation of H' proves that

$$col(G - F) \leqslant 3.$$

□

3 Counterclockwise orientations and Alon-Tarsi numbers of planar graphs

We recall a few more definitions and a few results from [11]. For a realizer (T_r, T_g, T_b) of a triangular graph G, a *colored path* is a path from an interior vertex of G to an exterior vertex of G that has only edges from one of the sets T_r, T_g, T_b. Every tree induced by sets T_r, T_g, T_b contains only one exterior vertex of G (consequence of Theorem 4.5 in [11]). We call these vertices *roots* and denote them by v_r, v_g, v_b. For every vertex, there are 3 unique colored paths that start at that vertex [11].

The colored paths in T_r, T_g, T_b are correspondingly called the *red path, green path*, and *blue path.* Colored paths that contain an interior vertex v of G as one of the ends are correspondingly denoted by $P_r(v), P_g(v), P_b(v)$. The path $P_i(v)$ ends in v_i for $i \in \{r, g, b\}$. Under the orientation given by the realizer, path $P_r(v)$ is a directed path from v to v_r. For an interior vertex v of G we see that $P_r(v), v_r v_b, P_b(v)$ forms a simple (i.e., not directed) cycle C. The subgraph of G bounded by C is called *the green region of* v and is denoted by $R_g(v)$. Regions of other colors are defined in an analogous way.

For interior vertices u, v of G, Lemma 5.2 in [11] implies that

$$u \in R_g(v) \implies R_g(u) \text{ is a subgraph of } R_g(v). \tag{1}$$

In other words, green regions are partially ordered by the relation of being a subgraph.

We are ready to prove the main ingredient of our argument.

Proposition 15. *For every triangular graph G there exists an augmented orientation (G, w, D) without nonempty Eulerian structures and with the maximum augmented out-degree at most 4.*

Proof. From Proposition 12 we know that there exists an internal 3-orientation L_G of G in which all cycles are oriented counterclockwise. Such an orientation can be extended to external edges in such a way that all directed cycles are oriented counterclockwise and the outer face is not a cycle. Let D be such an extension. We have that the orientation L_G corresponds to some realizer (T_r, T_g, T_b). Let w be the assignment of strengths to the edges of G in which the edges of T_r have strength 2 and all the other edges have strength 1. Then (G, w, D) is an augmented orientation.

Suppose for a contradiction that there exists a nonempty Eulerian structure H in (G, w, D). Consider a vertex v of H with a minimal green region R. We know that H does not contain any other vertex or edge of R.

First, we consider the case where v is an interior vertex of G. There are three edges directed away from v colored red, green, and blue, respectively (colors are given by the Schynder labelling). We know that blue and red edges oriented away from v cannot be in H, because v is a vertex with a minimal green region (see property (1)). Then, only the green edge can leave v in H. As H does not have isolated vertices, that edge belongs to H. Then, the augmented out-degree of v in H is equal to 1, so the augmented in-degree of v in H is also equal to 1. Red edges contribute 2 to augmented in-degrees so there cannot be any red edges incoming to v in H. There cannot also be any green edges incoming to v in H, due to the minimality of the green region of v. Then, the only edge of H that is oriented towards v must be blue. Since H is an Eulerian structure, it contains a simple directed cycle C that contains v. We see that cycle C must contain one green edge directed away from v in D and one blue edge directed towards v in D. Furthermore, C does not contain vertices from the green region of v. Therefore, C must be oriented clockwise. But in orientation D all directed cycles are oriented counterclockwise. This is a contradiction.

The other case, where v is an outer-vertex, is simpler. The vertices on the outer face do not have edges outgoing to the interior vertices, so the only cycle containing them could be the outer face, but the outer face is not a cycle. Therefore, v cannot belong to a directed cycle, which contradicts the fact that every Eulerian structure contains a directed cycle. □

Proposition 15 enables an alternative way of proving the main result of Zhu from [14].

Theorem 16. *The Alon-Tarsi number of a planar graph is at most 5.*

Proof. Let G be a triangulation of a plane graph F. From Proposition 12 there exists an internal 3-orientation D in which all directed cycles are oriented counterclockwise. Then, by Proposition 15, there exists an augmented orientation (G, w, D) without nonempty Eulerian structures. The maximal out-degree of a vertex in (G, w, D) is at most 4. From Lemma 8 we obtain

$$4 \geqslant \alpha(W_{G,w}) \geqslant \alpha(f_G) = AT(G) - 1.$$

Therefore, $AT(F)$ is at most 5 as well. □

4 Counterclockwise orientation in Thomassen's proof

The original proof of Theorem 16 from [14] followed the ideas of the famous proof of Thomassen of the fact that the choice number of a planar graph is at most 5 [12]. It is interesting that the list coloring algorithm that is implicitly given in Thomassen's work can be easily modified to construct a useful augmented orientation. We describe below the modified procedure.

