On Symmetric bi-Cayley Graphs of Prime Valency on Nonabelian Simple Groups

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Abstract

Let Γ be a bipartite graph, and let $\mathrm{Aut}\Gamma$ be the full automorphism group of the graph Γ. A subgroup $G \leqslant \mathrm{Aut}\Gamma$ is said to be bi-regular on Γ if G preserves the bipartition and acts regularly on both parts of Γ, while the graph Γ is called a bi-Cayley graph of G in this case. A subgroup $X \leqslant \mathrm{Aut}\Gamma$ is said to be bi-quasiprimitive on Γ if the bipartition-preserving subgroup of X is a quasiprimitive group on each part of Γ.

In this paper, a characterization is given for the connected bi-Cayley graphs on nonabelian simple groups which have prime valency and admit bi-quasiprimitive groups.

Mathematics Subject Classifications: 05C25, 20B25

1 Introduction

All (di)graphs considered in this paper are assumed to be finite and simple, unless otherwise stated.

For a graph Γ, we use $V\Gamma$, $E\Gamma$ and $\mathrm{Aut}\Gamma$ to denote the vertex set, edge set and full automorphism, respectively. A subgroup R of $\mathrm{Aut}\Gamma$ is said to be bi-regular on Γ if R is semiregular on $V\Gamma$ with exact two orbits. If $\mathrm{Aut}\Gamma$ admits a bi-regular group R then Γ is called a bi-Cayley graph over R, refer to [19]. (Note, some authors have used the term semi-Cayley instead, see [9] for example.) In the past three decades, bi-Cayley graphs have been involved deeply in many fields of graph theory and played an important role. In particular, many research works have been published about bi-Cayley graphs regarding their strong regularity [9, 20, 30, 31], automorphism [26, 40], semisymmetry

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[27], classification [5, 19, 32, 39], connectivity [3, 25], extendability [17, 29] and spectrum [16].

In this paper, we deal with bi-Cayley graphs in some narrow sense as in [11], where the term bi-Cayley graph were first used to name those bi-partite graphs which admit bi-regular groups. Thus a bi-Cayley graph of R is isomorphic to a bipartite graph $\mathrm{BCay}(R, S)$ with vertex set $R \times \{0, 1\}$ such that (x, i) and (y, j) are adjacent if and only if $i \neq j$ and $yx^{-1} \in S$, where $S \subseteq R$. Up to graph isomorphism, the subset S may be chosen so that it does not contain the identity 1 of R, refer to [27, Lemma 2.2].

Let R be a group and $S \subseteq R \setminus \{1\}$. The Cayley digraph $\mathrm{Cay}(R, S)$ is a directed graph with vertex set R such that $x \in R$ is adjacent to $y \in R$ if and only if $yx^{-1} \in S$. For the case where S is inverse closed, $\mathrm{Cay}(R, S)$ can be viewed as a graph and called a Cayley graph of R. For a (di)graph Γ, the standard double cover $\Gamma^{(2)}$ of Γ is a bipartite graph with vertex set $V \times \mathrm{C}_2$ such that $\{(u_1, 0), (u_2, 1)\}$ is an edge if and only if u_1 is adjacent to u_2 in Γ. Thus every bi-Cayley graph $\mathrm{BCay}(R, S)$ with $1 \notin S$ is in fact the standard double cover of the Cayley digraphs $\mathrm{Cay}(R, S)$.

In the literature, Cayley (di)graphs on nonabelian simple groups have received considerable attention, and a quite number of papers have addressed questions regarding their normality and symmetry, see [10, 12, 13, 14, 21, 24, 28, 36, 38] for references. This and the fact that bi-Cayley graphs are standard double covers of Cayley digraphs provide us the main motivation to investigate bi-Cayley graphs on nonabelian simple groups under certain limitations. In this paper, we give a characterization for bi-Cayley graphs on nonabelian simple groups which have prime valency and admit bi-quasiprimitive groups.

Let Γ be a connected graph, and $X \leqslant \mathrm{Aut}\Gamma$. An arc in Γ is an ordered pair of adjacent vertices, and a 2-arc is a triple (α, β, γ) of vertices with $\alpha \neq \gamma$ and $\{\alpha, \beta\}, \{\beta, \gamma\} \in E\Gamma$. The graph Γ is called X-vertex-transitive, X-edge-transitive, X-symmetric or $(X, 2)$-arc-transitive if X acts transitively on the vertex set, the edge set, the arc set or the 2-arc set of Γ, respectively. Assume that Γ is bipartite, and let X^+ be the bipartition preserving subgroup of X. Then X is said to be bi-quasiprimitive on Γ if X^+ induces a quasiprimitive permutation group on each part of Γ. Recall that a permutation group is quasiprimitive if its non-trivial normal subgroups are all transitive.

The main result of this paper is stated as follows.

Theorem 1. *Let T be a nonabelian simple group, and let Γ be a connected bi-Cayley graph on T of prime valence p. Assume that Γ is X-symmetric and X is bi-quasiprimitive on Γ, where $T < X \leqslant \mathrm{Aut}\Gamma$. Then one of the following cases holds:*

(1) $T \trianglelefteq X$ and X is almost simple;

(2) $X = X^+ \times \langle o \rangle$, X^+ is an almost simple group, o is an involution, and Γ is the standard double cover of the complete graph K_{p+1} or some X^+-symmetric Cayley graph Σ on T of valency p;

(3) Γ is isomorphic to one of the five graphs given in Example 5;

(4) $p \geqslant 5$, $X = \mathrm{S}_n$ and $T = \mathrm{A}_{n-1}$, where n is divisible by p.

Remark 2. For Theorem 1 (2), if $T \trianglelefteq X^+$ then $T \trianglelefteq X$, if $T \ntrianglelefteq X^+$ then the graph Σ is either the complete graph K_{p+1} or described as in [10, Theorem 1.1], [24, Theorem 1.1], [37, Theorem 5.1] and [38, Theorems 1.3 and 1.4]. As for Theorem 1 (4), some explanations are given at the end of this paper.

To end this section, we give some notations which are used in this paper. For the group-theoretic terminology not defined here we refer the reader to [6, 35]. For two groups K and H, denoted by $K.H$ an arbitrary extension of K by H, and by $K{:}H$ a semidirect product of K by H. For a group G, use $\mathrm{soc}(G)$ to denote the socle of G.

