A note on Ramsey numbers involving large books

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Abstract

For graphs G and H, the Ramsey number $R(G,H)$ is the minimum integer N such that any red/blue edge coloring of K_N contains either a red G or a blue H. Let $\chi(G)$ be the chromatic number of G, and $s(G)$ the minimum size of a color class over all proper vertex colorings of G with $\chi(G)$ colors. A connected graph H is called G-*good* if $R(G,H) = (\chi(G)-1)(|H|-1)+s(G)$. For graphs G and H, it is shown $K_p + nH$ is (K_2+G)-*good*, where n is double-exponential in terms of $p, |G|, |H|$, and $K_p + nH$ is C_{2m+1}-*good* for $n \geqslant (100q)^{8q^3}$, where $q = \max\{m, p, |H|\}$. Both proofs are short and avoid using the regularity method.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction

For graphs G and H, the Ramsey number $R(G,H)$ is the minimum N such that any red/blue edge coloring of K_N contains either a red G or a blue H.

For vertex disjoint graphs G_1 and G_2, denote by $G_1 \cup G_2$ the union of G_1 and G_2, and $G_1 + G_2$ the graph obtained from $G_1 \cup G_2$ by adding new edges to connect G_1 and G_2 completely. Call $G_1 \cup G_2$ and $G_1 + G_2$ the union and the joint of G_1 and G_2, respectively. Let mG be the union of m disjoint copies of G. Call $B_p(n) = K_p + nK_1$ a p-*book*, in which the given p-clique is called the *base* and the n additional vertices are called the *pages*. Books play central roles in Ramsey theory, and many important questions and results concern the Ramsey numbers of books versus other natural classes of graphs, see, e.g., [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

Let $\chi(G)$ be the chromatic number of G, and $s(G)$ the minimum size of a color class over all proper vertex colorings of G with $\chi(G)$ colors. Burr [2] observed a general lower bound as

$$R(G,H) \geqslant (\chi(G)-1)(|H|-1)+s(G) \tag{1}$$

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for a connected graph H of order $|H| \geqslant s(G)$. Burr defined H to be G-*good* if (1) becomes an equality. A well-known result of Chvátal [3] says $R(K_m, T_n) = (m-1)(n-1)+1$, in which T_n is a tree of order n. Hence, T_n is K_m-*good*.

Ramsey goodness has been extensively studied in the literature. An extremely general result of Nikiforov and Rousseau [21] says, roughly speaking, that H is G-*good* whenever G is "small" and H is "poorly connected" and "large". This theorem resolved the majority of the open problems on Ramsey goodness, but it comes with a major caveat: Nikiforov and Rousseau's proof uses Szemerédi's regularity lemma, and as such, the quantitative bounds involved in the definition of "large" above are truly enormous. A certain special case of the Nikiforov-Rousseau theorem is as follows.

Theorem 1 ([10, 17, 21])**.** *Let G and H be graphs, and p a positive integer. If n is sufficiently large, then $K_p + nH$ is $(K_2 + G)$-good.*

Recently, Fox, He, and Wigderson [9] found an alternative proof technique which avoids the regularity lemma, and allowed them to prove certain special cases of the Nikiforov-Rousseau theorem with much stronger quantitative bounds. Particularly, they proved the following result.

Theorem 2 ([9])**.** *If n is double-exponential in terms of p and $|G|$, then $B_p(n) = K_p + nK_1$ is $(K_2 + G)$-good. Particularly, if $n \geqslant 2^{p^{10m}}$, then $B_p(n)$ is K_m-good.*

Furthermore, their theorem [9] allows one to take $|G| = \delta n$ for some small $\delta > 0$, as long as G satisfies certain structural properties, namely, only one of its color classes can be this large, and the remaining ones must be of constant size. Both proofs are direct and avoid using the regularity method. They proposed a problem by writing "It would be interesting to find a direct proof of Nikiforov-Rousseau theorem without regularity lemma, as this would likely lead to superior quantitative bounds.". In this note, we shall answer them affirmatively by giving a direct and short proof for Theorem 1 without using regularity lemma.

Theorem 3. *Let G and H be graphs, and p a positive integer. If n is double-exponential in terms of p, $|G|$ and $|H|$, then $K_p + nH$ is $(K_2 + G)$-good.*

An easy corollary of Theorem 1 is as follows.

