On Disjunctive Rado Numbers for Some Sets of Equations

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Abstract

Given a set of linear equations $\mathscr{S}$ with positive integral parameters $a_1, \ldots, a_k$, $k \geqslant 2$, the disjunctive Rado number for the set $\mathscr{S}$ is the least positive integer $R = \mathscr{R}_d(\mathscr{S})$, if it exists, such that every 2-coloring χ of the integers in $[1, R]$ admits a monochromatic solution to at least one equation in $\mathscr{S}$. We give conditions for the existence of $\mathscr{R}_d(\mathscr{S})$, and also give general upper and lower bounds on $\mathscr{R}_d(\mathscr{S})$ when $\mathscr{S}$ is a set of additive equations $\{y = x + a_1, \ldots, y = x + a_k\}$. We also determine $\mathscr{R}_d(\mathscr{S})$ when $\max a_i$ is large enough, or when $a_1, \ldots, a_k$ form an arithmetic or geometric progression. We also give conditions for the existence of $\mathscr{R}_d(\mathscr{S})$ when $\mathscr{S}$ is a set of multiplicative equations $\{y = a_1 x, \ldots, y = a_k x\}$. Further, we give a general search-based algorithm to determine $\mathscr{R}_d(\mathscr{S})$ when $\mathscr{S}$ is a set of equations in two variables, give an upper bound on $\mathscr{R}_d(\mathscr{S})$ and an algorithm to determine solutions to $\mathscr{S}$. This algorithm runs in time $O(k a_k \log a_k)$ for the case of additive equations, which is exponentially better than the brute-force algorithm for the problem.

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1 Introduction

By an r-coloring of $\{1, \ldots, N\}$ we mean a mapping $\chi : \{1, \ldots, N\} \to \{1, \ldots, r\}$. In 1916, Schur [17] showed that for every positive integer r, there exists a least positive integer $s = s(r)$ such that for every r-coloring of the integers in the interval $[1, s]$, there exist $x, y, x + y \in [1, s]$ such that $\chi(x) = \chi(y) = \chi(x+y)$. Schur's theorem was generalized in a series of results in the 1930's by Rado [13, 14, 15] leading to a complete resolution to the following problem: characterize sets of linear homogeneous equations with integral coefficients $\mathscr{S}$

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such that for a given positive integer r, there exists a least positive integer $n = \mathscr{R}(\mathscr{S}; r)$ such that every r-coloring of the integers in the interval $[1, n]$ yields a monochromatic solution to the set $\mathscr{S}$. There has been a growing interest in the determination of the Rado numbers $\mathscr{R}(\mathscr{S}; r)$, particularly when $\mathscr{S}$ is a single equation and $r = 2$; for instance, see [1, 3, 4, 5, 6, 7, 9, 13, 14, 15]. When $r = 2$, we denote this number simply by $\mathscr{R}(\mathscr{S})$.

The problem of disjunctive Rado numbers was introduced by Johnson & Schaal in [8]. The 2-color disjunctive Rado number for the set of equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$ is the least positive integer N such that any 2-coloring of $\{1, \ldots, N\}$ admits a monochromatic solution to at least one of the equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$. Johnson & Schaal gave necessary and sufficient conditions for the existence of the 2-color disjunctive Rado number for the equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$ for all pairs of distinct positive integers a, b, and also determined exact values when it exists. The present authors provided alternate proofs for the same result; see [2]. Johnson & Schaal [8] also determined exact values for the pair of equations $ax_1 = x_2$ and $bx_1 = x_2$ whenever a, b are distinct positive integers. Lane-Harvard & Schaal [12] determined exact values of 2-color disjunctive Rado number for the pair of equations $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$ for all distinct positive integers a, b. Sabo, Schaal & Tokaz [16] determined exact values of 2-color disjunctive Rado number for $x_1 + x_2 - x_3 = c_1$ and $x_1 + x_2 - x_3 = c_2$ whenever c_1, c_2 are distinct positive integers. Kosek & Schaal [10] determined the exact value of 2-color disjunctive Rado number for the equations $x_1 + \cdots + x_{m-1} = x_m$ and $x_1 + \cdots + x_{n-1} = x_n$ for all pairs of distinct positive integers m, n.

In Section 2, we give some preliminaries and a theorem that characterizes $\mathscr{R}_d(\mathscr{S})$ whenever $\mathscr{S}$ is a set of equations in two variables; see Theorem 7.

In Section 3, we deal with the disjunctive Rado number $\mathscr{R}_d(\mathscr{A})$ for the set of additive equations $\mathscr{A} : \{y - x = a_1, \ldots, y - x = a_k\}$. We first relate $\mathscr{R}_d(\mathscr{A})$ with $\mathscr{R}_d(s\mathscr{A})$ for the related set of equations $s\mathscr{A} : \{y - x = sa_1, \ldots, y - x = sa_k\}$ for each $s > 1$. We also relate valid colorings for these two sets; see Theorem 11. We give an alternate proof of Johnson & Schaal's result in [8] for $k = 2$ in Theorem 12. We determine conditions for the existence of $\mathscr{R}_d(\mathscr{A})$ and give general upper and lower bounds for $\mathscr{R}_d(\mathscr{A})$; see Theorems 13, 15, 16. We also determine $\mathscr{R}_d(\mathscr{A})$ when $\max_i a_i$ is large enough; see Theorem 20. Finally, we determine the existence and value of $\mathscr{R}_d(\mathscr{A})$ when the numbers $a_1, \ldots, a_k$ form an arithmetic or a geometric progression; see Theorems 21, 22.

In Section 4, we deal with the disjunctive Rado number $\mathscr{R}_d(\mathscr{M})$ for the set of multiplicative equations $\mathscr{M} : \{y = a_1x, \ldots, y = a_kx\}$. We determine conditions for the existence of $\mathscr{R}_d(\mathscr{M})$ and give an upper bound on $\mathscr{R}_d(\mathscr{M})$ when it exists; see Theorem 23. Among other corollaries, we also independently derive Johnson & Schaal's result in [8] for the existence of $\mathscr{R}_d(\mathscr{M})$ for the case $k = 2$; see Corollary 24.

In Section 5 we give an algorithm for determining $\mathscr{R}_d(\mathscr{S})$ for any set of linear equations $\mathscr{S}$ in two variables provided that an upper bound on $\mathscr{R}_d(\mathscr{S})$ is known; see Algorithm 1. We transform our problem to one in Graph Theory to describe our algorithm, which is a combination of binary search and graph search. We prove the correctness of this algorithm and determine the running time in Theorem 27. Aided with bounds from Section 3, we also show that this algorithm runs in time $O(ka_k \log a_k)$ for the set of additive equations

$\mathscr{A}$ described above; see Corollary 28. This is exponentially faster than the brute-force algorithm, but still exponential in the input size $\Theta\big(\sum_i \log a_i\big)$. We then give an algorithm that determines all valid 2-colorings on $[1, \mathscr{R}_d(\mathscr{S})]$, provided that $\mathscr{R}_d(\mathscr{S})$ is known; see Algorithm 2. We prove the correctness of this algorithm in Theorem 29.

2 A general theorem for equations of two variables

We give some preliminaries in this section and then establish a theorem that characterizes disjunctive Rado numbers for sets of equations in two variables. One can obtain the existence criterion for sets $\mathscr{A}$ and $\mathscr{M}$ using this theorem, which we do in the respective sections for those sets.

For fundamental results on Ramsey theory on the integers, we refer the reader to the comprehensive text [11]. We only use standard definitions and basic results on Rado numbers, basic Graph Theory, and some simple search algorithms.

We denote the set of positive integers by $\mathbb{N}$. For integers $a < b$, we define the interval $[a, b] = \{a, a+1, \ldots, b\}$. Given a set S and an $r \in \mathbb{N}$, an r-coloring on S is a function χ from S to $[1, r]$. We work only with 2-colorings, and by a coloring χ we mean a 2-coloring henceforth (although some of our results can be generalized to any r).

Definition 1. For any set of equations $\mathscr{S}$ and for any $N \in \mathbb{N}$, $\chi : [1, N] \to \{1, 2\}$ is called a *valid coloring* for $\mathscr{S}$ if χ avoids monochromatic solution to every equation in $\mathscr{S}$.

Hence, for a valid coloring for $\mathscr{S}$, there are no numbers $x_1, \ldots, x_t \in [1, N]$, which satisfy some equation in $\mathscr{S}$ and for which $\chi(x_1) = \cdots = \chi(x_t)$.

Definition 2. For any set of equations $\mathscr{S}$, the *disjunctive Rado number* for $\mathscr{S}$, denoted by $\mathscr{R}_d(\mathscr{S})$, is the least positive integer, if it exists, for which there is no valid coloring for $\mathscr{S}$ on $[1, \mathscr{R}_d(\mathscr{S})]$. We define $\mathscr{C}_d(\mathscr{S})$ to be the set of all valid colorings for $\mathscr{S}$ on $[1, \mathscr{R}_d(\mathscr{S}) - 1]$, if $\mathscr{R}_d(\mathscr{S})$ exists. When the equations in $\mathscr{S}$ involve functions of two variables only, we denote this set by $\mathscr{S}_2$; so $\mathscr{S}_2 = \{f_1(x, y) = 0, \ldots, f_k(x, y) = 0\}$ where $f_1, \ldots, f_k$ are arbitrary functions in two variables.

We consider fixed but arbitrary distinct positive integers $a_1, \ldots, a_k$, where $k \geqslant 2$. Throughout this section, we assume that $a_1 < \cdots < a_k$. We define the set of additive equations $\mathscr{A} : \{y = x + a_1, \ldots, y = x + a_k\}$, and the set of multiplicative equations $\mathscr{M} : \{y = a_1 x, \ldots, y = a_k x\}$.

For integers a, b, m, we write $a \equiv b \pmod{m}$ if $m \mid (a - b)$. By $a \bmod m$, we mean the unique integer $b \in [1, m]$ such that $a \equiv b \pmod{m}$.[1] Therefore, the symbol mod has two distinct (but related) meanings; the difference will be clear from the context.

Suppose $\mathscr{S}$ and $\mathscr{T}$ are two sets of linear equations such that $\mathscr{S} \subset \mathscr{T}$. Then, since every equation in $\mathscr{S}$ is also in $\mathscr{T}$, the existence of $\mathscr{R}_d(\mathscr{S})$ implies the existence of $\mathscr{R}_d(\mathscr{T})$, by definition. We make frequent use of this simple observation, and record it as Lemma 3.

[1] This is slightly different from the standard use where $b \in [0, m-1]$. This change will help us simplify some of our results and proofs.

Lemma 3. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are two sets of linear equations such that $\mathscr{S} \subset \mathscr{T}$. If $\mathscr{R}_d(\mathscr{S})$ exists, then $\mathscr{R}_d(\mathscr{T})$ exists; moreover, $\mathscr{R}_d(\mathscr{T}) \leqslant \mathscr{R}_d(\mathscr{S})$.*

We now establish an existence theorem for an arbitrary set $\mathscr{S}_2 : \{f_1(x,y) = 0, \ldots, f_k(x,y) = 0\}$.