2 The groups X, X^+ and T

Let T be a nonabelian simple group, let Γ be a connected bi-Cayley graph on T of valency an odd prime p, and let $T < X \leqslant \mathrm{Aut}\Gamma$. Let Δ_1 and Δ_2 be the X^+-orbits on $V\Gamma$. In the following, we assume that Γ is X-symmetric and X is bi-quasiprimitive on Γ.

Since T is nonabelian simple, it is easily shown that $T \leqslant X^+$. If X^+ is unfaithful on Δ_1 or Δ_2 then Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{p,p}$, yielding $|T| = p$, which is impossible. Thus we may consider X^+ as a quasiprimitive group is faithful on each of Δ_1 and Δ_2.

For $\alpha \in V\Gamma$, let $X_\alpha = \{x \in X \mid \alpha^x = \alpha\}$ and $\Gamma(\alpha) = \{\beta \in V\Gamma \mid \{\alpha, \beta\} \in E\Gamma\}$, called the stabilizer of α in X and the neighborhood of α in Γ, respectively. It is easy to show that X_α is a subgroup of X^+.

Let $G = X^+$, and $\alpha \in \Delta_1 \cup \Delta_2$. Then $G = X_\alpha T$ is an exact factorization, where exact means that $X_\alpha \cap T = 1$. By [22, Theorem 1.7], as a quasiprimitive group on Δ_1 or Δ_2, $T \leqslant \mathrm{soc}(G)$ and one of the following holds:

(C1) $\mathrm{soc}(G) = T$;

(C2) $\mathrm{soc}(G) \cong T \times T$, and either $T \trianglelefteq G$ or $\mathrm{soc}(G)_\alpha \cong T$;

(C3) G is almost simple, $\mathrm{soc}(G) \neq T$ and $G = X_\alpha T$ is one of those exact factorizations in [2, Theorem 3] and [22, Theorem 1.2] with a simple factor T.

We next deal with the case (C3) by producing a possible list for (X, G, X_α, T) from [2, Theorem 3] and [22, Theorem 1.2] when X is also almost simple.

Denote by $X_\alpha^{[1]}$ the kernel of X_α acting on the neighborhood $\Gamma(\alpha)$ of α in Γ. Then $X_\alpha^{\Gamma(\alpha)} \cong X_\alpha/X_\alpha^{[1]}$. Let $\beta \in \Gamma(\alpha)$, and consider the action of $X_{\alpha\beta}$ on $\Gamma(\beta)$. We have

$$X_\alpha^{[1]}/(X_\alpha^{[1]} \cap X_\beta^{[1]}) \cong (X_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq X_{\alpha\beta}^{\Gamma(\beta)} = (X_\beta^{\Gamma(\beta)})_\alpha \cong (X_\alpha^{\Gamma(\alpha)})_\beta.$$

Note that $X_\alpha^{\Gamma(\alpha)}$ is a transitive permutation group on $\Gamma(\alpha)$ of degree p. Then $X_\alpha^{\Gamma(\alpha)}$ is known up to permutation isomorphism, refer to [8, page 99]:

(i) $X_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}_1(p)$;

(ii) $X_\alpha^{\Gamma(\alpha)} = \mathrm{A}_p$ or S_p, where $p \geqslant 7$;

(iii) $X_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}_2(11)$, $(X_\alpha^{\Gamma(\alpha)})_\beta \cong \mathrm{A}_5$ and $p = 11$;

(iv) $X_\alpha^{\Gamma(\alpha)} = \mathrm{M}_p$, $(X_\alpha^{\Gamma(\alpha)})_\beta = \mathrm{M}_{p-1}$ and $p \in \{11, 23\}$;

(v) $\mathrm{PSL}_d(q) \trianglelefteq X_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{P\Gamma L}_d(q)$, and $p = \frac{q^d-1}{q-1}$, where d is a prime.

Lemma 3. *Assume that X is almost simple and $\mathrm{soc}(X) = \mathrm{soc}(X^+) \neq T$. Then (X, G, H, T, p) is listed as in Table 1, where $G = X^+$, and $H = X_\alpha$ for some $\alpha \in V\Gamma$.*

Row	X	G	H	T	p	Remark
0	S_{12}	A_{12}	$\mathrm{C}_2 \times \mathrm{S}_3$	A_{11}	3	
1	S_{24}	A_{24}	S_4	A_{23}	3	
2	S_{48}	A_{48}	$\mathrm{C}_2 \times \mathrm{S}_4$	A_{47}	3	
3	S_n	A_n	$\lvert X_\alpha\rvert = n$	A_{n-1}	$\geqslant 5$	$p \mid n$, $n > 5, H$ has no cyclic Sylow 2-subgroup unless n is odd
4	$\mathrm{PGL}_2(11)$	$\mathrm{PSL}_2(11)$	C_{11}	A_5	11	
5	$\mathrm{PGL}_2(29)$	$\mathrm{PSL}_2(29)$	$\mathrm{C}_{29}{:}\mathrm{C}_7$	A_5	29	
6	$\mathrm{PGL}_2(59)$	$\mathrm{PSL}_2(59)$	$\mathrm{C}_{59}{:}\mathrm{C}_{29}$	A_5	59	
7	$\mathrm{M}_{12}.2$	M_{12}	D_{12}	M_{11}	3	
8	S_{11}	A_{11}	A_7	M_{11}	7	
9	S_{11}	A_{11}	M_{11}	A_7	11	
10	S_{12}	A_{12}	A_7	M_{12}	7	
11	S_n	A_n	A_{n-1}	$\lvert T\rvert = n$	$n-1$	
12	S_{q+1}	A_{q+1}	$\mathrm{PSL}_2(q)$	A_{q-2}	$p = q+1$	$q = 2^{2^s} > 2$
13	S_{q+1}	A_{q+1}	S_{q-2}	$\mathrm{PSL}_2(q)$	$p = q-2$	$q \equiv 3 \pmod 4$
14	$\Omega_8^+(2).2$	$\Omega_8^+(2)$	$\mathrm{C}_2^4{:}\mathrm{A}_5$	A_9	5	
15	$\Omega_8^+(2).2$	$\Omega_8^+(2)$	S_5	$\mathrm{Sp}_6(2)$	5	

Table 1: Almost simple groups X with $T \neq \mathrm{soc}(X^+)$

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Proof. By the foregoing analysis, either $X_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}_1(p)$ or $X_\alpha^{\Gamma(\alpha)}$ is insolvable.

