Crystal Isomorphisms and Mullineux Involution II

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Abstract

We present a new combinatorial and conjectural algorithm for computing the Mullineux involution for the symmetric group and its Hecke algebra. This algorithm is built on a conjectural property of crystal isomorphisms which reduces in fact to iterations of a very elementary procedure on sequences of integers.

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1 Introduction

Mullineux involution is an important map which has been originally defined by Mullineux [23] in the context of the modular representation theory of the symmetric group. More generally, it can be defined for the class of Hecke algebras of the symmetric group [2]. Let $n \in \mathbb{Z}_{>0}$ and $e \in \mathbb{Z}_{>1}$. Let η be a primitive e root of 1. The Hecke algebra of the symmetric group $\mathcal{H}_n(\eta)$ is defined as the associative unital $\mathbb{C}$-algebra with generators T_1, $\dots$, T_{n-1} and the following relations:

$$\begin{array}{rcll} (T_i-\eta)(T_i+1) & = & 0 & \text{for } i=1,\dots,n-1, \\ T_iT_{i+1}T_i & = & T_{i+1}T_iT_{i+1} & \text{for } i=1,\dots,n-2, \\ T_iT_j & = & T_jT_i & \text{if } |i-j|>1. \end{array}$$

It is known that the simple modules of this algebra are naturally labelled by the set of e-regular partitions $\mathrm{Reg}_e(n)$ with rank n (see §2.1 for the definition):

$$\mathrm{Irr}(\mathcal{H}_n(\eta)) = \{D^\lambda \mid \lambda \in \mathrm{Reg}_e(n)\}.$$

There is a $\mathbb{C}$-algebra automorphism $\sharp$ which can be defined on the generators of $\mathcal{H}_n(\eta)$ as follows. For all $i=1,\dots,n-1$, we have $T_i^\sharp = -\eta T_i$. This automorphism induces an involution:

$$m_e : \mathrm{Reg}_e(n) \to \mathrm{Reg}_e(n),$$

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defined as follows. For all $\lambda \in \operatorname{Reg}_e(n)$ there exists a unique $\mu \in \operatorname{Reg}_e(n)$ such that the module D^λ twisted by $\sharp$ is isomorphic to D^μ. Then we define $m_e(\lambda) := \mu$. If e is prime, this involution describes the structure of a simple $\mathbb{F}_e\mathfrak{S}_n$ -module twisted by the sign representation. If e is sufficiently large, or more generally if λ is an e-core, it is easy to see that $m_e(\lambda)$ is just the conjugate partition λ'.

The study of the Mullineux involution has a long story. A conjectural and combinatorial description of m_e (if e is prime) was first given by Mullineux [23] and proved later by Ford and Kleshchev [11]. Before this proof, Kleshchev gave a solution to the computation of the involution [19] (see also [1] and [3]). This solution may be rephrased in terms of crystal graph theory. Other algorithms were given by Xu [24, 25], or more recently by Fayers [6], and by the first author [15]. We also note that there exist different generalizations in the context of Ariki-Koike algebras [7, 17], affine Hecke algebras [22, 18], general linear groups [5] or rational Cherednik algebras [21, 10] and they are all connected with the above one. We also mention a recent conjecture by Bezrukavnikov on this involution in relation with Nabla operators and Haiman's $n!$ conjecture studied in [4].

All the above algorithms for computing the Mullineux involution have a common feature: they are recursive algorithms in n. The algorithms to compute the Mullineux image of a partition λ of rank n requires the computation of the Mullineux involution $m_e(\mu)$ for $|\mu| < n$. The aim of this paper is to present a conjectural algorithm which is recursive in e. This conjecture is in fact built on the description of the Mullineux involution by Kleshchev in terms of crystal graphs together with the concept of crystal isomorphisms described in [16]. The conjecture reduces in fact to a purely combinatorial conjecture on sequence of integers which can be stated without any mention to crystals and in a very simple way. Assuming the conjecture true, we prove a theorem (see Theorem 18) which shows that it is possible to compute m_e from the datum of m_{2e}. As m_e corresponds to the conjugation of partitions if e is sufficiently large, the algorithm follows.

The paper is organized as follows. We first recall in Section 2 several elementary combinatorial notions on partitions and crystals. This section ends with a presentation of Kleshchev's solution to the Mullineux problem. The third section explains the notion of crystal isomorphism. We next write down the conjectural combinatorial property previously evoked, Conjecture 11, which can be rephrased in the context of crystal isomorphisms. The last section presents several new results around this notion and states the conjectural algorithm for computing the Mullineux involution.

2 Mullineux involution for Hecke algebras

We first start with the definition of several elementary notions. Then we present the Kleshchev solution to the computation of the Mullineux involution.

2.1 Partitions and Young diagrams

A *partition* is a non increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of nonnegative integers. The *rank* of the partition is by definition the number $|\lambda| = \sum_{1 \leqslant i \leqslant m} \lambda_i$. We say that λ is a partition of n, where $n = |\lambda|$. The unique partition of 0 is the *empty partition* $\emptyset$. We denote by $\Pi(n)$ the set of partitions of n. For $e \in \mathbb{Z}_{>1}$, we say that λ is an *e-regular partition* if no nonzero part of λ can be repeated e or more times. The set of e-regular partitions of rank n is denoted by $\mathrm{Reg}_e(n)$. Given a partition $\lambda \in \Pi(n)$, its *Young diagram* $[\lambda]$ is the set:

$$[\lambda] = \left\{(a, b) \mid 1 \leqslant a \leqslant m,\ 1 \leqslant b \leqslant \lambda_a\right\} \subset \mathbb{N} \times \mathbb{N}.$$

The elements of this set are called the *nodes* of λ. The *e-residue* (or more simply, residue) of a node $\gamma \in [\lambda]$ is by definition $\mathrm{res}(\gamma) = b - a + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$. For $j \in \mathbb{Z}/e\mathbb{Z}$, we say that γ is a *j-node* if $\mathrm{res}(\gamma) = j$. In addition, γ is called a *removable j-node* for λ if the set $[\lambda] \setminus \{\gamma\}$ is the Young diagram of some partition μ. In this case, we also say that γ is an *addable j-node* for μ.

Let $\gamma = (a, b)$ and $\gamma' = (a', b')$ be two addable or removable j-nodes of the same partition λ. Then we write $\gamma > \gamma'$ if $a < a'$. Let $w_j(\lambda)$ be the word obtained by reading all the addable and removable j-nodes in increasing order and by encoding each addable j-node with the letter A and each removable j-node with the letter R. Then deleting as many subwords RA in this word as possible, we obtain a new word $\widetilde{w}_j(\lambda) = A \cdots AR \cdots R$. The node corresponding to the rightmost A (if it exists) is called the *good addable j-node* and the node corresponding to the leftmost R (if it exists) is called the *good removable j-node*.

2.2 Level 1 Fock space

Let $\mathcal{F}$ be the $\mathbb{C}$-vector space with basis given by all the partitions. It is called the (level 1) *Fock space*. There is an action of $\mathcal{U}(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}$ which makes $\mathcal{F}$ into an integrable module of level 1. For $i \in \mathbb{Z}$, the Kashiwara operators $\widetilde{e}_{i+e\mathbb{Z},e}$ and $\widetilde{f}_{i+e\mathbb{Z},e}$ are then defined as follows.

