An Ore-type condition for hamiltonicity in tough graphs and the extremal examples

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Abstract

Let G be a t-tough graph on $n \geqslant 3$ vertices for some $t > 0$. It was shown by Bauer et al. in 1995 that if the minimum degree of G is greater than $\frac{n}{t+1} - 1$, then G is hamiltonian. In terms of Ore-type hamiltonicity conditions, the problem was only studied when t is between 1 and 2, and recently the second author proved a general result. The result states that if the degree sum of any two nonadjacent vertices of G is greater than $\frac{2n}{t+1} + t - 2$, then G is hamiltonian. It was conjectured in the same paper that the "$+t$" in the bound $\frac{2n}{t+1} + t - 2$ can be removed. Here we confirm the conjecture. The result generalizes the result by Bauer, Broersma, van den Heuvel, and Veldman. Furthermore, we characterize all t-tough graphs G on $n \geqslant 3$ vertices for which $\sigma_2(G) = \frac{2n}{t+1} - 2$ but G is non-hamiltonian.

Keywords. Ore-type condition; toughness; hamiltonian cycle.

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1   Introduction

We consider only finite simple graphs. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $v \in V(G)$, $S \subseteq V(G)$, and $H \subseteq G$. Then $N_G(v)$ denotes the set of neighbors of v in G, $d_G(v) := |N_G(v)|$ is the degree of v in G, and $\delta(G) := \min\{d_G(v) : v \in V(G)\}$ is the minimum degree of G. Define $\deg_G(v, H) = |N_G(v) \cap V(H)|$, $N_G(S) = (\bigcup_{x \in S} N_G(x)) \setminus S$, and we write $N_G(H)$ for $N_G(V(H))$. Let $N_H(v) = N_G(v) \cap V(H)$ and $N_H(S) = N_G(S) \cap V(H)$. Again, we write $N_H(R)$ for $N_H(V(R))$ for any subgraph R of G. We use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. For notational simplicity we

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write $G - x$ for $G - \{x\}$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in G with one endvertex in V_1 and the other endvertex in V_2. For two integers a and b, let $[a, b] = \{i \in \mathbb{Z} : a \leqslant i \leqslant b\}$.

Throughout this paper, if not specified, we will assume t to be a nonnegative real number. The number of components of a graph G is denoted by $c(G)$. The graph G is said to be t-tough if $|S| \geqslant t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geqslant 2$. The toughness $\tau(G)$ is the largest real number t for which G is t-tough, or is ∞ if G is complete. This concept was introduced by Chvátal [7] in 1973. It is easy to see that if G has a hamiltonian cycle then G is 1-tough. Conversely, Chvátal [7] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed t-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so t_0 must be at least $\frac{9}{4}$ if Chvátal's toughness conjecture is true.

Chvátal's toughness conjecture has been verified for certain classes of graphs including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs [2]. The classes also include $2K_2$-free graphs [6, 15, 13], and R-free graphs for $R \in \{P_2 \cup P_3, P_3 \cup 2P_1, P_2 \cup kP_1\}$ [16, 9, 17, 12, 19], where $k \geqslant 4$ is an integer. In general, the conjecture is still wide open. In finding hamiltonian cycles in graphs, sufficient conditions such as Dirac-type and Ore-type conditions are the most classic ones.

Theorem 1.1 (Dirac's Theorem [8]). *If G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{n}{2}$, then G is hamiltonian.*

Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G)$ and they are nonadjacent$\}$ if G is noncomplete, and define $\sigma_2(G) = \infty$ otherwise. Ore's Theorem, as a generalization of Dirac's Theorem, is stated below.

Theorem 1.2 (Ore's Theorem [11]). *If G is a graph on $n \geqslant 3$ vertices with $\sigma_2(G) \geqslant n$, then G is hamiltonian.*

Analogous to Dirac's Theorem, Bauer, Broersma, van den Heuvel, and Veldman [4] proved the following result by incorporating the toughness of the graph.

Theorem 1.3 (Bauer et al. [4]). *Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\delta(G) > \frac{n}{t+1} - 1$, then G is hamiltonian.*

A natural question here is whether we can find an Ore-type condition involving the toughness of G that generalizes Theorem 1.3. Various theorems were proved prior to Theorem 1.3 by only taking $\tau(G)$ between 1 and 2 [10, 3, 5]. Let G be a t-tough graph on $n \geqslant 3$ vertices. The author showed in [14] that if $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, then G is hamiltonian. It was also conjectured in [14] that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is the right bound. In

this paper, we confirm the conjecture. For any odd integer $n \geqslant 3$, the complete bipartite graph $G := K_{\frac{n-1}{2}, \frac{n+1}{2}}$ is $\frac{n-1}{n+1}$-tough and satisfies $\sigma_2(G) = n - 1 = \frac{2n}{1 + \frac{n-1}{n+1}} - 2$. However, G is not hamiltonian. Thus, the degree sum condition that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is best possible for a t-tough graph on at least three vertices to be hamiltonian. In fact, for any odd integers $n \geqslant 3$, any graph from the family $\mathcal{H} = \{H_{\frac{n-1}{2}} + \overline{K}_{\frac{n+1}{2}} : H_{\frac{n-1}{2}} \text{ is any graph on } \frac{n-1}{2} \text{ vertices}\}$ is an extremal graph, where "+" represents the join of two graphs. We also show that $\mathcal{H}$ is the only family of extremal graphs.

Theorem 1. *Let G be a t-tough graph on $n \geqslant 3$ vertices. Then the following statements hold.*

(a) *If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then G is hamiltonian.*

(b) *If $\sigma_2(G) = \frac{2n}{t+1} - 2$ and G is not hamiltonian, then $G \in \mathcal{H}$.*

The remainder of this paper is organized as follows: in Section 2, we introduce some notation and preliminary results, and in Section 3, we prove Theorem 1.

2 Preliminary results

Let G be a graph and λ be a positive integer. Following [18], a cycle C of G is a D_λ-*cycle* if every component of $G - V(C)$ has order less than λ. Clearly, a D_1-cycle is just a hamiltonian cycle. We denote by $c_\lambda(G)$ the number of components of G with order at least λ, and write $c_1(G)$ just as $c(G)$. Two subgraphs H_1 and H_2 of G are *remote* if they are disjoint and there is no edge of G joining a vertex of H_1 with a vertex of H_2. For a subgraph H of G, let $d_G(H) = |N_G(H)|$ be the degree of H in G. We denote by $\delta_\lambda(G)$ the minimum degree of a connected subgraph of order λ in G. Again $\delta_1(G)$ is just $\delta(G)$.