Case 1. Assume that $X_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{AGL}_1(p)$. By [34, Theorem 4.7], either $p = 3$ or $X_\alpha^{[1]} \cap X_\beta^{[1]} = 1$, where $\beta \in \Gamma(\alpha)$. For $p = 3$, we have $X_\alpha \cong \mathrm{C}_3$, S_3, $\mathrm{C}_2 \times \mathrm{S}_3$, S_4 or $\mathrm{C}_2 \times \mathrm{S}_4$, refer to [1, 18C, page 126]. Noting that $X_\alpha = ((X_\alpha^{[1]} \cap X_\beta^{[1]}).(X_\alpha^{[1]})^{\Gamma(\beta)}).X_\alpha^{\Gamma(\alpha)}$ and $(X_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (X_\beta^{\Gamma(\beta)})_\alpha$, if $X_\alpha^{[1]} \cap X_\beta^{[1]} = 1$ then $X_\alpha = \mathrm{C}_{l'}.(\mathrm{C}_p{:}\mathrm{C}_l) = (\mathrm{C}_{l'} \times \mathrm{C}_p).\mathrm{C}_l$, where $l' \mid l \mid (p-1)$. Now X_α is solvable, and the almost simple group $G = X^+$ has an exact factorization $G = HT$ with $H = X_\alpha$ solvable. Then T is one of the groups K described as [2, Theorem 3]. Since T is a nonabelian simple group, (iii) of [2, Theorem 3] does not occur here. If (v) of [2, Theorem 3] holds then we get Rows 4-7 of Table 1 by comparing X_α and those H listed in [2, Table 4].

Suppose that (i) of [2, Theorem 3] holds. Then $\mathrm{soc}(G) = \mathrm{A}_n$, $T = \mathrm{A}_{n-1}$ and H is transitive on $\Omega := \{1, 2, \ldots, n\}$, where $n \geqslant 6$. Since $|X : G| = 2$ and X is almost simple, either $G = \mathrm{A}_n$ and $X = \mathrm{S}_n$, or $n = 6$ and $X \neq \mathrm{S}_6$. For the latter case, Γ is a connected symmetric cubic graph of order 120, and then $|\mathrm{Aut}\Gamma| = 720$ by [4], it follows that $G = \mathrm{A}_6$ and $X = \mathrm{M}_{10}$ or $\mathrm{PGL}_2(9)$. Thus $G = \mathrm{A}_n$ for both cases, and $|H| = |G : T| = n$. In particular, p is a divisor of n as $H = X_\alpha$ acts transitively on $\Gamma(\alpha)$. Clearly, H is regular on Ω. Assume that H contains an element o of order 2^s. Then o is a product of $\frac{n}{2^s}$ cycles of length 2^s. If $s > 0$ then, since a 2^s-cycle is an odd permutation, $\frac{n}{2^s}$ must be even, and so $\langle o \rangle$ is not a Sylow 2-subgroup of H. Therefore either n is odd or H has no cyclic Sylow 2-subgroup; in particular, $n \neq 2p$. Now, if $p \geqslant 5$ then we have Row 3 of Table 1, if $p = 3$ then one of Rows 0-2 of Table 1 holds.

Suppose that (ii) of [2, Theorem 3] holds. Then we have $G = \mathrm{A}_{r^a}$, $T = \mathrm{A}_{r^a-2}$, $H \lesssim \mathrm{A\Gamma L}_1(r^a)$, and H is a 2-homogeneous subgroup of A_{r^a} in the natural action, where r is a prime. In particular, H has a unique minimal normal subgroup, say C_r^a. If $p = 3$ then, since $H = X_\alpha \cong \mathrm{C}_3$, S_3, $\mathrm{C}_2 \times \mathrm{S}_3$, S_4 or $\mathrm{C}_2 \times \mathrm{S}_4$, we have $r^a \leqslant 4$, which is impossible as T is a nonabelian simple group. Thus $p \geqslant 5$, and so H has a minimal normal subgroup C_p. It follows that $p = r^a$, yielding $p = r$, $a = 1$ and $H \cong \mathrm{AGL}_1(p)$. Then H contains a $(p-1)$-cycle, which is an odd permutation. Thus we have a contradiction as H is a subgroup of A_p.

Suppose finally that (iv) of [2, Theorem 3] holds, that is, $\mathrm{soc}(G) = \mathrm{PSp}_{2m}(q)$, $H \cap \mathrm{soc}(G) \leqslant q^m{:}(q^m-1).m$ and $T \cap \mathrm{soc}(G) = \Omega^-_{2m}(q)$, where $m \geqslant 3$ and $q = 2^f$. Since T is a nonabelian simple group, we have $T = \Omega^-_{2m}(q) \leqslant \mathrm{soc}(G)$. This leads an exact factorization $\mathrm{soc}(G) = (H \cap \mathrm{soc}(G))T$. In particular, $|H \cap \mathrm{soc}(G)| = q^m(q^m - 1)$, and $|X_\alpha|$ is divisible by $q^m(q^m - 1)$. If $p = 3$ then we have $q^m - 1 = 3$, yielding $m = 2 < 3$, a contradiction. Therefore, $p \geqslant 5$. Pick a Sylow 2-subgroup Q of $H \cap \mathrm{soc}(G)$, and let $P = \mathrm{C}_2^{mf} \trianglelefteq q^m{:}(q^m-1).m$. Then $|Q| = |P| = 2^{mf}$. Write $m = 2^s m_0$ for an odd integer m_0. Then every Sylow 2-subgroup of $q^m{:}(q^m-1).m$ has order 2^{mf+s}. Noting that PQ is a 2-subgroup of $q^m{:}(q^m-1).m$, it follows that $|PQ|$ is a divisor of 2^{mf+s}. Since $|PQ| = \frac{|P||Q|}{|P\cap Q|} = \frac{2^{2mf}}{|P\cap Q|}$, we conclude that $|P \cap Q|$ has a divisor 2^{mf-s}. Thus $H = X_\alpha$ has a subgroup C_2^{mf-s}. Recall that $H = X_\alpha = (\mathrm{C}_{l'} \times \mathrm{C}_p).\mathrm{C}_l$, where $l' \bigm| l \bigm| p-1$. Checking the orders of elementary abelian 2-subgroups of H, we conclude that $2^{mf-s} \leqslant 4$. Then, since $m > 2$, we have $m = 4$ and $q = 2$, and so $|H \cap \mathrm{soc}(G)| = q^m(q^m - 1) = 2^4 \cdot 15$. In particular, 5 is the largest prime divisor of $|X_\alpha|$. This forces that Γ has valency $p = 5$, and then $|H| = |X_\alpha| = 5l'l$ with $l' \bigm| l \bigm| 4$, a contradiction.