Definition 4. Define relation $\mathcal{R}$ on $\mathbb{N}$ to be $\mathcal{R} = \{(x,y) : f_i(x,y) = 0 \text{ for some } i\}$, and let $\overline{\mathcal{R}}$ be its reverse; that is, $\overline{\mathcal{R}} = \{(x,y) : (y,x) \in \mathcal{R}\}$. We say that $\langle x_0, \ldots, x_m \rangle$ is a *closed $\mathscr{S}_2$-path* if $x_0, \ldots, x_{m-1}$ are distinct positive integers, $x_m = x_0$, and $(x_j, x_{j+1}) \in \mathcal{R} \cup \overline{\mathcal{R}}$ for each $j \in \{0, \ldots, m-1\}$.

Notice that $(x,y) \in \mathcal{R} \cup \overline{\mathcal{R}}$ implies that $f_i(x,y) = 0$ or $f_i(y,x) = 0$ for some i. Therefore, if there is a valid coloring $\Delta : [1,N] \to \{1,2\}$ where $N \geqslant x, y$, we must have $\Delta(x) \neq \Delta(y)$ by definition.

We now present a graph theoretic way of thinking about our problem for $\mathscr{S}_2$. For each $N \in \mathbb{N}$, let $G_N(\mathscr{S}_2) = ([1,N], E)$ be an undirected graph (possibly with self-loops but with no multi-edges), where $E = (\mathcal{R} \cup \overline{\mathcal{R}}) \cap [1,N]^2$. Similarly, let $G(\mathscr{S}_2) = (\mathbb{N}, E)$ be an undirected graph on the positive integers with edge set $E = \mathcal{R} \cup \overline{\mathcal{R}}$. Note that $G(\mathscr{S}_2)$ is an infinite graph. When $\mathscr{S}_2$ is clear from context, we denote these graphs simply as G_N and G, respectively.

Lemma 5. *$\mathscr{R}_d(\mathscr{S}_2)$ exists if and only if G is non-bipartite. Furthermore, $\mathscr{R}_d(\mathscr{S}_2)$ is the least integer N such that G_N is not bipartite, provided it exists.*

Proof. Suppose that $\chi : [1,N] \to \{1,2\}$ is a valid 2-coloring for $\mathscr{S}_2$. We claim that χ is a graph 2-coloring for G_N.[2] Indeed, if $(x,y) \in E(G_N)$, then by definition, there is an $i \in [1,k]$ such that $f_i(x,y) = 0$ or $f_i(y,x) = 0$. In either case, $\chi(x) \neq \chi(y)$. Similarly, if χ is a graph 2-coloring for G_N, then it is a valid 2-coloring for $\mathscr{S}_2$.

Suppose $\mathscr{R}_d(\mathscr{S}_2)$ exists, and denote it by R for brevity. Then for any $N < R$, there is a valid 2-coloring $\chi : [1,N] \to \{1,2\}$, which by our discussion is a valid 2-coloring of G_N. Conversely, since there is no valid coloring for $\mathscr{S}_2$ on $[1,R]$, the graph G_R must be non-bipartite.

A similar argument shows that G is bipartite if and only if there is a valid coloring $\chi : \mathbb{N} \to \{1,2\}$. Therefore, G is bipartite if and only if $\mathscr{R}_d(\mathscr{S}_2)$ does not exist. $\square$

We will also use the following well-known theorem that characterizes bipartite graphs.

Theorem 6. *An undirected graph G (possibly infinite and with self-loops but no multiple edges) is bipartite if and only if there are no odd cycles in G.*

We are now ready to present our theorem on the existence of $\mathscr{R}_d(\mathscr{S}_2)$.

[2]Here and elsewhere by a graph coloring we mean a proper graph vertex coloring. That is, for graph $H = (V,E)$, $\chi : V \to \{1,2\}$ is a graph coloring if and only if $\chi(x) \neq \chi(y)$ for all $(x,y) \in E$.

Theorem 7. *Let $\mathscr{S}_2$ be the set of equations $\{f_1(x,y)=0,\dots,f_k(x,y)=0\}$. Consider the set*

$$\mathbf{S}=\Big\{\max_{i\in[1,m]} x_i : \langle x_0,x_1,\dots,x_m\rangle \text{ is a closed } \mathscr{S}_2\text{-path and } m \text{ is odd}\Big\}.$$

Then, $\mathscr{R}_d(\mathscr{S}_2)$ exists if and only if $\mathbf{S}$ is nonempty. Furthermore, $\mathscr{R}_d(\mathscr{S}_2)=\min(\mathbf{S})$ if $\mathbf{S}$ is nonempty.

Proof. Observe first that closed $\mathscr{S}_2$-paths correspond to cycles in G. If $\langle x_0,\dots,x_m=x_0\rangle$ is a closed $\mathscr{S}_2$ path, then $(x_j,x_{j+1})\in\mathcal{R}\cup\overline{\mathcal{R}}$, and so $(x_j,x_{j+1})\in E(G)$ by definition. If $N=\max_i x_i$ for this closed path, then this cycle is also present in G_N since it is the subgraph of G induced by $[1,N]$. The converse is similarly true: each cycle in any G_N corresponds to a closed $\mathscr{S}_2$-path.

I. (Nonexistence): If $\mathbf{S}$ is empty, then we show that $\mathscr{R}_d(\mathscr{S}_2)$ does not exist. By Lemma 5, it is equivalent to show that G is bipartite. Suppose to the contrary that G is not bipartite. Then, by Theorem 6, G has an odd cycle $\langle x_0,x_1,\dots,x_m\rangle$, m odd, $x_m=x_0$, which is a closed $\mathscr{S}_2$-path by our discussion above. This implies that $\mathbf{S}$ is non-empty, a contradiction.

II. (Existence and Upper Bound): Suppose $\mathbf{S}$ is nonempty. Let $N=\min\mathbf{S}$; we show that $\mathscr{R}_d(\mathscr{S}_2)\leqslant N$. By definition, there exists a closed $\mathscr{S}_2$-path $\langle x_0,x_1,\dots,x_m\rangle$ with m odd and $N=\max_i x_i$. Note that this means each $x_i\in V(G_N)$. Also, closed $\mathscr{S}_2$-paths correspond to cycles in G, and therefore, $x_0,x_1,\dots,x_m=x_0$ is an odd cycle in G_N. From Theorem 6, G_N is not bipartite, and therefore from Lemma 5, we have that $N\geqslant\mathscr{R}_d(\mathscr{S}_2)$.

III. (Lower bound): We show that $\mathscr{R}_d(\mathscr{S}_2)\geqslant N$. By Lemma 5, it is enough to show that the graph G_{N-1} is bipartite. Suppose to the contrary that G_{N-1} is not bipartite, in which case there is an odd cycle $\langle x_0,\dots,x_m\rangle$ in G_{N-1} by Theorem 6. But from our observation, this cycle corresponds to a closed $\mathscr{S}_2$-path of odd length. We also have $\max_i x_i\leqslant N-1$, which implies that $N\neq\min(\mathbf{S})$, a contradiction. □

3 Set of additive equations $y=x+a_1,\dots,y=x+a_k$

It is easy to see that $\mathscr{R}_d(\{y=x+a\})$ does not exist for any $a\in\mathbb{N}$. Johnson & Schaal investigated the disjunctive Rado number in [8] for the additive set $\{y=x+a,y=x+b\}$ for distinct positive integers a,b. We investigate the disjunctive Rado number for the set $\mathscr{A}:\{y=x+a_1,\dots,y=x+a_k\}$, where $a_1<\dots<a_k$ and $k\geqslant 2$. Theorem 11 relates $\mathscr{R}_d(\mathscr{A})$ and valid colorings for $\mathscr{A}$ to the disjunctive Rado number and valid colorings for the related set of equations $s\mathscr{A}:\{y=x+sa_1,\dots,y=x+sa_k\}$ for any $s\in\mathbb{N}$. As warm-up to our results for $\mathscr{A}$, we use Theorems 7 and 11 to give an alternate proof of Johnson & Schaal's result for $\mathscr{R}_d(\mathscr{A})$ in Theorem 12; this proof was previously observed in [2]. We determine conditions for the existence of $\mathscr{R}_d(\mathscr{A})$ in Theorem 13, establish general upper and lower bounds on $\mathscr{R}_d(\mathscr{A})$ in Theorems 15 and 16 respectively, and determine $\mathscr{R}_d(\mathscr{A})$ for large enough a_k in Theorem 20. Near the end of this section, we determine $\mathscr{R}_d(\mathscr{A})$ when $a_1,\dots,a_k$ form an arithmetic or a geometric progression, in Theorems 21 and 22, respectively.

Notation 8. For each $i \in [1, k]$, let $\mathscr{A}_i$ denote the set of equations $\{y = x + a_1, \ldots, y = x + a_i\}$.

For $s \in \mathbb{N}$, let $s\mathscr{A}_i$ denote the related set of equations $\{y = x + sa_1, \ldots, y = x + sa_i\}$, and similarly let $\frac{\mathscr{A}_i}{s}$ denote the set of equations $\left\{y = x + \frac{a_1}{s}, \ldots, y = x + \frac{a_i}{s}\right\}$ whenever $s \mid a_j$ for each $j \in [1, i]$.

We write $g = \gcd(a_1, \ldots, a_k)$ and $f = \gcd(a_1, \ldots, a_{k-1})$.

Set $a_k = mf + a'_k$ where $m \in \mathbb{N} \cup \{0\}$ and $a'_k \in [1, f]$.

We first relate $\mathscr{R}_d(\mathscr{A})$ and $\mathscr{C}_d(\mathscr{A})$ to $\mathscr{R}_d(s\mathscr{A})$ and $\mathscr{C}_d(s\mathscr{A})$, respectively, for arbitrary $s \in \mathbb{N}$. We begin with some definitions that help us transform colorings from set $\mathscr{A}$ to set $s\mathscr{A}$ and vice-versa.

Definition 9. We begin with defining some algebra for our colorings.

(i) **Complement.** For any coloring χ, let $\overline{\chi}$ be its complement; that is, $\overline{\chi}(n) = ((1 + \chi(n)) \bmod 2)$ for all n in the domain of χ.

(ii) **The Expansion operator.** Given a coloring $\chi : [1, N] \to \{1, 2\}$, positive integer s, and positive integer $r \in [1, s]$, we let $\mathbf{E}_{s,r}\chi$ be a coloring on $[1, sN]$ defined as follows:

$$(\mathbf{E}_{s,r}\chi)(n) = \begin{cases} \chi(k+1) & \text{if } n = ks + r, 0 \leqslant k \leqslant N-1; \\ 2 & \text{otherwise.} \end{cases}$$

When $r = s$, we denote this by $s\chi$. Informally, this corresponds to expanding and shifting the coloring χ on $[1, N]$ to the larger domain $[1, sN]$, while maintaining relative distances. The 'expansion' is determined by s and the 'shift' is determined by r.

