Bounding Mean Orders of Sub-k-Trees of k-Trees

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Abstract

For a k-tree T, we prove that the maximum local mean order is attained in a k-clique of degree 1 and that it is not more than twice the global mean order. We also bound the global mean order if T has no k-cliques of degree 2 and prove that for large order, the k-star attains the minimum global mean order. These results solve the remaining problems of Stephens and Oellermann [*J. Graph Theory* 88 (2018), 61–79] concerning the mean order of sub-k-trees of k-trees.

Mathematics Subject Classifications: 05C05, 05C35

1 Introduction

In [10] and [11] Jamison considered the mean number of nodes in subtrees of a given tree. He showed that for trees of order n, the average number of nodes in a subtree of T is at least $(n+2)/3$, with this minimum achieved if and only if T is a path. He also showed that the average number of nodes in a subtree containing a root is at least $(n+1)/2$ and always exceeds the average over all unrooted subtrees. The mean subtree order in trees was further investigated, e.g. in [3, 7, 14, 20, 22], as well as extensions to arbitrary graphs [1, 4–6] and the mean order of the connected induced subgraphs of a graph [8, 9, 17–19].

In [16], Stephens and Oellermann extended the study to k-trees and families of sub-k-trees. A *k-tree* is a generalization of a tree that has the following recursive construction.

Definition 1 (**k-tree**)**.** Let k be a fixed positive integer.

1. The complete graph K_k is a k-tree.

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2. If T is a k-tree, then so is the graph obtained from T by joining a new vertex to all vertices of some k-clique of T.

3. There are no other k-trees.

Note that for $k = 1$ we have the standard recursive construction of trees. A *sub-k-tree* of a k-tree T is a subgraph that is itself a k-tree. Let $S(T)$ denote the collection of all sub-k-trees of T and let $N(T) := |S(T)|$ be the number of sub-k-trees. We denote by $R(T) = \sum_{X \in S(T)} |X|$ the total number of vertices in all sub-k-trees (where for a graph G the notation $|G|$ is used throughout to mean the number of vertices in G). The *global mean (sub-k-tree) order* is

$$\mu(T) = \frac{R(T)}{N(T)}.$$

For an arbitrary k-clique C of T, let $S(T; C)$ denote the collection of sub-k-trees containing C and let $N(T; C) := |S(T; C)|$. The *local clique number* is $R(T; C) = \sum_{X \in S(T;C)} |X|$ and the *local mean (sub-k-tree) order* is

$$\mu(T; C) = \frac{R(T; C)}{N(T; C)}.$$

The *degree* of C is the number of $(k + 1)$-cliques that contain C.
Stephens and Oellermann concluded their study of the mean order of sub-k-trees of k-trees with several open questions. Our main contribution here is to answer three of them.
It was conjectured by Jamison in [10] and proven by Vince and Wang in [20] that for trees of order n without vertices of degree 2—called *series-reduced* trees—the global mean subtree order is between $\frac{n}{2}$ and $\frac{3n}{4}$. For k-trees we provide similar asymptotically sharp bounds, answering [16, Problem 6].

Theorem 2. *For every k-tree T without k-cliques of degree 2, the global mean sub-k-tree order satisfies*

$$\frac{n + k}{2} - o_n(1) < \mu(T) < \frac{3n + k - 3}{4}.$$

These bounds are asymptotically sharp. In particular, for large k, $\frac{3n}{4}$ is not an upper bound. For large n, the k-star is the unique extremal k-tree for the lower bound.

Wagner and Wang proved in [21] that the maximum local mean subtree order occurs at a leaf or a vertex of degree 2. We prove an analogous result for k-trees, answering [16, Problem 4]. In contrast to the result for trees, it turns out that for $k \geqslant 2$, the maximum can only occur at a k-clique of degree 1, unless T is a k-tree of order $k + 2$.

Theorem 3. *Suppose that $k \geqslant 2$. For a k-tree T of order $n \neq k + 2$, if a k-clique C maximizes $\mu(T; C)$, then C must be a k-clique of degree 1. For $n = k + 2$, every k-clique C satisfies $\mu(T; C) = k + 1$.*

Lastly, Jamison [10] conjectured that for a given tree T and any vertex v, the local mean order is at most twice the global mean order of all subtrees in T. Wagner and Wang [21] proved that this is true. Answering [16, Problem 2] affirmatively, we show

Theorem 4. *The local mean order of the sub-k-trees containing a fixed k-clique C is less than twice the global mean order of all sub-k-trees of T.*

1.1 Related Results

A total of six questions were posed in [16]. Problems 1 and 3 were solved by Luo and Xu [13]. Regarding the first problem, Jamison [10] showed that for any tree T and any vertex v of T, the local mean order of subtrees containing v is an upper bound on the global mean order of subtrees of T. Stephens and Oellermann asked about a generalization to k-trees, to which Luo and Xu showed:

Theorem 5 ([13])**.** *For any k-tree T of order n with a k-clique C, we have $\mu(T;C) \geqslant \mu(T)$ with equality if and only if $T \cong K_k$.*

For the third problem, it was shown in [10] that paths have the smallest global mean subtree order. For k-trees we have:

Theorem 6 ([13])**.** *For any k-tree T of order n, we have*

$$\mu(T) \geqslant \frac{\binom{n-k+2}{3}}{\binom{n-k+1}{2} + (n-k)k + 1} + k$$

with equality if and only if T is a path-type k-tree.

Very recently, Li, Ma, Dong, and Jin [12] gave a partial proof of Theorem 3, showing that the maximum local mean order always occurs at a k-clique of degree 1 or 2, thus also solving [16, Problem 4]. They did this by combining the fact that the 1-characteristic trees of adjacent k-cliques can be obtained from each other by a partial Kelmans operation [12, Lem. 4.6], an inequality between local orders of neighboring vertices after performing a partial Kelmans operation [12, Thm. 3.3], as well as [16, Lem. 11] and [21, Thm. 3.2]. Theorem 3 also solves Problem 5.4 from [12], asking whether the maximum local mean order can ever occur at a k-clique of degree 2, but not at a k-clique of degree 1.
Outline: In Section 2, we go over definitions and notation. Theorems 2, 3, 4 are proven in Sections 3, 4, 5 respectively. We additionally address [16, Problem 5], which is a more general question asking what one can say about the local mean order of sub-k-trees containing a fixed r-clique for $1 \leqslant r \leqslant k$. We give a possible direction and partial results in the concluding section.

2 Notation and Definitions

The global mean sub-k-tree order $\mu(T)$ and the local mean sub-k-tree order $\mu(T;C)$ are defined in the introduction. The local mean order always counts the k vertices from C and it sometimes will be more convenient to work with the average number of additional vertices in a (uniform random) sub-k-tree of T containing C, in which case we use the notation $\mu^\bullet(T;C) = \mu(T;C) - k$. Moreover, the number of sub-k-trees not containing C

will be denoted by $\overline{N}(T; C) = N(T) - N(T; C)$ and the total number of vertices in the sub-k-trees that do not contain C will be denoted by $\overline{R}(T; C) = R(T) - R(T; C)$.
A *k-leaf* or *simplicial vertex* is a vertex belonging to exactly one $(k+1)$-clique of T, i.e., a vertex of degree k. A *simplicial k-clique* is a k-clique containing a k-leaf. Note that a k-clique of degree 1 is not necessarily simplicial. A *major k-clique* is a k-clique with degree at least 3. Two k-cliques are *adjacent* if they share a $(k-1)$-clique.
The *stem* of a k-tree T is the k-tree obtained by deleting all k-leaves from T.
Subclasses of trees generalize to subclasses of k-trees. We will reference two in particular: paths generalize to *path-type k-trees* that are either isomorphic to K_k or K_{k+1} or have precisely two k-leaves. Note that for every $n \geqslant k+4$ and $k \geqslant 2$, there are multiple non-isomorphic path-type k-trees of order n.
A *k-star* is either K_k or K_{k+1}, or it is the unique k-tree with $n-k$ simplicial vertices when $n \geqslant k+2$.
Furthermore, the combination of a k-star and a k-path is called a k-broom: take a k-path of a certain length and add some simplicial vertices to a simplicial k-clique of the k-path.

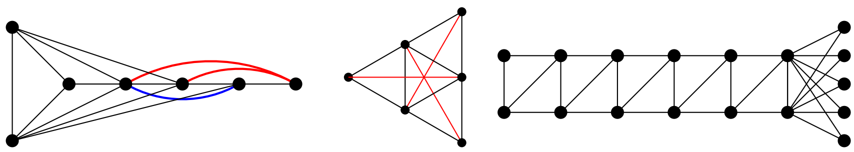


Figure 1: Examples of a 3-path, 3-star and 2-broom.