Case 2. Assume that $X_\alpha^{\Gamma(\alpha)}$ is insolvable. Then we may read out (X, G, X_α, T) from [22, Table 1.1]. Rows 5, 8, 12-14, 16-18 and 21 in [22, Table 1.1] are not in our consideration as the corresponding factorizations have no simple factor. By [7, Lemma 1.1], p is the largest prime divisor of $|X_\alpha|$ and $p^2 \nmid |X_\alpha|$. This excludes Rows 9-11 in [22, Table 1.1]. Noting that $|X : G| = 2$ and X is almost simple, Rows 15, 22 and 23 in [22, Table 1.1] are excluded.

For Row 4 in [22, Table 1.1], we have $X_\alpha = (\mathrm{A}_5 \times \mathrm{C}_3).\mathrm{C}_2$, forcing $X_\alpha^{\Gamma(\alpha)} = \mathrm{P\Gamma L}_2(4)$, $(X_\alpha^{\Gamma(\alpha)})_\beta \cong \mathrm{S}_4$ and $X_\alpha^{[1]} = \mathrm{C}_3$. By [34, Theorem 4.1], $X_\alpha^{[1]} \cap X_\beta^{[1]}$ is a 2-group for $\beta \in \Gamma(\alpha)$, and thus $X_\alpha^{[1]} \cap X_\beta^{[1]} = 1$. Then

$$\mathrm{C}_3 = X_\alpha^{[1]} \cong (X_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq (X_\beta^{\Gamma(\beta)})_\alpha \cong (X_\alpha^{\Gamma(\alpha)})_\beta \cong \mathrm{S}_4,$$

which is impossible.

Finally, inspecting the left 7 rows in [22, Table 1.1], we get Rows 3,8-15 of Table 1. □

3 Graphs arising from Table 1

It is well-known that every connected symmetric graph may be represented as a coset graph defined as follows.

Let X be a finite group, and $K < H < X$ such that H is core-free in X. Suppose that

(I) there exists an element $z \in \mathbf{N}_X(K) \setminus H$ such that $X = \langle H, z\rangle$, $z^2 \in K$ and $H^z \cap H = K$.

Define a graph $\mathrm{Cos}(X, H, K, z)$ on $[X : H] := \{Hx \mid x \in X\}$ such that $\{Hx, Hy\}$ is an edge if and only if $yx^{-1} \in HzH \setminus H$. Then $\mathrm{Cos}(X, H, K, z)$ is a connected X-symmetric graph of valency $k := |H : K|$, where X acts on $[X : H]$ by right multiplication, and the subgroups H, K and $\langle K, z\rangle$ serve as a vertex-stabilizer, an arc-stabilizer and an edge-stabilizer respectively.

It follows from (I) that z has even order, say $2^s m$ for $s > 0$ and odd m. Then $z = hz_0$, where h is a power of z^{2^s} and z_0 is a power of z^m. It is easily shown that $h \in K$, $HzH = Hz_0H$, $z_0 \in \mathbf{N}_X(K) \setminus H$, $X = \langle H, z_0\rangle$, $z_0^2 \in K$ and $H^{z_0} \cap H = K$. Thus, it is sufficient to consider those 2-elements z satisfying (I) when we determine the existence or construct connected symmetric graphs from a given triple (X, H, K).

For an automorphism $\phi \in \mathrm{Aut}X$, we have a bijection $Hx \mapsto H^\phi x^\phi$, $x \in X$, which in fact a graph isomorphism from $\mathrm{Cos}(X, H^\phi, K^\phi, z^\phi)$. Thus, in practices, we always choose the subgroup H up to the conjugation under $\mathrm{Aut}X$, while the subgroup K is chosen up to the conjugation under $\mathrm{Aut}(X, H) := \{\phi \in \mathrm{Aut}X \mid H^\phi = H\}$. Given a triple (X, H, K) and two elements z' and z'' satisfying the condition (I) above. If $Hz'H = Hz''H$ then $\mathrm{Cos}(X, H, K, z') = \mathrm{Cos}(X, H, K, z'')$, and if $z'' = (z')^x$ for some $\mathbf{N}_X(H)$ then $\mathrm{Cos}(X, H, K, z') \cong \mathrm{Cos}(X, H, K, z'')$. These observations will greatly help us deal with the triples (X, H, p) listed in Table 1.

Denote by (X_i, G_i, H_i, p_i) the quadruple described as in Row i of Table 1, where

$$i \in \{0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 14, 15\}.$$

It is easy to see that all subgroups of H_i with index p_i are conjugate. Note that H_i is recorded as in Table 1 up to isomorphism. Let n_i be the number of conjugacy classes of subgroups in G_i isomorphic to H_i. Then

$$n_0 = n_1 = n_2 = n_4 = n_5 = n_6 = n_8 = n_{10} = 1,\ n_9 = 2,\ n_7 = n_{14} = 3,\ n_{15} = 15.$$

For each representative for H_i up to conjugacy, still denoted by H_i, we fix a subgroup K_i of H_i with index p_i. Computation by GAP [15] shows that

(i) For $i \in \{0, 2, 7, 8, 10, 14, 15\}$, the normalizer $\mathbf{N}_{X_i}(K_i)$ does not contain 2-element z satisfying the condition (I);

(ii) For $i \in \{1, 4, 5, 6, 9\}$, up to isomorphism of graphs, the pair (X_i, H_i) produces a unique symmetric graph $\mathrm{Cos}(X_i, H_i, K_i, z_i)$ of valency p_i, where (X_i, H_i, K_i, z_i) is recorded as in Example 5.