- If λ has no good addable i-node then $\widetilde{f}_{i+e\mathbb{Z},e} \cdot \lambda = 0$.
- if λ has a good addable i-node γ then $\widetilde{f}_{i+e\mathbb{Z},e} \cdot \lambda = \mu$ where $[\mu] = [\lambda] \sqcup \{\gamma\}$.
- If λ has no good removable i-node then $\widetilde{e}_{i+e\mathbb{Z},e} \cdot \lambda = 0$.
- if λ has a good removable i-node γ then $\widetilde{e}_{i+e\mathbb{Z},e} \cdot \lambda = \mu$ where $[\mu] = [\lambda] \setminus \{\gamma\}$.

Using these operators one can construct the $\widehat{\mathfrak{sl}}_e$*-crystal graph* of $\mathcal{F}$, which is the graph with

- vertices: all the partitions λ of $n \in \mathbb{N}$,
- arrows: there is an arrow from λ to μ colored by $i \in \mathbb{Z}/e\mathbb{Z}$ if and only if $\widetilde{f}_{i+e\mathbb{Z},e} \cdot \lambda = \mu$, or equivalently if and only if $\lambda = \widetilde{e}_{i+e\mathbb{Z},e} \cdot \mu$.

Note that the definition makes sense for $e = \infty$. The corresponding graph, thus a $\mathfrak{sl}_\infty$-crystal graph, coincides with the Young graph, which describes the branching graph of the complex irreducible representations of symmetric groups.

2.3 Mullineux involution

We can first give an interpretation of the set of e-regular partitions using Kashiwara operators. The following result can be found for example in [20, §2.2].

Proposition 1. *A partition λ is an e-regular partition of n if and only if there exists $(i_1, \dots, i_n) \in \mathbb{Z}^n$ such that:*

$$\widetilde{f}_{i_1+e\mathbb{Z},e} \cdots \widetilde{f}_{i_n+e\mathbb{Z},e} \cdot \emptyset = \lambda.$$

In other words, the vertices in the connected component of the $\widehat{\mathfrak{sl}}_e$-crystal graph containing the empty partition are exactly the e-regular partitions. We thus have a subgraph of this crystal graph with vertices all these e-regular partitions.

Recall the definition of the Mullineux involution given in the introduction. The following result allows us to compute it in a purely combinatorial way thanks to the above results.

Theorem 2 (Kleshchev [19])**.** *Let λ be an e-regular partition and $(i_1, \dots, i_n) \in \mathbb{Z}^n$ such that:*

$$\widetilde{f}_{i_1+e\mathbb{Z},e} \cdots \widetilde{f}_{i_n+e\mathbb{Z},e} . \emptyset = \lambda.$$

Then, there exists an e-regular partition μ such that:

$$\widetilde{f}_{-i_1+e\mathbb{Z}e} \cdots \widetilde{f}_{-i_n+e\mathbb{Z},e} . \emptyset = \mu.$$

Moreover, we have $m_e(\lambda) = \mu$ where m_e is the Mullineux involution defined in the introduction.

If λ is a partition, every node of its Young diagram has an associated hook, defined as the set of nodes directly below or to its right including itself (using English notation for the Young diagram). A partition is called an *e-core* if it is empty or has no hook with $k.e$ nodes for every integer $k > 0$. Of course, if e is sufficiently large comparing to n ($e > n$), every partition of n is an e-core. If λ is an e-core, it is already contained in Mullineux's original paper [23] that $m_e(\lambda)$ is the conjugate partition of λ (defined as the partition obtained by interchanging rows and columns in the Young diagram of λ.)

Example 3. Let $e = 3$ and let $\lambda = (5, 2, 1, 1)$. This is a 3-regular partition. Then we have:

$$\widetilde{f}^2_{0+3\mathbb{Z},3} \widetilde{f}^2_{1+3\mathbb{Z},3} \widetilde{f}_{0+3\mathbb{Z},3} \widetilde{f}^2_{2+3\mathbb{Z},3} \widetilde{f}_{1+3\mathbb{Z},3} \widetilde{f}_{0+3\mathbb{Z},3} \emptyset = \lambda.$$

We get

$$\widetilde{f}^2_{0+3\mathbb{Z},3} \widetilde{f}^2_{2+3\mathbb{Z},3} \widetilde{f}_{0+3\mathbb{Z},3} \widetilde{f}^2_{1+3\mathbb{Z},3} \widetilde{f}_{2+3\mathbb{Z},3} \widetilde{f}_{0+3\mathbb{Z},3} \emptyset = (4, 2, 2, 1).$$

and thus $m_3(5, 2, 1, 1) = (4, 2, 2, 1)$. If $e = 6$ then $\lambda = (5, 2, 1, 1)$ is a 6-core and we have:

$$\widetilde{f}_{0+6\mathbb{Z},6} \widetilde{f}_{3+6\mathbb{Z},6} \widetilde{f}^2_{4+6\mathbb{Z},6} \widetilde{f}_{3+6\mathbb{Z},6} \widetilde{f}_{2+6\mathbb{Z},6} \widetilde{f}_{1+6\mathbb{Z},6} \widetilde{f}_{5+6\mathbb{Z},6} \widetilde{f}_{0+6\mathbb{Z},6} \emptyset = \lambda.$$

We obtain

$$\widetilde{f}_{0+6\mathbb{Z},6} \widetilde{f}_{3+6\mathbb{Z},6} \widetilde{f}^2_{2+6\mathbb{Z},6} \widetilde{f}_{3+6\mathbb{Z},6} \widetilde{f}_{4+6\mathbb{Z},6} \widetilde{f}_{5+6\mathbb{Z},6} \widetilde{f}_{1+6\mathbb{Z},6} \widetilde{f}_{0+6\mathbb{Z},6} \emptyset = (4, 2, 1, 1, 1),$$

which is the conjugate partition of λ, as expected.

The above algorithm is quite efficient but it may be difficult to compute the Mullineux image of an arbitrary partition of rank $n >> 0$. Indeed there is no simple canonical way to produce a path (that is a sequence of crystal operators) from an arbitrary e-regular partition to the empty one. In the following, we will study another way to compute this map without any use of the crystal and the Kashiwara operators.

3 Crystal isomorphisms for bipartitions

In this section, we quickly summarize the needed results to expose our algorithm. These results mainly concern certain expansions of the above discussion to the case of bipartitions.

3.1 Level 2 Fock space

From now we fix a *bicharge*, that is a couple $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. Let us denote by $\Pi^2(n)$ the set of pairs of partitions (*bipartitions*) (λ^1, λ^2) such that $|\lambda^1|+|\lambda^2| = n$. One can define the level 2 Fock space as the $\mathbb{C}$-vector space with basis indexed by all the elements of $\Pi^2(n)$ for $n \in \mathbb{Z}_{\geqslant 0}$. There is also a notion of crystal for this 2 Fock space with similar notions of Kashiwara operators $\widetilde{f}^{\mathbf{s}}_{i+e\mathbb{Z},e}$ and $\widetilde{e}^{\mathbf{s}}_{i+e\mathbb{Z},e}$. Importantly, the action of these operators on each bipartition really depends on the choice of $\mathbf{s}$.