For a given k-clique C in T, it is often useful to decompose T into C and the sub-k-trees that result from deleting C. Let $B_1, B_2, \ldots, B_d$ be the $(k+1)$-cliques that contain C, and let v_i be the vertex of $B_i \setminus C$. Moreover, let $C_{i,1}, \ldots, C_{i,k}$ be the k-subcliques of B_i other than C. The k-tree T can be *decomposed* into C and k-trees $T_{1,1}, \ldots, T_{d,k}$, rooted at $C_{1,1}, \ldots, C_{d,k}$ respectively, that are pairwise disjoint except for the vertices of the cliques $C_{i,j}$.
In a 1-tree, any two vertices are connected by a unique path. This fact generalizes to k-trees through the construction of the 1-characteristic tree of a k-tree T [13, 16]. For a k-clique C in T, a *perfect elimination ordering of T to C* is an ordering $v_1, v_2, \ldots, v_{n-k}$ of its vertices other than $V(C) = \{c_1, \ldots, c_k\}$ such that each vertex v_i is simplicial in the k-tree spanned by C and $v_j, 1 \leqslant j \leqslant i$. In [16] it is shown that for any $v \notin C$, there is a unique sequence of vertices that along with C induce a path-type k-tree $P(C, v)$ and that form a perfect elimination ordering of $P(C, v)$ to C. It is also proven that T can be written as $\bigcup P(C, v)$ where the union is taken over all k-leaves $v \in V$. Each k-tree $P(C, v)$ has an associated 1-tree $P'(C, v)$ where the vertices consist of a single vertex representing the entire clique C, along with the remaining non-C vertices of $P(C, v)$. The edges are consecutive pairs from the perfect elimination ordering. Taking $\bigcup P'(C, v)$ over all k-leaves v gives us what is called the 1-*characteristic tree* of T, which we will denote T'_C. See Figure 2 for an example.

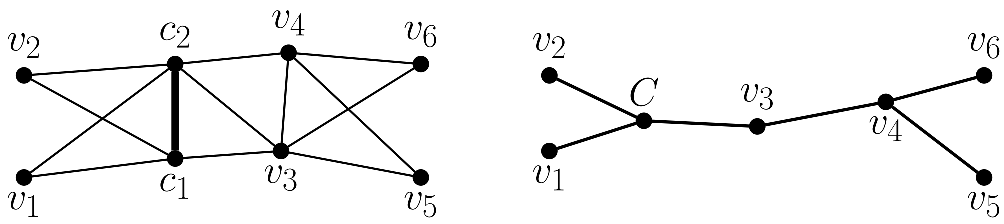


Figure 2: A 2-tree (left) with 2-clique $C = \{c_1, c_2\}$ and the 1-characteristic tree (right).

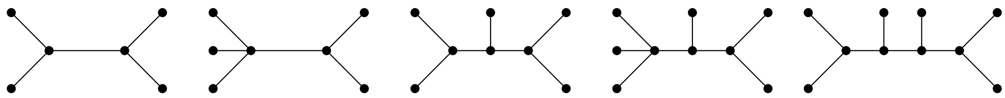


Figure 3: Series-reduced trees with minimum mean subtree order for $6 \leqslant n \leqslant 10$.

3 Excluding k-cliques of degree 2

In this section, we prove

Theorem 2. *For every k-tree T without k-cliques of degree 2, the global mean sub-k-tree order satisfies*

$$\frac{n+k}{2} - o_n(1) < \mu(T) < \frac{3n+k-3}{4}.$$

These bounds are asymptotically sharp. In particular, for large k, $\frac{3n}{4}$ is not an upper bound. For large n, the k-star is the unique extremal k-tree for the lower bound.

3.1 The lower bound

Among series-reduced trees of order n, for $4 \leqslant n \leqslant 8$ the star attains the maximum mean subtree order, but for $n \geqslant 11$ it attains the minimum mean subtree order. It can be derived from [7, Lem. 12] and an adapted version of [7, Cor. 11] (with 2 replaced by any $\varepsilon > 0$, at the cost of replacing 30 by n_ε) that this is indeed the case for n sufficiently large.[1]
In [2], it is shown that for $6 \leqslant n \leqslant 10$, the series-reduced trees attaining the minimum mean subtree order are those presented in Figure 3, and for $n \geqslant 11$, S_n is always the unique extremal graph.
In this subsection, we will prove that the above extremal statement generalizes to k-trees without a k-clique of degree 2.
First, we prove the following lemma, which states that every k-tree contains a $(k+1)$-clique C that plays the role of a centroid in a tree (a vertex or edge whose removal splits the tree into components of size at most $\frac{n}{2}$). Figure 4 is an example of a 2-tree demonstrating why we must take a $(k+1)$-clique and not a k-clique.

Lemma 7. *Any k-tree T (of order $n \geqslant k+1$) has a $(k+1)$-clique C such that the order of all components of $T \setminus C$ is at most $\left\lceil \frac{n-(k+1)}{2} \right\rceil$.*

[1] We thank John Haslegrave for this remark.

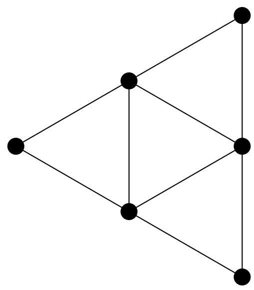


Figure 4: 2-tree with a centroid 3-clique but no centroid 2-clique.

Proof. Suppose for contradiction that such a $(k+1)$-clique does not exist. Consider the $(k+1)$-clique C for which the largest component of $T \setminus C$ has minimum order over all choices of C. By assumption, the largest component of $T \setminus C$ has order $n_0 > \left\lceil \frac{n-(k+1)}{2} \right\rceil$, and by the construction of a k-tree, there exist vertices $u \in C$ and v in the largest component T_0 of $T \setminus C$ such that u has no neighbor in T_0 and $v \cup (C \setminus u)$ forms a $(k+1)$-clique C'. Now $T \setminus C'$ has components whose sizes are bounded by $\max\{n - n_0 - k, n_0 - 1\} < n_0$, which gives a contradiction. Hence C satisfies the stated property. $\square$

The following lemma and its proof are similar to [15, Lemma 5.1].

Lemma 8. *For any k-tree T without k-cliques of degree 2, there is a k-clique C' for which $\mu(T; C') - \mu(T) \leqslant \frac{n^3}{2^{(n-k)/4+1}} = o_n(1)$.*

Proof. Take C as in Lemma 7. Let its vertices be $\{u_1, u_2, \ldots, u_{k+1}\}$.
For every component of $T \setminus C$, there is a unique vertex $u_i, 1 \leqslant i \leqslant k+1$, such that the component together with $C \setminus u_i$ forms a k-tree. Now we consider two cases that are handled analogously.
Case 1: There is some u_i such that the union of components that form a k-tree when adding $C' = C \setminus u_i$ has order at least $\frac{n-k}{2}$.
In this case, we consider the $r \leqslant n-k$ components of $T \setminus C'$ and for every $1 \leqslant j \leqslant r$, we let T_j be the union of such a component and C'.
We then apply the following claim:

Claim 9. *Let C' be a k-clique in a k-tree T of order n with no k-cliques of degree 2. Then T has at least $\frac{n-k+1}{2}$ simplicial vertices that are not in C'.*

Proof. Consider the 1-characteristic tree $T'_{C'}$, whose order is $n-k+1$. For every $u \in T \setminus C'$, either all the k-cliques containing u have degree 1, in which case u has degree 1 in $T'_{C'}$, or (at least) one of them has degree at least 3 and so does u in $T'_{C'}$. Thus $T'_{C'}$ is series-reduced and thus has at least $\frac{n-k+1}{2} + 1$ leaves. The latter implies that T has at least $\frac{n-k+1}{2}$ simplicial vertices that are not in C'. $\diamondsuit$

By the choice of C, for every $1 \leqslant j \leqslant r$, there are at least $\left\lfloor \frac{n-k+1}{2} \right\rfloor$ vertices that do not belong to T_j. By Claim 9, among them are at least $\frac{n-k}{4}$ simplicial vertices of T. Observe that given any sub-k-tree of T containing C', we can map it to its sub-k-tree intersection

with T_j. If two elements of $S(T;C')$ differ only in some subset of k-leaves of $\bigcup_{h\neq j} T_h$, then they map to the same element of $S(T_j;C')$. Thus, each element of $S(T_j;C')$ is mapped to at least $2^{\frac{n-k}{4}}$ times, and $N(T;C') \geqslant 2^{\frac{n-k}{4}} N(T_j;C')$.
Using the perfect elimination ordering, every sub-k-tree S in T_j not containing C' can be extended in a minimal way into a sub-k-tree containing C'. Furthermore, by considering the 1-characteristic tree $T'_{C'}$ of C', it is clear that there are no more than $|T_j| - k \leqslant \left\lceil \frac{n-(k+1)}{2} \right\rceil \leqslant \frac{n}{2}$ k-trees that extend to the same tree. Here we have used Lemma 7 again. Thus, if we define a map from $\overline{S}(T_j;C')$ to $S(T_j;C')$ using the minimal extension, every element of $S(T_j;C')$ is mapped to at most $\frac{n}{2}$ times.
Putting the previous two observations together, we have that the number of sub-k-trees containing C' is $N(T;C') \geqslant 2^{\frac{n-k}{4}} N(T_j;C') \geqslant \frac{1}{n} 2^{\frac{n-k}{4}+1} \overline{N}(T_j;C')$.
By summing over all j, we obtain that

$$rN(T;C') \geqslant \frac{1}{n} 2^{\frac{n-k}{4}+1} \sum_{j=1}^{r} \overline{N}(T_j;C') = \frac{1}{n} 2^{\frac{n-k}{4}+1} \overline{N}(T;C').$$