Then we have the following lemma.

Lemma 4. *Let Γ be a connected X-symmetric graph of prime valency with a vertex stabilizer H. Assume that (X, H) is one of the pairs listed in Rows* 0-2, 4-10, 14, 15 *of Table* 1. *Then Γ is isomorphic to one of the five graphs given in Example* 5.

Example 5. For each $i \in \{1, 4, 5, 6, 9\}$, the coset graph $\mathrm{Cos}(X_i, H_i, K_i, z_i)$ is a connected symmetric bi-Cayley graph on T_i of prime valency p, where

(1) $X_1 = \mathrm{S}_{24}$, $H_1 = \langle a_1, b_1\rangle \cong \mathrm{S}_4$, $K_1 = \langle c_1, d_1\rangle$, $T_1 = \mathrm{A}_{23}$, $p = 3$:

$$\begin{aligned}
z_1 =&(3,21)(5,23)(6,24)(7,16)(8,15)(9,20)(10,19)(11,22)(12,14),\\
a_1 =&(1,10,17,19)(2,9,18,20)(3,12,14,21)(4,11,13,22)(5,7,16,23)\\
&(6,8,15,24),\\
b_1 =&(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)(13,15)(14,16)(17,18)(19,21)\\
&(20,22)(23,24),\\
c_1 =&(1,15)(2,16)(3,13)(4,14)(5,18)(6,17)(7,9)(8,10)(11,12)(19,24)\\
&(20,23)(21,22),\\
d_1 =&(1,24)(2,23)(3,22)(4,21)(5,20)(6,19)(7,18)(8,17)(9,16)(10,15)\\
&(11,14)(12,13).
\end{aligned}$$

(2) $X_4 = \mathrm{PGL}_2(11)$, $H_4 = \langle a_4\rangle$, $K_4 = 1$, $T_4 \cong \mathrm{A}_5$, $p = 11$:

$$z_4 = (1,8)(2,5)(3,10)(4,6)(7,9),\ \ a_4 = (2,8,9,6,10,12,7,5,11,4,3).$$

(3) $X_5 = \mathrm{PGL}_2(29)$, $H_5 = \langle a_5, b_5\rangle$, $K_5 = \langle c_5\rangle$, $T_5 \cong \mathrm{A}_5$, $p = 29$:

$$\begin{aligned}
z_5 =&(1,2)(3,30)(4,29)(5,28)(6,27)(7,26)(8,25)(9,24)(10,23)(11,22)(12,21)\\
&(13,20)(14,19)(15,18)(16,17),\\
a_5 =&(3,19,7,23,11,27,15)(4,20,8,24,12,28,16)(5,21,9,25,13,29,17)\\
&(6,22,10,26,14,30,18),\\
b_5 =&(2,24,25,29,26,18,30,8,27,6,19,21,3,14,9,23,28,17,7,5,20,13,22,16,\\
&4,12,15,11,10),\\
c_5 =&(3,19,7,23,11,27,15)(4,20,8,24,12,28,16)(5,21,9,25,13,29,17)\\
&(6,22,10,26,14,30,18).
\end{aligned}$$

(4) $X_6 = \mathrm{PGL}_2(59)$, $H_4 = \langle a_6, b_6\rangle$, $K_6 = \langle c_6\rangle$, $T_6 \cong \mathrm{A}_5$, $p = 59$:

$$\begin{aligned} z_6 =& (1,2)(4,60)(5,59)(6,58)(7,57)(8,56)(9,55)(10,54)(11,53)(12,52)(13,51)\\ &(14,50)(15,49)(16,48)(17,47)(18,46)(19,45)(20,44)(21,43)(22,42)\\ &(23,41)(24,40)(25,39)(26,38)(27,37)(28,36)(29,35)(30,34)(31,33),\\ a_6 =& (2,27,28,19,29,33,20,45,30,11,34,52,21,14,46,25,31,9,12,7,35,37,53,\\ &42,22,39,15,3,47,55,26,18,32,44,10,51,13,24,8,6,36,41,38,60,54,17,\\ &43,50,23,5,40,59,16,49,4,58,48,57,56),\\ b_6 =& (3,13,23,33,43,53,5,15,25,35,45,55,7,17,27,37,47,57,9,19,29,39,49,\\ &59,11,21,31,41,51)(4,14,24,34,44,54,6,16,26,36,46,56,8,18,28,38,48,\\ &58,10,20,30,40,50,60,12,22,32,42,52),\\ c_6 =& (3,13,23,33,43,53,5,15,25,35,45,55,7,17,27,37,47,57,9,19,29,39,49,\\ &59,11,21,31,41,51)(4,14,24,34,44,54,6,16,26,36,46,56,8,18,28,38,48,\\ &58,10,20,30,40,50,60,12,22,32,42,52). \end{aligned}$$

(5) $X_9 = \mathrm{S}_{11}$, $H_9 = \langle a_9, b_9\rangle \cong \mathrm{M}_{11}$, $K_9 = \langle c_9, d_9\rangle$, $T_9 \cong \mathrm{A}_7$, $p = 11$:

$$\begin{aligned} z_9 =& (2,5)(4,9)(7,10),\ a_9 = (1,4,7,6)(2,11,10,9),\ b_9 = (1,10)(2,8)(3,11)(5,7),\\ c_9 =& (2,3,8,4)(5,6,9,10),\ d_9 = (1,5)(2,10,7,6,3,8,9,4). \end{aligned}$$

4 Proof of Theorem 1.1

Let T be a nonabelian simple group, and let Γ be a connected bi-Cayley graph on T of valency an odd prime p with bi-parts Δ_1 and Δ_2. In the following, assume that a subgroup $X \leqslant \mathrm{Aut}\Gamma$ is symmetric and bi-quasiprimitive on Γ, and $T < X^+$. Noting that $|\Delta_1| = |T| = |\Delta_2|$, since T is nonabelian simple, we conclude that Γ is a not a complete bipartite graph. In particular, X^+ is faithful on both Δ_1 and Δ_2.