To each $\boldsymbol{\lambda} := (\lambda^1, \lambda^2) \in \Pi^2(n)$ is associated its *Young diagram*:

$$[\boldsymbol{\lambda}] = \{(a, b, c) \mid a \geqslant 1,\ c \in \{1, 2\}, 1 \leqslant b \leqslant \lambda^c_a\}.$$

We define the *content* of a node $\gamma = (a, b, c) \in [\boldsymbol{\lambda}]$ as follows:

$$\mathrm{cont}(\gamma) = b - a + s_c,$$

and the residue $\mathrm{res}(\gamma)$ is by definition the content of the node taken modulo e. We will say that γ is an $i + e\mathbb{Z}$-node of $\boldsymbol{\lambda}$ when $\mathrm{res}(\gamma) \equiv i + e\mathbb{Z}$ (we will sometimes simply called it an i-node). Finally, we say that γ is *removable* when $\gamma = (a, b, c) \in [\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}]\backslash\{\gamma\}$ is the Young diagram of a bipartition. Similarly, γ is *addable* when $\gamma = (a, b, c) \notin [\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}] \cup \{\gamma\}$ is the Young diagram of a bipartition.

Let γ, γ' be two removable or addable i-nodes of $\boldsymbol{\lambda}$. We denote

$$\gamma \prec_{\mathbf{s}} \gamma' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \text{either} & b - a + s_c < b' - a' + s_{c'}, \\ \text{or} & b - a + s_c = b' - a' + s_{c'} \text{ and } c > c'. \end{cases}$$

For $\boldsymbol{\lambda}$ a bipartition and $i \in \mathbb{Z}/e\mathbb{Z}$, we can consider its set of addable and removable i-nodes. Let $w^{(e,\mathbf{s})}_i(\boldsymbol{\lambda})$ be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\prec_{\mathbf{s}}$, next by encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}^{(e,\mathbf{s})}_i(\boldsymbol{\lambda}) = A^p R^q$ for the word derived from $w^{(e,\mathbf{s})}_i(\boldsymbol{\lambda})$ by deleting as many of the factors RA as possible. In the following, we will sometimes write $\widetilde{w}_i(\boldsymbol{\lambda})$ and $w_i(\boldsymbol{\lambda})$ instead of $\widetilde{w}^{(e,\mathbf{s})}_i(\boldsymbol{\lambda})$ and $w^{(e,\mathbf{s})}_i(\boldsymbol{\lambda})$ if there is no possible confusion.

If $p > 0$, let γ be the rightmost addable i-node in $\widetilde{w}_i$. The node γ is called the *good addable i-node*. If $q > 0$, the leftmost removable i-node in $\widetilde{w}_i$ is called the *good removable i-node*. The definition of the Kashiwara operators $\widetilde{f}^{\mathbf{s}}_{i+e\mathbb{Z},e}$ and $\widetilde{e}^{\mathbf{s}}_{i+e\mathbb{Z},e}$ follows then exactly the same pattern as in §2.2. In the same spirit as in the above discussion, one can also define a certain subset of bipartitions $\Phi_{(\mathbf{s},e)}(n)$:

Definition 4. We say that (λ^1, λ^2) is an *Uglov bipartition* associated with $\mathbf{s} \in \mathbb{Z}^2$ if there exist $(i_1, \dots, i_n) \in \mathbb{Z}^n$ such that:

$$\widetilde{f}^{\mathbf{s}}_{i_1+e\mathbb{Z},e} \cdots \widetilde{f}^{\mathbf{s}}_{i_n+e\mathbb{Z},e}.(\emptyset, \emptyset) = (\lambda^1, \lambda^2).$$

We denote by $\Phi_{(e,\mathbf{s})}$ the set of Uglov bipartitions and by $\Phi_{(e,\mathbf{s})}(n)$ the set $\Phi_{(e,\mathbf{s})} \cap \Pi^2(n)$.

We make the three important following remarks.

Remark 5.

1. As explained in [16], the set $\Phi_{((s_1,s_2),e)}(n)$ provides a natural parametrization of the vertices for the crystal graph of the irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with weight $\Lambda_{s_1+e\mathbb{Z}} + \Lambda_{s_2+e\mathbb{Z}}$ (the $\Lambda_i's$ denoting the fundamental weights). The structure of this graph thus only depends on the choice of the bicharge modulo e. As a consequence, assume that $k \in \mathbb{Z}$ then there is a unique bijection:

$$\psi_{(e,(s_1,s_2+ke))} : \Phi_{(e,(s_1,s_2+ke))} \to \Phi_{(e,(s_1,s_2+(k+1)e))},$$

   preserving the rank of bipartitions and commuting with the Kashiwara operators, that is, for all $i \in \mathbb{Z}$ and $\lambda \in \Phi_{(e,(s_1,s_2+ke))}$, we have

$$\psi_{(e,(s_1,s_2+ke))}(\widetilde{f}^{(s_1,s_2+ke)}_{i+e\mathbb{Z},e}.\lambda) = \widetilde{f}^{(s_1,s_2+(k+1)e)}_{i+e\mathbb{Z},e}.\psi_{(e,(s_1,s_2+ke))}(\lambda),$$

   and

$$\psi_{(e,(s_1,s_2+ke))}(\widetilde{e}^{(s_1,s_2+ke)}_{i+e\mathbb{Z},e}.\lambda) = \widetilde{e}^{(s_1,s_2+(k+1)e)}_{i+e\mathbb{Z},e}.\psi_{(e,(s_1,s_2+ke))}(\lambda).$$

   This bijection may be computed thanks to a purely combinatorial algorithm given in section §4. This map is called a *crystal isomorphism*.

2. By [9, §6.2.16], in the case where $s_2 - s_1 > n - 1 - e$, the bijection $\psi_{(e,(s_1,s_2))}$ restricted to $\Phi_{(e,(s_1,s_2))}(n)$ is always the identity. We say that (s_1, s_2) is *very dominant* (comparing to n). This implies in particular that as soon as $s_2 - s_1 > n - 1 - e$, the set $\Phi_{(e,(s_1,s_2))}(n)$ only depends on the congruence class of (s_1, s_2) modulo e. Similarly, the action of the Kashiwara operators on the bipartitions of smaller rank only depends on the congruence class of (s_1, s_2) modulo e as soon as the above condition is satisfied. The set is then called the set of *Kleshchev bipartitions*. The set of Kleshchev bipartitions of rank n will be denoted by $\Phi^K_{(e,\mathbf{s})}(n)$ and we denote $\Phi^K_{(e,\mathbf{s})} := \sqcup_{n \geqslant 0} \Phi^K_{(e,\mathbf{s})}(n)$

3. One can define a bijection:

$$\widetilde{\psi}_{(e,(s_1,s_2))} : \Phi_{(e,(s_1,s_2))} \to \Phi^K_{(e,\mathbf{s})},$$

as follows. Let $n \in \mathbb{Z}_{\geqslant 0}$ and let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \in \Phi_{(e,(s_1,s_2))}$ of rank n. Assume that $k \in \mathbb{Z}_{>0}$ is such that $|s_2 + ke - s_1| > n - 1 - e$, then we define:

$$\widetilde{\psi}_{(e,(s_1,s_2))}(\lambda^1, \lambda^2) := \psi_{(e,(s_1,s_2+(k-1)e))} \circ \cdots \circ \psi_{(e,(s_1,s_2+e))} \circ \psi_{(e,(s_1,s_2))}(\lambda^1, \lambda^2).$$

Due to the above remark, this bijection does not depend on k.