Corollary 4. *Let graphs F and H be fixed. If F is a subgraph of $K_2 + G$ for some graph G such that*

$$\chi(F) = 2 + \chi(G), \text{ and } s(F) = 1,$$

then $K_p + nH$ is F-good.

Let C_{2m+1} be an odd-cycle of length $2m+1$. On the one hand, C_{2m+1} is a subgraph of $K_2 + P_{2m-1}$, where P_{2m-1} is a path of length $2m-2$. However, $\chi(C_{2m+1}) = 3$ and $\chi(K_2 + P_{2m-1}) = 4$. On the other hand, C_{2m+1} is a subgraph of $K_1 + K_{m,m}$, where $K_{m,m}$ is complete bipartite graph with m vertices in each part. Lin, Li and Dong [12] proved that $K_1 + nH$ is not $(K_1 + K_{m,m})$-*good*, Fan and Lin [7] proved that $B_p(n)$ is not $(K_1 + K_{m,m})$-*good*. Thus we shall consider the goodness of $K_p + nH$ for C_{2m+1}.

Theorem 5. *Let $m, p \geqslant 1$ be integers and H a graph with $h = |H|$. Let $q = \max\{m, p, h\}$. If $n \geqslant (100q)^{8q^3}$, then $K_p + nH$ is C_{2m+1}-good. Namely*

$$R(C_{2m+1}, K_p + nH) = 2(nh + p - 1) + 1.$$

2 A direct proof of Theorem 3

Proof of Theorem 3. Let Γ be a graph with $N = (\chi(G) + 1)(hn + p - 1) + 1$ vertices, where $h = |H|$ is the order of H. The proof of Theorem 3 follows by a reduction from Theorem 2. For the sake of completeness, we sketch the proof.

Firstly, a blowup variant of the Andrásfai-Erdős-Sós [1] theorem was proved, which says that if a graph has high minimum degree and does not contain some blowup of K_m, then it is $(m - 1)$-partite.

Secondly, they proved that, under the assumptions of the theorem, most vertices of Γ have degree at least $(1 - 1/(m - 1) - o(1))N$ (This is the only place where a double-exponential dependence is needed).

Thirdly, they applied the blowup variant of the Andrásfai-Erdős-Sós theorem, which proved that except for the small number of low-degree vertices, Γ is $(m - 1)$-partite.

Finally, they used a careful averaging argument to show that under these assumptions, $\overline{\Gamma}$ must contain a copy of $B_p(n)$.

To complete the proof, it suffices to show the following lemma.

Lemma 6. *Let G and H be graphs. If $B_p(n) = K_p + nK_1$ is G-good for any fixed p when n is large, then $K_p + nH$ is G-good for any fixed p when n is large.*

Proof. Let $L = (\chi(G) - 1)(hn + p - 1) + s(G)$. From (1), we shall show that $R(G, K_p + nH) \leqslant L$ for large n. Assume that the edges of K_L are colored red and blue and there is no red G. Let $w \geqslant p + (h - 1)(R(G, K_h) - 1)$ be an integer. From the assumption, $K_w + (hn + p - w)K_1$ is *G-good* for large n, so we have a blue $K_w + (hn + p - w)K_1$. Let W be the vertex set of K_w and X be the set of other $hn + p - w$ vertices. We shall complete the proof by giving a blue $K_p + nH$. Since $R(G, K_h) < |X|$ for large n, we can find a blue K_h in X. Let us delete such blue K_h's one by one. Then the number of the remaining vertices in X is at most $R(G, K_h) - 1$. By deleting a vertex from the remaining vertices and $h - 1$ vertices from W, we have some other blue K_h, this step needs to be repeated, potentially $R(G, K_h) - 1$ times. Then the number of vertices left in W is at least $w - (h - 1)(R(G, K_h) - 1) \geqslant p$, and the difference between this number and p is a multiple of h. We then delete some K_h from W and stop when the number of remaining vertices in W is just p, thus we have a blue $K_p + nK_h$ hence a blue $K_p + nH$.

Hence the proof of Theorem 3 is completed. □

3 Proof of Theorem 5

To show the result, we need a lemma from [9].