Lemma 2.1 ([16]). *Let $t > 0$ and G be a non-complete n-vertex t-tough graph. Then $|W| \leqslant \frac{n}{t+1}$ for every independent set W in G.*

Denote by $\overrightarrow{C}$ an orientation of C. We assume that the orientation is clockwise throughout the rest of this paper. For $x \in V(C)$, denote the immediate successor of x on $\overrightarrow{C}$ by x^+ and the immediate predecessor of x on $\overrightarrow{C}$ by x^-. We use $N_C^+(x)$ to denote the set of immediate successors for vertices from $N_C(x)$. For $u, v \in V(C)$, $u\overrightarrow{C}v$ denotes the segment of $\overrightarrow{C}$ starting at u, following $\overrightarrow{C}$ in the orientation, and ending at v. Likewise, $u\overleftarrow{C}v$ is the opposite segment of $\overrightarrow{C}$ with endpoints as u and v. Let $\text{dist}_{\overrightarrow{C}}(u, v)$ denote the length of the path $u\overrightarrow{C}v$. For any vertex $u \in V(C)$ and any positive integer k, define

$$L_u^+(k) = \{v \in V(C) : \text{dist}_{\overrightarrow{C}}(u, v) \in [1, k]\}$$

to be the set of k consecutive successors of u. Hereafter, all cycles under consideration are oriented, and we will not distinguish between the notation C and $\vec{C}$.

The following lemma provides a way of extending a cycle C provided that the vertices outside C have many neighbors on C. The proof follows from Lemma 2.1 and is very similar to the proof of Lemma 10 in [16]: if we assume instead that C cannot be extended by including x, then $N_C^+(x) \cup \{x\}$ is an independent set in G.

Lemma 2.2. *Let $t > 0$ and G be an n-vertex t-tough graph, and let C be a non-hamiltonian cycle of G. If $x \in V(G) \setminus V(C)$ satisfies $\deg_G(x, C) > \frac{n}{t+1} - 1$, then G has a cycle C' such that $V(C') = V(C) \cup \{x\}$.*

A path P connecting two vertices u and v is called a (u, v)-*path*, and we write uPv or vPu in order to specify the two endvertices of P. Let uPv and xQy be two paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx.

For an integer $\lambda \geqslant 1$, if a graph G contains a $D_{\lambda+1}$-cycle C but no D_λ-cycle, then $V(G) \setminus V(C) \neq \emptyset$. Furthermore, $G - V(C)$ has a component of order λ. The result below with $d_G(H)$ replaced by $\delta_\lambda(G)$ and H replaced by any component of $G - V(C)$ with order λ was proved in [4, Corollary 7(a)].

Lemma 2.3 ([14]). *Let G be a t-tough 2-connected graph of order n. Suppose G has a D_{s+1}-cycle but no D_s-cycle for some integer $s \geqslant 1$. Let C be a D_{s+1}-cycle of G such that C minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in [1, s]$ with $p > q$. Then $n \geqslant (t + |V(H)|)(d_G(H) + 1)$ for any component H of $G - V(C)$.*

The lemma below is the key to get rid of the "$+t$" in the lower bound $\frac{2n}{t+1} + t - 2$ on $\sigma_2(G)$ for guaranteeing the existence of a hamiltonian cycle [14].

Lemma 2.4. *Let G be a t-tough 2-connected graph of order n. Suppose that G has a $D_{\lambda+1}$-cycle but no D_λ-cycle for some integer $\lambda \geqslant 1$. Let C be a cycle of G. Then $G - V(C)$ has a component H with order at least λ such that $\deg_G(x, C) \leqslant \frac{n}{t+1} - \lambda$ for some $x \in V(H)$.*

Proof. Since G has no D_λ-cycle, it is clear that $G - V(C)$ has a component of order at least λ. We suppose to the contrary that for each component H with order at least λ of $G - V(C)$ and each $x \in V(H)$, we have $\deg_G(x, C) > \frac{n}{t+1} - \lambda$. Among all cycles C' of G that satisfy the two conditions below, we may assume that C is one that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p \geqslant \lambda$ and any q with $q < p$.

(1) Each component of $G - V(C)$ either has order at most $\lambda - 1$, or

(2) the component H has order at least λ such that for each $x \in V(H)$, we have $\deg_G(x, C) > \frac{n}{t+1} - \lambda$.

We take a component H with order at least λ and assume that $N_C(H)$ has size k for some integer $k \geqslant 2$, and that the k neighbors are $v_1, \ldots, v_k$ and appear in the same order along $\vec{C}$. Note that $k > \frac{n}{t+1} - \lambda$ by our assumption. For each $i \in [1,k]$, and each $v \in V(v_i^+ \vec{C} v_{i+1}^-)$, where $v_{k+1} := v_1$, we let $\mathcal{C}(v)$ be the set of components of $G - V(C)$ that have a vertex joining to v by an edge in G. As $N_C(H) \cap V(v_i^+ \vec{C} v_{i+1}^-) = \emptyset$, we have $H \notin \mathcal{C}(v)$. Let $w_i^* \in V(v_i^+ \vec{C} v_{i+1}^-)$ be the vertex with $\text{dist}_{\vec{C}}(v_i, w_i^*)$ minimum such that

$$\sum_{\substack{D \in \bigcup_{v \in V(v_i^+ \vec{C} w_i^*)} \mathcal{C}(v)}} |V(D)| + |V(v_i^+ \vec{C} w_i^*)| \geqslant \lambda.$$

If such a vertex w_i^* exists, let $L_{v_i}^*(\lambda)$ be the union of the vertex set $V(v_i^+ \vec{C} w_i^*)$ and all those vertex sets of graphs in $\bigcup_{v \in V(v_i^+ \vec{C} w_i^*)} \mathcal{C}(v)$; if such a vertex w_i^* does not exist, let $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$. Note that when w_i^* exists, by its definition, $w_i^* \in V(v_i^+ \vec{C} v_{i+1}^-)$. Thus $V(v_i^+ \vec{C} w_i^*) \cap V(v_j^+ \vec{C} w_j^*) = \emptyset$ if both w_i^* and w_j^* exist for distinct $i, j \in [1,k]$. If w_i^* exists, for any $r \in [1, \text{dist}_{\vec{C}}(v_i, w_i^*)]$, let v_i^r be the vertex from $v_i^+ \vec{C} w_i^*$ such that $\text{dist}_{\vec{C}}(v_i, v_i^r) = r$. Then define

$$L_{v_i}^*(\lambda, r) = \left(\bigcup_{\substack{D \in \bigcup_{v \in V(v_i^+ \vec{C} v_i^r)} \mathcal{C}(v)}} V(D) \right) \cup V(v_i^+ \vec{C} v_i^r).$$

We will show that we can make the following assumptions:

(a) If for some $i \in [1,k]$, it holds that $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$, then $\text{dist}_{\vec{C}}(v_i, v_j) \geqslant \lambda + 1$ for any $j \in [1,k]$ with $j \neq i$. Thus the vertex w_i^* exists for each $i \in [1,k]$.