Lemma 6. *One of the following holds:*

(1) $X = X^+ \times \langle o\rangle$, X^+ is almost simple and $T \leqslant \mathrm{soc}(X^+)$, where o is an involution;

(2) X is almost simple, and $T \leqslant \mathrm{soc}(X^+) = \mathrm{soc}(X)$.

Proof. Recalling that X^+ is a quasiprimitive group on Δ_1, by [22, Theorem 1.7], $T \leqslant \mathrm{soc}(X^+)$, and either $\mathrm{soc}(X^+)$ is simple or $\mathrm{soc}(X^+) = T \times L$, where $L \cong T$.

Suppose that $\mathrm{soc}(X^+) = T \times L$. For $\alpha \in \Delta_1$, we have $T \times L = \mathrm{soc}(X^+) = T\mathrm{soc}(X^+)_\alpha$, yielding $\mathrm{soc}(X^+)_\alpha \cong L \cong T$. In particular, X_α is insolvable, and hence Γ is $(X,2)$-arc-transitive. On the other hand, appealing to the [33, Theorem 2.3], either $\mathrm{soc}(X^+)$ is regular on Δ_1 or $\mathrm{soc}(X^+)_\alpha \leqslant H \times K$ for some $H < T$ and $K < L$, a contradiction. Therefore, $\mathrm{soc}(X^+)$ is simple.

Assume that X has a minimal normal subgroup N such that $N \not\leqslant X^+$. Then $N \cap X^+ = 1$, and $X = X^+N$ as $|X : X^+| \leqslant 2$. It follows that $X = X^+ \times N$ and $|N| = 2$. Then part (1) of the lemma follows.

Now assume that every minimal normal subgroup of X is contained in X^+. Then each minimal normal subgroup of X has at most two orbits on $V\Gamma = \Delta_1 \cup \Delta_2$. By [23, Theorem1.1], $\mathrm{soc}(X)$ is the unique minimal normal subgroup of X. Since $\mathrm{soc}(X^+)$ is characteristic in X^+, we know $\mathrm{soc}(X^+) \trianglelefteq X$ due to $X^+ \trianglelefteq X$. Since $\mathrm{soc}(X^+)$ is simple, one has $\mathrm{soc}(X) = \mathrm{soc}(X^+)$, and part (2) of this lemma occurs. □

Lemma 7. *Assume that $X = X^+ \times \langle o \rangle$, where o is an involution. Then Γ is isomorphic to the standard double cover of some X^+-symmetric Cayley graphs of valency p on T.*

Proof. Pick $\delta_1 \in \Delta_1$. Then $\delta_2 := \delta_1^o \in \Delta_2$, $\Delta_1 = \{\delta_1^g \mid g \in T\}$ and $\Delta_2 = \{\delta_2^g \mid g \in T\}$. Let $S = \{g \mid \delta_2^g \in \Gamma(\delta_1)\}$. Then $|S| = p$. Since X_{δ_1} acts transitively on $\Gamma(\delta_1)$, we have $|X_{\delta_1} : X_{\delta_1\beta}| = p$ for each $\beta \in \Gamma(\delta_1)$. In particular, $X_{\delta_1} \neq X_\beta$. Thus $\delta_2 \notin \Gamma(\delta_1)$ as $X_{\delta_1} = X_{\delta_2}$, yielding $1 \notin S$. For $g \in S$, since $\delta_2^g \in \Gamma(\delta_1)$, we have $\delta_1^{g^{-1}} \in \Gamma(\delta_2)$, and so $\delta_2^{g^{-1}} = (\delta_1^{g^{-1}})^o \in \Gamma(\delta_2^o) = \Gamma(\delta_1)$. This yields that $S = S^{-1}$. Then we have a Cayley graph $\Sigma = \mathrm{Cay}(T, S)$. Define

$$\phi : V\Gamma \to V\Sigma \times \mathrm{C}_2,\ \delta_1^{o^i g} \mapsto (g, i).$$

It is easily shown ϕ is an isomorphism from Γ to $\Sigma^{(2)}$. Then the only thing left is to equip Σ with X^+ as an arc-transitive graph.

Since T is regular on Δ_1, for any given $g \in T$ and $x \in X^+$, there is a unique $g_x \in T$ such that $\delta_1^{gx} = \delta_1^{g_x}$. By a routine examination, we get a faithful action of X^+ on T by

$$g^x := g_x,\ g \in T, x \in X^+,$$

while T acts on $V\Sigma$ by right multiplication, and X_{δ_1} fixes the vertex δ_1 and acts transitively on S. This completes the proof. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By Lemmas 6 and 7, either one of Theorem 1 (1) and (2) holds, or X is almost simple and $T < \mathrm{soc}(X^+) = \mathrm{soc}(X)$. For the latter case, by Lemma 4, either (3) of Theorem 1 holds or (X, H) is one of the pairs described as in Rows 3, 11, 12 and 13 of Table 1. For Row 11 of Table 1, X^+ acts 2-transitively on each Δ_1 and Δ_2, this forces that Γ is the standard double cover of K_{p+1}, desired as in Theorem 1 (2). If Row 3 of Table 1 holds for (X, H), then have Theorem 1 (4).

Next we assume that (X, H) is one of the pairs in Rows 12 and 13 of Table 1. Let $\{\alpha, \beta\} \in E\Gamma$, $H = X_\alpha$ and $K = X_{\alpha\beta}$. Write $\Gamma = \mathrm{Cos}(X, H, K, z)$, where z satisfies the condition (I) in Section 3. In particular, $X_{\{\alpha,\beta\}} = K\langle z \rangle = K.\mathrm{C}_2$.

Case 1. Suppose that (X, H) is described as in Row 12 of Table 1. Then $H = \mathrm{PSL}_2(q)$, $X = \mathrm{S}_{q+1}$ and $X^+ = \mathrm{A}_{q+1}$, where $q = 2^{2^s} > 2$ and $p = q + 1$. Considering the natural action of S_{q+1} on $\Omega = \{1, 2, \ldots, q + 1\}$, the vertex-stabilizer H is a sharply 3-transitive subgroup of S_{q+1}, and the arc-stabilizer K is the stabilizer of some point, say $q + 1$, in Ω.