Lemma 7. *Let r, s, t, p be positive integers with $s \leqslant t$ and $2p \leqslant t$, and let G be any graph. Let Γ be a G-free graph with $N \geqslant \binom{t}{s}^r \frac{t}{2ps} R(G, K_s)$ vertices which contains $K_r(t)$ as an induced subgraph, with parts $V_1, \dots, V_r$. If $\overline{\Gamma}$ does not contain a book $B_p(n)$ with at least $(1 - 4ps/t)N/r$ vertices, then Γ contains an induced copy of $K_{r+1}(s)$ with parts $W_0, \dots, W_r$, where $W_i \subset V_i$ for every $1 \leqslant i \leqslant r$.*

Proof of Theorem 5. Let $N = 2(nh + p - 1) + 1$ and $n \geqslant (100q)^{8q^3}$, where $q = \max\{m, p, h\}$.

Claim 1. $R(C_{2m+1}, B_p(n)) = 2(n + p) - 1$, namely, $B_p(n)$ is C_{2m+1}-*good*.

Proof of Claim 1. From (1), we shall show that $R(C_{2m+1}, B_p(n)) \leqslant 2(n + p) - 1$. Suppose for the sake of contradiction that there is a C_{2m+1}-free graph Γ on $2(n + p) - 1$ vertices such that $\overline{\Gamma}$ does not contain a $B_p(n)$. Let $\epsilon = 1/(8p)$ so that $1 - 4p\epsilon = 1/2$. Let $t_2 = mp$ and $t_1 = t_2/\epsilon = 8mp^2$. Since $n \geqslant (100q)^{8q^3}$, we have

$$R(C_{2m+1}, K_{t_1}) < R(K_{t_1}, K_{t_1}) < 4^{t_1} < 4^{8q^3} < 2(n + p) - 1,$$

and so Γ contains an independent set of order t_1. By Lemma 7, applied with $s = t_2,\ t = t_1$, and $G = C_{2m+1}$. Observe that

$$\binom{t_1}{t_2} \frac{t_1}{2pt_2} R(C_{2m+1}, K_{t_2}) \leqslant (e/\epsilon)^{t_2} \frac{t_1}{2pt_2} R(C_{2m+1}, K_{t_2}) \leqslant 2(n + p) - 1$$

for $n \geqslant (100q)^{8q^3}$. So either $\overline{\Gamma}$ contains a p-book with at least $(1-4p\epsilon)(2(n+p)-1) \geqslant n+p$ vertices, in which case we are done, or Γ contains $K_{mp,mp}$ as an induced subgraph. Denote $V(K_{mp,mp}) = V_1' \cup V_2'$, where $|V_1'| = |V_2'| = mp$. Denote $U = V(\Gamma) \setminus V(K_{mp,mp})$, and partition U into U_1, U_2, U_3, U_4 as follows, in which $E(u, V_i')$ denote the edge set between u and V_i' for $i = 1, 2$.

$$\begin{aligned}
U_1 &= \{u \in U |\ E(u, V_1') \neq \emptyset,\ E(u, V_2') = \emptyset\},\\
U_2 &= \{u \in U |\ E(u, V_1') = \emptyset,\ E(u, V_2') \neq \emptyset\},\\
U_3 &= \{u \in U |\ E(u, V_1') \neq \emptyset,\ E(u, V_2') \neq \emptyset\},\\
U_4 &= \{u \in U |\ E(u, V_1') = \emptyset,\ E(u, V_2') = \emptyset\}.
\end{aligned}$$

We claim that $U_3 = \emptyset$. Suppose to the contrary, that there exist two edges ux, uy with $u \in U_3,\ x \in V_1'$ and $y \in V_2'$. We shall construct a C_{2m+1} as follows. Choose $m - 1$ vertices $\{y_1, \dots, y_{m-1}\}$ from $N(x) \cap (V_2' \setminus y)$, and choose $m - 1$ vertices $\{x_1, \dots, x_{m-1}\}$ from $\bigcap_{i=1}^{m-1} N(y_i) \cap N(y) \cap (V_1' \setminus x)$. Then $uxy_1x_1y_2x_2 \cdots y_{m-1}x_{m-1}yu$ is an odd-cycle of length $2m + 1$, a contradiction.

Since $|V_1'|+|V_2'|+|U_1|+|U_2|+|U_4| = 2(n+p)-1$, we have that either $|V_1'|+|U_2|+|U_4| \geqslant p + n$ or $|V_2'| + |U_1| + |U_4| \geqslant p + n$, which implies $\overline{\Gamma}$ contains a $B_p(n)$ in any case, a contradiction. □

By Lemma 6, we know that $K_p + nH$ is C_{2m+1}-*good* for $n \geqslant (100q)^{8q^3}$. □

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