3.2 Mullineux map

There exists a Mullineux type map in the case of bipartitions. Let $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$ and let $-\mathbf{s} := (-s_1, -s_2)$. Our Mullineux map will be a map:

$$\mathcal{M}_{(e,\mathbf{s})} : \Phi^K_{(e,\mathbf{s})} \to \Phi^K_{(e,-\mathbf{s})},$$

which is uniquely defined as follows. Let $\boldsymbol{\lambda} \in \Phi^K_{(e,\mathbf{s})}(n)$. Let $n \in \mathbb{Z}_{>0}$. Let $\mathbf{s}^1 = (s_1, s_2 + ke)$ be a very dominant bicharge such that $\mathbf{s}^1 \equiv \mathbf{s} + e\mathbb{Z}$ and let $\mathbf{s}^2$ be a very dominant bicharge such that $\mathbf{s}^2 \equiv -\mathbf{s} + e\mathbb{Z}$. There exists $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ such that:

$$\widetilde{f}^{\mathbf{s}^1}_{i_1+e\mathbb{Z},e} \cdots \widetilde{f}^{\mathbf{s}^1}_{i_n+e\mathbb{Z},e}.\boldsymbol{\emptyset} = \boldsymbol{\lambda}.$$

Then it is shown in [7, §2] that there exists $\boldsymbol{\mu} \in \Phi^K_{(e,\mathbf{s}^2)}$ such that:

$$\widetilde{f}^{\mathbf{s}^2}_{-i_1+e\mathbb{Z},e} \cdots \widetilde{f}^{\mathbf{s}^2}_{-i_n+e\mathbb{Z},e}.\boldsymbol{\emptyset} = \boldsymbol{\mu}.$$

We denote $\mathcal{M}_{(e,(s_1,s_2))}(\boldsymbol{\lambda}) := \boldsymbol{\mu}$. Then it is proved in [17, Prop. 4.2] that $\boldsymbol{\mu} = (m_e(\lambda^1), m_e(\lambda^2))$. In the following section, we will use this property to deduce our conjectural algorithm.

4 Explicit computations and a combinatorial property

In this section, we explain how one can compute the above crystal isomorphisms. Our main conjecture is relied on a combinatorial conjectural property of these maps. This property can in fact be settled in a completely general framework.

4.1 A combinatorial map

We recall here results from [16]. Let e be a positive integer. For r a positive integer, we denote by $\mathcal{P}^r$ the set of strictly increasing sequences of non-negative integers with r parts. Let m_1 and m_2 be two positive integers such that $m_1 \leqslant m_2$.

Let $(X_1, X_2) \in \mathcal{P}^{m_1} \times \mathcal{P}^{m_2}$. Set

$$X_1 = (a_1, \ldots, a_{m_1}),\ X_2 = (b_1, \ldots, b_{m_2}).$$

We define an injection $\varphi : X_1 \to X_2$ as follows.

- We set
$$\varphi(a_1) = \max\{b_j \mid j = 1, \ldots, m_2, b_j \leqslant a_1\},$$
if it exists. Otherwise, we set
$$\varphi(a_1) = b_{m_2}$$

- We repeat this procedure with $(a_2,\dots,a_{m_1})$ and $X_2\setminus\{\varphi(a_1)\}$ and thus associate to each element of X_1 a unique element in X_2.

We now define a map:

$$\Psi_{(e,(m_1,m_2))}:\mathcal{P}^{m_1}\times\mathcal{P}^{m_2}\to\mathcal{P}^{m_1}\times\mathcal{P}^{m_2+e},$$

with $(Y_1,Y_2):=\Psi_{(e,(m_1,m_2))}(X_1,X_2)$:

$$Y_1=\{\varphi(a_j)\mid j=1,\dots,m_1\},$$

$$Y_2=\{a_j+e\mid j=1,\dots,m_1\}\cup\{b_j+e\mid j=1,\dots,m_2;b_j\notin Y_1\}\cup\{0,1\dots,e-1\},$$

where we reorder these two sets so that $Y_1\in\mathcal{P}^{m_1}$ and $Y_2\in\mathcal{P}^{m_2+e}$.

Remark 6. The map is injective and $(\Psi_{(e,(m_1,m_2))})^{-1}$ may be computed as follows. Assume that $(Y_1,Y_2):=\Psi_{(e,(m_1,m_2))}(X_1,X_2)$ then take Y_2' be the set $\{y-e\mid y\in Y_2\setminus\{0,1,\dots,e-1\}\}$. Then define $(a_1,\dots,a_{m_1}):=Y_1$ and $(b_1\dots,b_{m_2}):=Y_2'$. we define an injection $\varphi':Y_1\to Y_2'$ as follows.

- We set
$$\varphi'(a_1)=\min\{b_j\mid j=1,\dots,m_2,b_j\geqslant a_1\},$$
if it exists. Otherwise, we set
$$\varphi'(a_1)=b_1.$$

- We repeat this procedure with $(a_2,\dots,a_{m_1})$ and $Y_2'\setminus\{\varphi'(a_1)\}$ and thus associate to each element of Y_1 a unique element in Y_2'. Then we have
$$X_1=\{\varphi'(a_j)\mid j=1,\dots,m_1\},$$
$$X_2=\{a_j\mid j=1,\dots,m_1\}\cup\{b_j\mid j=1,\dots,m_2;b_j\notin X_1\}.$$
(after reordering the elements)

Remark 7. In the case where $X_1\subset X_2$, it follows from the above definition that

$$\Psi_{(e,(m_1,m_2))}(X_1,X_2)=(X_1,\{0,\dots,e-1\}\cup X_2+e).$$

4.2 Connection with crystal isomorphisms

Assume that $\mathbf{s}=(s_1,s_2)\in\mathbb{Z}^2$ and assume in addition that $s_1\leqslant s_2$ (we only need this case in the following but note that there is an analogue description of the crystal isomorphisms if $s_1\geqslant s_2$, see [16]). Let $\boldsymbol{\lambda}=(\lambda^1,\lambda^2)$ be a bipartition of n in $\Phi_{(e,(s_1,s_2))}$. One can assume that there exists an integer m such that $\lambda^1=(\lambda^1_1,\dots,\lambda^1_{m+s_1})$ and $\lambda^2=(\lambda^2_1,\dots,\lambda^2_{m+s_2})$, adding parts equal to 0 if necessary. For $j=1,\dots,m+s_1$, we set

$$\beta^1_j=\lambda^1_j-j+s_1+m.$$

For $j = 1, \dots, m + s_2$, we set

$$\beta_j^2 = \lambda_j^2 - j + s_2 + m.$$

We then define

$$X_1^{s_1,m}(\lambda^1) := (\beta_{s_1+m}^1, \dots, \beta_1^1) \in \mathcal{P}^{m+s_1}$$

and

$$X_2^{s_2,m}(\lambda^2) := (\beta_{s_2+m}^2, \dots, \beta_1^2) \in \mathcal{P}^{m+s_2}.$$

