Metric Dimension of a Direct Product of Three Complete Graphs

Briana Foster-Greenwood[a] Christine Uhl[b]

Submitted: Sep 19, 2023; Accepted: Apr 1, 2024; Published: Apr 19, 2024

Abstract

Classical Hamming graphs are Cartesian products of complete graphs, and two vertices are adjacent if they differ in exactly one coordinate. Motivated by connections to unitary Cayley graphs, we consider a generalization where two vertices are adjacent if they have no coordinate in common. This generalization is equivalent to a direct product of complete graphs. Metric dimension of classical Hamming graphs is known asymptotically, but, even in the case of hypercubes, few exact values have been found. In contrast, we determine the metric dimension for the entire diagonal family of 3-dimensional generalized Hamming graphs. Our approach is constructive and made possible by first characterizing resolving sets in terms of forbidden subgraphs of an auxiliary edge-colored hypergraph.

Mathematics Subject Classifications: 05C69, 05C12

1 Introduction

Consider an $m \times n$ chessboard with some cells occupied by **landmarks**. A landmark in cell (i, j) **sees** all other cells that are in row i or column j. Is it possible to place landmarks on the board so that each unoccupied cell is seen by a different (possibly empty) set of landmarks? What is the minimum number of landmarks required? What if the puzzle is played in higher dimensions on an $n_1 \times \cdots \times n_r$ board, where a landmark sees all other cells that share at least one coordinate with the landmark's cell?

This optimization puzzle is equivalent to finding the metric dimension of a generalized Hamming graph. After providing background on metric dimension and Hamming graphs, we solve the 2-dimensional puzzle using known results and then devote the rest of the paper to solving 3-dimensional puzzles on $n \times n \times n$ boards.

[a]Department of Mathematics and Statistics, California State Polytechnic University, Pomona, California, U.S.A. (brianaf@cpp.edu).
[b]Department of Mathematics, St. Bonaventure University, St. Bonaventure, New York, U.S.A. (cuhl@sbu.edu).

1.1 Metric dimension

Let G be a finite connected graph with vertex set V. For vertices $x, y \in V$, define the distance $d(x, y)$ to be the length of the shortest path between x and y in G. Given a subset of vertices $W \subseteq V$, whose elements are referred to as **landmarks**, we say W is a **resolving set** (or W **resolves** G) provided that for every pair of distinct vertices $x, y \in V - W$, there exists a landmark $w \in W$ such that $d(x, w) \neq d(y, w)$. A minimum size resolving set is a **metric basis** for G. The **metric dimension** of G, denoted $\dim G$, is the size of a metric basis.

In the context of graph theory, metric dimension was introduced independently by Harary and Melter [6] and Slater [16] in the 1970s. Bounds and values for metric dimension and its variants have been found for many graph families. See [18] for a nice survey paper on metric dimension and some applications, which include source localization (detecting the source of spread in networks), detecting network motifs, and embedding biological sequence data. See [11] for a survey on the many variants of metric dimension.

1.2 Hamming graphs

While there are many generalizations of Hamming graphs, we adopt the definition from [15]. For an r-tuple of positive integers $n_1, \ldots, n_r$ and a set of distances $K \subseteq \{1, 2, \ldots, r\}$, let the **generalized Hamming graph** $HG(n_1, \ldots, n_r; K)$ be the graph with vertex set $V = \{(x_1, \ldots, x_r) \mid 1 \leqslant x_i \leqslant n_i\}$ and adjacency defined by $x \sim y$ if and only if there exists $k \in K$ such that x and y differ in exactly k coordinates.

As noted in [15], generalized Hamming graphs include Cartesian products of complete graphs

$$HG(n_1, \ldots, n_r; 1) \cong K_{n_1} \square \cdots \square K_{n_r}.$$

In particular, this includes the classical Hamming graphs $H(d, q)$ which are the Cartesian products of d copies of the complete graph K_q (with the case of $q = 2$ yielding hypercubes, also known as binary Hamming graphs). Cáceres et al. [3] determine the metric dimension of a Cartesian product of two complete graphs and bound the metric dimension of the Cartesian product of a complete graph with another graph. Research on resolvability in Hamming graphs includes, for instance, work on coin-weighing problems (e.g., [5], [13], [4], [14]), asymptotic results for metric dimension of general Cartesian powers [8], and an integer linear programming approach for testing resolvability [12]. For more details and references, see the survey [18]. We also note that Junnila et al. [9] determine the minimum size of self-locating-dominating codes for the Hamming graphs $H(3, q)$, which is a different problem but has a similar flavor to metric dimension.