(b) $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are pairwise remote for any distinct $i, j \in [1,k]$.

With Assumptions (a) and (b), we can reach a contradiction as follows: note that $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are remote for any distinct $i, j \in [1,k]$ and H and $G[L_{v_i}^*(\lambda)]$ are remote for any $i \in [1,k]$. Let $S = V(G) \backslash \left((\bigcup_{i=1}^k L_{v_i}^*(\lambda)) \cup V(H) \right)$. Then $|S| \leqslant n - (k+1)\lambda$ and $c(G - S) = k + 1$. As G is t-tough, we get

$$n - (k+1)\lambda \geqslant |S| \geqslant t \cdot c(G - S) = t(k+1),$$

giving $k \leqslant \frac{n}{t+\lambda} - 1$. Since $n \geqslant (\lambda + t)(2t + 1)$ by Lemma 2.3 (G has a $D_{\lambda+1}$-cycle C' such that $G - V(C')$ has a component H' of order λ, and $d_G(H') \geqslant 2t$ by G being t-tough),

we get

$$\frac{n}{t+1} - \lambda - \left(\frac{n}{t+\lambda} - 1\right) = \frac{(\lambda-1)(n-(t+1)(t+\lambda))}{(t+1)(t+\lambda)} \geqslant 0,$$

and so $k \leqslant \frac{n}{t+\lambda} - 1 \leqslant \frac{n}{t+1} - \lambda$. This gives a contradiction to $k > \frac{n}{t+1} - \lambda$. Thus we are only left to show Assumptions (a) and (b). We show that if any one of the assumptions is violated, then we can decrease $c_p(G - V(C))$ for some $p \geqslant \lambda$.

For Assumption (a), if $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$ for some $i \in [1,k]$ but $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_j) \leqslant \lambda$ for some $v_j \in N_C(H)$ with $j \neq i$, then there must exist two consecutive indices $i, j \in [1,k]$ such that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_j) \leqslant \lambda$. Thus we may just assume $j = i+1$, where the index is taken modulo k. Let $v_i^*, v_{i+1}^* \in V(H)$ such that $v_i v_i^*, v_{i+1} v_{i+1}^* \in E(G)$, and let P be a (v_i^*, v_{i+1}^*)-path in H. Let $C_1 = v_i \overleftarrow{C} v_{i+1} v_{i+1}^* P v_i^* v_i$.

Note that every component of $G - V(C)$ not having any vertex joining to a vertex from $v_i^+ \overrightarrow{C} v_{i+1}^-$ in G is still a component of $G - V(C_1)$. Those components automatically satisfy Conditions (1) and (2) as listed in the beginning of this proof. Vertices in $v_i^+ \overrightarrow{C} v_{i+1}^-$ are contained in a distinct component of $G - V(C_1)$, and the component has order at most $\lambda - 1$ by the assumption that $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$ and $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \leqslant \lambda$. Finally, as any vertex from each component of $H - V(v_{i+1}^* P v_i^*)$ is not adjacent in G to any vertex from $v_i^+ \overrightarrow{C} v_{i+1}^-$, we know that components of $H - V(v_{i+1}^* P v_i^*)$ are components of $G - V(C_1)$, and that $\deg_G(w, C_1) > \frac{n}{t+1} - \lambda$ for any $w \in V(H - V(v_{i+1}^* P v_i^*))$. Hence each component of $G - V(C_1)$ either has order at most $\lambda - 1$ or is a component of order at least λ such that each vertex from the component has in G more than $\frac{n}{t+1} - \lambda$ neighbors on C_1. However, $c_{|V(H)|}(G - V(C_1)) < c_{|V(H)|}(G - V(C))$ and $c_q(G - V(C_1)) = c_q(G - V(C))$ for any $q > |V(H)||$, contradicting the choice of C. Therefore we have Assumption (a), which implies that the vertex w_i^* exists for each $i \in [1,k]$.

For Assumption (b), suppose it is false. Then there exist distinct $i, j \in [1,k]$ such that $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are not remote. By the definition of remote subgraphs, we have either $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) \neq \emptyset$ or $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) = \emptyset$ but $E_G(L_{v_i}^*(\lambda), L_{v_j}^*(\lambda)) \neq \emptyset$. In order to achieve a contradiction, we first show the following general claim, call it Claim ($*$).

Claim ($*$): For any $r \in [1, \mathrm{dist}_{\overrightarrow{C}}(v_i, w_i^*)]$ and $s \in [1, \mathrm{dist}_{\overrightarrow{C}}(v_j, w_j^*)]$, if $L_{v_i}^*(\lambda, r) \cap L_{v_j}^*(\lambda, s) = \emptyset$, then $E_G(L_{v_i}^*(\lambda, r), L_{v_j}^*(\lambda, s)) = \emptyset$.

Suppose otherwise that $E_G(L_{v_i}^*(\lambda, r), L_{v_j}^*(\lambda, s)) \neq \emptyset$. Since $L_{v_i}^*(\lambda, r) \cap L_{v_j}^*(\lambda, s) = \emptyset$, $E_G(L_{v_i}^*(\lambda, r), L_{v_j}^*(\lambda, s)) \neq \emptyset$ implies that there exist $y \in V(v_i^+ \overrightarrow{C} w_i^*) \cap L_{v_i}^*(\lambda, r)$ and $z \in V(v_j^+ \overrightarrow{C} w_j^*) \cap L_{v_j}^*(\lambda, s)$ such that $yz \in E(G)$. We choose $y \in V(v_i^+ \overrightarrow{C} w_i^*) \cap L_{v_i}^*(\lambda, r)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_i, y)$ minimum and $z \in V(v_j^+ \overrightarrow{C} w_j^*) \cap L_{v_j}^*(\lambda, s)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_j, z)$ minimum such that $yz \in E(G)$. By the choice of y and z, we have $E_G(V(v_i^+ \overrightarrow{C} y^-), V(v_j^+ \overrightarrow{C} z^-)) = \emptyset$.