Since $r \leqslant n$, we have that $N(T;C') \geqslant \frac{1}{n^2} 2^{\frac{n-k}{4}+1} \overline{N}(T;C')$. This implies that $\frac{\mu(T)}{\mu(T;C')} \geqslant 1 - \frac{n^2}{2^{\frac{n-k}{4}+1}}$. The result follows now from $\mu(T;C') \leqslant n$.
Case 2: For every u_i the union of components in $T \setminus C$ that form a k-tree when adding $C \setminus u_i$ has order smaller than $\frac{n-k}{2}$.
Let T_i, $1 \leqslant i \leqslant k+1$, be the k-trees obtained above when adding $C \setminus u_i$. By Claim 9, for each i there are at least $\frac{n-k}{4}$ k-leaves not belonging to T_i. Thus, the same computations apply and in particular we have that $N(T;C') \geqslant \frac{1}{n^2} 2^{\frac{n-k}{4}+1} \overline{N}(T;C')$ as before. Now for $C' = C \setminus u_1$, we conclude by inclusion monotonicity [13, Thm. 33] that $\mu(T;C') - \mu(T) \leqslant \mu(T;C) - \mu(T) \leqslant \frac{n^3}{2^{(n-k)/4+1}}$. □

Proof of Theorem 2, lower bound. Take a k-clique C' which satisfies Lemma 8. Let $T'_{C'}$ be the 1-characteristic tree of T with respect to C'. By [13, Thm. 33] and [7, Lem. 12], we conclude that $\mu(T;C') = \mu(T'_{C'};C') + k - 1 \geqslant \frac{n+k}{2} + \frac{i-1}{10}$, where i is the number of internal vertices in $T'_{C'}$. By Lemma 8, we conclude.
For sharpness, observe that if T is not a k-star, we have $i \geqslant 2$ and the lower bound inequality is strict. When T is a k-star, it contains no k-cliques of degree 2 provided that $n > k+2$. As computed in [16] (page 64), $\mu(T) = \frac{R(T)}{N(T)} = \frac{(n+k)2^{n-k-1}}{2^{n-k}+(n-k)k} = \frac{n+k}{2} - o_n(1)$. □

3.2 The upper bound

In this subsection, we generalize to k-trees the statement that a series-reduced tree has average subtree order at most $\frac{3n}{4}$ by giving a lower bound for the number of k-leaves and proving that k-leaves belong to at most half of the sub-k-trees. This idea was also used in [7]. Note that the upper bound is slightly larger than $\frac{3n}{4}$ for larger k, which intuitively can be explained by the fact that the smallest sub-k-tree already has k vertices, and more precisely the vertices in the base k-clique will all be major vertices.

Proof of Theorem 2, upper bound. Our upper bound will come from the observation that $\mu(T) = \sum_{v \in T} p(v)$ where $p(v)$ is the fraction of sub-k-trees containing v. We will specifically consider when v is a k-leaf and bound the corresponding terms in the summation.

We first prove that the 1-characteristic tree T'_C of a k-tree T without k-cliques of degree 2 is a series-reduced tree, for any k-clique C of T. Indeed, given a k-clique C, there is either exactly one vertex adjacent to C or at least 3. As such, the degree of C in T'_C is not 2. For any other vertex $v \in T \setminus C$, either it is a leaf or some vertex w was added to a k-clique C', where v is the latest vertex of C'. In the latter case, note that v was added to some k-clique of the form $C' \cup \{u\} \setminus \{v\}$. This implies that u and w are adjacent to all of C', and since T has no k-cliques of degree 2, there must be at least another vertex w' with this property. Now v is adjacent to w, w', and some earlier vertex, so its degree in T'_C is at least 3.

Thus T'_C has at least $\frac{n-k+1}{2} + 1$ leaves, which implies that T contains at least this many k-leaves (here one must also observe that if C has degree 1 in T'_C, some vertex of C is simplicial in T).

Now fix a k-leaf v. Since $n \geqslant k+2$, there is a vertex u (different from v) such that $N(v) \cup \{u\}$ spans a K_{k+1}. Define a function f on sub-k-trees containing v such that $f(C') = (C' \setminus \{v\}) \cup \{u\}$ for a k-clique C' and $f(T') = T' \setminus v$ otherwise. We can check that f maps k-cliques to k-cliques and is in fact an injection from sub-k-trees containing v to sub-k-trees not containing v. Indeed, because v is a k-leaf, $C' \cup \{u\}$ is not a $(k+1)$-clique and thus not a sub-k-tree. Hence there does not exist a $(k+1)$-clique C'' for which $C'' \setminus \{v\} = (C' \setminus \{v\}) \cup \{u\}$.

This implies that every k-leaf belongs to at most half of the sub-k-trees in T. Remembering that there are at least $\frac{n-k+3}{2}$ k-leaves, the global mean order of T is then

$$\mu(T) = \sum_{v \text{ non-}k\text{-leaf}} p(v) + \sum_{v \ k\text{-leaf}} p(v) \leqslant n - \frac{1}{2} \cdot \frac{n-k+3}{2} = \frac{3n+k-3}{4}.$$

For sharpness, let $n = 2s+3-k$ for an integer s. We construct T by first constructing a caterpillar T' which consists of a path-type k-tree P_s^{k+1} on vertices $v_1, v_2, \ldots, v_s$, for which every $k+1$ consecutive vertices form a clique, and adding a k-leaf connected to every k consecutive vertices. To obtain T, we extend T' by adding two k-leaves, which are connected to $\{v_1, \ldots, v_k\}$ and $\{v_{s-k+1}, \ldots, v_s\}$ respectively. Note that T has a "stem" of s vertices and the number of k-leaves is $\ell + 2 = (s-k+1) + 2 = s-k+3$.

This k-tree T has the property that none of the k-cliques has degree 2: if a k-clique contains one of the k-leaves, its degree is 1, since every k-leaf only belongs to one $(k+1)$-clique. Otherwise, a k-clique consists entirely of vertices of the stem. If they are consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+k-1}$, the degree of the clique is 3: the clique can be extended to a $(k+1)$-clique by adding v_{i-1}, v_{i+k}, or the additional k-leaf that is adjacent to it. If the k-clique consists of stem vertices that are non-consecutive, then it must be a subset of a clique of $k+1$ consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+k}$ (otherwise its first and last vertex would not be adjacent). This only extends to one $(k+1)$-clique (by adding the missing vertex), so the degree is 1.

For every $1 \leqslant i \leqslant s-k$, there are i sub-k-trees of the stem each of order $s+1-i$. Each of these can be extended by adding any subset of the $\ell+1-i$ neighboring k-leaves, or even one or two more if some of the end-vertices of the stem are involved.
Now we can compute that

$$\begin{aligned} N(T) &= (k(n-k)+1-\ell) + 2^{\ell+2} + 2\sum_{i=2}^{\ell} 2^i + \sum_{i=3}^{s-k+1} (i-2)\cdot 2^{\ell+1-i} \\ &\sim 9 \cdot 2^{\ell}. \end{aligned}$$

The first expression $k(n-k)+1-\ell$ counts the number of simplicial k-cliques different from the ones consisting of k consecutive vertices in the stem. $2^{\ell+2}$ is the number of sub-k-trees containing the whole stem. The third term counts the number of sub-k-trees containing v_1 or v_s but not both and at least k vertices of the stem. The last summation counts the sub-k-trees containing at least k vertices of the stem and none of v_1 and v_s. We can also compute $R(T)$ by summing the total size of the respective sub-k-trees.

$$\begin{aligned} R(T) &= (k(n-k)+1-\ell)k + 2^{\ell+2}\left(s+\frac{\ell+2}{2}\right) + 2\sum_{i=2}^{\ell} 2^i \left(k+i-2+\frac{i}{2}\right) \\ &\quad + \sum_{i=3}^{s-k+1} (i-2)\cdot 2^{\ell+1-i}\left(s+1-i+\frac{\ell+1-i}{2}\right) \\ &\sim 2^{\ell+2}\left(s+\frac{\ell+2}{2}\right) + 2^{\ell+2}(k-2) + 3\cdot 2^{\ell+1}(\ell-1) + 2^{\ell}\left(s+1+\frac{\ell+1}{2}\right) - 15\cdot 2^{\ell-1} \\ &= \left(5s+\frac{17}{2}\ell+4k-16\right)2^{\ell} \\ &= \left(\frac{27}{4}n+\frac{9}{4}k-\frac{111}{4}\right)2^{\ell}. \end{aligned}$$

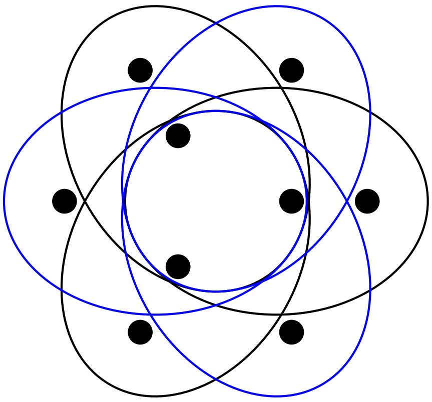
Finally, we conclude that $\mu(T) = \frac{R(T)}{N(T)} \sim \frac{3}{4}n + \frac{1}{4}k - \frac{37}{12}$, which is only $\frac{7}{3}$ away from the upper bound. These computations have also been verified in https://github.com/StijnCambie/AvSubOrder_ktree/blob/main/M_comb_ktree.mw. $\square$

4 The maximum local mean order

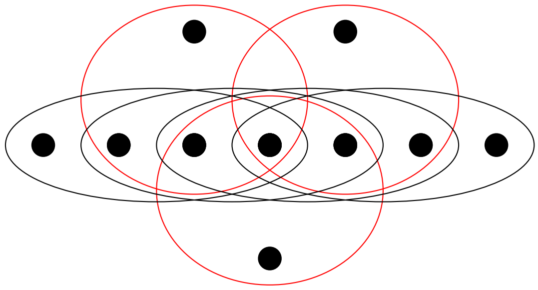
In this section, we prove

Theorem 3. *Suppose that $k \geqslant 2$. For a k-tree T of order $n \neq k+2$, if a k-clique C maximizes $\mu(T;C)$, then C must be a k-clique of degree 1. For $n = k+2$, every k-clique C satisfies $\mu(T;C) = k+1$.*

We consider the k-clique in a k-tree T for which the local mean order is greatest. Our aim is to show that its degree cannot be too large, specifically at most 2. To do so,



(a) A k-star



(b) A k-caterpillar.

