On the Anti-Ramsey Threshold for Non-Balanced Graphs

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Abstract

For graphs G, H, we write $G \xrightarrow{\text{rb}} H$ if for every proper edge-coloring of G there is a rainbow copy of H, i.e., a copy where no color appears more than once. Kohayakawa, Konstadinidis and the last author proved that the threshold for $G(n,p) \xrightarrow{\text{rb}} H$ is at most $n^{-1/m_2(H)}$. Previous results have matched the lower bound for this anti-Ramsey threshold for cycles and complete graphs with at least 5 vertices. Kohayakawa, Konstadinidis and the last author also presented an infinite family of graphs H for which the anti-Ramsey threshold is asymptotically smaller than $n^{-1/m_2(H)}$. In this paper, we devise a framework that provides a richer family of such graphs.

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1 Introduction

We say that a graph G has the *anti-Ramsey property* $G \xrightarrow{\text{rb}} H$ if for every proper edge-coloring of G there is a rainbow copy of H. The study of anti-Ramsey properties can be traced back to a question of Spencer, mentioned by Erdős in [6]: Does there exist a graph with arbitrarily large girth such that in every proper edge-coloring there is a rainbow cycle? Rödl and Tuza answered this question affirmatively [16] by showing that for some $\varepsilon > 0$, the random graph $G = G(n,p)$ with $p = n^{\varepsilon-1}$, with high probability, has few small cycles and $G \xrightarrow{\text{rb}} C_k$, for large k.

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As $G \xrightarrow{\text{rb}} H$ is an increasing property, it admits a threshold function[3], which we denote by $\mathrm{p}_H^{\text{rb}} = \mathrm{p}_H^{\text{rb}}(n)$. Kohayakawa, Konstadinidis and the last author [9] proved that, for any fixed graph H, we have $\mathrm{p}_H^{\text{rb}} \leqslant n^{-1/m_2(H)}$, where

$$m_2(H) := \max\left\{\frac{e(J)-1}{v(J)-2} : J \subseteq H, v(J) \geqslant 3\right\} \tag{1}$$

is the *maximum* 2-*density* of H. Furthermore, if the maximum in (1) is attained with H, then we say that H is 2-*balanced*.

Nenadov, Person, Škorić and Steger [14] showed that in fact $\mathrm{p}_H^{\text{rb}} = n^{-1/m_2(H)}$ if H is a cycle with at least 7 vertices or a complete graph with at least 19 vertices. This result was extended for cycles and complete graphs with at least 5 vertices, respectively, in [2] and [12].

Apart from complete graphs and cycles, not much is known about p_H^{rb} for other graphs H. One might feel compelled to conjecture that indeed this threshold is determined by the maximum 2-density for every graph that contains a cycle, specially because of 'standard' Ramsey threshold results such as the classical one from Rödl and Ruciński [15]. However, as proved in [10], this is not the case for a fairly large family of graphs H, whose threshold is asymptotically smaller than $n^{-1/m_2(H)}$.

Note that every proper-coloring of a triangle is rainbow and therefore the threshold for the event $G(n,p) \xrightarrow{\text{rb}} K_3$ is the same as the threshold for the appearance of triangles in $G(n,p)$, which is a local property. But for some other graphs H, such as complete graph and cycles with more than 3 vertices, it turns out that the property $G(n,p) \xrightarrow{\text{rb}} H$ seems to be related with more global aspects of the host graph. In this paper we explore the interplay of these two cases.

The family of graphs J for which is known that the threshold p_J^{rb} is asymptotically smaller than $n^{-1/m_2(J)}$ consists of graphs J obtained by 'attaching' a triangle to an edge of a graph H with $1 < m_2(H) < 2$, a result that we extend by allowing different graphs to be attached to H. Given graphs H and F with disjoint vertex sets and edges $u_1u_2 \in E(H)$ and $v_1v_2 \in E(F)$, an *amalgamation of* F *and* H *with* $u_1u_2 = v_1v_2$ is the graph obtained by identifying the vertices $u_1 = v_1$ and $u_2 = v_2$ (see Figure 1). We denote by $F \oplus H$ the family of all amalgamations of F and H.

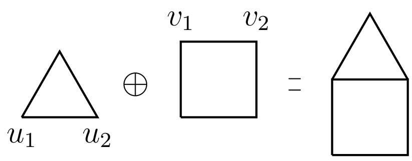

Figure 1: An example of the amalgamation obtained by identifying u_1u_2 and v_1v_2.

An interesting fact about amalgamations is that we do not increase the maximum 2-density of a graph by amalgamating it to a sparser graph. In fact, one can check that

$$\text{for any } J \in F \oplus H, \text{ we have } m_2(J) = \max\{m_2(F), m_2(H)\}. \tag{2}$$

In our main theorem, we apply a strategy to find rainbow copies of J in a proper edge-coloring of $G(n,p)$, which, under some conditions on F, typically works even when p is much smaller than $n^{-1/m_2(J)}$. For that, we need to define the following parameter of graphs, where S is a 2-balanced graph with $m_2(S) \geqslant m_2(H)$:

$$\beta(H,S) = \frac{1}{e(S)}\left(v(S) - 2 + \frac{1}{m_2(H)}\right).$$

We remark that the definition of $\beta(H,S)$ comes from a more general defined in [11], which generalizes the maximum 2-density to pairs of graphs. Also, note that

$$\frac{1}{m_2(S)} \leqslant \beta(H,S) \leqslant \frac{1}{m_2(H)}, \tag{3}$$

and both inequalities above are strict when $m_2(S) > m_2(H)$. We are now ready to state our main theorem.

Theorem 1. *Let H, F be graphs with $1 < m_2(H) < m_2(F)$ and let S be a 2-balanced graph S such that $S \xrightarrow{\text{rb}} F$. For any graph $J \in F \oplus H$, there exists $C > 0$ such that if $p \geqslant Cn^{-\beta(H,S)}$, then*

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \xrightarrow{\text{rb}} J\right) = 1.$$

The proof of Theorem 1 is an extension of the ideas developed in [10]. However, since the class of graphs considered here is more general, our proof requires more technicalities and a considerable improvement on their techniques. Note that Theorem 1 is useful when $\beta(H,S) > 1/m_2(J)$, since the result from [9] already implies that with high probability $G(n,p) \xrightarrow{\text{rb}} J$ for $p \geqslant Cn^{-1/m_2(J)}$. Furthermore, if $m_2(S) = m_2(F)$, then (2) and (3) give that

$$\beta(H,S) > 1/m_2(S) = 1/m_2(F) = 1/m_2(J).$$

Therefore, Theorem 1 implies that, given graphs H and F with $1 < m_2(H) < m_2(F)$, if there is a 2-balanced graph S with $m_2(S) = m_2(F)$ such that $S \xrightarrow{\text{rb}} F$, then for every $J \in F \oplus H$, we have $\mathrm{p}_J^{\text{rb}} \ll n^{-1/m_2(J)}$.

For any $t \in \mathbb{N}$, we define the *t-book graph* B_t as the graph composed by t triangles which all intersect on exactly one edge. In Section 6 we observe that $B_{3t-2} \xrightarrow{\text{rb}} B_t$ and, since $m_2(B_t) = 2$, the hypothesis of Theorem 1 are satisfied for every positive integer t and every graph H with $1 < m_2(H) < 2$ (e.g., cycles of length at least 4). Furthermore, since $m_2(B_{3t-2}) = m_2(B_t)$, we obtain as corollary the following theorem that give us an infinite family of graphs J such that $n^{-1/m_2(J)}$ is not the threshold for $G(n,p) \xrightarrow{\text{rb}} J$.

Corollary 2. *Let $t \in \mathbb{N}$ and let H be a graph with $1 < m_2(H) < 2$. Then for any $J \in B_t \oplus H$ we have $\mathrm{p}_J^{\text{rb}} \ll n^{-1/m_2(J)}$.*

The paper is organized as follows: in Section 2 we give an overview of the proof of Theorem 1 and in Section 3 we recall some definitions and results on the Regularity Method. In Section 4 we explore properties of proper colorings of $G(n,p)$ and the complete proof of Theorem 1 is given in Section 5. We finish by deducing Corollary 2 from Theorem 1 in Section 6.