By [13, Th. 4.6] (see also the generalization in [16]), we get:

Proposition 8. *Keeping the above notations, We have*

$$\psi_{(e,(s_1,s_2))}(\lambda^1, \lambda^2) = (\mu^1, \mu^2),$$

where (μ^1, μ^2) *is the unique bipartition of* n *such that*

$$\Psi_{(e,(s_1+m,s_2+m))}(X_1^{s_1,m}(\lambda^1), X_2^{s_2,m}(\lambda^2)) = (X_1^{s_1,m}(\mu^1), X_2^{s_2+e,m}(\mu^2)).$$

Remark 9. In the case where $X_1^{s_1,m}(\lambda^1) \subset X_2^{s_2,m}(\lambda^2)$, by Remark 7, we obtain:

$$\psi_{(e,(s_1,s_2))}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

Remark 10. As noted in [16], the algorithm to compute (μ^1, μ^2) from the datum of (λ^1, λ^2) in fact does not depend on e. This can be explained as follows. The map $\psi_{(e,(s_1,s_2))}$ can be seen as the composition of two crystal isomorphisms, the first one:

$$\Phi_{(e,(s_1,s_2))} \to \Phi_{(e,(s_2,s_1))},$$

is the restriction of the crystal isomorphism

$$\Phi_{(\infty,(s_1,s_2))} \to \Phi_{(\infty,(s_2,s_1))},$$

and thus do not depends on e, and the second :

$$\Phi_{(e,(s_2,s_1))} \to \Phi_{(e,(s_1,s_2+e))},$$

sends (λ^1, λ^2) to (λ^2, λ^1).

4.3 Computing the map $\widetilde{\psi}_{(e,(s_1,s_2))}$

Assume that $s_1 \leqslant s_2$. To compute $\widetilde{\psi}_{(e,(s_1,s_2))}$, as explained in Remark 5 (3), we have to fix $n \in \mathbb{Z}_{\geqslant 0}$ and compute $\widetilde{\psi}_{(e,(s_1,s_2))}|_{\Phi_{(e,\mathbf{s})}(n)}$. If $k \in \mathbb{Z}_{>0}$ is such that $|s_2 + ke - s_1| > n - 1 - e$, we have to compose k crystal isomorphisms:

$$\widetilde{\psi}_{(e,(s_1,s_2))}|_{\Phi_{(e,\mathbf{s})}(n)} := \psi_{(e,(s_1,s_2+(k-1)e))} \circ \cdots \circ \psi_{(e,(s_1,s_2+e))} \circ \psi_{(e,(s_1,s_2))}|_{\Phi_{(e,\mathbf{s})}(n)} \qquad (1)$$

However, in most of the cases, if we want to compute the image of a particular bipartition $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$ under $\widetilde{\psi}_{(e,(s_1,s_2))}$, one can be considerably more efficient thanks to the following remark. Let $\boldsymbol{\lambda} \in \Phi_{(e,\mathbf{s})}(n)$ and $h := \max\{i \in \mathbb{Z}_{>0} \mid \lambda_i^2 \neq 0\} + 1$. Assume that

$$\lambda_1^1 - 1 + s_1 \leqslant s_2 - h, \tag{2}$$

then we have for all relevant m, we have $X_1^{s_1,m}(\lambda^1) \subset X_2^{s_2,m}(\lambda^2)$. By Remarks 7 and 9, this implies that

$$\psi_{(e,(s_1,s_2))}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

But now we also have $\lambda_1^1 - 1 + s_1 \leqslant s_2 + e - h$ and thus we obtain

$$\psi_{(e,(s_1,s_2+e))}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

By an immediate induction, we deduce that for all $k \geqslant 0$, we have:

$$\psi_{(e,(s_1,s_2+ke))}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

In this case, we thus simply have:

$$\widetilde{\psi}_{(e,(s_1,s_2))}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

Of course, a similar result holds for $(\widetilde{\psi}_{(e,(s_1,s_2))})^{-1}$: if $(\lambda^1, \lambda^2) \in \Phi_{(e,\mathbf{s})}^K$ satisfies the above property, then we have for all $k \geqslant 0$ that $(\lambda^1, \lambda^2) \in \Phi_{(e,(s_1,s_2+ke))}$ and

$$(\psi_{(e,(s_1,s_2+ke))})^{-1}(\lambda^1, \lambda^2) = (\lambda^1, \lambda^2).$$

Note that the difference between the two charges of the multicharges appearing in 1 grows as we compose the isomorphisms. Thus, condition 2 must be satisfied after some compositions. We can now state our combinatorial conjecture

4.4 A combinatorial conjecture

Our main conjecture is the following one:

Conjecture 11. Let $X \in \mathcal{P}^m$. Set

$$(X_1^0, X_2^0) = (X, X)$$

and for $k > 0 \in \mathbb{Z}$

$$(X_1^k, X_2^k) = \Psi_{(e,(0,(k-1)e))} \circ \ldots \circ \Psi_{(e,(0,e))} \circ \Psi_{(e,(0,0))}(X, X) \in \mathcal{P}^m \times \mathcal{P}^{m+ke}.$$

Then if k is even, we have $X_1^k \subset X_2^k$.

We can establish the conjecture in the case $k = 2$. Note that if $X_1 \subset X_2$ then φ is the identity. We thus have that

$$\Psi_{(e,(0,0))}(X, X) = (X, X + e \cup \{0, 1, \dots, e-1\}).$$

Now we set

$$\Psi_{(e,(0,e))}(X, X + e \cup \{0, 1, \dots, e-1\}) = (Y_1, Y_2).$$

From the above procedure, the elements of Y_1 are some elements of $X+e\cup\{0, 1, \dots, e-1\}$ and Y_2 is given by $\{0, 1, \dots, e-1\}$ together with all the elements of $X+e$ and other elements of $X + 2e \cup \{e, e+1, \dots, 2e-1\}$. Now if $y_1 \in Y_1$, either $y_1 \leqslant e-1$ and then clearly $y_1 \in Y_2$ or $y_1 \in X + e$ by definition of the combinatorial procedure and therefore we also have $y_1 \in Y_2$ since $X + e \subset Y_2$. We thus have $Y_1 \subset Y_2$.