Figure 5: Sketch of k-trees with mean sub-k-tree order roughly $\frac{n}{2}$ and $\frac{3n}{4}$ for $k = 3$.

we prove that if a k-clique C has degree at least 2, then there is a neighboring k-clique whose local mean order is not smaller, with strict inequality when C has degree at least 3. From this, it already follows that any k-clique attaining the maximum has degree at most 2. Moreover, in the case where the maximum is attained at a k-clique of degree 2, we will show that one can always move along maximizing k-cliques without doubling back, which means that one always ends up in at least one k-clique with degree 1 in which the maximum is attained.

The proof requires some technical inequalities, which we prove first. We start with a generalization of [21, Lemma 2.1] to the k-tree case as follows.

Lemma 10. *For a sub-k-tree T and a k-clique C, we have*

$$R(T; C) \leqslant \frac{N(T; C)^2 + (2k-1)N(T; C)}{2}.$$

Equality holds if and only if T is a path-type k-tree and C is a simplicial k-clique.

Proof. The statement is clearly true when $T = C$, so assume $|T| > k$. Observe that every sub-k-tree T' of T containing C has a vertex v (not belonging to C) which is simplicial within T' (consider a perfect elimination order where C is taken as the base k-clique). This implies that $T' \backslash v$ is a sub-k-tree containing C as well.

If there exists a sub-k-tree containing C of order $\ell > k$, then the above implies that there must also exist a sub-k-tree of order $\ell - 1$ containing C. Thus, if we list the sub-k-trees in $S(T; C)$ from smallest to largest order, we see that

$$R(T; C) \leqslant k + (k+1) + \cdots + (k + N(T; C) - 1) = \frac{1}{2}(N(T; C)^2 + (2k-1)N(T; C))$$

as desired.

The equality case is clear, since every sub-k-tree $T' \neq C$ of T must have exactly one k-leaf not belonging to C, and equality is attained when T is a path-type k-tree. $\square$

Using the previous lemma, we can now bound the local mean order in terms of the number of sub-k-trees.

Lemma 11. *For a k-tree T and one of its k-cliques C, we have*

$$k + \frac{\log_2 N(T;C)}{2} \leqslant \mu(T;C) \leqslant \frac{N(T;C) + (2k-1)}{2}.$$

The minimum occurs if and only if T is a k-star and C its base k-clique. The maximum is attained exactly when T is a path-type k-tree and C is simplicial.

Proof. The upper bound follows immediately from Lemma 10 by dividing both sides of the inequality by $N(T;C)$. The equality cases are the same.
By [16, Thm. 12] we have $\mu(T;C) \geqslant \frac{|T|+k}{2}$. Since a sub-$k$-tree is determined by its vertices, we also have

$$N(T;C) \leqslant 2^{|T|-k}. \tag{1}$$

Combining these two inequalities, we get $\mu(T;C) \geqslant \frac{\log_2 N(T;C)}{2} + k$.
To attain the lower bound, equality must hold for (1). This is the case if and only if T is a k-star and C its base k-clique. Indeed, for a k-star of order n with base clique C, we have $N(T;C) = 2^{n-k}$ and $\mu(T;C) = \frac{n+k}{2}$. In the other direction, every vertex together with C needs to form a k-tree and thus a $(k+1)$-clique, which is possible only for a k-star. □

We will also make use of the following elementary inequality, which can be considered as the opposite statement of the inequality between the arithmetic mean and geometric mean (AM-GM).

Lemma 12. *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}_{\geqslant 1}$, and let $P = \prod_i x_i$. Then $\sum_i x_i \leqslant P + (n-1)$. Furthermore, equality is attained if and only if $x_i = P$ for some i and $x_j = 1$ for all $j \neq i$.*

Proof. This can be proven in multiple ways. The most elementary way is to observe that if $x_i, x_j > 1$, then $x_i x_j + 1 > x_i + x_j$ since $(x_i - 1)(x_j - 1) > 0$. Repeating this with pairs of elements which are strictly larger than 1 gives the result. Alternatively, one could consider the variables $\alpha_i = \log(x_i) \geqslant 0$. Since their sum is the fixed constant $\log P$ and the exponential function is convex, as a corollary of Karamata's inequality $\sum_i \exp(\alpha_i)$ is maximized when all except one are equal to 0. □

We are now ready to prove Theorem 3. Recall that T can be decomposed into C and k-trees $T_{1,1}, \ldots, T_{d,k}$ rooted at $C_{1,1}, \ldots, C_{d,k}$ respectively that are pairwise disjoint except for the vertices of the cliques $C_{i,j}$. Let B_i denote the $(k+1)$-clique that contains C as well as $C_{i,1}, \ldots, C_{i,k}$, and let v_i be the vertex of $B_i \setminus C$. Finally, set $N_{i,j} = N(T_{i,j}; C_{i,j})$ and $\mu^\bullet_{i,j} = \mu^\bullet(T_{i,j}; C_{i,j})$.

Proof of Theorem 3. We first observe that

$$N(T;C) = \prod_{i=1}^{d} \left(1 + \prod_{j=1}^{k} N_{i,j}\right). \tag{2}$$

This is because a sub-k-tree S of T containing C is specified as follows: given $1 \leqslant i \leqslant d$, choose the sub-$k$-tree intersection of S with $T_{i,1}, \ldots, T_{i,k}$. There are $\prod_{j=1}^{k} N_{i,j}$ ways to do this if $v_i \in S$ and one if $v_i \notin S$. We do this independently for each i, resulting in the product above.

Next, we would like to express the local mean order at C in terms of the quantities $N_{i,j}$ and $\mu^{\bullet}_{i,j}$: it is given by

$$\mu^{\bullet}(T; C) = \sum_{i=1}^{d} \frac{1 + \sum_{j=1}^{k} \mu^{\bullet}_{i,j}}{1 + \prod_{j=1}^{k} N_{i,j}^{-1}}. \tag{3}$$

To see this, note that we can interpret $\mu(T; C)$ as the expected size of a random sub-k-tree chosen from $S(T; C)$, which can then be written as the sum of the expected sizes of the intersection with each component $T_{i,j}$ in the decomposition. We have that $\frac{1}{1+\prod_{j=1}^{k} N_{i,j}^{-1}} = \frac{\prod_{j=1}^{k} N_{i,j}}{1+\prod_{j=1}^{k} N_{i,j}}$ is the probability that a randomly chosen sub-k-tree of T that contains C also contains v_i (by the same reasoning that gave us (2)). Once v_i is included, it adds 1 to the number of vertices, and an average total of $\sum_{j=1}^{k} \mu^{\bullet}_{i,j}$ is added from the extensions in $T_{i,1}, \ldots, T_{i,k}$.

Without loss of generality, we can assume that $N_{1,1} = \min_{i,j} N_{i,j}$. We want to compare $\mu^{\bullet}(T; C)$ to $\mu^{\bullet}(T; C_{1,1})$ and prove that $\mu^{\bullet}(T; C_{1,1}) \geqslant \mu^{\bullet}(T; C)$ provided that $d \geqslant 2$, with strict inequality if $d > 2$. Observe that this will be enough to prove our claim: no clique with degree greater than 2 can attain the maximum local mean order, and starting from any clique, we may repeatedly apply the inequality above to obtain a sequence of neighboring cliques whose local mean orders are weakly increasing and the last of which is a degree-1 k-clique. (More precisely, for the first step replacing C with $C' := C_{1,1}$ and considering the decomposition $\{C'_{i,j}\}$ and $\{T'_{i,j}\}$ with respect to C', the clique adjacent to C' with equal or larger $\mu^{\bullet}$ cannot be the original clique C as C will not correspond to $\min_{i,j} N'_{i,j}$. So in repeatedly applying the inequality, we will obtain a sequence of distinct cliques which must terminate but can only terminate once we have reached a clique of degree 1.)