2 Overview of the proof of the main result

Following the approach from [9], we randomly partition the colors of a proper coloring of $G = G(n,p)$ into finitely many classes. We show that, with high probability, the spanning subgraph induced by the colors in each of those classes has some *quasi-random* properties. In order to find a rainbow copy of a graph $J \in F \oplus H$, our strategy is to find a copy of H in which the color of each of its edges belong to different classes (which implies that it is a rainbow copy of H), and such that it can be extended on an edge $e \in E(H)$ to a rainbow copy of F where all of its edges have their colors in the same class as the one that contains the color of the edge e.

In order to find such rainbow copy of H that can be extended to a rainbow copy of J, we start our proof by fixing an *equipartition* of $V(G)$ into $v(S) + e(S)(v(H) - 2)$ sets as follows:

$$V(G) = \left(\bigcup_{i=1}^{v(S)} V_i \right) \cup \bigcup_{e \in E(S)} \left(\bigcup_{k=3}^{v(H)} U_k^e \right).$$

For each $e = v_i v_j \in E(S)$, we simply set $U_1^e = V_i$ and $U_2^e = V_j$. Then the first step of our strategy is to find many edge-disjoint *transversal* copies of S in $V_1 \cup \cdots \cup V_{v(S)}$ such that the colors of their edges belong all to the same class of our random partition of the colors. Since $S \xrightarrow{\text{rb}} F$, in each of those copies of S, there is a rainbow copy of F. For each edge $e \in E(F)$, we can show that the edges of those copies of F that belongs to the bipartite graph $G[U_1^e, U_2^e]$ are well distributed and induce a subgraph with density at least $Bn^{2-1/m_2(H)}$, for some $B > 0$. The next step is to extend those copies of F to a rainbow copy of J. This is done by showing that we can find a transversal copy of H in $U_1^e \cup \cdots \cup U_{v(H)}^e$ that extends those copies of F to a copy of J and such that the colors of each of its edges belong to different classes of our random partition of the colors.

3 Tool box

For any graph G and disjoint subsets $U, V \subseteq V(G)$, let us define the *density* of the pair (U, V) in G as

$$d_G(U,V) = \frac{e_G(U,V)}{|U||V|},$$

where $e_G(U,V)$ denotes the number of edges across U and V. We suppress G from the notation whenever it is clear from context. For any $\mu, p \in \mathbb{R}$, we say that G is (μ, p)-*upper* *uniform* if

$$d_G(U,V) \leqslant (1+\mu)p,$$

for every disjoint pair of sets $U, V \subseteq V(G)$ with $|U| \geqslant |V| \geqslant \mu v(G)$. If G is a bipartite graph with parts (U,V), then we say that G is (ε,p)-*regular* if

$$|d_G(U,V) - d_G(U',V')| \leqslant \varepsilon p,$$

for all $U' \subseteq U$ and $V' \subseteq V$ with $|U'| \geqslant \varepsilon|U|$ and $|V'| \geqslant \varepsilon|V|$.

The next lemma states that large induced graphs of regular subgraphs are still regular. The proof is straightforward by checking the definition.

Lemma 3. *Let $p \in (0,1]$ and $0 < \varepsilon < \mu < 1/2$. If $G[V_1,V_2]$ is an (ε,p)-regular bipartite graph, then for every $V_1' \subseteq V_1$, with $|V_1'| \geqslant \mu|V_1|$, the graph $G[V_1',V_2]$ is $(\varepsilon/\mu,p)$-regular. Furthermore, we have $d(V_1',V_2) \geqslant d(V_1,V_2) - \varepsilon p$.*

The next lemma states that regular bipartite graphs contain regular subgraphs with any given (sufficiently large) number of edges. A proof can be found in [7, Lemma 4.3].

Lemma 4. *For $\varepsilon \in (0,1/6)$, there exists $C = C(\varepsilon) > 0$ such that the following holds. Let $G = G[V_1,V_2]$ be an (ε,d_G)-regular bipartite graph, where $d_G = d_G(V_1,V_2)$. For all $Cn \leqslant m \leqslant e(G)$, there exists a spanning subgraph $H = H[V_1,V_2]$ of G with m edges which is $(2\varepsilon,d_H)$-regular, where $d_H = e(H)/|V_1||V_2|$.*

The following lemma can be found in [10, Lemma 6]. It states that upper uniform bipartite graphs contain a bipartite subgraph which is regular and has the same density.

Lemma 5. *For $\varepsilon \in (0,1/2)$ and $\gamma \in (0,1)$, there exists $\mu > 0$ such that the following holds for all $p \in (0,1]$. Let $G = G[V_1,V_2]$ be a (μ,p)-upper uniform bipartite graph, with $|V_1| = |V_2|$ and $d(V_1,V_2) \geqslant \gamma p$. There exist $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$, with $|V_1'|,|V_2'| \geqslant \mu|V_1|$, such that $G[V_1',V_2']$ is (ε,p)-regular with density at least γp.*

Let $\eta > 0$. We say that a graph G has the *discrepancy property* $\mathrm{DISC}(\eta)$ if for any subsets $V_1, V_2 \subseteq V(G)$, we have

$$\left| e_G(V_1,V_2) - \frac{\mathrm{vol}(V_1)\,\mathrm{vol}(V_2)}{\mathrm{vol}(V(G))} \right| \leqslant \eta\,\mathrm{vol}(V(G)),$$

where $\mathrm{vol}(X) := \sum_{x\in X} d_G(x)$, for any $X \subseteq V(G)$. Roughly speaking, if a graph G has the $\mathrm{DISC}(\eta)$ property, then its edges are almost uniformly distributed. The next lemma builds a bridge between discrepancy and classical regularity (see [10], Lemma 4).

Lemma 6. *For every $\varepsilon, \mu > 0$ there exist $\eta, \delta > 0$ such that the following holds. Let $p \in (0,1]$ and G be an n-vertex graph which satisfies*

(1) the discrepancy property $\mathrm{DISC}(\eta)$*;*

(2) $|d_G(v) - pn| \leqslant \delta pn$, for every $v \in V(G)$.

Then, for any disjoint subsets $V_1, V_2 \subseteq V(G)$ such that $|V_1|, |V_2| \geqslant \mu n$, the graph $G[V_1, V_2]$ is (ε, p)-regular.

In what follows, we will state Janson's inequality [8]. Suppose that $A_1, \ldots, A_t$ are t events in a given probability space, all of them with probability p. For each $i \in [t]$, let X_i be the indicator random variable for the event A_i. Let $X = X_1 + \cdots + X_t$ and let

$$\Delta = \sum_{i \sim j} \mathbb{P}(A_i \cap A_j),$$

where $i \sim j$ indicates that $i \neq j$ and $A_i \cap A_j \neq \emptyset$. Then, we have the following.

Lemma 7 (Janson's inequality [8])**.** *For every $\varepsilon > 0$, we have*

$$\mathbb{P}\left(X \leqslant (1-\varepsilon)\mu\right) \leqslant \exp\left\{-\frac{\varepsilon^2 \mu}{2(1+\Delta/\mu)}\right\},$$

where $\mu = \mathbb{E}[X]$. In particular, if $\Delta = o(\mu)$, then $X \geqslant (1-\varepsilon)\mu$, with high probability.

We end this section with one last probabilistic tool.

Lemma 8 (McDiarmid's inequality [13])**.** *Let $X_1, \ldots, X_M$ be independent random variables, with X_i taking values on a finite set A_i for each $i \in [M]$. Suppose that $f : \prod_{i=1}^{M} A_i \to \mathbb{R}$ satisfies $|f(x) - f(x')| \leqslant c_i$ whenever the M-tuple x and x' differ only in the ith element. If Y is the random variable given by $Y = f(X_1, \ldots, X_M)$, then, for any $a > 0$,*

$$\mathbb{P}\left(|Y - \mathbb{E}(Y)| > a\right) \leqslant 2 \exp\left\{-\frac{2a^2}{\sum_{i=1}^{M} {c_i}^2}\right\}.$$

4 Pseudorandomness and isolated copies

Given a proper edge-coloring of $G = G(n,p)$, we will consider a random partition of the colors used in this coloring. In this section we focus on exploring properties of the spanning subgraph of G generated by each color class of this partition. In particular, the proof of Theorem 1 reduces to an application of an embedding lemma in a sparse setting, formerly known as KŁR conjecture, which was proved in [5, 1]. Therefore, our goal is to guarantee that the graphs that we construct fit in the requirements of this embedding lemma.