(iii) **The Contraction operator.** Let the coloring $\mathbf{C}_{s,r}\chi$ on $\left[1, \left\lfloor \frac{N-r}{s} \right\rfloor + 1\right]$ be defined as follows:

$$(\mathbf{C}_{s,r}\chi)(n) = \chi\big(s(n-1) + r\big) \ \forall\, n.$$

Informally, this corresponds to contracting the coloring χ to a smaller domain while maintaining relative distances, where the 'contraction' is determined by s. It is easy to see that $\mathbf{C}_{s,r}\mathbf{E}_{s,r}\chi = \chi$ for each coloring χ, but the converse is not always true.

(iv) **Addition.** For colorings χ_1, χ_2 on $[1, N]$, let $\chi_1 + \chi_2$ be the element-wise addition of χ_1 and χ_2 modulo 2:

$$(\chi_1 + \chi_2)(n) = \big(\chi_1(n) + \chi_2(n)\big) \bmod 2 \quad \forall\, n.$$

This addition operation is clearly associative. For brevity, we denote $\chi_1 + \cdots + \chi_m$ by $\sum_{j=1}^{m} \chi_j$.

Lemma 10. *Let $s \in \mathbb{N}$.*

(i) *Suppose Δ_j is a coloring on $[1, N]$ for each $j \in [1, s]$. Then, for each $n \in [1, sN]$,*

$$\left(\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j\right)(n) = \Delta_{(n \bmod s)}\left(\left\lceil \frac{n}{s} \right\rceil\right).$$

(ii) *Suppose Δ is a coloring on $[1, sN]$. Then,*

$$\Delta = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j \text{ where } \Delta_j = \mathbf{C}_{s,j}\Delta \text{ for each } j \in [1, s].$$

Proof. (i) For $n \in [1, sN]$, write $n = ms + r$, where $r \in [1, s]$. Then, $m + 1 = \left\lceil \frac{n}{s} \right\rceil$. By definition, $(\mathbf{E}_{s,r}\Delta_r)(n) = \Delta_r(m + 1)$. So it suffices to prove that $(\mathbf{E}_{s,r}\Delta_r)(n) = 2$ for each $j \neq r$. Since $s \nmid (n - j)$ in this case, the result follows from definition of the expansion operator.

(ii) Let n, m, r be as defined in part (i). By definition, $\Delta_r(m + 1) = (\mathbf{C}_{s,r}\Delta)(m + 1) = \Delta(sm + r) = \Delta(n)$. From part (i), $\left(\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j\right)(n) = \Delta_r(m + 1)$. Therefore, $\left(\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j\right)(n) = \Delta(n)$ for all n.

□

We now use the machinery we have developed to prove the following theorem.

Theorem 11. *Let $a_1, \ldots, a_k, s$ be positive integers, $k \geqslant 2$.*

(i) *$\mathscr{R}_d(s\mathscr{A})$ exists if and only if $\mathscr{R}_d(\mathscr{A})$ exists. Furthermore, if both $\mathscr{R}_d(\mathscr{A})$ and $\mathscr{R}_d(s\mathscr{A})$ exist, then*

$$\mathscr{R}_d(s\mathscr{A}) = s\Big(\mathscr{R}_d(\mathscr{A}) - 1\Big) + 1.$$

(ii) *If both $\mathscr{R}_d(\mathscr{A})$ and $\mathscr{R}_d(s\mathscr{A})$ exist, then*

$$\mathscr{C}_d(s\mathscr{A}) = \left\{\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j : \Delta_j \in \mathscr{C}_d(\mathscr{A}) \text{ for each } j \in [1, s]\right\},$$

and consequently, $\left|\mathscr{C}_d(s\mathscr{A})\right| = \left|\mathscr{C}_d(\mathscr{A})\right|^s$.

Proof. (i) We break our proof into two parts. We will first show that the existence of $\mathscr{R}_d(s\mathscr{A})$ implies the existence of $\mathscr{R}_d(\mathscr{A})$, and that $\mathscr{R}_d(s\mathscr{A}) \geqslant s\Big(\mathscr{R}_d(\mathscr{A}) - 1\Big) + 1$ given the existence of $\mathscr{R}_d(s\mathscr{A})$. We will then prove that whenever $\mathscr{R}_d(\mathscr{A})$ exists,

$\mathscr{R}_d(s\mathscr{A}) \leqslant s\Big(\mathscr{R}_d(\mathscr{A})-1\Big)+1$, completing the proof of both the existence part and the desired equality.

Suppose first that $\mathscr{R}_d(s\mathscr{A})$ exists; denote this by R'. We show that $\mathscr{R}_d(\mathscr{A}) \leqslant \lceil \frac{R'}{s} \rceil$, and therefore that it exists. Let $\Delta : \big[1, \lceil \frac{R'}{s} \rceil\big] \to \{1,2\}$ be an arbitrary coloring and define $\Delta' = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta$. Then, by definition, the domain of Δ' is $\big[1, s\lceil \frac{R'}{s} \rceil\big] \supseteq [1, R']$. Therefore, Δ' admits a monochromatic solution $(x, x+sa_i)$ to some equation $y = x + sa_i$ in $s\mathscr{A}$; that is,

$$\Delta'(x) = \Delta'(x+sa_i)$$

for some $x, x+sa_i \in [1, R']$. From Lemma 10, this implies that

$$\Delta\Big(\Big\lceil \frac{x}{s} \Big\rceil\Big) = \Delta\Big(\Big\lceil \frac{x}{s} \Big\rceil + a_i\Big).$$

Since $\lceil \frac{x}{s} \rceil, \lceil \frac{x}{s} \rceil + a_i \in \big[1, \lceil \frac{R'}{s} \rceil\big]$, Δ is not valid for $\mathscr{A}$, proving our claim that $\mathscr{R}_d(\mathscr{A}) \leqslant \lceil \frac{R'}{s} \rceil$.

Denote $\mathscr{R}_d(\mathscr{A})$ by R for brevity. Let $\Delta : [1, R-1] \to \{1,2\}$ denote a valid coloring of set $\mathscr{A}$. Therefore, for each $i \in [1,k]$,

$$\Delta(x) \neq \Delta(x+a_i) \tag{1}$$

whenever $x, x+a_i \in [1, R-1]$.

We claim that $\Delta' = \displaystyle\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta$ is a valid coloring for the set $s\mathscr{A}$. Indeed, by definition, the domain of Δ' is $[1, s(R-1)]$, and from Lemma 10 part (i), for each $i \in [1,k]$

$$\Delta'(y+sa_i) = \Delta\Big(\Big\lceil \frac{y+sa_i}{s} \Big\rceil\Big) = \Delta\Big(\Big\lceil \frac{y}{s} \Big\rceil + a_i\Big) \neq \Delta\Big(\Big\lceil \frac{y}{s} \Big\rceil\Big) = \Delta'(y)$$

whenever $y, y+sa_i \in [1, s(R-1)]$ by eqn. (1). Therefore

$$\mathscr{R}_d(s\mathscr{A}) \geqslant s\Big(\mathscr{R}_d(\mathscr{A})-1\Big)+1. \tag{2}$$

We now assume that R exists, and prove that R' exists and is at most $s(R-1)+1$. We must show that every coloring of $[1, s(R-1)+1]$ admits a monochromatic solution to at least one of the equations in $s\mathscr{A}$.

Consider an arbitrary coloring $\chi' : [1, s(R-1)+1] \to \{1,2\}$. Let $\chi = \mathbf{C}_{s,1}\chi'$. By definition, the domain of χ is $\Big[1, \Big\lfloor \frac{s(R-1)}{s} \Big\rfloor + 1\Big] = [1, R]$, and since $R = \mathscr{R}_d(\mathscr{A})$, χ admits a monochromatic solution to at least one of the equations in $\mathscr{A}$. Thus there exists $x, y \in [1, R]$ such that $y - x = a_i$ for some $i \in [1,k]$ and $\chi(x) = \chi(y)$. But now $s(x-1)+1, s(y-1)+1 \in [1, s(R-1)+1]$ satisfy $\big(s(y-1)-1\big) - \big(s(x-1)-1\big) = sa_i$

and by definition, $\chi'(s(x-1)+1) = \chi(x) = \chi(y) = \chi'(s(y-1)+1)$. Hence χ' admits a monochromatic solution to at least once equation in $s\mathscr{A}$, and so

$$\mathscr{R}_d(s\mathscr{A}) \leqslant s\Big(\mathscr{R}_d(\mathscr{A})-1\Big)+1. \tag{3}$$

The desired equality follows from eqn. (2) and eqn. (3).

(ii) For brevity, denote the set $\left\{\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j : \Delta_j \in \mathscr{C}_d(\mathscr{A}) \text{ for each } j \in [1,s]\right\}$ by C. Let colorings $\Delta_j \in \mathscr{C}_d(\mathscr{A})$ for $j \in [1,s]$; that is, each Δ_j is a valid coloring for set $\mathscr{A}$ on $[1, R-1]$. We show that $\Delta = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j$ is a valid coloring for $s\mathscr{A}$ on $[1, s(R-1)]$, so that

$$\mathscr{C}_d(s\mathscr{A}) \supseteq C. \tag{4}$$

By definition, the domain of Δ is $[1, s(R-1)]$. Suppose $x', y' \in [1, s(R-1)]$ such that $y'-x' = sa_i$ for some i. Since $s \mid (y'-x')$, write $y' = sy+r$ and $x' = sx+r$, where $r \in [1,s]$. By Lemma 10, part (i), $\Delta(x') = \Delta_r(x+1)$ and $\Delta(y') = \Delta_r(y+1)$. But $(y+1)-(x+1) = a_i$, so that $\Delta_r(x) \neq \Delta_r(y)$, since Δ_r is a valid coloring for set $\mathscr{A}$. Therefore, $\Delta(x') \neq \Delta(y')$, proving that Δ is a valid coloring for set $s\mathscr{A}$.

Now assume that $\Delta \in \mathscr{C}_d(s\mathscr{A})$. Define $\Delta_j = \mathbf{C}_{s,j}\Delta$ for $j \in [1,s]$. Then, by Lemma 10, part (ii), we have $\Delta = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j$. We claim that each $\Delta_j \in \mathscr{C}_d(\mathscr{A})$, so that

$$\mathscr{C}_d(s\mathscr{A}) \subseteq C. \tag{5}$$

Suppose $x, y \in [1, R-1]$ such that $y-x = a_i$ for some i. For any $j \in [1,s]$, let $x' = s(x-1)+j$ and $y' = s(y-1)+j$, so that $x', y' \in [1, s(R-1)]$ and $y'-x' = sa_i$. Since Δ is valid for set $s\mathscr{A}$, $\Delta(x') \neq \Delta(y')$. Then, from Lemma 10, part (i), we have $\Delta(x') = \Delta_j(x)$ and $\Delta(y') = \Delta_j(y)$, so that $\Delta_j(x) \neq \Delta_j(y)$, proving that Δ_j is valid on $[1, R-1]$.