4.1 Algorithm description

The input of the procedure is a 2-connected near triangulation with two distinguished vertices v_1, v_2 that are counter-clockwise consecutive on the outer cycle. The edge between the distinguished vertices is already oriented; the other edges are not. The procedure constructs an orientation of all the edges of the given graph. Its behaviour depends on whether the outercycle has a chord. The cases are described in the following paragraphs.

The outer cycle has a chord – recursive step. Let $v_a v_b$ be a chord of the outercycle. It divides the outer cycle into two cycles C_1 and C_2, where C_1 contains both v_1, v_2. Cycles C_1 and C_2 together with their interiors, determine two subgraphs of the current graph denoted by H_1, H_2. The only common vertices of H_1 and H_2 are v_a and v_b. The procedure is run recursively first in subgraph H_1 with vertices v_1, v_2, and then in subgraph H_2 with distinguished vertices v_b, v_a (note that edge $v_a v_b$ is oriented in the run of the procedure on H_1).

There is no chord on the outer cycle – orienting step. If there is no chord on the outer cycle, we focus on the vertex that immediately follows v_2 in counter-clockwise order of the outercycle. We denote that vertex by v_3 and call it *the central vertex* of this step. In this step, the procedure orients and assigns strengths to the edges of the current graph adjacent to v_3 according to the following rules:

- edges of the outer cycle are oriented away from v_3 and are given strength 1,
- all the other edges are oriented towards v_3 and are given strength 2.

Finally, vertex v_3 is removed from the graph. If there are still some vertices beside v_1, v_2 left, the procedure is recursively called on the remaining graph with the same distinguished vertices v_1, v_2.

4.2 Constructed orientation

The procedure being recursive has to be defined on near triangulations. We are interesting however in running it on triangular graphs. To start the procedure, we choose two vertices of the outer triangle as distinguished vertices. Then, the first step of the procedure is the orienting step in which the central vertex is the third vertex of the outer triangle. Then, in the run of the procedure, the following invariants are kept. Note that in some sense they mimic Thomassen's conditions on the lengths of lists.

(I1) At the beginning of each step, no inner edge of the current near triangulation is oriented.

(I2) At the beginning of each but the first step, every not-distinguished vertex of the current outer cycle has exactly one outgoing edge. Moreover, that edge does not lie on the current outer cycle and is doubled.

(I3) For every vertex v, once v is removed in an orienting step, all the edges adjacent to v oriented in the later steps are oriented towards v.

We make a few observations about the orientation constructed by the procedure.

Proposition 17. *The procedure run on a triangular graph constructs an internal 3-orientation.*

Proof. Consider an internal vertex v. Invariant (I2) guarantees that at the beginning of the (orienting) step in which vertex v is removed from the graph, it has exactly one outgoing double edge. During that step, exactly two edges are oriented away from v. Altogether, its out-degree becomes 3. By (I3), if an edge adjacent to v is oriented in one of the later steps, it is always oriented towards v. Therefore, the out-degree of v stays 3 until the end of the procedure. □

Proposition 18. *The orientation constructed by the procedure does not contain a clockwise oriented triangle.*

Proof. Suppose for a contradiction, that a clockwise oriented triangle has been constructed. Consider the first orienting step of the procedure in which a vertex of the triangle has been removed. We call it *the current step*. Denote the vertex removed in this step by v. In the final orientation there are precisely 3 edges oriented away from v. At the beginning of the current step, exactly one of these edges is already oriented. Moreover it is oriented towards a vertex that was removed earlier. By the choice of v, this edge cannot belong to the triangle. By the same token, no edge from v to a distinguished vertex of the current step can belong to the triangle. That results from the fact that, beside the first two distinguished vertices, any vertex that becomes distinguished in the second recursive call of a recursive step is first removed in the first recursive call of that step.

We are left with only one candidate for the edge of the triangle that is outgoing from v. It is the edge directed towards the (counter-clockwise) next vertex of the outer cycle of the current step. We denote it by v'. Note that v' cannot be one of the distinguished vertices of the current step, since it would imply that it has been removed in an earlier orienting step. Let v'' be the third vertex of the triangle (beside v and v'). Vertex v'' cannot be an interior vertex of the near triangulation of the current step, since then the triangle would be counter-clockwise oriented. Moreover, by the choice of v, vertex v'' cannot have been removed before v. Therefore, v'' must belong to a subgraph that is going to be dealt with in the second recursive call of some earlier recursive step of the procedure, while v and v' belong to the subgraph that is oriented in the first recursive call of that step. However, in such a case, vertex v' is going to be removed before v'', in an orienting step executed within the near triangulation of the current step. That (by invariant (I3)) implies that the edge between v' and v'' is going to be directed towards v'. It contradicts the fact that the triangle is clockwise oriented. □

Proposition 19. *Edges doubled by the procedure run on a triangular graph form one of the trees of the realizer corresponding to the constructed internal 3-orientation.*