Let m and n be positive integers with $m \leqslant n^2$ and let $\varepsilon > 0$ and $p \in [0,1]$. Suppose that H is a graph with $V(H) = [h]$. Consider h disjoint sets $V_1, \ldots, V_h$, each of size n, and for each $ij \in E(H)$, add m edges between the pair (V_i, V_j) in a way that the resulting bipartite graph is (ε, p)-regular. We denote by $\mathcal{G}(H, n, m, p, \varepsilon)$ the collection of all graphs obtained in this way. In most applications, p will be of order $O(m/n^2)$ and sometimes we will have $p = m/n^2$. But we would like to point out that the main role of p here is controlling the sparse regularity. We say that a copy of H in $G \in \mathcal{G}(H, n, m, p, \varepsilon)$ is *transversal* if the vertex corresponding to i in the copy of H is in V_i. We denote by $\mathcal{G}^*(H, n, m, p, \varepsilon)$ the set of all graphs $G \in \mathcal{G}(H, n, m, p, \varepsilon)$ that do not contain a transversal copy of H. Now we are ready to state the embedding result we use in our proof.

Theorem 9 (KŁR conjecture [1, 5])**.** *For every graph H and every positive ϑ, there exist positive constants B, n_0 and ε such that the following holds. For every $n \in \mathbb{N}$ with $n \geqslant n_0$ and $m \in \mathbb{N}$ with $m \geqslant Bn^{2-1/m_2(H)}$, we have*

$$\left|\mathcal{G}^*(H, n, m, m/n^2, \varepsilon)\right| \leqslant \vartheta^m \binom{n^2}{m}^{e(H)}.$$

4.1 Random partition of the colors

In this subsection, we prove that, in a typical outcome of $G(n,p)$, the hypothesis of Lemma 6 is met by the graphs induced by the colors assigned to each edge of H. Let G be a graph and consider a proper edge-coloring $c : E(G) \to \mathbb{N}$ of G. Fix a positive integer T. To each color $i \in \mathbb{N}$, we assign to i an element $\sigma(i) \in [T]$ chosen uniformly at random from $[T]$. For each $t \in [T]$, let G_t be the spanning subgraph of G with edge set

$$E(G_t) = \{e \in E(G) : \sigma(c(e)) = t\},$$

that is, $E(G_t)$ corresponds to the edges of G for which their color was assigned to t.

In this strategy we have to consider two probability spaces when dealing with G_t: the one which defines $G\,(n, p)$ and the one which defines the random assignment σ for a fixed proper edge-coloring of $G\,(n, p)$. To avoid confusion, we use $\mathbb{P}$ and $\mathbb{E}$ to refer to the distribution of $G\,(n, p)$ and for a fixed proper-coloring of $c : E(G) \to \mathbb{N}$, we use $\mathbb{P}_\sigma$ and $\mathbb{E}_\sigma$ to refer to the distribution of the random assignment of colors σ. Our aim in this subsection is to show that G_t satisfies the two requirements of Lemma 6, which are the concentration of degrees and $\mathrm{DISC}(\eta)$. Let us start with the degree distribution of G_t.

Lemma 10. *Let $\delta > 0$ and T be a positive integer. If $p \gg (\log n)/n$, then the following holds for $G = G(n,p)$ with high probability. For any proper edge-coloring $c : E(G) \to \mathbb{N}$ of G, we have for a random assignment $\sigma : \mathbb{N} \to [T]$ that*

$$\mathbb{P}_\sigma\left(\forall v \in V(G),\ d_{G_t}(v) = (1 \pm \delta)\frac{pn}{T}\right) = 1 - o(1),$$

for every $t \in [T]$.

We omit the proof of Lemma 10, since it follows from a straightforward Chernoff's bound argument, together with the fact that edges touching each vertex have distinct colors.

Let us now focus on proving that G_t has the $\mathrm{DISC}(\eta)$ property. A straightforward proof that a random graph satisfies $\mathrm{DISC}(\eta)$ can be tricky, since any concentration inequality we obtain has to be stronger than the number of choices of subsets of the vertex set. Luckily for us, the works of Chung and Graham [4] relate this property with the distribution of circuits of even length. Given a graph G, we say that a sequence $C = (v_1, \ldots, v_\ell)$ of vertices of G is an *ℓ-circuit* if $v_i v_{i+1} \in E(G)$, for every $i \in [\ell - 1]$, and $v_1 v_\ell \in E(G)$. The *weight* of an ℓ-circuit $C = (v_1, \ldots, v_\ell)$ is given by

$$w(C) = \prod_{i=1}^{\ell} \frac{1}{d_G(v_i)}.$$

We denote by $\mathcal{C}_\ell(G)$ the collection of all ℓ-circuits of G. We say that G has the $\mathrm{CIRCUIT}_\ell(\eta)$ property if

$$\sum_{C \in \mathcal{C}_\ell(G)} w(C) = 1 \pm \eta.$$

The following lemma from [4] shows that CIRCUIT essentially implies DISC.

Lemma 11. *For every $\eta > 0$ and positive integer ℓ, if G has the $\mathrm{CIRCUIT}_{2\ell}(\eta)$ property, then G has the $\mathrm{DISC}(\eta^{1/2\ell})$ property.*

Since the degrees in G_t are concentrated around pn/T for every $t \in [T]$, we basically have to show that the number of circuits of some even length ℓ is close to $(pn/T)^\ell$ in G_t. In principle, it is not clear even how to compute the expectation of this value, since the edges are not selected independently. We simplify this problem in two steps. First we call upon the result stated below and proved in [9, Corollary 4.9], which shows that for certain values of p, almost all ℓ-circuits are actually cycles. For that, we denote by $\mathcal{C}'_\ell(G)$ the number of ℓ-cycles in a graph G.

Lemma 12. *Let $\ell \geqslant 2$ be an integer and $\delta > 0$. If $p \gg n^{-1+1/\ell}$, then with high probability $G = G(n,p)$ satisfies*

$$|\mathcal{C}_{2\ell}(G)| \leqslant (1+\delta)|\mathcal{C}'_{2\ell}(G)|.$$

Our next aim is to show that almost all cycles in a proper edge-coloring of $G(n,p)$ are in fact rainbow. If we assume that to be true, it is easy to see that the expected number of ℓ-cycles in each G_t is roughly $(pn/T)^\ell$, since each color is independently assigned to a class. In order to prove such statement we count the number of non-rainbow cycles basically by counting the shortest path whose first and last edges have the same color and then by completing them into cycles. Therefore, the following special case of a classical result of Spencer [17, Theorem 2] is fairly convenient.

Lemma 13. *Let $\ell \geqslant 2$ and $G = G(n,p)$, with $p^\ell n^{\ell-1} \gg \log n$. Then with high probability, for every pair of vertices $u, v \in V(G)$, there are $\Theta(p^\ell n^{\ell-1})$ paths of length ℓ connecting u to v in G.*

We remark that the values of p needed to apply Lemma 13 are lower for longer paths. This fact plays an important role in the proof of Lemma 14, which we are now ready to state.

Lemma 14. *Let ℓ be an integer and $p^{\lceil \ell/2 \rceil} n^{\lceil \ell/2 \rceil - 1} \gg \log n$. With high probability, in every proper edge-coloring of $G = G(n,p)$ there are $O(p^{\ell-1} n^{\ell-1})$ non-rainbow ℓ-cycles.*

Proof. Let $G = G(n,p)$ be as in the statement. Notice that as an straightforward application of Chernoff's inequality (which we will omit the details here), it follows that with high probability we have $d(v) \leqslant 2pn$, for every $v \in V(G)$. Fix now a proper edge-coloring of G.

We say that a path in G is *color-tied* if the first and last edges have the same color. Note that every non-rainbow ℓ-cycle must contain a color-tied path of length at most $\lfloor \ell/2 \rfloor + 1$, by considering the shortest path between edges with the same color. We will give an upper bound for the number of non-rainbow ℓ-cycles by giving an upper bound for the number of color-tied paths and then by counting in how many ways these paths can be extended into an ℓ-cycle in G.