Then K a sharply 2-transitive subgroup of S_q acting on $\Omega_0 = \{1, 2, \ldots, q\}$. It is easy to see that $\mathbf{N}_X(K)$ fixes the point $q+1$, and thus $\mathbf{N}_X(K) = \mathbf{N}_{\mathrm{S}_q}(K)$. Note $K = E{:}C$ with $E \cong \mathrm{C}_2^{2^s}$ and $C \cong \mathrm{C}_{2^{2^s}-1}$. Then E is a characteristic subgroup of K, and E is regular on Ω_0. Then $\mathbf{N}_{\mathrm{S}_q}(K) \leqslant \mathbf{N}_{\mathrm{S}_q}(E)$. Viewing Ω_0 as the 2^s-dimensional vector space over the field of order 2, it follows that $\mathbf{N}_{\mathrm{S}_q}(K) \leqslant \mathbf{N}_{\mathrm{S}_q}(E) = E{:}\mathrm{GL}_{2^s}(2)$. Then

$$\mathbf{N}_{\mathrm{S}_q}(K) = \mathbf{N}_{\mathrm{S}_q}(K) \cap \mathbf{N}_{\mathrm{S}_q}(E) = E{:}(\mathbf{N}_{\mathrm{S}_q}(K) \cap \mathrm{GL}_{2^s}(2)) \leqslant E{:}\mathbf{N}_{\mathrm{GL}_{2^s}(2)}(C).$$

By [18, page 187, II.7.3], we write $\mathbf{N}_{\mathrm{GL}_{2^s}(2)}(C) = C{:}D$, where $D \cong \mathrm{C}_{2^s}$. Then $z \in \mathbf{N}_X(K) = \mathbf{N}_{\mathrm{S}_q}(K) \leqslant E{:}(C{:}D)$. Write $z = ecd$, where $e \in E$, $c \in C$ and $d \in D$. Then $K.\mathrm{C}_2 = K\langle z\rangle = K\langle d\rangle$, which forces that d is an involution. Thus $|C\langle d\rangle| = 2|C|$. On the other hand, $\mathbf{N}_H(C)$ has order $2|C|$. Noting that $C{:}D$ has a unique subgroup of order $2|C|$, we get $C\langle d\rangle = \mathbf{N}_H(C)$. Then $\langle H, z\rangle \leqslant \langle H, e, c, d\rangle = \langle H, d\rangle = H \neq X$, a contradiction.

Case 2. Suppose that (X, H) is described as in Row 13 of Table 1. Then $p = q-2$ and $K \cong \mathrm{S}_{q-3}$. Consider the natural action of S_{q+1} on $\Omega = \{1, 2, \ldots, q+1\}$. Since $\mathrm{S}_{q-2} \cong H < \mathrm{A}_{q+1}$, either H has three orbits on Ω with length 1, 2 and $q-2$, or $q = 7$ and H has two orbits on Ω with length 2 and 6.

Assume that $q = 7$ and H has two orbits on Ω, say Ω_1 and Ω_2 of length 6 and 2, respectively. In this case, K acts transitively on Ω_1 and fixes Ω_2 setwise. It follows that $\mathbf{N}_X(K)$ fixes Ω_2 setwise. Then $\langle H, z\rangle$ is not transitive on Ω, a contradiction.

Assume H has three orbits on Ω say, without of generality, $\Omega_1 = \{1, 2, \ldots, q-2\}$, $\Omega_2 = \{q-1, q\}$ and $\Omega_3 = \{q+1\}$. Then Ω_2 and Ω_3 are K-orbits. Noting that $4 \leqslant q-3 \neq 5$, we conclude that K fixes one point in Ω_1, say $q-2$, and acts transitively on $\Omega_1 \setminus \{q-2\}$. Then $\mathbf{N}_X(K)$ fixes Ω_2 setwise. Thus $\langle H, z\rangle \neq \mathrm{S}_{q+1}$, a contradiction. □

We end this paper by some remarks on Theorem 1 (4).

Remark 8. Suppose that H is a regular subgroup of the alternating group A_n, where n is divisible by a prime $p \geqslant 5$. Then all regular subgroups isomorphic to H are conjugate in S_n, see [37, Lemma 4.6]. It is easily shown that H is core-free in S_n. Suppose further that H contains a subgroup of index p, and there exists a 2-element $z \in \mathrm{S}_n$ satisfying the condition (I) given in Section 3. Then we have a connected S_n-symmetric graph $\mathrm{Cos}(\mathrm{S}_n, H, K, z)$, which has valency p and vertex set $[\mathrm{S}_n : H]$. Clearly, $z \notin \mathrm{A}_n$, and A_n has two orbits on $[\mathrm{S}_n : H]$, say $[\mathrm{A}_n : H]$ and $[\mathrm{A}_n : H]z := \{Hxz \mid x \in \mathrm{A}_n\}$. It follows that Σ is a bipartite graph with the bipartition $([\mathrm{A}_n : H], [\mathrm{A}_n : H]z)$.

Consider the natural action of S_n on $\{1, 2, \ldots, n\}$, and view S_{n-1} as the stabilizer of n in S_n. Then we have exact factorizations $\mathrm{S}_n = H\mathrm{S}_{n-1}$ and $\mathrm{A}_n = H\mathrm{A}_{n-1}$. By $\mathrm{A}_n = H\mathrm{A}_{n-1}$, we know that A_{n-1} acts regularly on $[\mathrm{A}_n : H]$. By $\mathrm{S}_n = H\mathrm{S}_{n-1}$, there exist unique $h \in H$ and $z_0 \in \mathrm{S}_{n-1}$ such that $z = hz_0$. Then $\mathrm{A}_n = H^{z_0}\mathrm{A}_{n-1}$ and $[\mathrm{A}_n : H]z = [\mathrm{A}_n : H]z_0$. Noting that $Hz = Hz_0$ and H^{z_0} is the vertex-stabilizer of Hz_0 in A_n, it follows that A_{n-1} is regular on $[\mathrm{A}_n : H]z$. Therefore $\mathrm{Cos}(\mathrm{S}_n, H, K, z)$ is an S_n-symmetric bi-Cayley graph of A_{n-1}. Clearly, $HzH = Hz_0H$, $\langle H, z_0\rangle = \mathrm{S}_n$ and $H^{z_0} \cap H = H^z \cap H = K$. In addition, if further $z_0^2 \in K$ then $z_0 \in \mathbf{N}_{\mathrm{S}_n}(K)$, and so we may use z_0 instead of the element z in $\mathrm{Cos}(\mathrm{S}_n, H, K, z)$.