Let us first express $\mu^{\bullet}(T; C_{1,1})$ in terms of the $N_{i,j}$ and $\mu^{\bullet}_{i,j}$ as well. First, we have

$$N(T; C_{1,1}) = N_{1,1} + \prod_{j=1}^{k} N_{1,j} \prod_{i=2}^{d} \left(1 + \prod_{j=1}^{k} N_{i,j}\right).$$

The reasoning is similar to (2): there are $N_{1,1}$ sub-k-trees that contain $C_{1,1}$, but not the full $(k+1)$-clique B_1, and the remaining product counts sub-k-trees containing B_1. We also have

$$\mu^{\bullet}(T; C_{1,1}) = \frac{N_{1,1}\mu^{\bullet}_{1,1}}{N(T; C_{1,1})} + \left(1 - \frac{N_{1,1}}{N(T; C_{1,1})}\right)\left(1 + \sum_{j=1}^{k} \mu^{\bullet}_{1,j} + \sum_{i=2}^{d} \frac{1 + \sum_{j=1}^{k} \mu^{\bullet}_{i,j}}{1 + \prod_{j=1}^{k} N_{i,j}^{-1}}\right),$$

using the fact that $\frac{N_{1,1}}{N(T; C_{1,1})}$ is the probability that a random sub-k-tree containing $C_{1,1}$ does *not* contain B_1. Let us now take the difference $\mu^{\bullet}(T; C_{1,1}) - \mu^{\bullet}(T; C)$: we have, after some manipulations,

$$\begin{aligned}\mu^\bullet(T;C_{1,1})-\mu^\bullet(T;C) &= \frac{N_{1,1}}{N(T;C_{1,1})}\left(\mu^\bullet_{1,1}-\mu^\bullet(T;C)\right)\\ &\quad+\left(1-\frac{N_{1,1}}{N(T;C_{1,1})}\right)\left(1+\sum_{j=1}^k \mu^\bullet_{1,j}-\frac{1+\sum_{j=1}^k \mu^\bullet_{1,j}}{1+\prod_{j=1}^k N_{i,j}^{-1}}\right)\\ &= \frac{N_{1,1}}{N(T;C_{1,1})}\left(\mu^\bullet_{1,1}-\mu^\bullet(T;C)\right)\\ &\quad+\left(1-\frac{N_{1,1}}{N(T;C_{1,1})}\right)\left(\frac{1+\sum_{j=1}^k \mu^\bullet_{1,j}}{1+\prod_{j=1}^k N_{1,j}}\right).\end{aligned}$$

Now let $F := \frac{1+\sum_{j=1}^k \mu^\bullet_{1,j}}{1+\prod_{j=1}^k N_{1,j}}$. We want to show that $\mu^\bullet(T;C_{1,1})-\mu^\bullet(T;C)\geqslant 0$, i.e., that

$$\frac{N_{1,1}}{N(T;C_{1,1})}\mu^\bullet_{1,1}+F\geqslant \frac{N_{1,1}}{N(T;C_{1,1})}\left(\mu^\bullet(T;C)+F\right).$$

Equivalently,

$$\mu^\bullet_{1,1}+\frac{N(T;C_{1,1})}{N_{1,1}}\cdot F\geqslant \mu^\bullet(T;C)+F. \tag{4}$$

Using the previous computations, we know that

$$\frac{N(T;C_{1,1})}{N_{1,1}}=1+\prod_{j=2}^k N_{1,j}\prod_{i=2}^d\left(1+\prod_{j=1}^k N_{i,j}\right).$$

So the left-hand side of (4) is equal to

$$\mu^\bullet_{1,1}+F+\frac{\prod_{j=2}^k N_{1,j}}{1+\prod_{j=1}^k N_{1,j}}\left(1+\sum_{j=1}^k \mu^\bullet_{1,j}\right)\prod_{i=2}^d\left(1+\prod_{j=1}^k N_{i,j}\right).$$

We can subtract F from both sides and use (3) to replace $\mu^\bullet(T;C)$. Taking into account that $\mu^\bullet_{1,1}\geqslant 0$, it is sufficient to prove

$$\frac{\prod_{j=2}^k N_{1,j}}{1+\prod_{j=1}^k N_{1,j}}\left(1+\sum_{j=1}^k \mu^\bullet_{1,j}\right)\prod_{i=2}^d\left(1+\prod_{j=1}^k N_{i,j}\right)\geqslant \sum_{i=1}^d \frac{1+\sum_{j=1}^k \mu^\bullet_{i,j}}{1+\prod_{j=1}^k N_{i,j}^{-1}}. \tag{5}$$

We first prove (5) for $d=2$, in which case it can be rewritten as

$$\begin{aligned}&\frac{\prod_{j=1}^k N_{1,j}}{N_{1,1}(1+\prod_{j=1}^k N_{1,j})}\left(1+\sum_{j=1}^k \mu^\bullet_{1,j}\right)\left(1-N_{1,1}+\prod_{j=1}^k N_{2,j}\right)\\ &\qquad\geqslant \frac{\prod_{j=1}^k N_{2,j}}{1+\prod_{j=1}^k N_{2,j}}\left(1+\sum_{j=1}^k \mu^\bullet_{2,j}\right).\end{aligned}$$

Write $\prod_{j=1}^k N_{1,j} = N_{1,1}^{k-1} y$ and $\prod_{j=1}^k N_{2,j} = N_{1,1}^{k-1} z$ where $y, z \geqslant N_{1,1}$. By Lemma 11, we have $\mu^\bullet_{1,j} \geqslant \frac{1}{2}\log_2 N_{1,j}$ and $\mu^\bullet_{2,j} \leqslant \frac{1}{2}(N_{2,j}-1)$. Applying Lemma 12 to the numbers $\frac{N_{2,j}}{N_{1,1}}$, $1 \leqslant j \leqslant k$, gives us that $\sum_{j=1}^k N_{2,j} \leqslant (k-1)N_{1,1} + z$. Hence we find that it suffices to prove that

$$\frac{y\left(1 - N_{1,1} + N_{1,1}^{k-1} z\right)}{N_{1,1}(1 + N_{1,1}^{k-1} y)} \left(1 + \frac{(k-1)\log_2 N_{1,1} + \log_2 y}{2}\right) \geqslant \frac{z}{1 + N_{1,1}^{k-1} z}\left(1 + \frac{(k-1)N_{1,1} + z - k}{2}\right).$$

We note that the left-hand side is strictly increasing in y and the right-hand side is independent of y, which implies that it is sufficient to prove the equality when $y = N_{1,1}$. That is, we want to prove that for every $z \geqslant y \geqslant 1$

$$\frac{1 - y + y^{k-1} z}{1 + y^k}\left(1 + \frac{k}{2}\log_2 y\right) \geqslant \frac{z}{1 + y^{k-1} z}\left(1 + \frac{(k-1)y + z - k}{2}\right). \tag{6}$$

If $y = N_{1,1} = 1$, (6) reduces to $\frac{z}{2} \geqslant \frac{z}{2}$.
If $y = N_{1,1} \geqslant 2$, $z \geqslant y$ and $k \geqslant 2$ imply that $2 + (k-1)y + z - k \leqslant kz$. Together with $\log_2 y \geqslant 1$, we conclude that it is sufficient to prove that

$$\frac{1 - y + y^{k-1} z}{1 + y^k}\frac{k+2}{2} \geqslant \frac{z}{1 + y^{k-1} z}\frac{kz}{2}$$

which is equivalent to

$$(1 - y + y^{k-1} z)(k+2)(1 + y^{k-1} z) \geqslant (1 + y^k) k z^2.$$

For $k = 2$, the difference between the two sides is an increasing function in y, and for $y = 2$ it reduces to $3z^2 - 2 \geqslant 0$, which holds.
For $k \geqslant 3$, the inequality is immediate, using that $y^{2(k-1)} z^2 \geqslant (1.5y^k + 1) z^2 \geqslant y^k z + y^k z^2 + z^2$ (remember that $z \geqslant y \geqslant 2$).
Once (5) has been verified for $d = 2$, we can apply induction to prove it for $d \geqslant 3$. Let $C = \frac{\prod_{j=2}^k N_{1,j}}{1 + \prod_{j=1}^k N_{1,j}}\left(1 + \sum_{j=1}^k \mu^\bullet_{1,j}\right)$, $g_i = 1 + \prod_{j=1}^k N_{i,j}$ and $f_i = \frac{1 + \sum_{j=1}^k \mu^\bullet_{i,j}}{1 + \prod_{j=1}^k N_{i,j}^{-1}}$. We can then rewrite (5) as

$$C \prod_{i=2}^d g_i \geqslant \sum_{i=1}^d f_i$$

By the induction hypothesis, we have

$$\frac{C}{g_m}\prod_{i=2}^d g_i \geqslant \left(\sum_{i=1}^d f_i\right) - f_m$$

for every $2 \leqslant m \leqslant d$. Summing over m, we obtain that

$$C \prod_{i=2}^{d} g_i \left(\sum_{m=2}^{d} \frac{1}{g_m} \right) \geqslant f_1 + (d-2) \sum_{i=1}^{d} f_i.$$

Since we have $g_m \geqslant 2$ for every m, the conclusion now follows as

$$\sum_{m=2}^{d} \frac{1}{g_m} \leqslant \frac{(d-1)}{2} \leqslant d-2,$$

and since $f_1 > 0$, inequality (5) is even strict in the case that $d > 2$.
We conclude that if C has degree $d \geqslant 2$, then $\mu^\bullet(T; C) \leqslant \mu^\bullet(T; C_{1,1})$. Equality can only be attained when $d = 2$. Looking back over the proof of (5), we also see that for equality to hold, all $N_{i,j}$ except for $N_{2,2}$ (up to renaming) must be equal to 1, and due to Lemma 11, $T_{2,2}$ has to be a path-type k-tree.
For $n \geqslant k+3$, we compute that in a path-type k-tree T the maximum among the degree-1 k-cliques is attained by a central one, which implies that no degree-2 k-cliques can be extremal. Let B_1 be the unique $(k+1)$-clique containing C, and assume that $T \setminus B_1$ contains components of size a and b. Thus $a + b = n - (k+1)$. Then $\mu(T; C) = \frac{k+(a+1)(b+1)\frac{n+k}{2}}{(a+1)(b+1)+1} = \frac{n+k}{2} - \frac{n-k}{2((a+1)(b+1)+1)}$, and this is maximized if and only if $|a-b| \leqslant 1$. The latter can also be derived from considering the 1-characteristic tree. This is illustrated in Figure 6 below. We emphasize that the maximizing k-clique of degree 1 is not simplicial. When $n \leqslant k+2$, every k-clique has the same local mean sub-k-tree order, and so the (unique) degree-2 k-clique when $n = k+2$ is the only case where equality can occur at a k-clique with degree 2. This concludes the proof of Theorem 3. $\square$

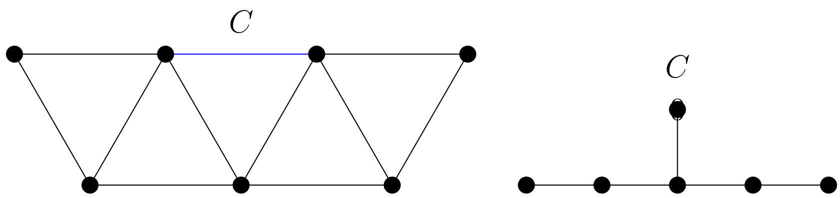


Figure 6: 2-path for which the maximum local mean is attained in a non-simplicial clique C, with its 1-characteristic tree T'_C.