Proof. Discussed invariants imply that double edges always form a forest that spans all internal vertices. Moreover, when the procedure is run on a triangular graph, that forest is, in fact, a tree that is rooted in the unique vertex of the outer triangle that has not been distinguished in the first call. It remains to verify that all the doubled edges belong to one of the trees of the realizer. It is easily checked that in the orienting step of a vertex v, the edges that become doubled in this step are exactly all the edges between two outgoing edges of v that are not doubled. That implies that all doubled edges indeed get the same color (i.e. are going to belong to the same tree of the realizer). Moreover, the color is the same as the color of the unique outgoing doubled edge of v. □

It has been observed in [2] (Corollary 1.5.2) that whenever there is a clockwise oriented cycle in an orientation of a triangular graph, this orientation also contains a clockwise oriented triangle. Therefore, Proposition 18 implies that the orientation constructed by the procedure is in fact the counterclockwise internal 3-orientation in which the edges of one of the trees of corresponding realizers are doubled. As we already explained (Proposition 15 and the proof of Theorem 16), that orientation certifies that $AT(G) \leqslant 5$. Interestingly, the above procedure can be also viewed as a specific realisation of the algorithm of Brehm from [2] designed for generating realizers of planar graphs.

5 Final thoughts

We showed that Zhu's strengthening of the result of Thomassen can be derived from the independent developments on the realizers of planar graphs started by Schnyder. Although stemming from different lines of research, these two proofs are not really different. We showed that a natural modification of Thomassen's procedure can be used to construct an orientation that certifies that the Alon-Tarsi number of the graph is at most 5.

A slight technical generalization of the method of Alon and Tarsi was used in our proof. It has a property that, while it allows proving $AT(G) \leqslant 5$, it does not help to find a monomial of f_G that certifies that property. At the same time, after careful strengthening of some edges, the problem of counting Eulerian structures becomes much easier. Indeed, in some sense, all nontrivial Eulerian subgraphs of the orientations corresponding to monomials of f_G certifying $AT(G) \leqslant 5$ cancel themselves. We expect that augmented orientations may also be useful in analysing graph polynomials for other graph classes. Let us also note that our modification of f_G into $W_{G,w}$ by strengthening edges is just one of the possible options. Polynomial $W_{G,w}$ used in our proof can be viewed as f_G multiplied by another polynomial defined by a specific spanning tree of the underlying graph. It is tempting to look for other useful examples of such multipliers that depend on other substructures of the studied graphs.

The bound from Theorem 1 has also recently been given a short proof by Gu and Zhu [5]. They managed to greatly simplify the original argument with a technique that is somewhat similar to the extension of Alon-Tarsi method used in the current paper. As these two ideas seem to be related, it is rational to take both perspectives into account when considering potential generalizations of graph polynomials described above. That paper (i.e. [5]) also contains a simplified proof of a recent result of Grytczuk and Zhu [4] that every planar graph G contains a matching M such that $AT(G - M) \leqslant 4$. It would also be interesting to describe this matching in terms of Schnyder realizers.

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References

[1] Noga Alon and Michael Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.

[2] Enno Brehm. 3-orientation and schnyder 3-tree-decompositions construction and order structure. *Diploma Thesis at Freie Universität Berlin*, 2000.

[3] Hubert de Fraysseix and Patrice Ossona de Mendez. On topological aspects of orientations. *Discrete Math.*, 229(1-3):57–72, 2001.

[4] Jarosław Grytczuk and Xuding Zhu. The Alon-Tarsi number of a planar graph minus a matching. *J. Combin. Theory Ser. B*, 145:511–520, 2020.

[5] Yangyan Gu and Xuding Zhu. The Alon-Tarsi number of planar graphs–a simple proof. arXiv:2203.16308, 2022.

[6] Tommy R. Jensen and Bjarne Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

[7] Ringi Kim, Seog-Jin Kim, and Xuding Zhu. The Alon-Tarsi number of subgraphs of a planar graph. arXiv:1906.01506, 2019.

[8] Yuri V. Matiyasevich. A criterion for vertex colorability of a graph stated in terms of edge orientations. arXiv:0712.1884, 2007.

[9] Julius Petersen. Die Theorie der regulären graphs. *Acta Mathematica*, 15:193–220, 1891.

[10] Uwe Schauz. A paintability version of the combinatorial Nullstellensatz, and list colorings of k-partite k-uniform hypergraphs. *Electron. J. Combin.*, 17(1):#R176, 2010.

[11] Walter Schnyder. Embedding planar graphs on the grid. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*, pages 138–148, 1990.

[12] Carsten Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994.

[13] Margit Voigt. List colourings of planar graphs. *Discrete Math.*, 120(1-3):215–219, 1993.

[14] Xuding Zhu. The Alon-Tarsi number of planar graphs. *J. Combin. Theory Ser. B*, 134:354–358, 2019.