Eqn. (4) and eqn. (5) together give the desired equality.

We now determine the cardinality of C. Lemma 10, part (i) implies that

$$\sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta'_j \quad \Longleftrightarrow \quad \Delta_j = \Delta'_j \text{ for each } j \in [1,s]. \tag{6}$$

We give a natural bijection $\phi : \mathscr{C}_d(\mathscr{A})^s \to C$. For $(\Delta_1, \ldots, \Delta_s) \in \mathscr{C}_d(\mathscr{A})^s$, define

$$\phi(\Delta_1, \ldots, \Delta_s) = \sum_{j=1}^{s} \mathbf{E}_{s,j}\Delta_j.$$

Note that ϕ is a surjection, by definition of C. Eqn. (6) implies that it is also an injection, proving that ϕ is a bijection and therefore proving that $\left|\mathscr{C}_d(s\mathscr{A})\right| = |\mathscr{C}_d(\mathscr{A})|^s$. □

We are now ready to give an alternate proof of Johnson & Schaal's result from [8]; that is, for the case $k = 2$.

Theorem 12. [8, Theorem 1]
For distinct positive integers a_1, a_2 *with* $\gcd(a_1, a_2) = g$, *let* $\mathscr{A} : \{y = x + a_1, y = x + a_2\}$. *Then*

$$\mathscr{R}_d(\mathscr{A}) = \begin{cases} a_1 + a_2 - g + 1 & \text{if } \frac{a_1}{g} + \frac{a_2}{g} \text{ is odd;} \\ \text{does not exist} & \text{if } \frac{a_1}{g} + \frac{a_2}{g} \text{ is even.} \end{cases}$$

Proof. We note that $\mathscr{R}_d(\mathscr{A})$ exists if and only if there exists a closed $\mathscr{A}$-path $\langle x_0, x_1, \ldots, x_m = x_0\rangle$ of odd length, by Theorem 7.

Suppose $\langle x_0, x_1, \ldots, x_m = x_0\rangle$ is a closed path of odd length exists, so that for each j, $x_j = x_{j-1} \pm a$ for $a \in \{a_1, a_2\}$. Therefore, there exist $\lambda_1, \lambda_2 \in \mathbb{Z}$ such that $x_m = x_0 + \lambda_1 a_1 + \lambda_2 a_2$ and $\lambda_1 + \lambda_2 \equiv m \pmod 2$ is odd. So $\lambda_1 \frac{a_1}{g} + \lambda_2 \frac{a_2}{g} = 0$. If both $\frac{a_1}{g}$ and $\frac{a_2}{g}$ are odd, then

$$0 = \lambda_1 \cdot \frac{a_1}{g} + \lambda_2 \cdot \frac{a_2}{g} \equiv \lambda_1 + \lambda_2 \equiv 1 \pmod 2,$$

a contradiction. Therefore, if $\mathscr{R}_d(\mathscr{A})$ exists, at least one of $\frac{a_1}{g}, \frac{a_2}{g}$ is even. Since not both $\frac{a_1}{g}, \frac{a_2}{g}$ can be even, $\mathscr{R}_d(\mathscr{A})$ does not exist if $\frac{a_1}{g} + \frac{a_2}{g}$ is even.

We now show that $\mathscr{R}_d(\mathscr{A}) = a_1 + a_2 - g + 1$ when $\frac{a_1}{g} + \frac{a_2}{g}$ is odd. We first prove the result for the case $g = 1$. The result can then be extended to an arbitrary g using Theorem 11, part (ii).

(Upper Bound): To prove the upper bound, we will show that there is a closed $\mathscr{A}$-path $\langle x_0, \ldots, x_{a_1+a_2}\rangle$ with $\max_{i\in[1,a_1+a_2]} x_i \leqslant a_1 + a_2$. Theorem 7 then implies the result since $a_1 + a_2$ is odd.

Define the sequence x_i for $i \in [0, a_1 + a_2]$ where $x_0 = 1$, and for $i \geqslant 1$,

$$x_i = \begin{cases} x_{i-1} + a_2 & \text{if } x_{i-1} \leqslant a_1, \\ x_{i-1} - a_1 & \text{if } x_{i-1} > a_1. \end{cases}$$

Clearly, each $x_i \in [1, a_1 + a_2]$, so that $\max_i x_i \leqslant a_1 + a_2$. We will show that $x_i \neq x_j$ for $i, j \in [0, a_1 + a_2 - 1]$, $i \neq j$, and $x_{a_1+a_2} = x_0 = 1$. Thus, we will have a closed $\mathscr{A}$-path of odd length $a_1 + a_2$.

Suppose $x_i = x_j$ for some distinct $i, j \in [0, a_1 + a_2 - 1]$, and assume without loss of generality that $i < j$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{N} \cup \{0\}$ satisfying $\lambda_1 + \lambda_2 = j - i$ and $x_j = x_i + \lambda_2 a_2 - \lambda_1 a_1 = x_i$, or that $\lambda_2 a_2 - \lambda_1 a_1 = 0$. Since $\lambda_1 + \lambda_2 = j - i$, we get $\lambda_1(a_1 + a_2) = (j - i)a_2$. Since $\gcd(a_1 + a_2, a_2) = 1$, we have that $(a_1 + a_2) \mid (j - i)$ which is not possible since $i, j \in [0, a_1 + a_2 - 1]$, $i \neq j$.

We now show that $x_{a_1+a_2} = x_0 = 1$. There exist $\mu_1, \mu_2 \in \mathbb{N} \cup \{0\}$ such that $\mu_1 + \mu_2 = a_1+a_2$ and $x_{a_1+a_2} = 1+\mu_2 a_2-\mu_1 a_1 \in [1, a_1+a_2]$. Therefore, we get $\mu_1 \in \left[a_2-1+\frac{1}{a_1+a_2}, a_2\right]$. Since $\mu_1, \mu_2 \in \mathbb{N}$, we have $\mu_1 = a_2$, $\mu_2 = a_1$, so that $x_{a_1+a_2} = x_0$.

(Lower Bound) If $\mathscr{R}_d(\mathscr{A}) < a_1+a_2$, then there exists a closed $\mathscr{A}$-path $\langle x_0, x_1, \ldots, x_m = x_0\rangle$ where $\max_i x_i < a_1 + a_2$ and m is odd. Now $\max_i x_i < a_1 + a_2$ implies that $m < a_1 + a_2$ since $x_0, \ldots, x_{m-1}$ are distinct. Since $x_m = x_0$, there exist $\lambda_1, \lambda_2 \in \mathbb{N} \cup \{0\}$ satisfying $\lambda_1 + \lambda_2 = m$ such that either $x_m = x_0 + \lambda_1 a_2 - \lambda_2 a_1$ or $x_m = x_0 - \lambda_1 a_2 + \lambda_2 a_1$. In both cases we have $\lambda_1 a_2 - \lambda_2 a_1 = 0$. This combined with $\lambda_1 + \lambda_2 = m$ gives us $\lambda_1 = \frac{ma_2}{a_1+a_2}$. As $\gcd(a_1 + a_2, a_2) = 1$, $(a_1 + a_2) \mid m$, a contradiction, since $m < a_1 + a_2$. □

The next theorem gives a necessary and sufficient condition for the existence of $\mathscr{R}_d(\mathscr{A})$. The key idea is to reduce the existence of $\mathscr{R}_d(\mathscr{A})$ to that of $\mathscr{R}_d\left(\frac{\mathscr{A}}{g}\right)$ using Theorems 11, 12.

Theorem 13. *Let $k \geqslant 2$. Then*

$$\mathscr{R}_d(\mathscr{A}) \text{ exists if and only if at least one of } \tfrac{a_1}{g}, \ldots, \tfrac{a_k}{g} \text{ is even.}$$

Proof. We have

$$\mathscr{R}_d(\mathscr{A}) \text{ exists if and only if } \mathscr{R}_d\left(\frac{\mathscr{A}}{g}\right) \text{ exists}$$

by Theorem 11. Therefore it suffices to prove the result under the assumption $g = 1$.

Suppose $a_1, \ldots, a_k$ are all odd. The coloring $\Delta : \mathbb{N} \to \{1, 2\}$ given by $\Delta(x) = x \bmod 2$ is clearly a valid coloring of $\mathbb{N}$.

Now suppose at least one of $a_1, \ldots, a_k$ is even, say a_i. Since $g = 1$, at least one of $a_1, \ldots, a_k$ must be odd; say a_j, $j \neq i$. Let $\mathscr{B}$ denote the set of equations $\{y = x + a_i, y = x + a_j\}$. Since a_i and a_j have opposite parity, $\frac{a_i}{\gcd(a_i,a_j)}$ and $\frac{a_j}{\gcd(a_i,a_j)}$ have opposite parity, and so $\mathscr{R}_d(\mathscr{B})$ exists by Theorem 12. Therefore, $\mathscr{R}_d(\mathscr{A})$ exists by Lemma 3. □

Remark 14. We remark that the nonexistence of $\mathscr{R}_d(\mathscr{A})$ when each a_i/g is odd also follows from Theorem 7, by showing that there is no closed $\mathscr{A}$-path of odd length on $\mathbb{N}$.

Theorems 11, 13 allow us to assume $g = 1$ without loss of generality, which is what we assume for the rest of this section unless stated otherwise. We have the following general upper bound using Theorems 12, 13.

Theorem 15. *Let $k \geqslant 2$ and $g = 1$. If $\mathscr{R}_d(\mathscr{A})$ exists, then*

$$\mathscr{R}_d(\mathscr{A}) \leqslant a_1 + a_k.$$

Proof. Since $g = 1$ and $\mathscr{R}_d(\mathscr{A})$ exists, Theorem 13 tells us that at least one of $a_1, \ldots, a_k$ is even. Since $g = 1$, not all $a_1, \ldots, a_k$ can be even, and therefore at least one of $a_1, \ldots, a_k$ is odd. Therefore, there exists some $j > 1$ such that a_1 and a_j have opposite parity. Denote the set of equations $\{y = x + a_1, y = x + a_j\}$ by $\mathscr{B}$ for brevity. Since a_1, a_j have opposite parity, $\frac{a_1}{\gcd(a_1,a_j)}$, $\frac{a_j}{\gcd(a_1,a_j)}$ also have opposite parity. Therefore, from Theorem 12, we have that $\mathscr{R}_d(\mathscr{B}) \leqslant a_1 + a_j - \gcd(a_1, a_j) + 1 \leqslant a_1 + a_k$. Hence,

$$\mathscr{R}_d(\mathscr{A}) \leqslant \mathscr{R}_d(\mathscr{B}) \leqslant a_1 + a_k$$

by Lemma 3. $\square$

We use the definitions introduced in Notation 8. The next theorem gives a lower bound on $\mathscr{R}_d(\mathscr{A})$ provided that $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist. Here and elsewhere, for a given set $S \subseteq \mathbb{Z}$, we define the indicator function $\delta_S : \mathbb{Z} \to \{0, 1\}$ as follows: $\delta_S(x) = 1$ if and only if $x \in S$.