To count the number of color-tied paths of length k, for a fixed $k \in [\lfloor \ell/2 \rfloor + 1]$, we first choose an ordered pair (v_0, v_1) such that $v_0v_1 \in E(G)$. For $i = 1, \ldots, k-2$, we inductively extend the path $v_0v_1 \cdots v_i$ by choosing a vertex v_{i+1} from the neighborhood of v_i. We then let the last vertex v_k be the only neighbour of v_{k-1} such that v_0v_1 has the same color as $v_{k-1}v_k$. Therefore, the number of color-tied of length k is at most $4pn^2 \cdot (2pn)^{k-2} \cdot 1 = O(p^{k-1}n^k)$, which is smaller than the number of paths in G of length k by a factor of $\Omega(pn)$.

Now let $v_0v_1 \ldots v_k$ be a color-tied path of length $k \in [\lfloor \ell/2 \rfloor + 1]$. Since $p^{\ell-k}n^{\ell-k-1} \gg p^{\lfloor \ell/2 \rfloor - 1}n^{\lfloor \ell/2 \rfloor - 2} \gg \log n$, then Lemma 13 give us that, with high probability, there are $\Theta(p^{\ell-k}n^{\ell-k-1})$ paths of length $\ell - k$ connecting v_0 to v_k. Therefore, the number of non-rainbow cycles is at most

$$\sum_{k=1}^{\lfloor \ell/2 \rfloor + 2} O\left(p^{k-1}n^k \cdot p^{\ell-k}n^{\ell-k-1}\right) = O(p^{\ell-1}n^{\ell-1}),$$

which conclude the proof of the lemma. $\square$

Now we put all pieces together to prove that G_t has the DISC(η) property.

Lemma 15. *Let $\eta, \beta \in (0,1)$ and T be a positive integer. If $p \gg n^{-\beta}$, then the following holds for $G = G(n,p)$ with high probability. For any proper edge-coloring $c : E(G) \to \mathbb{N}$ of G we have*

$$\mathbb{P}_\sigma\big(G_t \textit{ satisfies } \mathrm{DISC}(\eta)\big) = 1 - o(1),$$

for every $t \in [T]$.

Proof. Let $\ell > 1/(1-\beta)$ and $\eta' = \eta^{2\ell}$. Note that $p^\ell n^{\ell-1} \gg \log n$ and $pn \gg \log n$. By Lemma 10, for any $\delta > 0$, with high probability we have

$$\mathbb{P}_\sigma\left(d_{G_t}(v) = (1 \pm 3\delta)\frac{pn}{T}\right) = 1 - o(1),$$

for all $v \in V(G)$ and all $t \in [T]$. Therefore, by choosing δ small enough as a function of η' and ℓ, we have

$$\sum_{C \in \mathcal{C}_{2\ell}(G_t)} w(C) = \sum_{C \in \mathcal{C}_{2\ell}(G_t)} \prod_{v \in V(C)} \frac{1}{d_{G_t}(v)} \;=\; |\mathcal{C}_{2\ell}(G_t)| \left(1 \pm \frac{\eta'}{3}\right)\left(\frac{T}{pn}\right)^{2\ell}, \tag{4}$$

where $\mathcal{C}_{2\ell}(G_t)$ denotes the set of 2ℓ-circuits in G_t.

Lemma 11 implies that if G_t has the $\text{CIRCUIT}_{2\ell}(\eta')$ property, then it has the property $\text{DISC}(\eta)$. Therefore, it is enough to prove that

$$\mathbb{P}_\sigma\big(G_t \text{ satisfies } \text{CIRCUIT}_{2\ell}(\eta')\big) = 1 - o(1),$$

By the definition of the $\text{CIRCUIT}_{2\ell}(\eta')$ property, we have to show that $\sum w(C) = 1 \pm \eta'$. By (4), it is sufficient to prove that for every $t \in [T]$, we have

$$\mathbb{P}_\sigma\left(|\mathcal{C}_{2\ell}(G_t)| = \left(1 \pm \frac{\eta'}{3}\right)\left(\frac{pn}{T}\right)^{2\ell}\right) = 1 - o(1). \tag{5}$$

Recall that $\mathcal{C}'_{2\ell}(G_t) \subset \mathcal{C}_{2\ell}(G_t)$ is the collection of 2ℓ-cycles in G_t. Let us turn our attention to $G(n,p)$. Since $p \gg n^{-1+1/2\ell}$, the numbers of 2ℓ-circuits and of 2ℓ-cycles in $G(n,p)$ are with high probability asymptotically equal. More precisely, by choosing $\delta \leqslant \eta'/(12T^{2\ell})$ and observing that $|\mathcal{C}'_{2\ell}(G)| \leqslant 2(pn)^{2\ell}$, Lemma 12 implies that with high probability, we have

$$|\mathcal{C}_{2\ell}(G) \setminus \mathcal{C}'_{2\ell}(G)| \leqslant \frac{\eta'}{12T^{2\ell}} \cdot |\mathcal{C}'_{2\ell}(G)| \leqslant \frac{\eta'}{6}\left(\frac{pn}{T}\right)^{2\ell}. \tag{6}$$

Therefore, we can disregard the 2ℓ-circuits that are not 2ℓ-cycles from the computation. That is, in order to prove (5), it is enough to show that

$$\mathbb{P}_\sigma\left(|\mathcal{C}'_{2\ell}(G_t)| = \left(1 \pm \frac{\eta'}{6}\right)\left(\frac{pn}{T}\right)^{2\ell}\right) = 1 - o(1), \tag{7}$$

for every $t \in [T]$.

In order to prove (7), fix $t \in [T]$ and for each $i \in c(E(G))$, let $A_i = \{0,1\}$ and let X_i be the indicator function for the event $\{\sigma(i) = t\}$ and set $Y = |\mathcal{C}'_{2\ell}(G_t)|$. Note that $Y = f(X_1, \dots, X_r)$, for some $f : \prod_{i=1}^r A_i \to \mathbb{R}$. Now, let c_i be the smallest real number for which $|f(x) - f(x')| \leqslant c_i$, whenever $x, x' \in \prod_{i=1}^r A_i$ differ only on the ith coordinate. By double counting the pairs (i,e) such that $i \in c(E(G))$ and $e \in E(G)$ has color i and is contained in a 2ℓ-cycle, we obtain

$$\sum_{i=1}^r c_i \leqslant 2\ell|\mathcal{C}'_{2\ell}(G)|. \tag{8}$$

Moreover, since $pn^{1-2/(2\ell-1)} \to \infty$, Lemma 13 implies that the number of 2ℓ-cycles in G containing a given edge $e \in G$ is at most $Dp^{2\ell-1}n^{2\ell-2}$, for some large constant $D > 0$. Since each color $i \in [r]$ induces a matching in G, it follows that $c_i \leqslant Dp^{2\ell-1}n^{2\ell-1}$. Together with (8), we obtain that

$$\sum_{i=1}^r c_i^2 \leqslant Dp^{2\ell-1}n^{2\ell-1}\sum_{i=1}^r c_i \leqslant 2\ell Dp^{2\ell-1}n^{2\ell-1}|\mathcal{C}'_{2\ell}(G)|.$$

To finish the proof of (7), we have to calculate the expectation of $Y = |\mathcal{C}'_{2\ell}(G_t)|$. Since each edge of G is in G_t with probability $1/T$, for each $C \in \mathcal{C}'_{2\ell}(G)$, we have $\mathbb{P}_\sigma(C \in$

$G_t) \geqslant 1/T^{2\ell}$ (with equality, if C is rainbow). Since $(pn)^{2\ell} \gg 1$, we know that with high probability, $|\mathcal{C}'_{2\ell}(G)| = (1 - o(1))(pn)^{2\ell}$. Since $p^\ell n^{\ell-1} \gg \log n$, we can use Lemma 14 to get that, with high probability, almost all of the 2ℓ-cycles in G are rainbow and, therefore, they contribute with $1/T^{2\ell}$ to the expectation of Y. In conclusion, with high probability, we have that

$$\mathbb{E}_\sigma(Y) = |\mathcal{C}'_{2\ell}(G)| \left(\frac{1}{T^{2\ell}} + o(1)\right) = \left(1 \pm \frac{\eta'}{12}\right)\left(\frac{pn}{T}\right)^{2\ell}.$$

Finally, by McDiarmid's inequality (Lemma 8),

$$\mathbb{P}_\sigma\left(|Y - \mathbb{E}[Y]| > \frac{\eta'}{12}\left(\frac{pn}{T}\right)^{2\ell}\right) \leqslant 2\exp\left\{-\frac{\Omega((pn)^{4\ell})}{\sum_i^r {c_i}^2}\right\} = 2\exp\{-\Omega(pn)\} = o(1).$$

Therefore, with probability tending to 1 (under $\mathbb{P}_\sigma$) equation (7) holds, which, together with with (6), implies (5) and finishes the proof. $\square$

4.2 Isolated copies of S

Given a graph G on n vertices, we call a copy of a graph S in G *isolated* if it does not share an edge with any other copy of S in G. Let G^S be the spanning subgraph of G induced by all the edges that belong to some isolated copy of S in G. Since G^S will play a role in the application of Theorem 9, the following issue arises. Theorem 9 is a counting result that states that most pseudo-random blow-ups of a graph H will have a transversal copy of H. We would like to apply this fact to a blow-up of H with edges in G^S. However, it is not clear, at first glance, that the edges of G^S are well distributed. The following lemma, which first appeared at [11] (see their Lemma 13), give us that. For $E \subseteq E(K_n)$, we write $E \sqsubseteq G^S$ if $E \subseteq G^S$ and if no two edges in E belong to the same isolated copy of S.