In the following, it will be convenient to write the image of an element $(X_1, X_2) \in \mathcal{P}^{m_1} \times \mathcal{P}^{m_2}$ under a map $\Psi_{(e,\mathbf{s})}$ as $\begin{pmatrix} X_2 \\ X_1 \end{pmatrix}$ instead of (X_1, X_2). This is what we are going to do in the following example. Assume that

$$X = \{0, 3, 5, 6, 10, 12, 15, 18, 20\},$$

and $e = 3$. We check that

$$\Psi_{(3,(0,0))}(X, X) = \begin{pmatrix} 0 & 1 & 2 & 3 & 6 & 8 & 9 & 13 & 15 & 18 & 21 & 23 \\ 0 & 3 & 5 & 6 & 10 & 12 & 15 & 18 & 20 & & & \end{pmatrix}$$

$$\Psi_{(3,(0,3))} \circ \Psi_{(3,(0,0))}(X, X) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 & 8 & 9 & 13 & 15 & 18 & 21 & 23 & 24 & 26 \\ 0 & 2 & 3 & 6 & 8 & 9 & 13 & 15 & 18 & & & & & & \end{pmatrix}$$

and the set below is included in the set above, as claimed by the conjecture. Then by applying $\Psi_{(3,(0,6))}$ we get

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 11 & 12 & 16 & 18 & 21 & 24 & 26 & 27 & 29 \\ 0 & 2 & 3 & 6 & 8 & 9 & 13 & 15 & 18 & & & & & & & & & \end{pmatrix}$$

and the action of $\Psi_{(3,(0,9))}$ then gives

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 & 12 & 16 & 18 & \cdots \\ 0 & 2 & 3 & 6 & 7 & 9 & 11 & 12 & 18 & & & & & & \end{pmatrix}$$

which yet again satisfies the inclusion property.

Note that in the assumptions of the conjecture, we really need k to be even. In the case when k is odd, the assertion is wrong as we can see in the above example.

Remark 12.

- This conjecture has been checked for all couples $(X, X) = (X^{0,m}(\lambda), X^{0,m}(\lambda))$ with λ an arbitrary partition of rank n with $n \leqslant 40$ (and e arbitrary). A proof for the conjecture has already been obtained by M.Fayers when $e = 2$ [8].

- For any $X \in \mathcal{P}^m$, there exists an integer k_0 such that $X_1^k \subset X_2^k$ for any $k \geqslant k_0$. This corresponds to the asymptotic case where (X_1^k, X_2^k) is the symbol of a Kleshshev bipartition.

4.5 A reformulation of the conjecture

In fact, the previous sequence of elements $\begin{pmatrix} X_2^k \\ X_1^k \end{pmatrix}, k \geqslant 0$ is completely determined by the sequence $(X_1^k), k \geqslant 0$. Indeed, we have by definition $X_2^0 = X$. Now if we assume that the sequences $X_2^0, X_2^1, \ldots, X_2^{k-1}$ and $X_1^0, X_1^1, \ldots, X_1^{k-1}, X_1^k$ are determined, we have by definition of our algorithm

$$X_2^k = \{0, \ldots, e-1\} \cup \left(\left(\left(X_2^{k-1} \setminus X_1^k\right) \cup X_1^{k-1}\right) + e\right).$$

For any integer k, we have the equivalence

$$X_1^k \subset X_2^k \Longleftrightarrow X_1^{k+1} = X_1^k$$

Our conjecture is thus equivalent to the assertion

$$X_1^{k+1} = X_1^k \text{ for any even } k \geqslant 0.$$

It is also interesting to observe that each list $X_1^k := (a_1, \ldots, a_l)$ determines an e-regular partition $\lambda^k := (\lambda_1^k, \ldots, \lambda_l^k)$ where $\lambda_j^k := a_{l-j+1} - (l-j)$ for $j = 1, \ldots, l$. We thus get the following proposition

Proposition 13. *Conjecture 11 holds if and only if the sequence of e-regular partitions $(\lambda^k), k \geqslant 0$ satisfies $\lambda^k = \lambda^{k+1}$ for any even integer k.*

5 Conjectural consequences on crystal isomorphisms

We first establish some elementary results concerning e-regular partitions and then explain our conjectural algorithm.

Proposition 14. *Let λ be an e-regular partition and consider a sequence $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ such that:*

$$\widetilde{f}_{i_1+\mathbb{Z}e,e} \cdots \widetilde{f}_{i_n+\mathbb{Z}e,e}.\emptyset = \lambda.$$

Then we have

$$(\widetilde{f}^{(0,0)}_{i_1+\mathbb{Z}e,e})^2 \cdots (\widetilde{f}^{(0,0)}_{i_n+\mathbb{Z}e,e})^2(\emptyset, \emptyset) = (\lambda, \lambda),$$

and in particular we have $(\lambda, \lambda) \in \Phi_{(e,(0,0))}$.

Proof. Let $\lambda \in \operatorname{Reg}_e(n)$. By Proposition 1, there exists $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ such that:

$$\widetilde{f}_{i_1+\mathbb{Z}e,e} \cdots \widetilde{f}_{i_n+\mathbb{Z}e,e}.\emptyset = \lambda.$$

We set

$$\widetilde{\lambda} := \widetilde{f}_{i_1+\mathbb{Z}e,e} \cdots \widetilde{f}_{i_{n-1}+\mathbb{Z}e,e}.\emptyset.$$

By induction, we have that

$$(\widetilde{f}^{(0,0)}_{i_1+\mathbb{Z}e,e})^2 \ldots (\widetilde{f}^{(0,0)}_{i_{n-1}+\mathbb{Z}e,e})^2(\emptyset, \emptyset) = (\widetilde{\lambda}, \widetilde{\lambda}).$$

Assume that

$$w_{i_n+e\mathbb{Z}}(\widetilde{\lambda}) = Z_1, \dots Z_m.$$

where for all $i = 1, \dots, m$, $Z_i \in \{A, R\}$ corresponds to a node (a_i, b_i). Then we have:

$$w_{i_n+e\mathbb{Z}}(\widetilde{\lambda}, \widetilde{\lambda}) = T_1, \dots, T_{2m},$$

where $T_{2i-1} = Z_i$ correspond to the node $(a_i, b_i, 2)$ for $i = 1, \dots, m$ and $T_{2i} = Z_i$ for $i = 1, \dots, m$ corresponds to the node $(a_i, b_i, 1)$. It follows that if (a_k, b_k) is a good addable $i_n + e\mathbb{Z}$-node for $\widetilde{\lambda}$ then $(a_i, b_i, 2)$ is a good addable $i_n + e\mathbb{Z}$-node for $(\widetilde{\lambda}, \widetilde{\lambda})$ and $(a_i, b_i, 1)$ is a good addable $i_n + e\mathbb{Z}$-node for $(\widetilde{\lambda}, \lambda)$. We conclude that

$$(\widetilde{f}^{(0,0)}_{i_1+\mathbb{Z}e,e})^2 \dots (\widetilde{f}^{(0,0)}_{i_n+\mathbb{Z}e,e})^2(\emptyset, \emptyset) = (\lambda, \lambda)$$

as required. □

The following result comes from [14, Lemma 3.2.12] (see also [12] for an similar result, but for a different realization of the Fock space).

Proposition 15. *Let λ be an e-regular partition and let $(i_1, \dots, i_n) \in \mathbb{Z}^n$ be such that:*

$$\widetilde{f}_{i_1+\mathbb{Z}e,e} \cdots \widetilde{f}_{i_n+\mathbb{Z}e,e}.\emptyset = \lambda.$$

Then we have:

$$\widetilde{f}^{(0,e)}_{i_1+2\mathbb{Z}e,2e}\widetilde{f}^{(0,e)}_{i_1+e+2\mathbb{Z}e,2e} \dots \widetilde{f}^{(0,e)}_{i_n+2\mathbb{Z}e,2e}\widetilde{f}^{(0,e)}_{i_n+e+2\mathbb{Z}e,2e}(\emptyset, \emptyset) = (\lambda, \lambda),$$

and in particular we have $(\lambda, \lambda) \in \Phi_{(2e,(0,e))}$.