Theorem 16. *Let $k \geqslant 2$ and $g = 1$. If $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, then*

$$\mathscr{R}_d(\mathscr{A}) \geqslant a_k + f.$$

Proof. Let $x_j = ja_k \bmod f$ for $j \in [1, f]$. Since $g = \gcd(a_k, f) = 1$, $x_i \neq x_j$ for $i \neq j$, and therefore $\{x_j : j \in [1, f]\} = [1, f]$. Let $S = [1, f - a'_k]$. We define a coloring $\Delta : [1, a_k + f - 1] \to \{1, 2\}$ and show that it is valid for $\mathscr{A}$:

$$\Delta(x_1) = \Delta(a'_k) = 2, \tag{7}$$

$$\Delta(x_j) = \big(\Delta(x_{j-1}) + \delta_S(x_{j-1}) + m\big) \bmod 2 \qquad \text{for } j \in [2, f], \tag{8}$$

$$\Delta(x) = (1 + \Delta(x - f)) \bmod 2 \qquad \text{for } x \in [f + 1, a_k + f - 1]. \tag{9}$$

Equations (7) and (8) define Δ on $[1, f]$ iteratively and eqn. (9) defines it iteratively on $[f + 1, a_k + f - 1]$. It is easy to see that Δ is well-defined on $[1, a_k + f - 1]$. By Theorem 13, $\frac{a_j}{f}$ is odd for each $j \in [1, k - 1]$, and so $(x + a_j) - x$ is an odd multiple of f for each x. This, along with eqn. (9) implies that $\Delta(x) \neq \Delta(x + a_j)$ for $j \in [1, k - 1]$ whenever $x, x + a_j \in [1, a_k + f - 1]$.

It remains to prove that $\Delta(x) \neq \Delta(x + a_k)$ for $x \in [1, f - 1]$. Since $x_f = f$ and $x \in [1, f - 1]$, we have $x = x_j$ for some $j \in [1, f - 1]$. And so, by eqn. (8),

$$\Delta\big((x + a_k) \bmod f\big) - \Delta(x) \equiv \delta_S(x) + m \pmod 2.$$

Also, $\Delta(x + a_k) = \Delta(x + mf + a_k{}') = \big(\Delta(x + a_k{}') + m\big) \bmod 2$. Notice that $x + a_k{}' = ((x + a_k{}') \bmod f)$ if $x \in S$ and $x + a_k{}' = f + \big((x + a_k{}') \bmod f\big)$ otherwise. This gives us

$$\Delta(x + a_k) - \Delta\big((x + a_k{}') \bmod f\big) \equiv m + 1 + \delta_S(x) \pmod 2.$$

Since $a_k \equiv a_k{}' \pmod f$, the two equations give us $\Delta(x+a_k)-\Delta(x) \equiv 1+2m+2\delta_S(x) \equiv 1 \pmod 2$, proving our claim. $\square$

In Theorem 19, we prove that the valid coloring defined in the above theorem is the only valid coloring on $[1, a_k + f - 1]$. But we need to establish more structure on our valid colorings to do that; we prove Lemmas 17 and 18 to that end.

Given $N \in \mathbb{N}$ and $B \subseteq \mathbb{N}$, an (N, B)-path from x to y is a sequence of integers $\langle x_0, x_1, \ldots, x_{n-1}, x_n \rangle$ where $|x_i - x_{i-1}| \in B$ for each $i \in [1, n]$, $x_0, x_1, \ldots, x_n \in [1, N]$, $x_0 = x$ and $x_n = y$. We call n the length of this path. Note that this is an equivalence relation on $[1, N]$: (i) there is an (N, B)-path from x to x for each $x \in [1, N]$, (ii) there is an (N, B)-path from x to y if and only if there is an (N, B)-path from y to x, and (iii) there is an (N, B)-path from x to z if there is an (N, B)-path from x to y and an (N, B)-path from y to z.

Given integers $b_1, \ldots, b_m$, not all zero, we know that $\gcd(b_1, \ldots, b_m)$ is an integer linear combination of numbers b_i; that is, $\gcd(b_1, \ldots, b_m) = \sum_{i=1}^{m} \lambda_i b_i$, where each $\lambda_i \in \mathbb{Z}$. Our next lemma proves a similar result for intervals of $\mathbb{N}$, provided that the interval is long enough.

Lemma 17. *Suppose B is a nonempty set of positive integers and let $b = \gcd(B)$. Then, given an integer $N \geqslant \min(B) + \max(B)$, there is an (N, B)-path from x to $x + b$ for each $x \in [1, N - b]$.*

Proof. First note that it is equivalent to prove that there is an (N, B)-path from x to $x \bmod b$ for each $x \in [1, N]$. Let $B = \{b_1, \ldots, b_m\}$ such that $b_1 < \cdots < b_m$. We induct on m. For $m = 1$, the result follows trivially since $b = b_1$. Assume that the statement is true for all sets of size $m - 1$.

Let $b' = \gcd(b_1, \ldots, b_{m-1})$, so that $b = \gcd(b_m, b')$. Suppose $x \in [1, N - b]$, and let $y = x \bmod b', y' = (x + b) \bmod b'$. Define $y_j = (y + j b_m) \bmod b'$ for $j \in \left[0, \frac{b'}{b} - 1\right]$. Then, $Y = \left\{y_j : j \in \left[0, \frac{b'}{b} - 1\right]\right\} = \{z \in [1, b'] : z \equiv y \pmod{b}\}$. Since $y' \leqslant b'$ and $y' - y \equiv (x + b) - x \equiv 0 \pmod{b}$, $y' \in Y$. Trivially, $y_0 = y$.

By the induction hypothesis, there is an $\big(N, B - \{b_m\}\big)$-path from x to y and from $(x + b)$ to y'. We will show that there is an (N, B)-path from y_j to y_{j+1} for each j, and therefore that there is an (N, B)-path from y to y', proving our claim.

Since each $y_j \leqslant b'$, we have $y_j + b_m \leqslant N$, and therefore there is an (N, B)-path from y_j to $y_j + b_m$. Note that $y_{j+1} = (y_j + b_m) \bmod b'$ and $N \geqslant b_1 + b_m \geqslant b_1 + b_{m-1}$. Then, by the induction hypothesis, there is an $\big(N, B - \{b_m\}\big)$-path from y_{j+1} to $(y_j + b_m) \bmod b'$. Therefore, there is an (N, B)-path from y_j to y_{j+1}. □

Lemma 18. *Let $k \geqslant 2$ and $N \geqslant a_1 + a_{k-1}$, and suppose $\Delta : [1, N] \to \{1, 2\}$ is a valid coloring for $\mathscr{A}$. If $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, then for each $x \in [1, N - f]$,*

$$\Delta(x) \neq \Delta(x + f).$$

Proof. From Lemma 17, there is an $\big(N, \{a_1, \ldots, a_{k-1}\}\big)$-path from x to $x + f$. We prove that the length of any such path is odd, which implies our result.

Take any path from x to $x + f$, and correspondingly write $x + f = x + \sum_{j=1}^{k-1} \lambda_j a_j$ for some integers λ_j, so that $1 = \displaystyle\sum_{j=1}^{k-1} \left(\lambda_j \cdot \frac{a_j}{f}\right)$. Since $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, each $\dfrac{a_j}{f}$ is odd by Theorem 13 for $j \in [1, k - 1]$. Therefore,

$$1 = \sum_{j=1}^{k-1} \left(\lambda_j \cdot \frac{a_j}{f}\right) \equiv \sum_{j=1}^{k-1} \lambda_j \equiv \sum_{j=1}^{k-1} |\lambda_j| \pmod{2}.$$

Since $\sum_j |\lambda_j|$ is the length of the path, $\Delta(x) \neq \Delta(x+f)$. □

Theorem 19. *Let $k \geqslant 2$, $N \geqslant a_k + f - 1$, $a_k \geqslant a_{k-1} + a_1 - f + 1$ and $g = 1$. If $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, then there exists at most one (up to taking complement) valid coloring for $\mathscr{A}$ on $[1, N]$.*

Proof. Let $\chi : [1, N] \to \{1, 2\}$ be a valid coloring for $\mathscr{A}$. From Lemma 18, we have that $\chi(x) = \big(1 + \chi(x+f)\big) \bmod 2$ for all $x \in [1, N-f]$ since $a_{k-1} + a_1 \leqslant a_k + f - 1 \leqslant N$. Therefore, $\chi(x)$ is uniquely determined by $\chi(x \bmod f)$ for each $x \in [1, N]$. We further prove that $\chi(x)$ is uniquely determined by $\chi(a'_k)$ for each $x \in [1, f]$, which implies the theorem.

To this end, let $x_j = ja_k \bmod f$ for $j \in [1, f]$. Since $\gcd(f, a_k) = g = 1$, $X = \{x_j : j \in [1, f]\} = [1, f]$. Since $x_j \in [1, f-1]$ for each $j \neq f$, $x_j + a_k \leqslant N$, and therefore from Lemma 18 for $j \neq f$,

$$\chi(x_j + a_k) = \chi(x_j + a'_k + mf) \equiv \chi(x_j + a'_k) + m \pmod 2.$$

Since χ avoids a monochromatic solution to $x + a_k = y$, $\chi(x_j + a_k) = \big(1 + \chi(x_j)\big) \bmod 2$. This gives us $\chi(x_j + a'_k) = \big(\chi(x_j) + m + 1\big) \bmod 2$. Now

$$x_j + a'_k = \begin{cases} (x_j + a'_k) \bmod f & \text{if } x_j \leqslant f - a'_k, \\ f + \big((x_j + a'_k) \bmod f\big) & \text{if } x_j > f - a'_k. \end{cases}$$

This, combined with Lemma 18 gives us

$$\begin{aligned} \chi(x_{j+1}) &= \chi\big((x_j + a_k) \bmod f\big) \\ &= \chi\big((x_j + a'_k) \bmod f\big) \\ &= \begin{cases} \big(\chi(x_j) + m + 1\big) \bmod 2, & \text{if } x_j \leqslant f - a'_k, \\ \big(\chi(x_j) + m\big) \bmod 2, & \text{if } x_j > f - a'_k. \end{cases} \end{aligned}$$

Hence, $\chi(x_{j+1}) = \big(\chi(x_j) + m + \delta_{[1, f - a_k{}']}(x_j)\big) \bmod 2$ for each $j \neq f$. Thus, χ is uniquely determined on $X = [1, f]$ and so also on $[1, N]$ by $\chi(a'_k)$ (or indeed by $\chi(x)$ for any $x \in [1, f]$). Therefore, χ is unique up to the value of $\chi(a'_k)$, which is either 1 or 2. □

In the next theorem, we determine $\mathscr{R}_d(\mathscr{A})$ when a_k is large enough, and therefore for all but finitely many values of a_k, for any given collection of positive integers $a_1, \ldots, a_{k-1}$.