Let $v_i^*, v_j^* \in V(H)$ such that $v_i v_i^*, v_j v_j^* \in E(G)$, P be a (v_i^*, v_j^*)-path in H, and let $C_1 = v_i \overleftarrow{C} z y \overrightarrow{C} v_j v_j^* P v_i^* v_i$. Note that no vertex of H is adjacent in G to any vertex of $v_i^+ \overrightarrow{C} y^-$ or $v_j^+ \overrightarrow{C} z^-$ by the fact that $V(v_i^+ \overrightarrow{C} y^-) \subseteq V(v_i^+ \overrightarrow{C} w_i^*)$ and $V(v_j^+ \overrightarrow{C} z^-) \subseteq V(v_j^+ \overrightarrow{C} w_j^*)$ and Assumption (a). By the assumption that $L_{v_i}^*(r) \cap L_{v_j}^*(s) = \emptyset$ and the definitions of $L_{v_i}^*(\lambda)$ and $L_{v_j}^*(\lambda)$, we know that $v_i^+ \overrightarrow{C} y^-$ and $v_j^+ \overrightarrow{C} z^-$ are respectively contained in distinct components of $G - V(C_1)$ that each have order at most $\lambda - 1$. By the same reasoning as in proving Assumption (a), we know that each component of $G - V(C_1)$ has order at most $\lambda - 1$ or is a component such that each vertex from the component has in G more than $\frac{n}{t+1} - \lambda$ neighbors on C_1. However, $c_{|V(H)|}(G - V(C_1)) < c_{|V(H)|}(G - V(C))$ and $c_q(G - V(C_1)) = c_q(G - V(C))$ for any $q > |V(H)|$, contradicting the choice of C. Thus Claim $(*)$ holds.

Now let us get back to proving Assumption (b) by contradiction. Assume first that $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) \neq \emptyset$. Then there exist $v \in V(v_i^+ \overrightarrow{C} w_i^*)$ and $u \in V(v_j^+ \overrightarrow{C} w_j^*)$ such that $\mathcal{C}(v) \cap \mathcal{C}(u) \neq \emptyset$, we then further choose v closest to v_i and u closest to v_j along $\overrightarrow{C}$ such that $\mathcal{C}(v) \cap \mathcal{C}(u) \neq \emptyset$. Thus for any $w_i \in V(v_i^+ \overrightarrow{C} v^-)$ and any $w_j \in V(v_j^+ \overrightarrow{C} u^-)$, it holds that $\mathcal{C}(w_i) \cap \mathcal{C}(w_j) = \emptyset$. Let $D \in \mathcal{C}(v) \cap \mathcal{C}(u)$ and $v', u' \in V(D)$ such that $vv', uu' \in E(G)$, and P' be a (v', u')-path of D. Let $v_i^*, v_j^* \in V(H)$ such that $v_i v_i^*, v_j v_j^* \in E(G)$, and let P be a (v_i^*, v_j^*)-path in H. Then $C_1 = v_i v_i^* P v_j^* v_j \overleftarrow{C} v v' P' u' u \overrightarrow{C} v_i$ is a cycle. Since each of $V(v_i^+ \overrightarrow{C} v^-)$ and $V(v_j^+ \overrightarrow{C} u^-)$ contains at most $\lambda - 1$ vertices and they are proper subsets of $V(v_i^+ \overrightarrow{C} w_i^*)$ and $V(v_j^+ \overrightarrow{C} w_j^*)$ respectively, by Assumption (a) above, we have $N_C(H) \cap (V(v_i^+ \overrightarrow{C} v^-) \cup V(v_j^+ \overrightarrow{C} u^-)) = \emptyset$. By the choices of v and u that for any $w_i \in V(v_i^+ \overrightarrow{C} v^-)$ and any $w_j \in V(v_j^+ \overrightarrow{C} u^-)$, it holds that $\mathcal{C}(w_i) \cap \mathcal{C}(w_j) = \emptyset$, Claim $(*)$ implies that the components of $G - V(C_1)$ that respectively contain $v_i^+ \overrightarrow{C} v^-$ and $v_j^+ \overrightarrow{C} u^-$ are disjoint. Since $V(v_i^+ \overrightarrow{C} v^-)$ is a proper subset of $V(v_i^+ \overrightarrow{C} w_i^*)$ and $V(v_j^+ \overrightarrow{C} u^-)$ is a proper subset of $V(v_j^+ \overrightarrow{C} w_j^*)$, it follows by the definitions of $L_{v_i}^*(\lambda)$ and $L_{v_j}^*(\lambda)$ that the components of $G - V(C_1)$ that respectively contain $v_i^+ \overrightarrow{C} v^-$ and $v_j^+ \overrightarrow{C} u^-$ have order at most $\lambda - 1$. By the same reasoning as in proving Assumption (a), we know that each component of $G - V(C_1)$ has order at most $\lambda - 1$ or is a component such that each vertex from the component has in G more than $\frac{n}{t+1} - \lambda$ neighbors on C_1. However, $c_{|V(H)|}(G - V(C_1)) < c_{|V(H)|}(G - V(C))$ and $c_q(G - V(C_1)) = c_q(G - V(C))$ for any $q > |V(H)|$, contradicting the choice of C. Thus we must have $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) = \emptyset$. Applying Claim $(*)$ again with $r = \mathrm{dist}_{\overrightarrow{C}}(v_i, w_i^*)$ and $s = \mathrm{dist}_{\overrightarrow{C}}(v_j, w_j^*)$, we have $E_G(L_{v_i}^*(\lambda), L_{v_j}^*(\lambda)) = \emptyset$. Therefore, $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are remote, contradicting our assumption. Thus Assumption (b) holds. $\qquad \square$

3 Proof of Theorem 1

Let G be a t-tough graph on $n \geqslant 3$ vertices such that $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2$. We may assume that G is not a complete graph. Thus G is $\lceil 2t \rceil$-connected as it is t-tough. Suppose to the contrary that G is not hamiltonian.

Claim 1. *We may assume that G is 2-connected.*

Proof. Since $t > 0$, G is connected. Assume to the contrary that G has a cut vertex x. By considering the degree sum of two vertices respectively from two components of $G - x$, we know that $\sigma_2(G) \leqslant n - 1$. On the other hand, G has a cut vertex implies $t \leqslant \frac{1}{2}$ and so $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2 \geqslant \frac{4n}{3} - 2$. If $\sigma_2(G) > \frac{4n}{3} - 2$, then we get a contradiction to $\sigma_2(G) \leqslant n - 1$ as $n \geqslant 3$. Thus we assume $\sigma_2(G) = \frac{4n}{3} - 2$, which contradicts $\sigma_2(G) \leqslant n - 1$ if $n \geqslant 4$. Thus $n = 3$ and so $G = P_3$, but this implies $G \in \mathcal{H}$. $\square$

Since G is 2-connected, Lemma 2.3 implies

$$n \geqslant (t + 1)(\lceil 2t \rceil + 1).$$

Also as G is 2-connected, G contains cycles. Let $\lambda \geqslant 0$ be the integer such that G admits no D_λ-cycle but a $D_{\lambda+1}$-cycle. Then we choose C to be a longest $D_{\lambda+1}$-cycle that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in [1, \lambda]$ with $p > q$. As G is not hamiltonian, we have $\lambda \geqslant 1$. Thus $V(G) \setminus V(C) \neq \emptyset$. Since C is not a D_λ-cycle but a $D_{\lambda+1}$-cycle, $G - V(C)$ has a component H of order λ. Let

$$W = N_C(H) \quad \text{and} \quad \omega = |W|.$$

Since G is a connected t-tough graph, it follows that $\omega \geqslant \lceil 2t \rceil$. On the other hand, Lemma 2.3 implies that $\omega \leqslant \frac{n}{t+\lambda} - 1$.