Lemma 16. *Let F be a graph and $G = G(n,p)$, with $p = p(n) \in (0,1]$. If $q = n^{v(F)-2}p^{e(F)}$, then for any $E \subseteq E(K_n)$ we have that*

$$\mathbb{P}\left(E \sqsubseteq E(G^S)\right) \leqslant q^{|E|}.$$

Later in the proof of Theorem 1, we will need to prove that G^S, and hence G^S_t, is (μ, q)-upper uniform for $q = O(n^{v(S)}p^{e(S)})$. This is stated in the next lemma. The proof is a straightforward application of Lemma 16 and it can be found in [11, Lemma 14].

Lemma 17. *Let S be a graph, $\beta > 0$ be such that $\beta < (v(S) - 1)/e(S)$, and let $\mu > 0$. Then with high probability, for $G = G(n,p)$ with $p = n^{-\beta}$, the graph G^S is (μ, q)-upper-uniform, where $q = 6e(S)n^{v(S)-2}p^{e(S)}$.*

For disjoint sets $V_1, \ldots, V_{v(S)} \subseteq V(G)$, we denote by $Z_G(V_1, \ldots, V_{v(S)})$ the number of *transversal* copies of S in $G[V_1, \ldots, V_{v(S)}]$, i.e., copies of S in G with one vertex in each V_i with $i \in [v(S)]$. Let $Y_G(V_1, \ldots, V_{v(S)})$ be the number of transversal copies of S that are also isolated. We may omit the sets $V_1, \dots, V_{v(S)}$ from the notation if they are clear from the context. Note that for $G = G(n,p)$ and for disjoint linear-sized sets $V_1, \dots, V_{v(S)} \subseteq V(G)$, we have that

$$\mathbb{E}[Z(V_1, \dots, V_{v(S)})] = \Theta\left(n^{v(S)}p^{e(S)}\right).$$

In the next lemma, which can also be found in [11], we prove that in a typical outcome of $G = G(n,p)$, a positive proportion of transversal copies of S in $G[V_1, \dots, V_{v(S)}]$ are actually isolated.

Proposition 18. *Let S be a 2-balanced graph and $\beta > 0$ such that $1/m_2(S) < \beta \leqslant (v(S)-1)/e(S)$. For every $\mu > 0$, the following holds with high probability for $G = G(n,p)$, with $p = n^{-\beta}$. For every family of pairwise disjoint sets $V_1, \dots, V_{v(S)} \subseteq V(G)$, with $|V_i| \geqslant \mu n$, we have*

$$Y_G(V_1, \dots, V_{e(S)}) \geqslant \frac{1}{2}\mathbb{E}Z_G(V_1, \dots, V_{e(S)}).$$

In particular, $Y_G(V_1, \dots, V_{e(S)}) = \Omega\left(n^{v(S)}p^{e(S)}\right)$.

Proof. Let $V_1, \dots V_{v(S)} \subset V(G)$ be a fixed family of pairwise disjoint sets. In order to apply Janson's inequality (Lemma 7) to the random variable $Z = Z_G(V_1, \dots, V_{e(S)})$, note that

$$\Delta = \sum_{K_2 \subseteq J \subset S} O\left(n^{2v(S)-v(J)}p^{2e(S)-e(J)}\right) = o(\mathbb{E}Z). \tag{9}$$

Indeed, since S is 2-balanced, for every $K_2 \subseteq J \subset S$, we have that

$$\frac{v(S)-v(J)}{e(S)-e(J)} \leqslant \frac{1}{m_2(S)}.$$

Consequently, by our choice of p, we get

$$n^{v(S)-v(J)}p^{e(S)-e(J)} = o(1).$$

And since $\mathbb{E}Z = \Omega(n^{v(S)}p^{e(S)})$ (here, we use the fact that $|V_i| \geqslant \mu n$, for each $i \in [v(S)]$), we have that (9) follows. Then, by Janson's inequality, we have that

$$\mathbb{P}\left(Z \leqslant \frac{3}{4}\mathbb{E}Z\right) = \exp(-\Omega(\mathbb{E}Z)).$$

Since $\mathbb{E}Z \gg n$, we can take an union bound over all the choices of $V_1, \dots, V_{v(S)}$ (which is at most $2^{v(S)n}$) and guarantee that with high probability, we have $Z \geqslant 3\mathbb{E}Z/4 = \Omega(n^{v(S)}p^{e(S)})$, for any choice of $V_1, \dots, V_{v(S)}$.

Let X be the number of non-isolated copies of S in G. Notice that $\mathbb{E}X = O(\Delta)$. Therefore, by Markov's inequality and by (9), we have

$$\mathbb{P}\left(X \geqslant \frac{1}{4}\mathbb{E}Z\right) \leqslant \frac{4\mathbb{E}X}{\mathbb{E}Z} = \frac{o(\mathbb{E}Z)}{\mathbb{E}Z} = o(1).$$

Then, with probability $1 - o(1)$, we have

$$Y_G(V_1, \ldots, V_{e(S)}) \geqslant Z - X \geqslant \frac{3}{4}\mathbb{E}Z - \frac{1}{4}\mathbb{E}Z = \frac{1}{2}\mathbb{E}Z,$$

for any choice of $V_1, \ldots, V_{v(S)}$ as in the statement. $\square$

In the next lemma, we show that many transversal copies are isolated, even in a typical outcome of G_t, for each $t \in [T]$. In particular, the bipartite graph $G_t^S[V_i, V_j]$, for $ij \in E(S)$, has $\Omega(n^{v(S)}p^{e(S)}) = \Omega(n^{2-1/m_2(H)})$ many edges, since each edge in $G_t^S[V_i, V_j]$ comes from an isolated copy of S and by our choice of p.

Lemma 19. *Let S be a 2-balanced graph and $\beta > 0$ such that $1/m_2(S) < \beta \leqslant (v(S) - 1)/e(S)$. For every $\mu > 0$ and integer $T > 0$, there exists $\alpha > 0$ such that the following holds. With high probability for every proper edge-coloring of $G = G(n, p)$, with $p = n^{-\beta}$, and for a fixed family of disjoint sets $V_1, \ldots, V_{v(S)} \subseteq V(G)$ with $|V_i| \geqslant \mu n$, for $i \in [v(S)]$, we have*

$$\mathbb{P}_\sigma\left(Y_{G_t}(V_1, \ldots, V_{e(S)}) \geqslant \alpha n^{v(S)} p^{e(S)}\right) = 1 - o(1),$$

for every $t \in [T]$.