Our algorithm is now built on the following result which assumes Conjecture 11.

Proposition 16. *Assume that Conjecture 11 is true then for all e-regular partitions λ of n and $k \in \mathbb{Z}_{>0}$, we have:*

$$\psi_{(2e,(0,(2k+1)e))} \circ \dots \circ \psi_{(2e,(0,3e))} \circ \psi_{(2e,(0,e))}(\lambda, \lambda) = \psi_{(e,(0,(2k+1)e))} \circ \dots \circ \psi_{(e,(0,e))} \circ \psi_{(e,(0,0))}(\lambda, \lambda).$$

In particular, we have

$$\widetilde{\psi}_{(2e,(0,e))}(\lambda, \lambda) = \widetilde{\psi}_{(e,(0,0))}(\lambda, \lambda).$$

Proof. Let λ be an e-regular partition of n. By Prop. 14 and 15, we have that $(\lambda, \lambda) \in \Phi_{(2e,(0,e))} \cap \Phi_{(e,(0,0))}$. By Remark 7, we have

$$\psi_{(e,(0,0))}(\lambda, \lambda) = (\lambda, \lambda) \in \Phi_{(e,(0,e))}.$$

We thus have $(\lambda, \lambda) \in \Phi_{(e,(0,e))} \cap \Phi_{(2e,(0,e))}$. Now, the algorithm to compute the image of a bipartition under the crystal isomorphism $\psi_{(e,(s_1,s_2))}$ does not depend on e but only on the pair (s_1, s_2). This implies that:

$$\psi_{(e,(0,e))}(\lambda, \lambda) = \psi_{(2e,(0,e))}(\lambda, \lambda).$$

We then argue by induction. Assume that

$$\begin{array}{rcl}(\mu^1,\mu^2) & := & \psi_{(2e,(0,(2k-1)e))}\circ\cdots\circ\psi_{(2e,(0,3e))}\circ\psi_{(2e,(0,e))}(\lambda,\lambda)\\ & = & \psi_{(e,(0,(2k-1)e))}\circ\cdots\circ\psi_{(e,(0,e))}\circ\psi_{(e,(0,0))}(\lambda,\lambda).\end{array}$$

We use Conjecture 11 and Remark 7 to deduce that:

$$\psi_{(e,(0,2ke))}(\mu^1,\mu^2)=(\mu^1,\mu^2).$$

Again, the algorithm for computing the crystal isomorphisms implies that:

$$\psi_{(2e,(0,(2k+1)e))}(\mu^1,\mu^2)=\psi_{(e,(0,(2k+1)e))}(\mu^1,\mu^2),$$

and we are done. □

Remark 17. Note that in fact to prove the above result, we only need to prove Conjecture 11 in the case where $X\in\mathcal{P}^m$ is such that there is no $i\in X$ such that $i,i+1,\dots,i+e-1$ are in X (this corresponds to the β-sets associated with e-regular partitions).

Assuming that the conjecture 11 is true, we will now be able to obtain our algorithm. To do this, we yet need a remarkable property of the Mullineux map. As our Mullineux map is defined on the set of Kleshchev bipartitions, we will use the map $\widetilde{\psi}_{(e,\mathbf{s})}$ which "takes to the very dominant world" (see Remark 5 (3)).

Theorem 18. *Assume that Conjecture 11 is satisfied. Let λ be an e-regular partition and denote $(\mu^1,\mu^2):=\widetilde{\psi}_{(e,(0,0))}(\lambda,\lambda)$. We have:*

$$\begin{array}{rcl}\widetilde{\psi}_{(e,(0,0))}(m_e(\lambda),m_e(\lambda)) & = & \widetilde{\psi}_{(2e,(0,e))}(m_e(\lambda),m_e(\lambda))\\ & = & (m_{2e}(\mu^1),m_{2e}(\mu^2))\\ & = & (m_e(\mu^1),m_e(\mu^2)).\end{array}$$

Proof. Let λ be an e-regular partition. There exists $(i_1,\dots,i_n)\in\mathbb{Z}^n$ such that:

$$\widetilde{f}_{i_1+\mathbb{Z}e,e}\cdots\widetilde{f}_{i_n+\mathbb{Z}e,e}.\emptyset=\lambda.$$

By Proposition 14, we have that $(\lambda,\lambda)\in\Phi_{(e,(0,0))}$ and we have:

$$\widetilde{f}^2_{i_1+\mathbb{Z}e,e}\cdots\widetilde{f}^2_{i_n+\mathbb{Z}e,e}.(\emptyset,\emptyset)=(\lambda,\lambda),$$

and by Proposition 15, we have $(\lambda,\lambda)\in\Phi_{(2e,(e,0))}$. Now by Proposition 16, we have that

$$\widetilde{\psi}_{(e,(0,0))}(\lambda,\lambda)=\widetilde{\psi}_{(2e,(0,e))}(\lambda,\lambda).$$

Set $\boldsymbol{\mu}:=(\mu^1,\mu^2):=\widetilde{\psi}_{(e,(0,0))}(\lambda,\lambda)$. Then by §4, we have $\mathcal{M}_{(e,(0,0))}(\boldsymbol{\mu})=(m_e(\mu^1),m_e(\mu^2))$ and $\mathcal{M}_{(2e,(0,e))}(\boldsymbol{\mu})=(m_{2e}(\mu^1),m_{2e}(\mu^2))$. If we argue exactly as above with the bipartition $(m_e(\lambda),m_e(\lambda))$, we get:

$$\widetilde{\psi}_{(e,(0,0))}(m_e(\lambda),m_e(\lambda))=\widetilde{\psi}_{(2e,(0,e))}(m_e(\lambda),m_e(\lambda)).$$

By definition we have:

$$(\widetilde{f}^{(0,0)}_{-i_1+\mathbb{Z}e})^2\cdots(\widetilde{f}^{(0,0)}_{-i_n+\mathbb{Z}e})^2(\emptyset,\emptyset)=(m_e(\lambda),m_e(\lambda)),$$

Choose $k\in\mathbb{Z}$ such that $(0,ke)$ is very dominant. We obtain

$$\widetilde{\psi}_{(e,(0,0))}(m_e(\lambda),m_e(\lambda))=(\widetilde{f}^{(0,ke)}_{-i_1+\mathbb{Z}e})^2\cdots(\widetilde{f}^{(0,ke)}_{-i_n+\mathbb{Z}e})^2(\emptyset,\emptyset)$$

and by definition we get:

$$\widetilde{\psi}_{(e,(0,0))}(m_e(\lambda),m_e(\lambda))=\mathcal{M}_{(e,(0,0))}(\boldsymbol{\mu}),$$

Similarly, with exactly the same argument we get:

$$\widetilde{\psi}_{(2e,(0,e))}(m_{2e}(\lambda),m_{2e}(\lambda))=\mathcal{M}_{(2e,(0,e))}(\boldsymbol{\mu}),$$

and thus the result follows. □

The algorithm can now be stated as follows. Let $n\in\mathbb{N}$.

1. If e is sufficiently large, we know the Mullineux image of any e-regular partition because then any e-regular partition is an e-core and thus its Mullineux image is its conjugate partition.