Claim 2.

$$\begin{cases} \lambda + \omega < \dfrac{n}{t+1} & \text{if } \lambda \geqslant 2, \\ \lambda + \omega \leqslant \dfrac{n}{t+1} & \text{if } \lambda = 1. \end{cases}$$

Proof. If $\lambda = 1$, then the assertion holds by $\omega \leqslant \frac{n}{t+\lambda} - 1$. Thus we assume $\lambda \geqslant 2$ and assume to the contrary that $\lambda + \omega \geqslant \frac{n}{t+1}$. Then we have $n \leqslant (\lambda + \omega)(t+1)$. By Lemma 2.3, we have $n \geqslant (\lambda + t)(\omega + 1)$. Thus we have

$$(\lambda + t)(\omega + 1) \leqslant (\lambda + \omega)(t + 1),$$

which implies $\lambda\omega + \lambda + t\omega + t \leqslant \lambda t + \lambda + t\omega + \omega$ and so $(\lambda - 1)\omega \leqslant (\lambda - 1)t$. Since $\lambda \geqslant 2$, we get $\omega \leqslant t$, a contradiction to $\omega \geqslant 2t$. Note that the argument above for $\lambda \geqslant 2$ holds for all components of $G - V(C)$ as Lemma 2.3 holds for all components of $G - V(C)$. $\square$

Claim 3. *If $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2$, then H is the only component of $G - V(C)$.*

Proof. Suppose $H^* \neq H$ is another component of $G - V(C)$. Then we have $d_G(x) + d_G(y) \geqslant \sigma_2(G)$ for any $x \in V(H)$ and $y \in V(H^*)$. Since $d_G(x) \leqslant \lambda + \omega - 1$ and $d_G(y) \leqslant |V(H^*)| + |N_C(H^*)| - 1$, Claim 2 implies that $|V(H^*)| + |N_C(H^*)| > \sigma_2(G) - (\frac{n}{t+1} - 1) + 1 \geqslant \frac{n}{t+1}$ if $\lambda \geqslant 2$. Repeating exactly the same argument for $|V(H^*)| + |N_C(H^*)|$ as in the proof of Claim 2 leads to a contradiction.

Thus we assume $\lambda = 1$. We get the same contradiction as above if $\sigma_2(G) > \frac{2n}{t+1} - 2$ or $\lambda + \omega < \frac{n}{t+1}$. Thus we have $\sigma_2(G) = \frac{2n}{t+1} - 2$ and $\omega = \frac{n}{t+1} - 1$ by Claim 2. Then H and H^* each contains only one vertex, say x and y, respectively. We first claim that the vertex y is adjacent in G to at most one vertex from W^+. For otherwise, suppose there are distinct $u, v \in W^+$ such that $yu, yv \in E(G)$. Then $C^* = u^- \overleftarrow{C} v y u \overrightarrow{C} v^- x u^-$ is a $D_{\lambda+1}$-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$. This contradicts the choice of C.

We then claim that the set W^+ is an independent set in G. For otherwise, suppose there are distinct $u, v \in W^+$ such that $uv \in E(G)$. Then $C^* = u^- \overleftarrow{C} v u \overrightarrow{C} v^- x u^-$ is a $D_{\lambda+1}$-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$. This contradicts the choice of C.

Now let $S = V(G) \setminus (W^+ \cup V(H) \cup V(H^*))$. Then $c(G - S) \geqslant \omega + 1$. However

$$\frac{|S|}{c(G - S)} \leqslant \frac{n - \omega - 2}{\omega + 1} = \frac{\frac{tn}{t+1} - 1}{\frac{n}{t+1}} < t,$$

a contradiction.

Therefore, H is the only component of $G - V(C)$. $\qquad\square$

Since H is the only component of $G - V(C)$, every vertex $v \in V(C) \setminus W$ is only adjacent in G to vertices on C. As vertices from $V(C) \setminus W$ are nonadjacent in G with vertices from H, we have

$$\deg_G(v, C) \geqslant \sigma_2(G) - (\omega + \lambda - 1) \quad \text{for any } v \in V(C) \setminus W. \tag{1}$$

We construct the vertex sets L_u^+ for each $u \in W$ as follows:

$$L_u^+ = \begin{cases} \{v \in V(C) : \mathrm{dist}_{\overrightarrow{C}}(u, v) < \frac{n}{t+1} - \omega + 1\} & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2; \\ \{v \in V(C) : \mathrm{dist}_{\overrightarrow{C}}(u, v) \leqslant \frac{n}{t+1} - \omega + 1\} & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2. \end{cases}$$

Claim 4. *The following statements hold.*

(a) *If $\sigma_2(G) = \frac{2n}{t+1} - 2$, then for any two distinct vertices $u, v \in W$, we have $\mathrm{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$ and $E_G(L_u^+, L_v^+) = \emptyset$.*

(b) *If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then for any two distinct vertices $u, v \in W$, we have $\mathrm{dist}_{\overrightarrow{C}}(u, v) > \frac{n}{t+1} - \omega + 1$ and $E_G(L_u^+, L_v^+) = \emptyset$.*

Proof. We only show Claim 4(a), as the proof for Claim 4(b) follows the same argument by just using the strict inequality. Let $u^* \in N_H(u), v^* \in N_H(v)$ and P be a (u^*, v^*)-path of H. For the first part of the statement, it suffices to show that when we arrange the vertices of W along $\overrightarrow{C}$, for any two consecutive vertices u and v from the arrangement, we have $\text{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$. Note that $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$ for such pairs of u and v. Assume to the contrary that there are distinct $u, v \in W$ with $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$ and $\text{dist}_{\overrightarrow{C}}(u, v) < \frac{n}{t+1} - \omega + 1$. Let $C^* = u\overleftarrow{C}vv^*Pu^*u$. Since H has order λ and $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$, $H - V(P)$ is a union of components of $G - V(C^*)$ that each is of order at most $\lambda - 1$ and $u^+\overrightarrow{C}v^-$ is a component of $G - V(C^*)$ of order less than $\frac{n}{t+1} - \omega$ but at least λ (G has no D_λ-cycle). By (**1**), for each vertex $x \in V(u^+\overrightarrow{C}v^-)$, $\deg_G(x, C^*) > \sigma_2(G) - (\omega + \lambda - 1) - (\frac{n}{t+1} - \omega - 1) = \frac{n}{t+1} - \lambda$. This shows a contradiction to Lemma 2.4.