Theorem 20. *Let $k \geqslant 2$ and $g = 1$. If $a_k \geqslant a_{k-1} + a_1 - f + 1$, we have*

$$\mathscr{R}_d(\mathscr{A}) = \begin{cases} \mathscr{R}_d(\mathscr{A}_{k-1}) & \text{if } \mathscr{R}_d(\mathscr{A}_{k-1}) \text{ exists}, \\ a_k + f & \text{if } \mathscr{R}_d(\mathscr{A}_{k-1}) \text{ does not exist and } a_k - a_{k-1} \text{ is odd}, \\ \text{does not exist} & \text{if } \mathscr{R}_d(\mathscr{A}_{k-1}) \text{ does not exist and } a_k - a_{k-1} \text{ is even}. \end{cases}$$

Proof. We divide the proof into three cases.

Case I: Suppose $\mathscr{R}_d(\mathscr{A}_{k-1})$ exists. From Lemma 3, we have that $\mathscr{R}_d(\mathscr{A})$ exists and $\mathscr{R}_d(\mathscr{A}) \leqslant \mathscr{R}_d(\mathscr{A}_{k-1})$. Theorem 15 combined with Theorem 11 tells us that $\mathscr{R}_d(\mathscr{A}_{k-1}) \leqslant a_1 + a_{k-1} - f + 1$. Let $\Delta : [1, \mathscr{R}_d(\mathscr{A}_{k-1}) - 1] \to \{1, 2\}$ be a valid coloring for $\mathscr{A}_{k-1}$; that is, suppose Δ does not admit a monochromatic solution to any equation in $\mathscr{A}_{k-1}$. Since $a_k \geqslant a_{k-1} + a_1 - f + 1 \geqslant \mathscr{R}_d(\mathscr{A}_{k-1})$, Δ does not admit a monochromatic solution to $y = x + a_k$ either, and therefore $\mathscr{R}_d(\mathscr{A}) \geqslant \mathscr{R}_d(\mathscr{A}_{k-1})$, which proves our claim.

Case II: Suppose $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist and $a_k - a_{k-1}$ is odd. Theorem 16 provides the lower bound; we proceed to show that $\mathscr{R}_d(\mathscr{A}) \leqslant a_k + f$. Let $\chi : [1, a_k + f] \to \{1, 2\}$ be a valid coloring for $\mathscr{A}$. From Theorem 19, $\chi(n) = \Delta(n)$ for each $n \in [1, a_k + f - 1]$ for Δ defined in Theorem 16. We will show that $\chi(a_k - a_{k-1} + f) \neq \chi(f)$, which implies that one of $(a_k - a_{k-1} + f, a_k + f)$ and $(f, a_k + f)$ is a monochromatic solution to some equation in $\mathscr{A}$.

We first relate $\chi(f)$ and $\chi(a'_k)$. If f is even, so is a_{k-1}. If f is odd, then since $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, a_{k-1} is odd from Theorem 13. In either case, f and a_{k-1} have the same parity. Therefore, since $a_k - a_{k-1}$ is odd by assumption, f and a_k have opposite parity. Consequently, from eqn. (8), we have

$$\begin{aligned} \chi(f) &\equiv \chi(a'_k) + \sum_{i=1}^{f-1} \delta_S(x_i) + m(f-1) \\ &= \chi(a'_k) + (f - a'_k) + mf - m \\ &= \chi(a'_k) + f + a_k - 2{a_k}' + m \\ &\equiv \chi(a'_k) + m + 1 \pmod 2. \end{aligned}$$

The last congruence holds because f and a_k have the opposite parity. Finally,

$$\chi(a_k - a_{k-1} + f) \equiv 1 + \chi(a_k - a_{k-1}) \equiv \chi(a_k) \equiv \chi({a_k}') + m \equiv (\chi(f) + m + 1) + m \equiv \chi(f) + 1,$$

each taken mod 2. The first congruence holds from Lemma 17, the second congruence holds because χ is valid for $\mathscr{A}$ and the third congruence holds from Lemma 18. Therefore, $\chi(a_k - a_{k-1} + f) \neq \chi(f)$, proving our claim.

Case III: Suppose $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist and $a_k - a_{k-1}$ is even. If f is even, then so is a_{k-1} and therefore, a_k is also even by our assumption, which is a contradiction since $1 = g = \gcd(f, a_k)$. As $\mathscr{R}_d(\mathscr{A}_{k-1})$ does not exist, $\frac{a_j}{f}$ is odd for each $j \in [1, k-1]$ from Theorem 13. This implies that a_j is odd for each $j \in [1, k-1]$. Since $a_k - a_{k-1}$ is even, this implies that a_k is also odd. So, once again by Theorem 13, $\mathscr{R}_d(\mathscr{A})$ does not exist. □

We now determine $\mathscr{R}_d(\mathscr{A})$ when the numbers $a_1, \ldots, a_k$ form an arithmetic progression or a geometric progression.

Theorem 21. *Let a, d, k be positive integers, $k \geqslant 2$ and $\gcd(a, d) = 1$. Let $AP(a, d; k)$: $\{y = x + a, y = x + (a + d), \ldots, y = x + (a + (k-1)d)\}$. Then*

(i) $\mathscr{R}_d\big(AP(a,d;k)\big)$ *exists if and only if* d *is odd.*

(ii) *If* d *is odd, then*

$$\mathscr{R}_d\big(AP(a,d;k)\big) = \mathscr{R}_d\big(AP(a,d;2)\big) = 2a+d.$$

Proof. (i) Since $\gcd(a, a+d) = \gcd(a,d) = 1$, d is even implies a is odd, and so $a, a+d, \ldots, a+(k-1)d$ are all odd. On the other hand, if d is odd, at least one of $a, a+d$ must be even. The equivalence of existence now follows from Theorem 13.

(ii) Note that $\mathscr{R}_d\big(\mathrm{AP}(a,d;2)\big) = 2a+d$ by Theorem 12, that $\mathscr{R}_d\big(\mathrm{AP}(a,d;k)\big)$ exists when d is odd by part (i), and that $\mathscr{R}_d\big(\mathrm{AP}(a,d;k)\big) \leqslant \mathscr{R}_d\big(\mathrm{AP}(a,d;2)\big)$ by Lemma 3. Therefore, to show that $\mathscr{R}_d\big(\mathrm{AP}(a,d,k)\big) = 2a+d$, it suffices to provide a valid coloring of $[1, 2a+d-1]$ for $\mathrm{AP}(a,d;k)$.

Choose a valid coloring Δ of $[1, 2a+d-1]$ for the set of equations $\big\{y = x+a, y = x+(a+d)\big\}$. We claim that Δ is also a valid coloring of $[1, 2a+d-1]$ for the set of equations $\big\{y = x+a, y = x+a+d, \ldots, y = x+a+(k-1)d\big\}$. If this was not the case, there would exist $x, x+(a+id) \in [1, 2a+d-1]$ such that $\Delta(x) = \Delta(x+a+id)$, for some $i \in [2, k-1]$ (because Δ is a valid coloring for the set $\{y = x+a, y = x+(a+d)\}$). We may assume, without loss of generality, that $\Delta(x) = 2$.

From the fact that Δ is a valid coloring for the set $\{y = x+a, y = x+(a+d)\}$, so that no two elements in $[1, 2a+d-1]$ that differ by either a or $a+d$ can have the same color, we obtain $\Delta\big(x+(i-1)d\big) = 1$ from $\Delta(x+a+id) = 2$, and then $\Delta\big(x+a+(i-1)d\big) = 2$ from $\Delta\big(x+(i-1)d\big) = 1$. Hence $\Delta(x+a+id) = 2$ implies $\Delta\big(x+a+(i-1)d\big) = 2$. Repeating this argument yields $\chi(x+a) = 2 = \chi(x)$, thereby contradicting the validity of Δ for the set $\{y = x+a, y = x+(a+d)\}$.

This proves our claim that Δ is also a valid coloring of $[1, 2a+d-1]$ for $\mathrm{AP}(a,d;k)$, and also proves the theorem. □

Theorem 22. *Let* a, r, k *be positive integers,* $k \geqslant 2$. *Let* $GP(a,r;k) : \{y = x+a, y = x+ar, \ldots, y = x+ar^{k-1}\}$. *Then*

(i) $\mathscr{R}_d\big(GP(a,r;k)\big)$ *exists if and only if* r *is even.*

(ii) *If* r *is even, then*

$$\mathscr{R}_d\big(GP(a,r;k)\big) = \mathscr{R}_d\big(GP(a,r;2)\big) = ar+1.$$

Proof. (i) This is a direct consequence of Theorem 13.

(ii) By Theorem 11, part (ii), we must show $\mathscr{R}_d\big(\mathrm{GP}(1,r;k)\big) = r+1$. We induct on k. By Theorem 12, $\mathscr{R}_d\big(\mathrm{GP}(1,r;2)\big) = \mathscr{R}_d\big(\{y = x+1, y = x+r\}\big) = r+1$. Clearly, $r^{k-1} \geqslant r^{k-2}+1$ since $r \geqslant 2$. Therefore, from Theorem 20, $\mathscr{R}_d\big(\mathrm{GP}(1,r;k)\big) = \mathscr{R}_d\big(\mathrm{GP}(1,r;k-1)\big)$ which equals $\mathscr{R}_d\big(\mathrm{GP}(1,r;2)\big) = r+1$ by induction. □

4 Set of multiplicative equations $y = a_1x, \ldots, y = a_kx$

Let $a_1, \ldots, a_k$ be distinct positive integers, and let $\mathscr{M}$ denote the set of equations $\{y = a_1x, \ldots, y = a_kx\}$. Johnson & Schaal in [8] showed that for $k = 2$, $\mathscr{R}_d(\mathscr{M})$ exists if and only if $a_1 = c^s$ and $a_2 = c^t$ for some positive integers c, s, t with $\gcd(s, t) = 1$ and $s + t$ odd. We generalize their result on existence to arbitrary k using Theorem 7, and derive their result as a corollary.