Proof. Let $G = G(n, p)$, with $p = n^{-\beta}$ and $1/m_2(S) < \beta \leqslant (v(S) - 1)/e(S))$, and let $c : E(G) \to [r]$ be a proper edge-coloring of G, for some $r \in \mathbb{N}$. For an integer $T > 0$ consider a random partition of the colors into T classes. For each $i \in [r]$, let X_i be the indicator function for the event $\sigma(i) = t$ and observe that $Y_{G_t} = Y_{G_t}(V_1, \ldots, V_{e(S)})$ is a function of $X_1, \ldots, X_r$. For each $i \in [r]$, let $c_i = c_i(G)$ be the smallest real number such that if we change the value of X_i only, then the value of Y_{G_t} will be altered by at most c_i. Since the coloring of G is proper, by altering the value of X_i we add or remove at most a perfect matching from G_t, which implies that it will affect at most n isolated copies of S. Therefore, we have $c_i \leqslant n$. Furthermore, since a transversal isolated copy of S in G can be affected by at most $e(S)$ changes in the value of $X_1, \ldots, X_r$, we also have $\sum_{i=1}^r c_i \leqslant e(S) Y_G$. Hence,

$$\sum_{i=1}^{r} c_i^2 \leqslant n \sum_{i=1}^{r} c_i \leqslant e(S) \cdot n \cdot Y_G.$$

Furthermore, note that each copy of S in G belongs to G_t with probability $(1/T)^k$, where k is the number of colors that appears in such copy of S. In particular, such copy of S is in G_t with probability at least $(1/T)^{e(S)}$. Therefore,

$$\mathbb{E}_\sigma[Y_{G_t}] \geqslant \frac{Y_G}{T^{e(S)}}.$$

Note that, by Proposition 18, we have that $Y_G = \Omega(n^{v(S)} p^{e(S)})$. Therefore, Lemma 8 yields that

$$\mathbb{P}_\sigma\left[Y_{G_t} < \frac{1}{2}\mathbb{E}[Y_{G_t}]\right] \leqslant 2\exp\left\{-\frac{\mathbb{E}[Y_{G_t}]^2}{2\sum_{i\in[r]} {c_i}^2}\right\} = \exp\left\{-\Omega(n^{v(S)-1}p^{e(S)})\right\} = o(1).$$

The last equality follows from the fact that $\beta < (v(S)-1)/e(S)$. Consequently, with probability $1-o(1)$ under $\mathbb{P}_\sigma$, we have that

$$Y_{G_t} \geqslant \frac{1}{2}\mathbb{E}[Y_{G_t}] \geqslant \frac{Y_G}{2T^{e(S)}} \geqslant \alpha n^{v(S)}p^{e(S)},$$

for some $\alpha > 0$ that only depends on the graph S and the values of T and μ. $\square$

5 Proof of Theorem 1

Let H, F and S be as in the statement of Theorem 1. Let us say that $V(H) = \{u_1,\dots,u_h\}$, $V(S) = \{v_1,\dots,v_s\}$ and $V(F) = \{w_1,\dots,w_f\}$. Consider $J \in F \oplus H$ as an amalgamation of F and H with $u_1u_2 = w_1w_2$ and as in the statement of Theorem 1 and let $p \geqslant Cn^{-\beta(H,S)}$, where C is a sufficiently large constant. We consider an equipartition of $V = V(G))$ into $s + e(S)(h-2)$ sets as follows:

$$V = \left(\bigcup_{i=1}^{s} V_i\right) \cup \bigcup_{e\in E(S)} \left(\bigcup_{k=3}^{h} U_k^e\right). \tag{10}$$

Let $V_0 := \cup_{i=1}^s V_i$ and $n_0 := |V_0|$. For each $e = v_iv_j \in E(S)$, we define $U_1^e = V_i$ and $U_2^e = V_j$. The partition given by (10) can be pictured as a blow-up of S (namely, $(V_1,\dots,V_s)$) and $e(S)$ blow-ups of H (namely, $(U_1^e,\dots,U_h^e)$, for each edge $e = v_iv_j \in E(S)$), such that each $(U_1^e,\dots,U_h^e)$ together with $(V_1,\dots,V_s)$ make a blow-up of an amalgamation of H and S with $u_1u_2 = v_iv_j$. In the proof of Theorem 1, we will first look for several rainbow copies of F in each transversal isolated copy of S contained in the blow-up $(V_1,\dots,V_s)$ and then extend it to a rainbow copy of J using the transversal copies of H in the blow-up $(U_1^e,\dots,U_h^e)$, where e is the edge of S in which the edge w_1w_2 of F is associated to.

As sketched in Section 2, the proof has two steps: in the first step we want to find a transversal copy of S; in the other step, we want to find a transversal rainbow copy of H. These two steps are done in two subgraphs of $G(n,p)$ with different densities. For this reason, instead of working in $G(n,p)$, we work on a random graph G obtained in two rounds: in the first round, we sample a random graph G' as $G(n,p')$, where $p' := e(H)q$ and $q := 6e(S)n^{v(S)-2}p^{e(S)}$ (this choice of q is motivated by Lemmas 16 and 17). In the second round, we generate the random graph G as follows: each pair of vertices uv in V, independently of any other pair of vertices, becomes an edge in G with probability 1 if $uv \in E(G')$; and with probability $(p-p')/(1-p')$ if $uv \notin E(G')$. Therefore, a pair of vertices in V is an edge in G with probability p, which means that G has the same distribution as $G(n,p)$. Note that, since $p' < p$, we may consider a coupling between G'

and G such that $G' \subseteq G$. In order to prove Theorem 1, it suffices to prove that with high probability the random graph G satisfies $G \xrightarrow{\text{rb}} J$, for any $J \in F \oplus H$.

Now, let us define some constants. Our choice of constants is such that it allows us to apply all the lemmas in Sections 3 and 4. Let $\alpha = \alpha(H, S)$ be given by Lemma 19 with $\mu = |V_1|/n$ and $T = e(H)$. We may also assume that $\alpha \leqslant 1/(6e(S)s^f)$. One should keep in mind that α is the constant controlling the number of isolated transversal copies of S. Having α we define $\vartheta = (\alpha^2/4e)^{e(H)}$ and let B and ε be given by Theorem 9 applied with H and ϑ. The constant ε controls the *density error* in the regularity of G and B is the constant that controls p. Let $\gamma = \alpha^2$ and consider $\mu > 0$ as the constant given by Lemma 5 applied with ε and γ. Notice that μ controls the *uniformity*, and γ controls the density of small bipartite subgraphs. Let δ and η be given by Lemma 6 applied with ε and $\mu/2$. By considering a smaller value of η, we may also suppose that $\eta \leqslant \mu^2/(2^8 s^4 h^2)$. The constant δ controls the *degree error* in G' and η is the parameter for the *discrepancy property* of G'. Here it is a list of the associations between the constants just for quick reference:

$$H, S \xrightarrow{\text{Lemma 19}} \alpha, \qquad H, \alpha \xrightarrow{\text{explicit}} \vartheta, \qquad H, \vartheta \xrightarrow{\text{Theorem 9}} \varepsilon, B,$$
$$\alpha \xrightarrow{\text{explicit}} \gamma, \qquad \varepsilon, \gamma \xrightarrow{\text{Lemma 5}} \mu, \qquad \varepsilon, \mu \xrightarrow{\text{Lemma 6}} \delta, \eta.$$

Let $c : E(G) \to \mathbb{N}$ be a proper coloring of $E(G)$. We will consider a *random assignment* $\sigma : \mathbb{N} \to E(H)$ of the colors as follows: for each color $i \in \mathbb{N}$, we assign independently and uniformly at random an edge $\sigma(i) \in E(H)$. For each $t \in E(H)$, let G_t be the spanning subgraph of G with edge set

$$E(G_t) = \{e \in E(G) : \sigma(c(e)) = t\},$$

that is, $E(G_t)$ is the subset of edges of $E(G)$ for which their color was randomly assigned by σ to t. In the same way, we define G'_t (that is, G'_t is the subgraph of G' with edges whose color is assigned to t by σ). Note that G_t and G'_t are random graphs like the one analyzed in Section 4.1.

Recall from Section 4.2 that $Y_{G_t} := Y_{G_t}(V_1, \ldots, V_s)$ denotes the number of transversal isolated copies of S in $G_t[V_1, \ldots, V_s]$. For a fixed G and a fixed proper edge-coloring $c : E(G) \to \mathbb{N}$ of G, we denote by $\mathcal{E}_1 = \mathcal{E}_1(G, c)$ the event where, in a random assignment $\sigma : \mathbb{N} \to E(H)$, we have $Y_{G_t} \geqslant \alpha n^{v(S)} p^{e(S)}$, for every $t \in E(H)$. By Lemma 19, with high probability, the random graph G is such that for every proper edge-coloring c of G, we have

$$\mathbb{P}_\sigma(\mathcal{E}_1) = 1 - o(1). \tag{11}$$

Let G^S be the spanning subgraph of G induced by all the edges that belong to transversal isolated copies of S in $V_1 \cup \cdots \cup V_s$. Let $\mathcal{P}$ be the event where the random graph G is such that G^S is (μ, q)-upper uniform. By Lemma 17, we have that $\mathcal{P}$ happens with high probability. Our next claim states that, if $\mathcal{P}$ happens, then for any assignment σ in $\mathcal{E}_1$, we have that for some $v_i v_j \in E(S)$, there is a large *fairly regular* and dense subgraph of $G_t[V_i, V_j]$ whose edges are all contained in distinct isolated rainbow copies of F and correspond to the edge $w_1 w_2$ of F.