For the second part of the statement, we assume to the contrary that $E_G(L_u^+, L_v^+) \neq \emptyset$. Applying the first part, we know that $\text{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$ and $\text{dist}_{\overrightarrow{C}}(v, u) \geqslant \frac{n}{t+1} - \omega + 1$ (exchanging the role of u and v). Thus $L_u^+ \cap L_v^+ = \emptyset$. We choose $x \in L_u^+$ with $\text{dist}_{\overrightarrow{C}}(u, x)$ minimum and $y \in L_v^+$ with $\text{dist}_{\overrightarrow{C}}(v, y)$ minimum such that $xy \in E(G)$. By this choice of x and y, it follows that $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \emptyset$. Let $C^* = u\overleftarrow{C}yx\overrightarrow{C}vv^*Pu^*u$. Since H is of order λ and no vertex of H is adjacent in G to any vertex of $u^+\overrightarrow{C}x^-$ or $v^+\overrightarrow{C}y^-$ by the first part of the statement, $H - V(P)$ is a union of components of $G - V(C^*)$ that each have order at most $\lambda - 1$. Also $u^+\overrightarrow{C}x^-$ and $v^+\overrightarrow{C}y^-$ are components of $G - V(C^*)$ that each have order less than $\frac{n}{t+1} - \omega$ but at least one of them has order at least λ.

Since $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \emptyset$, by (**1**), for each vertex $w \in V(u^+\overrightarrow{C}x^-) \cup V(v^+\overrightarrow{C}y^-)$, $\deg_G(w, C^*) > \frac{n}{t+1} - \lambda$. This shows a contradiction to Lemma 2.4. $\square$

By Claim 4, $G[L_u^+]$ and $G[L_v^+]$ are remote for any two distinct $u, v \in W$. Furthermore, H is remote with $G[L_u^+]$ for any $u \in W$. In addition, we have $|L_u^+| \geqslant \frac{n}{t+1} - \omega$ if $\sigma_2(G) = \frac{2n}{t+1} - 2$, and $|L_u^+| > \frac{n}{t+1} - \omega$ if $\sigma_2(G) > \frac{2n}{t+1} - 2$. Let $S = V(G) \setminus \left((\bigcup_{u \in W} L_u^+) \cup V(H)\right)$. Then $c(G - S) = \omega + 1$ and

$$\begin{cases} |S| < n - \omega\left(\dfrac{n}{t+1} - \omega\right) - \lambda & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\[3mm] |S| \leqslant n - \omega\left(\dfrac{n}{t+1} - \omega\right) - \lambda & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2. \end{cases}$$

As G is t-tough and so $|S| \geqslant tc(G - S) = t(\omega + 1)$, we get

$$\begin{cases} n > \omega \left(\dfrac{n}{t + 1} - \omega + t \right) + \lambda + t & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\[2ex] n \geqslant \omega \left(\dfrac{n}{t + 1} - \omega + t \right) + \lambda + t & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2. \end{cases}$$

Claim 5. *It holds that $\sigma_2(G) = \frac{2n}{t+1} - 2$, $\lambda = 1$, and $\omega = \frac{n}{t+1} - 1$.*

Proof. Note that we have $\omega \leqslant \frac{n}{t+1} - \lambda \leqslant \frac{n}{t+1} - 1$ by Claim 2. Suppose to the contrary that $\sigma_2(G) > \frac{2n}{t+1} - 2$, $\lambda \geqslant 2$, or $\omega < \frac{n}{t+1} - 1$. Now we have

$$n \geqslant \omega \left(\frac{n}{t + 1} - \omega + t \right) + \lambda + t,$$

implying

$$\left(\frac{\omega}{t + 1} - 1 \right) n \leqslant \omega(\omega - t) - \lambda - t. \tag{2}$$

The inequality (2) cannot achieve equality when $\sigma_2(G) > \frac{2n}{t+1} - 2$, since we have $n > \omega \left(\frac{n}{t+1} - \omega + t \right) + \lambda + t$ under the current assumptions. If $\omega < t + 1$, then we have $\omega < 2$ because $2t \leqslant \omega < t + 1$ implies $t < 1$, a contradiction to Claim 1. Thus we have $\omega \geqslant t + 1$, implying $\frac{\omega}{t+1} - 1 \geqslant 0$. Then by Claim 2, we have

$$\left(\frac{\omega}{t + 1} - 1 \right) n \geqslant \left(\frac{\omega}{t + 1} - 1 \right) (\omega + \lambda)(t + 1). \tag{3}$$

Note that if $\lambda \geqslant 2$ or $\omega < \frac{n}{t+1} - 1$, then the inequality (3) cannot achieve the equality. By the assumption for the contrary, at least one of the inequalities (2) or (3) cannot achieve the equality. Therefore, combining (2) and (3), we get

$$\omega(\omega - t) - \lambda - t > \left(\frac{\omega}{t + 1} - 1 \right) (\omega + \lambda)(t + 1),$$

which implies

$$\begin{aligned} \omega^2 - \omega t - \lambda - t \ &> \ \omega(\omega + \lambda) - (\omega + \lambda)(t + 1) \\ &= \ \omega^2 + \omega\lambda - \omega t - \omega - \lambda t - \lambda. \end{aligned}$$

This gives $(\lambda - 1)t > (\lambda - 1)\omega$, leading to $0 < 0$ or $\omega < t$, a contradiction. $\qquad\square$

By Claim 5, Theorem 1(a) holds. In the rest of the proof, we show Theorem 1(b). Recall that H consists of a single vertex, and this assumption will be used in the remainder of the proof. Let

$$W^* = W^+ \cup V(H).$$

Since $u^+ \in L_u^+$ for each $u \in W$, Claim 4 implies that W^* is an independent set in G.

Claim 6. *Every vertex in $V(G) \setminus W^*$ is adjacent in G to at least two vertices from W^*.*

Proof. Suppose to the contrary that there exists $x \in V(G) \setminus W^*$ such that x is adjacent in G to at most one vertex from W^*. Let $S = V(G) \setminus (W^* \cup \{x\})$. Then $c(G - S) \geqslant \omega + 1$. However

$$\frac{|S|}{c(G-S)} \leqslant \frac{n - \omega - 2}{\omega + 1} = \frac{\frac{tn}{t+1} - 1}{\frac{n}{t+1}} < t,$$

a contradiction. $\qquad\square$

Claim 7. *For every $v \in W^+$, we have $\deg_G(v, C) = \frac{n}{t+1} - 1$ and v is not adjacent in G to any two consecutive vertices on C.*

Proof. Since $\sigma_2(G) = \frac{2n}{t+1} - 2$, we have $\deg_G(v, C) \geqslant \frac{n}{t+1} - 1$ for every $v \in W^+$. As W^* is an independent set in G, $v^+ \notin W^*$. By Claim 6, v^+ is adjacent in G to another vertex u from W^*. If $\{u\} = V(H)$, then $C^* = v^- \overset{\leftharpoonup}{C} v^+ u v^-$ is a $D_{\lambda+1}$-cycle of G with v being the only component of $G - V(C^*)$. Assume instead that $u \in W^+$. Let $V(H) = \{x\}$. Then $C^* = v^+ u \overset{\rightharpoonup}{C} v^- x u^- \overset{\leftharpoonup}{C} v^+$ is a $D_{\lambda+1}$-cycle of G with v being the only component of $G - V(C^*)$.