2. Assume that we know m_{2e}. Let λ be an e-regular partition. We compute:

$$(\mu^1,\mu^2):=\widetilde{\psi}_{(2e,(0,e))}(\lambda,\lambda).$$

3. Then we compute:

$$(\nu^1,\nu^2):=(\widetilde{\psi}_{(2e,(0,e))})^{-1}(m_{2e}(\mu^1),m_{2e}(\mu^2)).$$

4. We must have by the previous proposition

$$m_e(\lambda)=\nu^1=\nu^2.$$

Example 19. Take $e=3$ and the 3-regular partition $\lambda=(6,5,2,2,1,1)$. This is a partition of rank 17 and so the very dominant case is reached if $s_2-s_1>30$. To perform our algorithm, we must compute:

$$\widetilde{\psi}_{(2e,(0,0))}(\lambda,\lambda)=\psi_{(2e,(0,ke))}\circ\cdots\circ\psi_{(2e,(0,3e))}\circ\psi_{(2e,(0,e))}(\lambda,\lambda)$$

until we reach the "very dominant case". We consider the β-sets associated with the bipartition (λ,λ) with respect to the bicharge $(0,3)$:

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 5 & 7 & 8 & 12 & 14 \\ 1 & 2 & 4 & 5 & 9 & 11 & & & \end{pmatrix}$$

To compute $\psi_{(2e,(0,e))}(\lambda,\lambda)$, we need to apply the algorithm described in §4.1. We obtain:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 11 & 12 & 15 & 17 & 18 & 20 \\ 1 & 2 & 4 & 5 & 7 & 8 \end{pmatrix}$$

The associated bipartition is $((3,3,2,2,1,1),(6,5,5,4,1,1))$. In principle, we have to apply again the algorithm until the "very dominant case", but note that we are already in the case described in §4.3 so

$$\widetilde{\psi}_{(2e,(0,0))}(\lambda,\lambda)=(\mu^1,\mu^2)=((3,3,2,2,1,1),(6,5,5,4,1,1))$$

By induction, we know $m_6(3,3,2,2,1,1)=(6,4,2)$ (because $(3,3,2,2,1,1)$ is a 6-core) and $m_6(6,5,5,4,1,1)=(11,9,2)$. So now we have to compute $(\widetilde{\psi}_{(6,(0,0))})^{-1}$ for $((6,4,2),(11,9,2))$ starting from the very dominant case. In fact, using Remark 4.3 again, we see that $((6,4,2),(11,9,2))$ is in $\Phi_{(6,(0,9))}$ and that:

$$(\widetilde{\psi}_{(6,(0,0))})^{-1}((6,4,2),(11,9,2))=(\psi_{(6,(0,3))})^{-1}((6,4,2),(11,9,2))$$

To compute this latter expression, we use our (reversed) algorithm, we consider the following symbol:

$$\begin{pmatrix} 0 & 1 & 2 & 5 & 13 & 16 \\ 2 & 5 & 8 \end{pmatrix}.$$

This gives:

$$\begin{pmatrix} 0 & 1 & 2 & 5 & 8 & 16 \\ 2 & 5 & 13 \end{pmatrix}.$$

We get $((11,4,2),(11,4,2))$ and one can check that we indeed have $m_e(\lambda)=(11,4,2)$.

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References

[1] C. Bessenrodt and J.Olsson, On residue symbols and the Mullineux conjecture, *J. Algebraic Comb.* 7 (1998), 227-251.

[2] J. Brundan, Modular branching rules and the Mullineux map for Hecke algebras of type A, *Proc. London Math. Soc.* (3) 77 (1998), no. 3, 551–581.

[3] J. Brundan and J. Kujawa, A new proof of the Mullineux Conjecture, *J. Algebraic Combin.* 18 (2003), 13–39.

[4] P. Dimakis and G. Yue, Combinatorial wall-crossing and the Mullineux involution *J. Algebraic Combin.* volume 50, pages (2019), 49–72.

[5] O.Dudas and N.Jacon, Alvis-Curtis duality for finite general linear groups and a generalized Mullineux involution, *SIGMA* 14, (2018).

[6] M. Fayers, Regularisation, crystals and the Mullineux map, *J. Comb. Algebra* 6 (2022), 315–352.

[7] M. Fayers, Weights of multipartitions and representations of Ariki-Koike algebras II: canonical bases, *J. Algebra* 319 (2008) 2963–2978.

[8] M. Fayers, Private communication.

[9] M. Geck and N. Jacon, Representations of Hecke algebras at roots of unity. Algebra and Applications, 15. Springer-Verlag London, Ltd., London, 2011.

[10] T. Gerber, N. Jacon and E. Norton, Generalized Mullineux involution and perverse equivalences, *Pacific J. of Math.* Vol. 306 (2020), No. 2, 487–517.

[11] B. Ford, A. Kleshchev, A proof of the Mullineux conjecture, *Math. Z.* 226 (2) (1997) 267–308.

[12] H. Lin and J. Hu, Crystal of affine type A and Hecke algebras at a primitive $2l$th root of unity, *J. Algebra* Volume 589, 1 January 2022, Pages 51-81.

[13] N. Jacon, Crystal graphs of irreducible highest weight $U_q(\hat{sl}_e)$ -modules of level two and Uglov bipartitions, *J. Algebraic Comb.* Vol. 27, no. 2 (2008), 143-162

[14] N. Jacon, Two maps on affine type A crystals and Hecke algebras, *Electronic Journal of Combinatorics* Vol. 28, Issue 2, #P2.21, 2021.

[15] N. Jacon, Crystal isomorphisms and Mullineux involution I, *Combinatorial Theory* Volume 3, Issue 2 (2023)

[16] N. Jacon and C. Lecouvey, Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of higher level, *Algebras and Rep. theory* 13, 467-489, 2010.

[17] N. Jacon and C. Lecouvey, On the Mullineux involution for Ariki-Koike algebras, *J. Algebra* Vol. 321, Issue 8, 15 April 2009, Pages 2156–2170.

[18] N. Jacon and C. Lecouvey, Kashiwara and Zelevinsky involutions in affine type A, *Pacific J. of Math.* Vol. 243, No. 2, 2009, 287–311.

[19] A. Kleshchev, Branching rules for modular representations of symmetric groups III: Some corollaries and a problem of Mullineux, *J. Lond. Math. Soc.* 54 (1996) 25–38.

[20] A. Lascoux, B. Leclerc, and Jean-Yves Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Comm. Math. Phys.* 181:205–263, 1996.

[21] I.Losev, Supports of simple modules in cyclotomic Cherednik categories O, *Adv. Math.* 377 (2021), 107491.

[22] C. Mœglin and J.-L. Waldspurger, Sur l'involution de Zelevinski, *J. Reine Angew. Math.* 372 (1986), 136–177.

[23] G. Mullineux, Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups, *J. London Math. Soc.* (2) 20 (1979), no.1, 60–66.

[24] M. Xu, On p-series and the Mullineux conjecture, *Comm. Algebra* 27 (1999), no. 11, 5255–5265.

[25] M. Xu, On Mullineux' conjecture in the representation theory of symmetric groups. *Comm. Algebra* 25 (1997), no. 6, 1797–1803.