5 Bounding local mean order by global mean order

In this section, we prove

Theorem 4. *The local mean order of the sub-k-trees containing a fixed k-clique C is less than twice the global mean order of all sub-k-trees of T.*

As before, given a k-tree T and a k-clique C in T, we utilize the decomposition of T into C and k-trees $T_{1,1}, \ldots, T_{d,k}$ rooted at $C_{1,1}, \ldots, C_{d,k}$. Set $N_{i,j} = N(T_{i,j}; C_{i,j})$, $\overline{N}_{i,j} = \overline{N}(T_{i,j}; C_{i,j})$, $\mu_{i,j} = \mu(T_{i,j}; C_{i,j})$, $\overline{\mu}_{i,j} = \overline{\mu}(T_{i,j}; C_{i,j})$, $\mu^\bullet_{i,j} = \mu^\bullet(T_{i,j}; C_{i,j})$ and $R_{i,j} = R(T_{i,j}; C_{i,j})$, $\overline{R}_{i,j} = \overline{R}(T_{i,j}; C_{i,j})$.
Since a sub-k-tree not containing C needs to be a sub-k-tree of some k-tree $T_{i,j}$, we have

$$\overline{N}(T; C) = \sum_{i=1}^{d} \sum_{j=1}^{k} (N_{i,j} + \overline{N}_{i,j})$$

and

$$\overline{R}(T; C) = \sum_{i=1}^{d} \sum_{j=1}^{k} (R_{i,j} + \overline{R}_{i,j}).$$

Following the proof for trees, we show

Lemma 13. *For any k-tree T and k-clique $C \in T$,*

$$R(T; C) > \overline{N}(T; C).$$

Proof. Assume to the contrary that there exists a minimum counterexample T. Since the statement is true when $T = C$, we have $|T| > k$ and we can consider the decomposition as before.
Note that if $N_{i,j} = 1$, we have that $\overline{N}_{i,j} = 0$, and otherwise we have $\overline{N}_{i,j} \leqslant R_{i,j} = \mu_{i,j} N_{i,j} = (k + \mu^\bullet_{i,j}) N_{i,j}$.
We can rewrite $R(T; C) - \overline{N}(T; C)$ as

$$N(T; C) \left(k + \mu^\bullet(T; C)\right) - \overline{N}(T; C).$$

Thus it suffices to show that

$$N(T; C)(k + \mu^\bullet(T; C)) - \sum_{i=1}^{d} \sum_{j=1}^{k} N_{i,j} - \sum_{i=1}^{d} \sum_{\substack{j=1 \\ N_{i,j} > 1}}^{k} (k + \mu^\bullet_{i,j}) N_{i,j}$$

is nonnegative. Expanding using (2) and (3) gives an expression that is increasing in $\mu^\bullet_{i,j}$ for each i, j, and so it suffices to prove that this expression is nonnegative when each of these is 0.
Let f be a function on the positive integers defined by $f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 1 + kx & \text{if } x > 1. \end{cases}$
We now want to prove that

$$\left(k + \sum_{i=1}^{d} \frac{1}{1 + \prod_{j=1}^{k} N_{i,j}^{-1}} \right) N(T; C) \geqslant \sum_{i=1}^{d} \sum_{j=1}^{k} f(N_{i,j}). \tag{7}$$

When $d = 1$, this becomes $k + (k+1)\prod_{j=1}^{k} N_{1,j} \geqslant \sum_{j=1}^{k} f(N_{1,j})$. When increasing a value $N_{1,j}$ which is at least equal to 2, the left-hand side increases more than the right-hand side. As such, it is sufficient to consider the case where a of the terms $N_{1,j}$ equal 2, while the other $k-a$ terms equal 1. In this case, the desired inequality holds since $k + (k+1)2^a > k + 2ak$ for every integer $0 \leqslant a \leqslant k$.
Next, we consider the case $d \geqslant 2$. In this case, when $N_{i,j}$ increases by 1, the left-hand side of (7) increases by at least $2k$ and the right-hand side by at most $2k$. When all $N_{i,j}$ are equal to 1, the conclusion follows from $\left(k + \frac{d}{2}\right) 2^d > k2^d > dk$. □

We now bound the local mean order by the global mean order.

Proof of Theorem 4. Let T be a k-tree and C a k-clique in T. We want to prove that

$$\mu(T; C) < 2\mu(T).$$

We proceed by induction on the number of vertices in T. Note first that the inequality is trivial if $|T| \leqslant 2k$: since the mean is taken over sub-k-trees, which have at least k vertices each, we have $\mu(T) \geqslant k$. On the other hand, we clearly have $\mu(T; C) \leqslant |T|$, and both inequalities hold with equality only if $|T| = k$.
We thus proceed to the induction step, and assume that $|T| > 2k$. We have two cases with respect to C.
Case 1: C is simplicial.
Let v be a k-leaf in C, let C' denote the clique adjacent to C, and let $T' = T - \{v\}$. Moreover, let $N, \overline{N}, R$, and $\overline{R}$ denote $N(T'; C')$, $\overline{N}(T; C')$, $R(T'; C')$ and $\overline{R}(T'; C')$, respectively. We have

$$\mu(T; C) = \frac{N + R + k}{N + 1}$$

and

$$\mu(T) = \frac{2R + \overline{R} + N + k^2}{2N + \overline{N} + k}.$$

We want to prove that

$$(2N + \overline{N} + k)(2\mu(T) - \mu(T; C)) > 0,$$

which is equivalent to

$$2R + 2\overline{R} + 2k^2 - 2k - \frac{(N + R + k)(\overline{N} + k - 2)}{N + 1} > 0.$$

By the induction hypothesis, we have

$$2\frac{R + \overline{R}}{N + \overline{N}} = 2\mu(T') > \mu(T'; C') = \frac{R}{N},$$

so it is sufficient to prove that

$$\frac{R(N+\overline{N})}{N}+2k^2-2k-\frac{(N+R+k)(\overline{N}+k-2)}{N+1}>0.$$

Multiplying by $\frac{N+1}{N}$, this is seen to be equivalent to

$$R-\overline{N}+2k^2-3k+2-\frac{(k-3)R+k\overline{N}-k^2}{N}+\frac{\overline{N}R}{N^2}>0.$$

This can be broken up into three terms as follows:

$$\frac{(N^2-kN+3N+\overline{N})(R-\overline{N})}{N^2}+\Big(\frac{\overline{N}}{N}-k\Big)^2+\Big((k-1)(k-2)+\frac{k^2}{N}+\frac{3\overline{N}}{N}\Big)>0.$$

Note here that the second term is trivially nonnegative, and the last term trivially positive. Since $|T'|\geqslant 2k$, we have $N>k$ (one gets at least $k+1$ sub-k-trees containing C' by successively adding vertices); hence $N^2-kN+3N+\overline{N}>0$. Thus the first term is positive by Lemma 13, completing the induction step in this case.

Case 2: C is not simplicial.

By Theorem 3, we only need to consider the case where C has degree 1. Let v be the unique common neighbor of C, and let C_i, $1\leqslant i\leqslant k$, be the other k-cliques in the $(k+1)$-clique spanned by $C\cup\{v\}$.

Let T_i, $1\leqslant i\leqslant k$, be the sub-k-trees rooted at C_i (pairwise disjoint except for the vertices of the cliques C_i). Let $N_i=N(T_i;C_i)$, $\overline{N}_i=\overline{N}(T_i;C_i)$, $R_i=R(T_i;C_i)$, $\overline{R}_i=\overline{R}(T_i;C_i)$, $\mu_i=\mu(T_i;C_i)$ and $\mu_i^\bullet=\mu^\bullet(T_i;C_i)$. We can assume without loss of generality that $N_1\geqslant N_2\geqslant\cdots\geqslant N_j>1=N_{j+1}=\cdots=N_k$, where $j\geqslant 2$ since C is not simplicial.