Again, since G has no D_λ-cycle and v is the only vertex of G outside $V(C)$, it follows that v is not adjacent in G to any two consecutive vertices on C^*. Furthermore, we must have $\deg_G(v, C^*) = \frac{n}{t+1} - 1$. As if $\deg_G(v, C^*) > \frac{n}{t+1} - 1$, then $\{v\} \cup N_C(v)^+$ is an independent set of G with more than $\frac{n}{t+1}$ vertices, a contradiction to Lemma 2.1. The claim follows as $\deg_G(v, C) = \deg_G(v, C^*)$ and two neighbors of v that are consecutive on C will also be consecutive on C^* as v is not adjacent in G to any vertex from W^+. $\qquad\square$

Our goal is to show that $N_C(W^+) = N_C(H)$. To do so, we investigate how vertices in $N_C(W^+)$ are located along $\overset{\rightharpoonup}{C}$. We start with some definitions. A *chord* of C is an edge uv with $u, v \in V(C)$ and $uv \notin E(C)$. Two chords ux and vy of C that do not share any endvertices are *crossing* if the four vertices u, x, v, y appear along $\overset{\rightharpoonup}{C}$ in the order u, v, x, y or u, y, x, v. For two distinct vertices $x, y \in N_C(W^+)$, we say x and y form a *crossing* if there exist distinct vertices $u, v \in W^+$ such that ux and vy are crossing chords of C.

Claim 8. *For any two distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, it follows that x and y do not form any crossing.*

Proof. Suppose to the contrary that for some distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, the two vertices x and y form a crossing. Let $u, v \in W^+$ such that $xu, yv \in E(G)$. Assume, without loss of generality, that the four vertices u, v, x, y appear in the order u, v, x, y along $\overset{\rightharpoonup}{C}$. Let $V(H) = \{w\}$. Then $ux\overset{\leftharpoonup}{C}vy\overset{\rightharpoonup}{C}u^- wv^- \overset{\leftharpoonup}{C}u$ is a hamiltonian cycle of G, a contradiction to our assumption that G is not hamiltonian. $\qquad\square$

Claim 9. *For any vertex $v \in W^+$ and any two distinct $x, y \in N_C(v)$, $x\overrightarrow{C}y$ contains a vertex from W^+.*

Proof. By Claim 7, $x\overrightarrow{C}y$ has at least three vertices. Suppose to the contrary that $x\overrightarrow{C}y$ contains no vertex from W^+. We furthermore choose x and y so that $x\overrightarrow{C}y$ contains no other vertex from $N_C(v) \setminus \{x, y\}$. Assume that the three vertices v, x, y appear in the order v, x, y along $\overrightarrow{C}$. By Claim 6, each internal vertex of $x\overrightarrow{C}y$ is adjacent in G to a vertex from W^+. Then by our selection of x and y, we know that each internal vertex of $x\overrightarrow{C}y$ is adjacent in G to a vertex from $W^+ \setminus \{v\}$. Applying Claim 8, x^+ does not form a crossing with x, and so x^+ forms a crossing with y. Similarly, x^{++} does not form a crossing with x^+, and so forms a crossing with y. Continuing this argument for all the internal vertices of $x^{++}\overrightarrow{C}y$, we know that y^- forms a crossing with y, a contradiction to Claim 8. $\quad\square$

We assume that the ω neighbors of the vertex from $V(H)$ on C are $v_1, \ldots, v_\omega$ and they appear in the same order along $\overrightarrow{C}$. For each $i \in [1, \omega]$, let $I_i = V(v_i\overrightarrow{C}v_{i+1}) \setminus \{v_i\}$, where $v_{\omega+1} := v_1$.

Claim 10. *For every $v \in W^+$, it holds that $N_C(v) = W$.*

Proof. Since $x\overrightarrow{C}y$ contains a vertex from W^+ for any two distinct $x, y \in N_C(v)$ by Claim 9, it follows that no I_i can contain more than one vertex from $N_C(v)$. Since $\deg_G(v, C) = \omega = |W^+|$ by Claim 7 and $\{I_1, \ldots, I_\omega\}$ is a partition of $V(C)$, the Pigeonhole Principle implies that each I_i contains exactly one vertex from $N_C(v)$.

Assume to the contrary that $N_C(v) \neq W$. Let $i \in [1, \omega]$ be the index such that $\mathrm{dist}_{\overrightarrow{C}}(v, v_i)$ is largest and $vv_i \notin E(G)$. Note that the index i exists since $v^- \in W$ and $vv^- \in E(G)$. In particular, every vertex $u \in W \cap V(v_i^+\overrightarrow{C}v)$ is adjacent to v by the choice of i. Let z be the vertex in $N_C(v) \cap I_{i-1}$. We prove the four subclaims below. Let $V(H) = \{x\}$ in the rest arguments.

Claim A: $z = v_i^-$.

Proof of Claim A. Suppose otherwise that $z \neq v_i^-$. Then by Claim 6, z^+ is adjacent in G to at least two vertices from W^+. By Claim 8, $N_C(z^+) \cap W^+ \subseteq V(v_i^+\overrightarrow{C}v)$. Thus z^+ is adjacent in G to a vertex from $W^+ \cap V(v_i^+\overrightarrow{C}v^-)$ as z is the only neighbor of v from I_{i-1} in G. By repeating this procedure for all the vertices from $V(z^{++}\overrightarrow{C}v_i^-)$ iteratively, we conclude that v_i^- is adjacent in G to a vertex $u \in W^+ \cap V(v_i^+\overrightarrow{C}v^-)$. As $v_i^+v_i \in E(G)$ and $v_iv_i^- \in E(C)$, Claim 7 implies that v_i^+ is not adjacent in G to v_i^-. Thus we have $u \notin \{v_i^+, v\}$. However, since $u^-v \in E(G)$ by our choice of the index i, the cycle $x\overleftarrow{v^-C}uv_i^-\overleftarrow{C}vu^-\overleftarrow{C}v_ix$ is longer than C, a contradiction. Thus z must be v_i^-. $\quad\square$

Claim B: $v_{i+1} = v^-$.