Claim 20. *Suppose that $G \in \mathcal{P}$ and let $c : E(G) \to \mathbb{N}$ be a proper edge-coloring. For a random assignment σ in the event $\mathcal{E}_1$, the following holds for $t_0 = u_1u_2$. For some $e = v_iv_j \in E(S)$ there exist $W_1 \subseteq V_i$, $W_2 \subseteq V_j$ with $|W_1| = |W_2| \geqslant \mu|V_i|$ and a bipartite spanning subgraph $B_{t_0} \subseteq G_{t_0}[W_1, W_2]$ such that*

(1) For every $ab \in E(B_{t_0})$, with $a \in W_1$ and $b \in W_2$, there is an isolated rainbow copy of F in V_0 containing ab and such that all the edges have colors assigned to t by σ. Moreover, in this copy, the vertices a and b correspond to the vertices w_1 and w_2 of F, respectively.

(2) The graph B_{t_0} is $(2\varepsilon, q)$-regular and has $m = \gamma q|W_1||W_2|$ edges.

Proof. Since $S \xrightarrow{\text{rb}} F$, in each transversal isolated copy of S in $G_{t_0}[V_1, \ldots, V_s]$, we can find an transversal isolated rainbow copy of F. Note that there are at most s^f different ways for a copy of F to be transversal in $G_{t_0}[V_1, \ldots, V_s]$. As $\mathcal{E}_1$ holds, by the pigeon-hole principle, we have for some $i_1, \ldots, i_f \in [s]$ at least

$$\left(\frac{\alpha}{s^f}\right) n^{v(S)} p^{e(S)}$$

transversal isolated rainbow copies of F in $G_{t_0}[V_{i_1}, \ldots, V_{i_f}]$ with the corresponding copy of w_k belonging to V_{i_k}, for each $k \in [f]$. Let $i = i_1$ and $j = i_2$. We turn our attention to the bipartite graph $B = (V_i \cup V_j; E')$ induced by the edges in $G_{t_0}[V_i, V_j]$ contained in those transversal isolated rainbow copies of F. Observe that B already satisfies property (1). Furthermore, since each edge of B is in exactly one of those copies of F, and since $\alpha \leqslant 1/(6e(S)s^f)$ and $\gamma = \alpha^2$, we have

$$E(B) \geqslant \left(\frac{\alpha}{s^f}\right) n^{v(S)} p^{e(S)} = \frac{\alpha}{6e(S)s^f} qn^2 \geqslant \alpha^2 qn^2 \geqslant \gamma q|V_i||V_j|.$$

As $G \in \mathcal{P}$, the graph G^S is (μ, q)-upper uniform and so it is B. Moreover, since $d_B(V_i, V_j) \geqslant \gamma q$, Lemma 5 give us $W_1 \subseteq V_i$ and $W_2 \subseteq V_j$ with $|W_1| = |W_2| \geqslant \mu|V_i|$, such that the bipartite graph $B[W_1, W_2]$ is (ε, q)-regular with density at least γq. To finish the proof, we apply Lemma 4 to B and obtain a $(2\varepsilon, q)$-regular spanning subgraph $B_{t_0} \subseteq B$ with exactly $m = \gamma q|W_1||W_2|$ edges. This shows property (2). □

We may assume, without lost of generality, that the edge e in Claim 20 is the edge $e = v_1v_2$. Therefore, we obtain a subgraph $B_{t_0} \subseteq G_{t_0}[W_1, W_2]$ with density exactly γq satisfying conditions (1) and (2) of Claim 20.

In what follows, we will focus on the random graph G' and we will define two events $\mathcal{E}_2$ and $\mathcal{E}_3$ for which a random assignment $\sigma : \mathbb{N} \to E(H)$ in those events will guarantee that, for every $t \in E(H)$, the random graph G'_t satisfies, respectively, condition (1) and (2) of Lemma 6 with the chosen δ and η.

For a fixed G' and a fixed proper edge-coloring $c : E(G') \to \mathbb{N}$ of G', let $\mathcal{E}_2 = \mathcal{E}_2(G', c)$ be the event in which the random assignment $\sigma : \mathbb{N} \to E(H)$ give us that, for every

$t \in E(H)$,

$$d_{G'_t}(v) = (1 \pm \delta)qn, \text{ for every } v \in V. \tag{12}$$

By Lemma 10, with high probability, the random graph G' is such that for every edge-coloring c of G', we have

$$\mathbb{P}_\sigma(\mathcal{E}_2) = 1 - o(1). \tag{13}$$

For a fixed G' and a fixed proper edge-coloring $c : E(G') \to \mathbb{N}$ of G', let $\mathcal{E}_3 = \mathcal{E}_3(G', c)$ be the event in which the random assignment $\sigma : \mathbb{N} \to E(H)$ give us that for every $t \in E(H)$, G'_t has the property DISC(η). By Lemma 15, with high probability, the random graph G' is such that for every edge-coloring c of G', we have

$$\mathbb{P}_\sigma\big(\mathcal{E}_3\big) = 1 - o(1). \tag{14}$$

We now show that, for each $i \in [h] \setminus \{1, 2\}$, there exists $W_i \subseteq U_i^e$ of the same size as W_1 and such that, for each $t = u_i u_j \in E(H) \setminus \{t_0\}$ (note that t may contain u_1 or u_2, but not both at the same time), the bipartite graph $G'_t[W_i, W_j]$ is fairly regular and dense.

Claim 21. *For any assignment σ in the event $\mathcal{E}_2 \cap \mathcal{E}_3$, the following holds. For every $i \in [h] \setminus \{1, 2\}$, there exists $W_i \subset U_i^e$, with $|W_i| = |W_1|$, such that for each $t = u_i u_j \in E(H) \setminus \{t_0\}$, there exists a bipartite spanning subgraph $B_t \subseteq G'_t[W_i, W_j]$ that is $(2\varepsilon, q)$-regular and has exactly $m = \gamma q|W_i||W_j|$ edges.*

Proof. For each $i \in [h] \setminus [2]$, let W_i be an arbitrary subset of U_i^e of size $|W_1|$. Under the event $\mathcal{E}_2 \cap \mathcal{E}_3$, Lemma 6 guarantees that the bipartite graph $G'_t[W_i, W_j]$ is (ε, q)-regular, for each $t = u_i u_j \in E(H) \setminus \{u_1 u_2\}$. Now we are left to show that $G'_t[W_i, W_j]$ has density at least γq.

Since G'_t has the DISC(η) property, we have

$$e(G'_t[W_i, W_j]) \geqslant \frac{\mathrm{vol}(W_i)\mathrm{vol}(W_j)}{\mathrm{vol}(G'_t)} - \eta \cdot \mathrm{vol}(G'_t), \tag{15}$$

where the volume is over G'_t. By (12), we have $\mathrm{vol}(G'_t) < 2qn^2$. Moreover, the volumes $\mathrm{vol}(W_i)$ and $\mathrm{vol}(W_j)$ are at least $qn|W_i|/2$. Therefore, it follows from (15) that

$$e(G'_t[W_i, W_j]) \geqslant \left(\frac{qn}{2}\right)^2 \cdot \frac{|W_i||W_j|}{2qn^2} - 2\eta qn^2 \geqslant \frac{q}{16}|W_i||W_j|.$$

In the last inequality, we used that $|W_1| \geqslant (\mu/hs^2)n$ and $\eta \leqslant \mu^2/(2^8 s^4 h^2)$. As $\gamma \leqslant 1/16$, it follows that $G'_t[W_i, W_j]$ has density at least γq. We apply Lemma 4 to obtain a spanning bipartite subgraph $B_t \subseteq G'_t[W_i, W_j]$ that is $(2\varepsilon, q)$-regular and has $m = \gamma q|W_i||W_j|$ edges. $\square$

By Lemma 17, $G \in \mathcal{P}$ with high probability. Combining this with Lemmas 10, 15 and 19, with high probability for every proper edge-coloring of G, a random assignment σ is in the event $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ with probability $1 - o(1)$. There must exist an assignment $\sigma : \mathbb{N} \to E(H)$ such that the conclusion of Claims 20 and 21 hold. Hence we have that the $G_0 := \cup_{t \in E(H)} B_t$ is a spanning subgraph of $G[W_1, \ldots, W_h]$ and with the following properties:

(a) For all $i \in [h]$, we have $|W_i| = \tilde{n}$ for some $\tilde{n} \geqslant \mu |V_1|$.