We can now express the local and global mean in a similar way to Case 1. Here $N(T;C)=\prod_{i=1}^k N_i+1$, and all the sub-$k$-trees counted here, except for C, contain v. We have

$$\mu(T;C)=\frac{\prod_{i=1}^k N_i(1+\sum_{i=1}^k\mu_i^\bullet)}{\prod_{i=1}^k N_i+1}+k,$$

$$\mu(T)=\frac{\prod_{i=1}^k N_i(1+\sum_{i=1}^k\mu_i^\bullet+k)+\sum_{i=1}^k(R_i+\overline{R}_i)+k}{\prod_{i=1}^k N_i+\sum_{i=1}^k(N_i+\overline{N}_i)+1}.$$

In the remainder of this section, we will omit the bounds in products and sums if they are over the entire range from 1 to k: $\sum N_i$ and $\prod N_i$ mean $\sum_{i=1}^k N_i$ and $\prod_{i=1}^k N_i$ respectively. Then $\big(\prod N_i+\sum(N_i+\overline{N}_i)+1\big)\,(2\mu(T)-\mu(T;C))$ equals

$$(\prod N_i)(1+\sum\mu_i^\bullet+k)+k+2\sum(R_i+\overline{R}_i)-k\sum(N_i+\overline{N}_i)-\frac{(\prod N_i)(1+\sum\mu_i^\bullet)\sum(N_i+\overline{N}_i)}{\prod N_i+1}. \tag{8}$$

We want to show that this expression is positive. By induction, we know that $2(R_i+\overline{R}_i)>(k+\mu_i^\bullet)(N_i+\overline{N}_i)$, thus $2(R_i+\overline{R}_i)-k(N_i+\overline{N}_i)>\mu_i^\bullet(N_i+\overline{N}_i)$. It follows that (8) is greater than

$$(\prod N_i)(1+\sum\mu_i^\bullet+k)+k+\sum\mu_i^\bullet(N_i+\overline{N}_i)-\frac{(\prod N_i)(1+\sum\mu_i^\bullet)\sum(N_i+\overline{N}_i)}{\prod N_i+1}.$$

Multiplying by $\frac{\prod N_i+1}{\prod N_i}$ and observing that this factor is greater than 1, we find that (8) is indeed positive if we can prove that

$$(\prod N_i+1)(1+\sum\mu_i^\bullet+k)+k+\sum\mu_i^\bullet(N_i+\overline{N}_i)\geqslant(1+\sum\mu_i^\bullet)\sum(N_i+\overline{N}_i). \quad (9)$$

In particular, a potential counterexample would have to satisfy

$$\frac{(1+\sum\mu_i^\bullet)\sum(N_i+\overline{N}_i)}{\prod N_i+1}>1+\sum\mu_i^\bullet+k. \quad (10)$$

To simplify proving eq. (9), we first note that it is sufficient to consider the case where $k=j$. Once $N_i,\mu_i^\bullet$ and $\overline{N}_i$ are fixed for $1\leqslant i\leqslant j$, the terms that are dependent on k are $(\prod_{i\leqslant j}N_i+1)k+k$ on the left, and $(1+\sum_{i\leqslant j}\mu_i^\bullet)(k-j)$ on the right. The latter since if $N_i=1$, then T_i only consists of C_i, and thus $\overline{N}_i=\mu_i^\bullet=0$. Now since $\prod_{i\leqslant j}N_i+1\geqslant 1+\sum_{i\leqslant j}N_i>1+\sum_{i\leqslant j}\mu_i^\bullet$, the increase of the left side is larger than the increase on the right side. Here $\prod_{i\leqslant j}N_i\geqslant\sum_{i\leqslant j}N_i$ is true since the product of $j\geqslant 2$ numbers, each greater than or equal to 2, is at least equal the sum of the same j numbers. Moreover, $N_i>\mu_i^\bullet$ follows from Lemma 11.
So from now on, we can assume that $j=k$ and all N_i are at least equal to 2.
By lemma 13, $\overline{N}_i\leqslant R_i=\mu_iN_i=(k+\mu_i^\bullet)N_i$. Since (9) is a linear inequality in each $\overline{N}_i$ and the coefficient on the right-hand side is always greater than the coefficient on the left-hand side, we can reduce eq. (9) to a sufficient inequality that is only dependent on k, N_i and $\mu_i^\bullet$ for $1\leqslant i\leqslant k$ by taking $\overline{N}_i=(k+\mu_i^\bullet)N_i$. This will be assumed in the following.
If now all the parameters in (9) are fixed except for one $\mu_i^\bullet$, we have a linear inequality in $\mu_i^\bullet$: the quadratic terms stemming from $\mu_i^\bullet\overline{N}_i$ are equal on both sides and cancel. As such, it is sufficient to prove the inequality for the extremal values of $\mu_i^\bullet$. Here we use the trivial inequality $\mu_i^\bullet\geqslant 0$ as well as the upper bound $\mu_i^\bullet\leqslant\frac{N_i-1}{2}$, which is taken from lemma 11. So if (9) can be proven in the case where $\overline{N}_i=(k+\mu_i^\bullet)N_i$ and $\mu_i^\bullet\in\{0,\frac{N_i-1}{2}\}$ for all i with $1\leqslant i\leqslant k$, we are done.
For $2\leqslant k\leqslant 5$, this is achieved by exhaustively checking all 2^k cases that result (using symmetry, there are actually only $k+1$ cases to consider). See the detailed verifications in `https://github.com/StijnCambie/AvSubOrder_ktree`. So for the rest of the proof, we assume that $k\geqslant 6$, and we will use the slightly weaker bound $\mu_i^\bullet\leqslant\frac{N_i}{2}$ instead of $\mu_i^\bullet\leqslant\frac{N_i-1}{2}$ for $1\leqslant i\leqslant k$ in a few cases.
We distinguish two further cases, depending on the value of $\mu_1^\bullet$. In these cases, we will use the following two inequalities.

Claim 14. *Let $k\geqslant 6$ and $N_1\geqslant N_2\geqslant\ldots\geqslant N_k\geqslant 2$. Then*

$$\prod_{i=2}^k N_i\geqslant 3\sum_{2\leqslant i\leqslant k}N_i, \quad (11)$$

$$\frac{5}{3}(\prod_{i=1}^k N_i+1)\geqslant\left(\sum_{2\leqslant i\leqslant k}N_i-2\right)\sum_{1\leqslant i\leqslant k}N_i. \quad (12)$$

Proof. The first inequality, eq. (11), is true if all the N_i are equal to 2, since $2^{k-1} > 6(k-1)$ for every $k \geqslant 6$. Increasing some N_i by 1 increases the product by at least 2^{k-2}, while the sum increases by only 3. So the inequality holds by a straightforward inductive argument. Next, we prove eq. (12). When all N_i are equal to 2, it becomes $\frac{5}{3}(2^k+1) \geqslant 4(k-2)k$. This is easily checked for $k \in \{6,7\}$, and for $k \geqslant 8$, the stronger inequality $2^k \geqslant 4k^2$ can be shown by induction.

Now observe that the difference between the left- and right-hand sides is increasing with respect to N_1, since $\prod_{i\geqslant 2} N_i > \sum_{i\geqslant 2} N_i$ by the first inequality. It is also increasing in the other N_i's; for example, we can see this is true for N_2 since

$$\frac{5}{3}\prod_{i\neq 2} N_i \geqslant 5\sum_{i\neq 2} N_i > 2\sum_{1\leqslant i\leqslant k} N_i > \left(\sum_{2\leqslant i\leqslant k} N_i - 2\right) + \sum_{1\leqslant i\leqslant k} N_i.$$

In the first step, we have applied eq. (11) but replacing N_1 with N_2. Again, we may conclude using induction. ♢

Claim 15. *Given $\mu_1^\bullet = \frac{N_1}{2}$, it is sufficient to consider the case where $\mu_i^\bullet = \frac{N_i}{2}$ for all $1 \leqslant i \leqslant k$.*

Proof. Starting from any counterexample to (9), we can iteratively change $\mu_i^\bullet$ (considered as variables) for $1 \leqslant i \leqslant k$ based on the worst case of the linearization (to 0 if the coefficient on the left-hand side is greater and to $\frac{N_i}{2}$ if the coefficient on the right-hand side is greater) to obtain further counterexamples.

To show that it suffices to consider $\mu_i^\bullet = \frac{N_i}{2}$ for every $1 \leqslant i \leqslant k$, we prove that, given $\mu_1^\bullet = \frac{N_1}{2}$, the coefficient of $\mu_2^\bullet$ on the left-hand side in eq. (9) is not greater than the coefficient on the right-hand side. This then implies that $\mu_2^\bullet = \frac{N_2}{2}$ is indeed the worst case. For $3 \leqslant i \leqslant k$, we can argue in the same fashion.

Assume for sake of contradiction that the coefficient on the left-hand side is greater. Recall that $\overline{N}_2 = (k+\mu_2^\bullet)N_2$. After subtracting $\mu_2^\bullet(N_2+\overline{N}_2)$ from both sides, the coefficient of $\mu_2^\bullet$ on the left is $\prod N_i + 1$, while on the right it is $\sum_{i\neq 2}(N_i+\overline{N}_i) + (1+\sum_{i\neq 2}\mu_i^\bullet)N_2$. Thus we must have

$$\prod N_i + 1 > \sum_{i\neq 2}(N_i+\overline{N}_i) + (1+\sum_{i\neq 2}\mu_i^\bullet)N_2 \geqslant \sum_{i\neq 2}(N_i+\overline{N}_i) + (1+\mu_1^\bullet)N_2.$$

Using that

$$(1+\mu_1^\bullet)N_2 = \left(1+\frac{N_1}{2}\right)N_2 \geqslant \left(1+\frac{N_2}{2}\right)N_2 \geqslant (1+\mu_2^\bullet)N_2 = N_2 + \overline{N}_2 - kN_2$$

(recall here that we are assuming without loss of generality that $N_1 \geqslant N_2 \geqslant \cdots \geqslant N_k$), this implies that $\prod N_i + 1 \geqslant \sum(N_i+\overline{N}_i) - kN_2$. Adding kN_2 to both sides and multiplying both sides by $1+\sum\mu_i^\bullet$ results in

$$\left(\prod N_i + 1 + kN_2\right)(1+\sum\mu_i^\bullet) \geqslant (1+\sum\mu_i^\bullet)\sum(N_i+\overline{N}_i) > (1+\sum\mu_i^\bullet+k)(\prod N_i+1).$$