Let $n = pm$. By the above argument, it suffices to complete the following three steps for the existence and construction of graphs meeting Theorem 1 (4).

Step 1 Determine those groups of order n which is possible as a vertex-stabilizer of some symmetric graph of valency p;

Step 2 For a possible vertex-stabilizer H, consider the action of H on H by right multiplication, and determine whether or not H can be embedding in A_n as a regular subgroup;

Step 3 Consider the subgroups K of H with $|H : K| = p$ up to the conjugation under $\mathbf{N}_{\mathrm{S}_n}(H)$, calculate $\mathbf{N}_{\mathrm{S}_n}(K)$ and search for the elements z satisfying the condition (I) given in Section 3.

(1) If $n = p$ then there exist graphs meeting Theorem 1 (4). Let $a = (1, 2, \ldots, p)$, $z = (1, 2)$ and $H = \langle a \rangle$. Then $\langle a, b \rangle = \mathrm{S}_p$ and $H \cap H^z = 1$. Thus we have a connected S_p-symmetric graph $\Sigma = \mathrm{Cos}(\mathrm{S}_p, H, 1, z)$ of valency p, which is a bi-Cayley graph of A_{p-1}.

(2) If $p = 5$ and $H \cong \mathrm{A}_5$ then there are S_{60}-symmetric bi-Cayley graphs of A_{59} of valency 5. Note that A_5 has a permutation representation (induced by right multiplication on elements) of degree 60. This says that S_{60} has a regular subgroup $H \cong \mathrm{A}_5$. Since H is a nonablian simple group, no odd permutation is contained in H, forcing $H < \mathrm{A}_{60}$. Then we have an exact factorization $\mathrm{A}_{60} = HT$, where $T = \mathrm{A}_{59}$. Fix a subgroup K of H with $K \cong \mathrm{A}_4$. Calculation with GAP shows that there exists $z \in \mathrm{S}_{60}$ such that $z^2 \in K = H \cap H^z$ and $\langle H, z \rangle = \mathrm{S}_{60}$. Then we get a connected S_{60}-symmetric graph $\mathrm{Cos}(\mathrm{S}_{60}, H, K, z)$ of valency 5, which is a bi-Cayley graph of A_{59}.

(3) Assume that H is solvable. Then, as a vertex-stabilizer of some symmetric graph of valency p, we have $H \cong (\mathrm{C}_{l'} \times \mathrm{C}_p) : \mathrm{C}_l$, where $l' \bigm| l \bigm| (p-1)$. If $|H| = p$ then, by (1), the pair (S_p, H) produces connected symmetric bi-Cayley graphs of A_{p-1} with valency p.

Suppose next that $|H| > p$. By Lemma 3, H has no cyclic Sylow 2-subgroup unless n is odd. In the following, we consider only the existence of graphs when $p = 5$ or 7. Note that a subgroup of H with index p is a Hall p'-subgroup. Since H is solvable, all subgroups of H with index p are conjugate in H.

Let $p = 5$. We have $H \cong \mathrm{C}_2 \times \mathrm{D}_{10}$, $(\mathrm{C}_2 \times \mathrm{C}_5).\mathrm{C}_4$ or $\mathrm{C}_4 \times \mathrm{C}_5{:}\mathrm{C}_4$. Consider the action of H on H by right multiplication, and embed in A_n as a regular subgroup. Fix a subgroup $K < H$ with $|H : K| = 5$. For $H \cong (\mathrm{C}_2 \times \mathrm{C}_5).\mathrm{C}_4$ or $\mathrm{C}_4 \times \mathrm{C}_5{:}\mathrm{C}_4$, calculation with GAP shows that $|\mathbf{N}_{\mathrm{S}_n}(K)| = |\mathbf{N}_{\mathrm{A}_n}(K)|$, yielding $\mathbf{N}_{\mathrm{S}_n}(K) < \mathrm{A}_n$, and so there exists no element z satisfying the condition (I) given in Section 3. Thus let $H \cong \mathrm{C}_2 \times \mathrm{D}_{10}$. Calculation with GAP shows that there exist elements z satisfying (I). Therefore, the pair (S_{20}, H) with $H \cong \mathrm{C}_2 \times \mathrm{D}_{10}$ produces connected S_{20}-symmetric bi-Cayley graphs of A_{19} with valency 5.

Let $p = 7$. We have $H \cong \mathrm{C}_2 \times (\mathrm{C}_7{:}\mathrm{C}_2)$, $\mathrm{C}_7{:}\mathrm{C}_3$, $\mathrm{C}_3 \times (\mathrm{C}_7{:}\mathrm{C}_3)$, $\mathrm{C}_2 \times (\mathrm{C}_7{:}\mathrm{C}_6)$ or $\mathrm{C}_6 \times (\mathrm{C}_7{:}\mathrm{C}_6)$. Fix a subgroup $K < H$ with $|H : K| = 7$. By a similar argument as for the case $p = 5$, if $H \cong \mathrm{C}_2 \times (\mathrm{C}_7{:}\mathrm{C}_2)$, $\mathrm{C}_7{:}\mathrm{C}_3$, $\mathrm{C}_3 \times (\mathrm{C}_7{:}\mathrm{C}_3)$ or $\mathrm{C}_2 \times (\mathrm{C}_7{:}\mathrm{C}_6)$, then there exist elements z satisfying (I), and thus each pair (S_n, H) produces connected symmetric bi-Cayley graphs of A_{n-1} with valency 7. As for $H \cong \mathrm{C}_6 \times (\mathrm{C}_7{:}\mathrm{C}_6)$, by calculation with GAP, we know that $\mathbf{N}_{\mathrm{S}_{252}}(K)$ is of order 113747151468625920 and not contained in A_{252}, but we do not know if there are some elements $z \in \mathbf{N}_{S_{252}}(K)$ satisfying $z^2 \in K$ and $\langle H, z \rangle = S_{252}$. □

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