Theorem 23. *Let $p_1, \ldots, p_m$ be the set of primes in the prime factorization of $\prod_{i=1}^{k} a_i$. Consider the matrix $M = [s_{ij}]$ where s_{ij} is the largest power of p_i in a_j. Then $\mathscr{R}_d(\mathscr{M})$ exists if and only if there exists $\mathbf{t} = (t_1, \ldots, t_k)^\top \in \mathbb{Z}^k$ that satisfies*

(i) $M\mathbf{t} = \mathbf{0}$, *and*

(ii) $\sum_{i=1}^{k} t_i$ *is odd.*

Proof. We recall Definition 4. From Theorem 7, it is enough to show that there exists a closed $\mathscr{M}$-path of odd length if and only if there exists $\mathbf{t} = (t_1, \ldots, t_k)^\top$ that satisfies (i) and (ii).

Suppose first that a closed $\mathscr{M}$-path $\langle x_0, x_1, \ldots, x_m = x_0\rangle$ of odd length exists. Then $x_m = x_0 = x_0 \prod_{i=1}^{k} {a_i}^{t_i}$ for some integers $t_i \in \mathbb{Z}$ such that $\sum_{i=1}^{k} t_i$ is odd. So $\prod_{i=1}^{k} {a_i}^{t_i} = 1$. Therefore,

$$1 = \prod_{j=1}^{k} a_j^{t_j} = \prod_{j=1}^{k}\prod_{i=1}^{m} {p_i}^{s_{ij}t_j} = \prod_{i=1}^{m} {p_i}^{\sum_{j=1}^{k} s_{ij}t_j}.$$

This gives us $\sum_{j=1}^{k} s_{ij}t_j = 0$ for all i. As $\sum_{i=1}^{k} t_i$ is odd, we have a $\mathbf{t}$ which satisfies both (i) and (ii).

Conversely, suppose we have a $\mathbf{t} = (t_1, \ldots, t_k)^\top \in \mathbb{Z}^k$ which satisfies (i) and (ii). By a simple calculation, this implies that $\prod_{i=1}^{k} {a_i}^{t_i} = 1$. We now construct a closed $\mathscr{M}$-path of odd length using these t_i's.

Consider the sets $T_+ = \{i \,|\, t_i > 0\}$ and $T_- = \{i \,|\, t_i < 0\}$, and say $m = |T_+|$, $n = |T_-|$. We can shuffle the indices i so that $t_i > 0$ for $i \in [1, m]$ and $t_i < 0$ for $i \in [m+1, n]$. Let $p = \sum_{t \in T_+} t$ and $q = \sum_{t \in T_-} |t|$. For $i \leqslant p$, define $x_i = a_1x_{i-1}$ if $i \in [1, t_1]$, $x_i = a_2x_{i-1}$ for $i \in [t_1 + 1, t_1 + t_2]$ and so on. For $p < i \leqslant q$, define $x_i = \frac{x_{i-1}}{a_{m+1}}$ for $i \in [p + 1, p + t_{m+1}]$, $x_i = \frac{x_{i-1}}{a_{m+2}}$ for $i \in [p + t_{m+1} + 1, p + t_{m+1} + t_{m+2}]$ and so on. Now consider the $\mathscr{M}$-path $\langle 1 = x_0, x_1, \ldots, x_{p+q}\rangle$. We have

$$x_{p+q} = 1 \cdot \prod_{i=1}^{m} {a_i}^{t_i} \cdot \prod_{i=m+1}^{m+n} \frac{1}{{a_i}^{|t_i|}} = \prod_{i=1}^{m+n} {a_i}^{t_i} = \prod_{i=1}^{k} {a_i}^{t_i} = 1 = x_0.$$

Also, $\sum_{i=1}^{k} t_i = p+q$ is odd. So $\langle 1 = x_0, x_1, \ldots, x_{p+q} = 1 \rangle$ is a closed $\mathscr{M}$-path of odd length, proving our claim. □

We now obtain Johnson & Schaal's result on existence of $\mathscr{R}_d(\mathscr{M})$ for $k = 2$ as a corollary, but do not determine $\mathscr{R}_d(\mathscr{M})$ in this case.

Corollary 24. *When $k = 2$, $\mathscr{R}_d(\mathscr{M})$ exists if and only if there exist positive integers c, t_1, t_2 such that $a_1 = c^{t_1}$, $a_2 = c^{t_2}$ and t_1, t_2 having opposite parity.*

Proof. Suppose first that there exist such c, t_1, t_2. Then since $a_2^{t_1} = a_1^{t_2}$, it is easy to see that $\mathbf{t} = (t_1, -t_2)^\top$ satisfies $M\mathbf{t} = 0$ and clearly $t_1 - t_2 \equiv 1 \pmod 2$.

Suppose now that $\mathscr{R}_d(\mathscr{M})$ exists. Then, by Theorem 23, there exists $\mathbf{t} = (t_1, -t_2)^\top$ such that $M\mathbf{t} = 0$ and $t_1 - t_2 \equiv 1 \pmod 2$, or that

$$a_1^{t_1} = a_2^{t_2}.$$

We can assume without loss of generality that $t_1, t_2 > 0$ (otherwise $-\mathbf{t}$ satisfies the conditions as well).

For a prime p, if α_1, α_2 are the highest powers of p that divide a_1, a_2 respectively, then $\alpha_1 = qt_1$ and $\alpha_2 = qt_2$ for some integer $q \geqslant 0$. Therefore, $a_1 = c^{t_1}$ and $a_2 = c^{t_2}$ for some $c \in \mathbb{N}$, proving our claim. □

The following result gives the nonexistence of $\mathscr{R}_d(\mathscr{M})$ in some cases.

Corollary 25. *$\mathscr{R}_d(\mathscr{M})$ does not exist if any of the following conditions is satisfied:*

(i) *M has full column rank.*

(ii) *Each a_i is prime.*

(iii) *There exists a row in M with no even entries.*

Proof.

(i) If M has full column rank, then M is left-invertible. So, $\mathbf{t} = \mathbf{0}$ is the only solution for $M\mathbf{t} = \mathbf{0}$. Here $\sum_{i=1}^{k} t_i = 0$, which is even. So $\mathscr{R}_d(\mathscr{M})$ does not exist.

(ii) Follows directly from part (i), since $M = I_k$ in this case.

(iii) We have a row in M (say the ith row) in which all entries are odd; that is, s_{ij} is odd for all $j \in [1, k]$. Suppose $\mathbf{t} = (t_1, t_2, \ldots, t_k)^\top$ satisfies $M\mathbf{t} = \mathbf{0}$. Then $\sum_{j=1}^{k} s_{ij} t_j = 0$, and since s_{ij}'s are odd, we have

$$0 = \sum_{j=1}^{k} s_{ij} t_j \equiv \sum_{j=1}^{k} t_j \pmod 2.$$

So $\sum_{j=1}^{k} t_j$ is even and this implies that $\mathscr{R}_d(\mathscr{M})$ does not exist. □

5 Algorithms for determining $\mathscr{R}_d$ and valid colorings

Let $a_1, \ldots, a_k$, $k \geqslant 2$, be positive integers, with $a_1 < \ldots < a_k$, and let $\mathscr{A}$ be the set of equations $\{y = x + a_1, \ldots, y = x + a_k\}$, with $\mathscr{R}_d(\mathscr{A})$ the corresponding disjunctive Rado number. In this section, we give an algorithm to determine the disjunctive Rado number for a general set $\mathscr{S}_2$ of equations in two variables, assuming that there is an algorithm which returns all possible solutions (x, y) to a given equation in $\mathscr{S}_2$ on any interval $[1, N]$ for $N \in \mathbb{N}$ and that a theoretical upper bound is known when this number exists. This algorithm will reduce to an $O(ka_k \log a_k)$ time algorithm for set $\mathscr{A}$. We also present a related algorithm that gives all possible valid colorings for $[1, \mathscr{R}_d(\mathscr{S}_2) - 1]$, provided that $\mathscr{R}_d(\mathscr{S}_2)$ is known.

We now make precise the requirements for the given set of equations. Let $\mathscr{S}_2 = \{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ be a (finite) set of equations in two variables (say x and y). Suppose for $x \in [1, N]$, we are given a subroutine $S(N, x; \mathscr{S}_2)$ that returns the set

$$S(N, x; \mathscr{S}_2) = \{y : (x, y) \text{ satisfies some equation } \mathcal{E} \in \mathscr{S}_2, y \in [1, N]\},$$

that is, the set of all integers $y \in [1, N]$ so that (x, y) satisfies some equation $\mathcal{E}_i$ for $i \in [1, k]$. Let the running time of this subroutine be $T(N, x; \mathscr{S}_2)$. Suppose also that we are given an upper bound $\mathcal{U}(\mathscr{S}_2)$ on $\mathscr{R}_d(\mathscr{S}_2)$ whenever it exists; that is, we are guaranteed that if $\mathscr{R}_d(\mathscr{S}_2)$ exists, then $\mathscr{R}_d(\mathscr{S}_2) \leqslant \mathcal{U}(\mathscr{S}_2)$.

Our algorithm is a simple combination of binary search over the solution space and search over a graph. We first describe our problem graph theoretically.

For each $N \in \mathbb{N}$, recall that we defined the undirected graph $G_N(\mathscr{S}_2) = G_N$ as follows: the vertex set $V(G_N) = [1, N]$ and the edge set $E(G_N) = \{(x, y) : (x, y) \text{ satisfies some } \mathcal{E} \in \mathscr{S}_2\}$. Similarly, G is the graph on vertices $\mathbb{N}$ with the edge set correspondingly defined. Note that each G_N is an induced subgraph of G. The following lemma is an extension of Lemma 5 and easily follows; we omit its proof.

Lemma 26.

(i) *Every valid 2-coloring of $[1, N]$ for set $\mathscr{S}_2$ is a graph 2-coloring of $G_N(\mathscr{S}_2)$, and vice-versa. Similarly, every valid 2-coloring of $\mathbb{N}$ for $\mathscr{S}_2$ is a graph 2-coloring of $G(\mathscr{S}_2)$, and vice-versa.*

(ii) *$\mathscr{R}_d(\mathscr{S}_2)$ exists if and only if $G(\mathscr{S}_2)$ is not bipartite. Moreover, if it exists, then it is the least integer N for which $G_N(\mathscr{S}_2)$ is not bipartite.*

(iii) *If $G_N(\mathscr{S}_2)$ is not bipartite, then $G(\mathscr{S}_2)$ is not bipartite, and $G_M(\mathscr{S}_2)$ is not bipartite for any $M > N$.*

We now describe and analyze Algorithm 1, which tells us if $\mathscr{R}_d(\mathscr{S}_2)$ exists and determines its value when it does. Since we are working with an arbitrary but fixed $\mathscr{S}_2$, we omit it from our notation subsequently, unless specified otherwise.