In the second inequality, we applied (10) which we may do since we began with the assumption that we have a counterexample to (9). After simplification, we get that $N_2(1+\sum \mu_i^\bullet) > \prod N_i$, and thus

$$\sum_{i\neq 2} N_i > 1 + \frac{1}{2}\sum N_i \geqslant 1 + \sum \mu_i^\bullet > \prod_{i\neq 2} N_i.$$

Since $k \geqslant 6$, this is a clear contradiction to eq. (11). $\diamondsuit$

Having proven Claim 15, we are left with two cases to consider: $\mu_i^\bullet = \frac{N_i}{2}$ for all $1 \leqslant i \leqslant k$, or $\mu_1^\bullet = 0$. It is easy to conclude in the former case, except when $k = 6$ and at least 5 values N_i are equal to 2, which has to be handled separately. See https://github.com/StijnCambie/AvSubOrder_ktree/blob/main/2M-mu_j_large_case1.mw for details. The final remaining case is when $\mu_1^\bullet = 0$. We obtain two new inequalities by multiplying eq. (12) with $\frac{k+1}{2}$ and eq. (11) with $(1+\sum \mu_i^\bullet)\frac{N_1}{6}$, and use that $N_1 = \max\{N_i\}$ and $\mu_i^\bullet \leqslant \frac{N_i-1}{2}$.

$$\frac{5(k+1)}{6}(\prod N_i + 1) \geqslant \frac{k+1}{2}\left(\sum_{2\leqslant i\leqslant k} N_i - 2\right)\sum_{1\leqslant i\leqslant k} N_i \geqslant (1+\sum \mu_i^\bullet)\sum_{1\leqslant i\leqslant k}(1+k)N_i, \tag{13}$$

$$\frac{\prod N_i}{6}(1+\sum \mu_i^\bullet) \geqslant (1+\sum \mu_i^\bullet)\frac{N_1}{2}\sum_{2\leqslant i\leqslant k} N_i \geqslant (1+\sum \mu_i^\bullet)\sum_{2\leqslant i\leqslant k} N_i\mu_i^\bullet. \tag{14}$$

Summing these two inequalities together, we have that eq. (9) holds as a corollary of

$$(\prod N_i + 1)(1+\sum \mu_i^\bullet + k) \geqslant (1+\sum \mu_i^\bullet)\sum\left(N_i(1+k+\mu_i^\bullet)\right). \qquad \square$$

We remark that by considering a suitable k-broom, one can show that theorem 4 is sharp, as was also the case for trees.

6 Conclusion

This paper together with [13, Thm. 18 & 20] answers all of the open questions from [16] except for one, which was stated as rather general and open-ended:

Problem 16. For a given r-clique R, $1 \leqslant r < k$, what is the (local) mean order of all sub-k-trees containing R?

One natural version of this question is to consider the local mean sub-k-tree order over sub-k-trees that contain a fixed vertex. In this direction, we prove the following result, which can be considered as another monotonicity result related to [13, Thm. 23].

Theorem 17. *Let T be a k-tree, $k \geqslant 2$, C a k-clique of T, and v a vertex in C. Then $\mu(T; v) < \mu(T; C)$.*

Proof. The statement is trivially true if $|T| \leqslant k+1$. So assume that T with $|T| \geqslant k+2$ is a minimum counterexample to the statement. Recalling the decomposition into trees $T_{i,j}$ used earlier, note that all sub-k-trees containing v and not C are part of $T_{i,j}$ for some i, j. Without loss of generality, we can assume that $\mu(T_{1,1}; v) = \max_{i,j} \mu(T_{i,j}; v)$. It suffices to prove that $\mu(T_{1,1}; v) < \mu(T; C)$. Taking into account (3), it is sufficient to consider the case where C is a simplicial k-clique of T. Let u be the simplicial vertex of B_1. Let $T' = T \setminus \{u\}$ and $C' = B_1 \setminus \{u\}$.

Claim 18. *We have $\mu(T'; C') < \mu(T; C)$.*

Proof. Let $R = R(T'; C')$ and $N = N(T'; C')$. We now need to prove that $\mu(T; C) = \frac{2R+N+k}{2N+1} > \frac{R}{N} = \mu(T'; C')$, which is equivalent to $N + k > \frac{R}{N}$. The latter is immediate since $N \geqslant |T'| - (k-1)$ and $\frac{R}{N} = \mu(T'; C') \leqslant |T'|$. ♢

Since the sub-k-trees containing v are exactly those that contain C, or sub-k-trees of T' containing C', or k-cliques within B_1 different from C, we conclude that $\mu(T; v) < \mu(T; C)$. □

Luo and Xu [13, Ques. 35] also asked if for a given order, a k-tree attaining the largest global mean sub-k-tree order is necessarily a caterpillar-type k-tree. In contrast with the questions in [16], this question is still open for trees. We prove that the local version (proven in [3, Thm. 3]), which states that for fixed order, the maximum is attained by a broom, is also true for the generalization of k-trees. This is almost immediate by observations from [16].

Proposition 19. *If a k-tree T of order n and k-clique C of T attain the maximum possible value of $\mu(T; C)$, then T has to be a k-broom with C being one of its simplicial k-cliques.*

Proof. Let T'_C be the characteristic 1-tree of T with respect to C. Then by [16, Lem. 11] $\mu(T; C) = \mu(T'_C; C) + k - 1$. Since T'_C is a tree on $n - (k-1)$ vertices, by [3, Thm. 3], the maximum local mean subtree order is attained by a broom B. Since there is a k-broom T with C being a simplicial k-clique for which $T'_C \cong B$ with C as root, this maximum can be attained. Reversely, if $\mu(T'_C; C)$ is maximized, then T'_C is a broom where C is its root (and thus simplicial). □

As such, we conclude that the k-tree variants of many results on the average subtree order for trees are now also proven. The analogue of [10, Ques. (7.5)] can be considered as the only question among them where the answer is slightly different for k-trees: in contrast with the case of trees ($k = 1$), the maximum local mean sub-k-tree order cannot occur in a k-clique with degree 2 when $k \geqslant 2$ (with one small exception).

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References

[1] S. Cambie, G. Chen, Y. Hao, and N. Tokar. Decreasing the mean subtree order by adding k edges. *J. Graph Theory*, 105(3):357–366, 2024.

[2] S. Cambie and S. Wagner. Mean subtree order under contraction. *In preparation*, 2024.

[3] S. Cambie, S. Wagner, and H. Wang. On the maximum mean subtree order of trees. *European J. Combin.*, 97:Paper No. 103388, 19, 2021.

[4] B. Cameron and L. Mol. On the mean subtree order of graphs under edge addition. *J. Graph Theory*, 96(3):403–413, 2021.

[5] X. Chen, G. Wei, and H. Lian. Solution to a conjecture on the mean subtree order of graphs under edge addition. *J. Graph Theory*, 104(3):645–658, 2023.

[6] A. J. Chin, G. Gordon, K. J. MacPhee, and C. Vincent. Subtrees of graphs. *J. Graph Theory*, 89(4):413–438, 2018.

[7] J. Haslegrave. Extremal results on average subtree density of series-reduced trees. *J. Combin. Theory Ser. B*, 107:26–41, 2014.

[8] J. Haslegrave. The number and average size of connected sets in graphs with degree constraints. *J. Graph Theory*, 100(3):530–542, 2022.

[9] J. Haslegrave. The path minimises the average size of a connected induced subgraph. *Discrete Math.*, 345(5):Paper No. 112799, 7, 2022.

[10] R. E. Jamison. On the average number of nodes in a subtree of a tree. *J. Combin. Theory Ser. B*, 35(3):207–223, 1983.

[11] R. E. Jamison. Monotonicity of the mean order of subtrees. *J. Combin. Theory Ser. B*, 37(1):70–78, 1984.

[12] Z. Li, T. Ma, F. Dong, and X. Jin. On the maximum local mean order of sub-k-trees of a k-tree. arXiv:2309.11885, 2023.

[13] Z. Luo and K. Xu. On the Local and Global Mean Orders of Sub-k-Trees of k-Trees. *Electron. J. Combin.*, 30(1):Paper No. 1.43, 2023.

[14] Z. Luo, K. Xu, S. Wagner, and H. Wang. On the mean subtree order of trees under edge contraction. *J. Graph Theory*, 102(3):535–551, 2023.

[15] D. Ralaivaosaona and S. Wagner. On the distribution of subtree orders of a tree. *Ars Math. Contemp.*, 14(1):129–156, 2018.

[16] A. M. Stephens and O. R. Oellermann. The mean order of sub-k-trees of k-trees. *J. Graph Theory*, 88(1):61–79, 2018.

[17] A. Vince. The average size of a connected vertex set of a graph—explicit formulas and open problems. *J. Graph Theory*, 97(1):82–103, 2021.

[18] A. Vince. The average size of a connected vertex set of a k-connected graph. *Discrete Math.*, 344(10):Paper No. 112523, 4, 2021.

[19] A. Vince. A lower bound on the average size of a connected vertex set of a graph. *J. Combin. Theory Ser. B*, 152:153–170, 2022.

[20] A. Vince and H. Wang. The average order of a subtree of a tree. *J. Combin. Theory Ser. B*, 100(2):161–170, 2010.

[21] S. Wagner and H. Wang. On the local and global means of subtree orders. *J. Graph Theory*, 81(2):154–166, 2016.

[22] R. Wang. Extrema of local mean and local density in a tree. arXiv:2306.13422, 2023.