(b) For each $t = u_i u_j \in E(H)$, the bipartite graph $G_0[W_i, W_j]$ is $(2\varepsilon, q)$-regular, has $m = \gamma q \tilde{n}^2$ edges and all the colors on those edges are assigned to t.

(c) For every $ab \in G_0[W_1, W_2]$, with $a \in W_1$ and $b \in W_2$, there is a copy F_{ab} of F in $G[V_0]$ containing ab. Moreover, in these copies the vertices a and b correspond to the vertices w_1 and w_2, respectively.

(d) The graph F_{ab} is a rainbow graph whose colors are assigned to t_0.

Observe that properties (a) and (b) guarantee that $G_0 \in \mathcal{G}(H, \tilde{n}, m, \gamma q, 2\varepsilon)$.

Recall that $J \in H \oplus F$. Now we claim that if G_0 contains a transversal copy of H, then G contains a rainbow copy of J in the coloring c. In fact, as the edges in $G_0[W_i, W_j]$ only use colors assigned to $t = u_i u_j$, any transversal copy of H in G_0 is rainbow. Moreover, by properties (c) and (d), each transversal copy of H in G_0 can be extended to a copy of J in G by amalgamating a copy of F with H on the edge $t_0 = u_1 u_2$ that is rainbow and only uses colors assigned to t_0. Therefore, we conclude that the copy of J we found is also rainbow.

Let $\mathcal{G}$ be the family of h-partite graphs G_0, with $V(G_0) \subseteq V$, that may be construct as the following. Chose $W_1 \subseteq U_1^e, \ldots, W_h \subseteq U_h^e$ (for some $e \in E(H)$) as subsets of size $\tilde{n}$ to be the h parts of G_0; and for each $u_i u_j \in E(H)$, chose m edges to belong to $G_0[W_i, W_j]$ such that $G_0[W_i, W_j]$ is a $(2\varepsilon, q)$-regular graph. Let $\mathcal{G}^*$ be the family of graphs $G_0 \in \mathcal{G}$ that has no transversal copy of H. Therefore, $\mathcal{G}^*$ is the family of possible graphs G_0 that we can construct in G and that satisfies $(a)-(d)$ above, but that has not transversal copy of H.

Let $\mathcal{A}$ be the event in which G has a subgraph G_0 satisfying $(a)-(d)$. We have proved so far that $\mathbb{P}(\mathcal{A}) = 1 - o(1)$. The discussion above implies that if $G \in \mathcal{A}$ and if none of the graphs in $\mathcal{G}^*$ are contained in G, then $G \xrightarrow{\text{rb}} J$. Therefore, we can bound the probability that $G \overset{\text{rb}}{\nrightarrow} J$ by

$$\mathbb{P}\left(G \overset{\text{rb}}{\nrightarrow} J\right) \leqslant \mathbb{P}\left(\bigcup_{G_0 \in \mathcal{G}^*} \{G_0 \subseteq G\}\right) + \mathbb{P}(\neg \mathcal{A})$$

We already have that $\mathbb{P}(\neg \mathcal{A}) = o(1)$. It remains to show that other term in the inequality above is $o(1)$.

By Lemma 16 and properties (b)-(c), we have

$$\mathbb{P}\big(G_0[W_1, W_2] \sqsubseteq G^F\big) \leqslant q^m \qquad \text{and} \qquad \mathbb{P}\big(G_0[W_i, W_j] \subset G\big) \leqslant q^m$$

for every $\{i, j\} \subseteq [h]$ for which $u_i u_j \in E(H) \setminus \{t_0\}$. Therefore,

$$\mathbb{P}\left(\bigcup_{G_0 \in \mathcal{G}^*} \{G_0 \subseteq G\}\right) \leqslant |\mathcal{G}^*| \cdot q^{e(H)m}. \tag{16}$$

Since C is large enough and $p \geqslant Cn^{-\beta(H,S)}$, we have $m = \gamma q \tilde{n} \geqslant Bn^{2-1/m_2(H)}$. This together with the choice of ε allow us to apply Theorem 9 and obtain the that

$$|\mathcal{G}^*| \leqslant e(S) 2^{h\tilde{n}} \cdot \vartheta^m \binom{\tilde{n}^2}{m}^{e(H)} \leqslant 2^{e(H)m} \cdot \vartheta^m \binom{\tilde{n}^2}{m}^{e(H)}.$$

The term $e(S)2^{h\tilde{n}}$ (which is at most $2^{e(H)m}$, since $m \gg n$) comes from the possible choices of $W_1, \ldots, W_h$ to be the h parts of G_0. Combining with (16), we obtain that

$$\begin{aligned}
\mathbb{P}\big(G \overset{\text{rb}}{\nrightarrow} J\big) &\leqslant |\mathcal{G}^*| \cdot q^{e(H)m} + o(1) \\
&\leqslant 2^{e(H)m} \cdot \vartheta^m \binom{\tilde{n}^2}{m}^{e(H)} q^{e(H)m} + o(1) \\
&\leqslant \vartheta^m \left(\frac{2eq\tilde{n}^2}{m}\right)^{e(H)m} + o(1) \\
&\leqslant \vartheta^m \left(\frac{2e}{\alpha^2}\right)^{e(H)m} + o(1).
\end{aligned}$$

Since $\vartheta = (\alpha^2/4e)^{e(H)}$, we get $\mathbb{P}\big(G \overset{\text{rb}}{\nrightarrow} J\big) \leqslant 2^{-m} + o(1)$, which finishes our proof.

6 Book graphs

Recall that, for a positive integer t, the book graph B_t is the graph obtained by the amalgamation of t triangles along the same edge. In this short section we show that $B_{3t-2} \xrightarrow{\text{rb}} B_t$, which guarantee that $p_J^{\text{rb}} \ll n^{-1/m_2(J)}$, for any $J \in B_t \oplus H$, as discussed in the end of the introduction (see Corollary 2).

Lemma 22. *$B_{3t-2} \xrightarrow{\text{rb}} B_t$ for every $t \geqslant 1$.*

Proof. The base case $t = 1$ is trivial, since every proper edge coloring of a triangle is rainbow. We assume that the lemma holds for every integer up to $t - 1$ and we move one step in the induction. Let Φ be a proper-coloring of B_{3t-2} and let

$$V(B_{3t-2}) = \{u_1, u_2, v_1, \ldots, v_{3t-2}\}$$

where $\{u_1, u_2, v_i\}$ is a triangle, for each $i \in [3t-2]$, and $\{v_1, \ldots, v_{3t-2}\}$ is an independent set. By induction, we have that Φ induces a rainbow copy of B_{t-1}, which, without loss of generality, we assume to be induced by $\{u_1, u_2, v_1, \ldots, v_{t-1}\}$.

Let X be the set containing any v_k, with $t \leqslant k \leqslant 3t-2$, such that $\{u_1, u_2, \ldots, v_{t-1}, v_k\}$ does not induces a rainbow copy B_t. Since the coloring is proper, we have that $\Phi(u_i v_k)$ is different of $\Phi(u_1 u_2)$ for every $i \in \{1, 2\}$ and $v_k \in X$. Therefore, if v_k belongs to X, then we must have that $\Phi(u_i v_k) = \Phi(u_{3-i} v_\ell)$ for some $i \in \{1, 2\}$ and $\ell \in [t-1]$. For fixed $i \in \{1, 2\}$, since the coloring is proper, there can be at most $t-1$ indices k such that $\Phi(u_i v_k) = \Phi(u_{3-i} v_\ell)$, for some $\ell \in [t-1]$. Therefore we have that $|X| < 2t-2$ and we conclude that there exists a vertex that yields a rainbow copy of B_t. $\square$

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