Algorithm 1 Determine $\mathscr{R}_d(\mathscr{S}_2)$

```
procedure DisjunctiveRadoNumber(𝒮₂)
    L ← 1
    U ← 𝒰(𝒮₂)
    Construct(G_U(𝒮₂))
    if IsBipartite(G_U(𝒮₂)) then
        return ∞
    end if
    while L < U do
        n = ⌊(L+U)/2⌋
        Induce G_n(𝒮₂) from G_U(𝒮₂)
        if IsBipartite(G_n(𝒮₂)) then
            L ← n + 1
        else
            U ← n
        end if
    end while
    return L
end procedure
```

Theorem 27. *Let $\mathscr{S}_2$ be a finite set of equations in two variables. If we are given an upper bound $\mathcal{U}$ for $\mathscr{R}_d(\mathscr{S}_2)$, assuming it exists, and a subroutine $S(N,x)$ that runs in finite time $T(N,x)$, then Algorithm 1 determines whether $\mathscr{R}_d(\mathscr{S}_2)$ exists and runs in time*

$$O\Big(\max\Big\{\sum_{x=1}^{\mathcal{U}} T(\mathcal{U},x), \Big(\mathcal{U}+\sum_{x=1}^{\mathcal{U}}\big|S(\mathcal{U},x)\big|\Big)\log\mathcal{U}\Big\}\Big).$$

Proof. We first describe the subroutines used in the algorithm:

- Let $G_{\mathcal{U}} = (V_{\mathcal{U}}, E_{\mathcal{U}})$. Construct$(G_{\mathcal{U}})$ constructs the graph as an adjacency list using $S(\mathcal{U},x)$ in $O\big(|V_{\mathcal{U}}|+|E_{\mathcal{U}}|\big)$ steps. Since the set of edges in $G_{\mathcal{U}}$ is precisely the set $\bigcup_{x=1}^{\mathcal{U}}\big\{(x,y) : y \in S(\mathcal{U},x)\big\}$, Construct$(G_{\mathcal{U}})$ runs in time

$$O\Big(\mathcal{U}+\sum_{x=1}^{\mathcal{U}} T(\mathcal{U},x)\Big) = O\Big(\sum_{x=1}^{\mathcal{U}} T(\mathcal{U},x)\Big).$$

- Subroutine IsBipartite checks if a graph is bipartite, and runs in time $O\big(|V|+|E|\big)$ for any graph $G=(V,E)$. We omit the details of this well-known algorithm, which can be implemented through any standard search algorithm for a graph (breath-first search, for instance).

- Since G_n is the subgraph of $G_{\mathcal{U}}$ induced by vertex set $[1,n]$, Induce G_n from $G_{\mathcal{U}}$ can be implemented by enabling a flag for each vertex $x \in [1,n]$.

Since $\mathscr{R}_d(\mathscr{S}_2) \leqslant \mathcal{U}$ if it exists, Lemma 26, part (iii) tells us that $\mathscr{R}_d(\mathscr{S}_2) \leqslant \mathcal{U}$ if $G_{\mathcal{U}}$ is not bipartite, and does not exist otherwise.

In the latter case, we say that $\mathscr{R}_d(\mathscr{S}_2) = \infty$. In the former case, by Lemma 26, part (ii), we need to search for the least $n \in [1, \mathcal{U}]$ for which G_n is not bipartite. Part (iii) of the same lemma allows us to perform a binary search over this interval. This proves the correctness of the algorithm.

To analyze the running time, note that there are $O(\log \mathcal{U})$ steps in binary search, and each step consists of inducing $G_n = (V_n, E_n)$ and checking if it is bipartite for some $n \in [1, \mathcal{U}]$, which runs in time

$$O\big(|V_n| + |E_n|\big) = O\big(|V_{\mathcal{U}}| + |E_{\mathcal{U}}|\big) = O\bigg(\mathcal{U} + \sum_{x=1}^{\mathcal{U}} \big|S(\mathcal{U}, x)\big|\bigg).$$

Therefore, the running time of the algorithm is

$$O\bigg(\max\bigg\{\sum_{x=1}^{\mathcal{U}} T(\mathcal{U}, x), \bigg(\mathcal{U} + \sum_{x=1}^{\mathcal{U}} \big|S(\mathcal{U}, x)\big|\bigg) \log \mathcal{U}\bigg\}\bigg). \qquad \square$$

Corollary 28. *Algorithm 1 determines whether or not $\mathscr{R}_d(\mathscr{A})$ exists and determines its value when it exists in time*

$$O\big(k a_k \log a_k\big).$$

Proof. From Theorems 11 and 15, $\mathscr{R}_d(\mathscr{A}) \leqslant a_1 + a_k - \gcd(a_1, \ldots, a_k) + 1 \leqslant a_1 + a_k < 2a_k$, so we can choose $\mathcal{U}(\mathscr{A}) = 2a_k$.

Given $x \in [1, 2a_k]$, $|S(\mathcal{U}, x, \mathscr{A})| \leqslant k$ since each equation $x + a_i = y$, $i \in [1, k]$ admits at most one y in $S(\mathcal{U}, x; \mathscr{A})$. Determining this set requires one addition and comparison operation for each equation $y = x + a_i$, which requires $O(\log a_k)$ time. Therefore, $T(\mathcal{U}, x; \mathscr{A}) \leqslant O(k \log a_k)$ for each $x \leqslant 2a_k$. Therefore, by Theorem 27, the running time of Algorithm 1 for $\mathscr{A}$ is

$$O\big(\max\big\{a_k \cdot k \log a_k, a_k \cdot k \cdot \log a_k\big\}\big) = O\big(k a_k \log a_k\big). \qquad \square$$

We now present a related algorithm to generate all valid colorings for set $\mathscr{S}_2$ on $[1, \mathscr{R}_d(\mathscr{S}_2) - 1]$, when $\mathscr{R}_d(\mathscr{S}_2)$ exists and is known. For a 2-coloring χ, we let $\overline{\chi}$ to be the element-wise complement of χ. For $A, B \subseteq \mathbb{N}$, $A \cap B = \emptyset$ and 2-colorings χ_A, χ_B on A, B respectively, let $\chi_A \cup \chi_B$ be the 2-coloring $(\chi_A \cup \chi_B)$ defined on $A \cup B$ as follows:

$$(\chi_A \cup \chi_B)(n) = \begin{cases} \chi_A(n) & \text{if } n \in A, \\ \chi_B(n) & \text{if } n \in B. \end{cases}$$

Theorem 29. *Let $\mathscr{S}_2$ be a finite set of equations in two variables. If we are given an upper bound $\mathcal{U}$ for $\mathscr{R}_d(\mathscr{S}_2)$, assuming it exists, and a subroutine $S(N, x)$ that runs in finite time $T(N, x)$, then Algorithm 2 generates all valid colorings on $[1, \mathscr{R}_d(\mathscr{S}_2) - 1]$ when $\mathscr{R}_d(\mathscr{S}_2)$ exists. Moreover, if $G_{\mathscr{R}_d - 1}$ has ℓ components, then there are 2^ℓ valid colorings on $[1, \mathscr{R}_d(\mathscr{S}_2) - 1]$ for $\mathscr{S}_2$.*

Algorithm 2 Generate all valid colorings on $[1, \mathscr{R}_d(\mathscr{S}_2) - 1]$

procedure GenerateAllValidColorings($\mathscr{S}_2$)
  $R \leftarrow$ DisjunctiveRadoNumber($\mathscr{S}_2$)
  Construct$\big(G_{R-1}(\mathscr{S}_2)\big)$
  for components $C_1, \ldots, C_\ell$ of $G_{R-1}(\mathscr{S}_2)$ **do**
    $\Delta_j \leftarrow$ Get2Coloring(C_j)
  end for
  return $\left\{ \bigcup_j \chi_j : \chi_j \in \left\{\Delta_j, \overline{\Delta_j}\right\} \right\}$ ▷ Set of all combinations of valid colorings on each component.
end procedure

Proof. If $G = (V, E)$ is a connected bipartite graph, there is exactly one graph 2-coloring of V up to taking complements, and therefore exactly two graph 2-colorings of V. The subroutine IsBipartite can be easily modified to create a subroutine Get2Coloring which determines one of these 2-colorings. Since every component of a graph can be colored independently, a bipartite graph with ℓ components has exactly 2^ℓ 2-colorings. The correctness of the algorithm then follows by Lemma 26, part (i). □

Numerical examples. We give two numerical examples of $\mathscr{R}_d(\mathscr{A})$ using our algorithm.

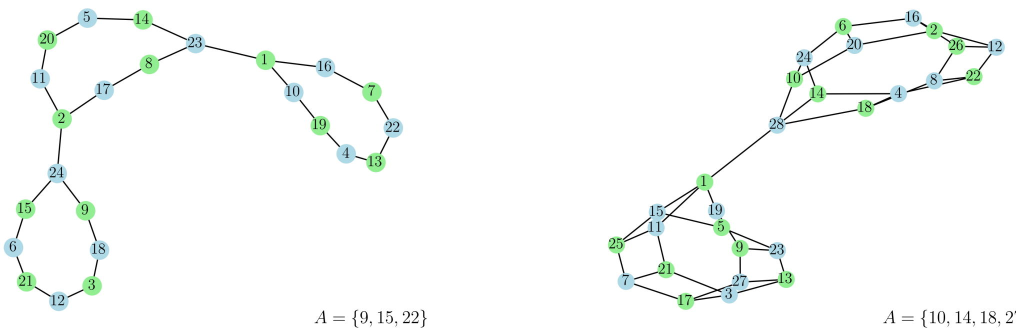

Figure 1: Two examples of valid 2-colorings of $[1, \mathscr{R}_d(\mathscr{A}) - 1]$ for set of equations $\mathscr{A}$: $\{x + a_i = y : i \in \{1, \ldots, k\}\}$ represented as 2-coloring of graph $G_{\mathscr{R}_d(\mathscr{A})-1}$. We denote $A = \{a_1, \ldots, a_k\}$.

Consider $k = 3$, and $(a_1, a_2, a_3) = (9, 15, 22)$. In this case $f = \gcd(9, 15) = 3$, and therefore, $a_3 \geqslant a_2 + a_1 - f + 1$, so that Theorem 20 applies. From Theorem 12 we have that $\mathscr{R}_d(\mathscr{A}_2)$ does not exist, since $\frac{9}{3} + \frac{15}{3}$ is even. Further, $a_3 - a_2$ is odd. The same theorem then tells us that $\mathscr{R}_d(\mathscr{A}) = 22 + 3 = 25$. This is confirmed by Algorithm 1. Algorithm 2 generates the unique (up to complement) 2-coloring of $[1, 24]$, as Figure 1 (left) depicts.

Now consider $k = 4$ and $(a_1, a_2, a_3, a_4) = (10, 14, 18, 27)$. It can be checked that Theorem 20 again applies. We first need to check if $\mathscr{R}_d(\mathscr{A}_3)$ exists. Theorem 13 tells us that $\mathscr{R}_d(\mathscr{A}_3)$ does not exist. Further, $a_4 - a_3$ is again odd, so by Theorem 20, $\mathscr{R}_d(\mathscr{A}) = 29$. Figure 1 (right) shows the unique valid coloring on $[1, 28]$.

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