Proof of Claim B. Suppose that $v_{i+1} \neq v^-$. Considering v_{i+1}^+ in the place of v and applying Claim A to it, v_{i+1}^+ must be adjacent to v_i or v_i^- (if $v_{i+1}^+ v_i \notin E(G)$, then i is the index such that $\mathrm{dist}_{\overrightarrow{C}}(v_{i+1}^+, v_i)$ is largest and $v_{i+1}^+ v_i \notin E(G)$). If $v_{i+1}^+ v_i^- \in E(G)$, then the cycle $x v^- \overleftarrow{C} v_{i+1}^+ v_i^- \overleftarrow{C} v v_{i+1} \overleftarrow{C} v_i x$ is longer than C, a contradiction. Thus we have $v_{i+1}^+ v_i \in E(G)$. We consider the vertex v^+. Since W^- is independent in G and v is adjacent to $v_i^- \in W^-$, we have $v \notin W^-$. Thus $v^+ \notin W$. Then by Claim 6, v^+ is adjacent in G to a vertex $u \in W^+ \setminus \{v\}$. However, the cycle

$$
\begin{cases}
x v_i \overrightarrow{C} v v_i^- \overleftarrow{C} u v^+ \overrightarrow{C} u^- x & \text{if } u \in V(v^+ \overrightarrow{C} v_{i-1}^+), \\
x v_i v_{i+1}^+ \overrightarrow{C} v v_i^- \overleftarrow{C} v^+ v_i^+ \overrightarrow{C} v_{i+1} x & \text{if } u = v_i^+, \\
x v^- \overleftarrow{C} u v^+ \overrightarrow{C} v_i^- v u^- \overleftarrow{C} v_i x & \text{if } u \in V(v_{i+1}^+ \overrightarrow{C} v^-),
\end{cases}
$$

is longer than C, a contradiction. $\qquad\square$

Claim C: $\omega \geqslant 4$.

Proof of Claim C. By Claim 1, we have $\omega \geqslant 2$. Suppose that $\omega \in [2, 3]$. First, suppose $\omega = 2$. Since $v \in W^+$ is adjacent to a vertex in W^- and W^+ is independent in G, we have $W^- \setminus W^+ \neq \emptyset$. Also a vertex $u \in W^- \setminus W^+$ is adjacent to all vertices in W^+ by Claim 6. Then $u^+ \in W$ and so $u^{++} \in W^+$ is adjacent to u^+ and u, contrary to Claim 7. Next, suppose $\omega = 3$. We let, without loss of generality, $v = v_1^+$. Then Claim B implies $v_1^+ v_3^- \in E(G)$. Note that since W^+ is independent in G, v_3^- must not be v_2^+. We also have $v_2^+ v_1 \notin E(G)$, as otherwise $v_1 v_2^+ \overrightarrow{C} v_3^- v_1^+ \overrightarrow{C} v_2 x v_3 \overrightarrow{C} v_1$ is a cycle longer than C. Applying Claim A to v_2^+, we get $v_2^+ v_1^- \in E(G)$. Similarly, $v_3^+ v_2 \notin E(G)$, as otherwise $v_3^+ v_2 \overleftarrow{C} v_1 x v_3 \overleftarrow{C} v_2^+ v_1^- \overleftarrow{C} v_3^+$ is a cycle longer than C. Applying Claim A to v_3^+, we get $v_3^+ v_2^- \in E(G)$. Then as the degrees of all vertices from W^+ are of degree 3 in G, Claims 3, 6, and 7 imply that the graph G is isomorphic to the Petersen graph. However, $3 = \omega = \frac{10}{t+1} - 1$ implies that G is $\frac{3}{2}$-tough, contradicting that the toughness of the Petersen graph is at most $\frac{4}{3}$ (in the Petersen graph, deleting two independent vertices from one 5-cycle and another two independent vertices that are non-neighbors of the first two deleted vertices from the second disjoint 5-cycle gives three components). Thus we have $\omega \geqslant 4$. $\qquad\square$

Claim D: For every $j \in [1, \omega]$, $|I_j| \neq 3$.

Proof of Claim D. Suppose that $|I_j| = 3$ for some $j \in [1, \omega]$. Then we have $v_j^+ v_{j+1}^- \in E(C)$, which implies $N_C(v_j^+) \neq W$. Applying Claim A to v_j^+, we get $v_j^+ v_{j-1}^- \in E(G)$. By symmetry of the orientation of C, we have $v_{j+1}^- v_{j+2}^+ \in E(G)$. Also we have $\omega \geqslant 4$ by Claim C, which implies $v_{j-1} \in V(v_{j+2}^+ \overrightarrow{C} v_j^-)$. Then the cycle $x v_{j-1} \overrightarrow{C} v_j^+ v_{j-1}^- \overleftarrow{C} v_{j+2}^+ v_{j+1}^- \overrightarrow{C} v_{j+2} x$ is longer than C, a contradiction. $\qquad\square$

We now show a contradiction. The vertex v^+ must not be in W since $vv_i^- \in E(G)$ by Claim A and W^- is independent in G. Thus v^+ is adjacent in G to a vertex $u \in W^+\setminus\{v\}$ by Claim 6. If $u \neq v_i^+$, then the cycle $xv_i\overrightarrow{C}vv_i^-\overleftarrow{C}uv^+\overrightarrow{C}u^-x$ is longer than C, a contradiction. Thus we have $u = v_i^+$. We consider the cycle $C^* = v^+\overrightarrow{C}v_ixv^-\overleftarrow{C}v_i^+v^+$ in G. Note that we have $V(C^*) = V(G) \setminus \{v\}$. Then since the length of C^* is equal to the length of C, we can apply Claim D to C^*. However, v^-, x, v_i, v_i^- are four consecutive vertices on C^* appearing in the order v^-, x, v_i, v_i^- and $v^-, v_i^- \in N_{C^*}(v)$, showing that C^* does not satisfy Claim D, a contradiction. This completes the proof of Claim 10. $\qed$

Claim 10 implies that $N_C(W^*) = W$. Thus every vertex from W^* is adjacent in G to every vertex from W. Therefore $t \leqslant \tau(G) \leqslant \frac{|W|}{|W^*|}$ as W^* is an independent set in G. Consequently, $|W| \geqslant t|W^*| = \frac{tn}{t+1}$ and so $W = V(G) \setminus W^*$ by noticing $|W^*| = \frac{n}{t+1}$. Thus G contains a spanning complete bipartite graph between W^* and W. On the other hand, since $|W^+| = |W| = \frac{n}{t+1} - 1$ and $V(G) = W^* \cup W = (W^+ \cup V(H)) \cup W$, we know that $2(\frac{n}{t+1} - 1) + 1 = n$ and so $t = \frac{n-1}{n+1}$. Thus $|W| = \frac{n-1}{2}$ and $|W^*| = \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Therefore, $G \in \mathcal{H}$. The proof of Theorem 1 is now complete. $\qed$

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