For purposes of the puzzle introduced at the beginning of this article, we are interested in the generalized Hamming graphs $HG(n_1, \ldots, n_r; r)$. These graphs are isomorphic to direct products (also called categorical or Kronecker products) of complete graphs

$$HG(n_1, \ldots, n_r; r) \cong K_{n_1} \times K_{n_2} \times \cdots \times K_{n_r},$$

in which two vertices are adjacent if they have no coordinates in common. For $r \geqslant 2$ and

$n_i \geqslant 3$, the graphs are connected with diameter two. In particular,

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \text{ and } y \text{ have no coordinates in common,} \\ 2 & \text{if } x \text{ and } y \text{ have at least one coordinate in common.} \end{cases}$$

(Note that for $r \geqslant 2$, if $n_1 = n_2 = 2$, the graph $HG(n_1, \ldots, n_r; r)$ is disconnected, and if $n_1 = 2$ and $n_2, \ldots, n_r \geqslant 3$, the graph is connected but has diameter 3.)

In terms of the puzzle, we see that, for $r \geqslant 2$ and $n_i \geqslant 3$, the cells of an $n_1 \times \cdots \times n_r$ board correspond to vertices in the graph $HG(n_1, \ldots, n_r; r)$, and a landmark w sees vertex x precisely when $d(w, x) = 2$. Also, x and y that are not landmarks will be resolved if and only if there is a landmark that sees x or y but not both, i.e., x and y are seen by different sets of landmarks. Therefore a resolving set for the graph $HG(n_1, \ldots, n_r; r)$ corresponds to a solution to the puzzle, and the metric dimension is the minimum number of landmarks required to solve the puzzle.

Note that, alternatively, we could use the graphs $HG(n_1, \ldots, n_r; 1, \ldots, r-1)$. These graphs also have diameter two, but the criteria for distance one and distance two are swapped. It follows that the graphs $HG(n_1, \ldots, n_r; r)$ and $HG(n_1, \ldots, n_r; 1, \ldots, r-1)$ have the same resolving sets and metric dimension, so we could use either graph to represent our puzzles.

The generalized Hamming graphs under consideration can be expressed in terms of various graph operations involving Cartesian products, complements, power graphs, and exact distance graphs. Given a graph G, we can construct the **graph complement** $\overline{G}$, the ***k*-th power graph** $G^{(k)}$, and the **exact distance-*k* graph** $G^{[\natural k]}$. All three graphs share the same vertex set as G. Distinct vertices are adjacent in the k-th power graph if they are at most distance k apart in G, while adjacency in the exact distance-k graph requires a distance of exactly k. By [15, Lemma 3.1], the graphs $HG(n_1, \ldots, n_r; r)$ and $HG(n_1, \ldots, n_r; 1, \ldots, r-1)$ are complements of each other and we have the isomorphisms

$$HG(n_1, \ldots, n_r; r) \cong K_{n_1} \times K_{n_2} \times \cdots \times K_{n_r} \cong (K_{n_1} \square \cdots \square K_{n_r})^{[\natural r]}$$

and

$$HG(n_1, \ldots, n_r; 1, \ldots, r-1) \cong (K_{n_1} \square \cdots \square K_{n_r})^{(r-1)}.$$

For structural results on exact distance graphs of product graphs, see [1]. The above isomorphisms will be useful in solving the 2-dimensional puzzle.

1.3 Two- and three-dimensional puzzles

By the remarks in the previous subsection, we can solve the 2-dimensional puzzle on an $m \times n$ board by finding the metric dimension and minimum resolving sets of the direct product $HG(m, n; 2) \cong K_m \times K_n$ or Cartesian product $HG(m, n; 1) \cong K_m \square K_n$. Choosing the latter, we can apply [3, Theorem 6.1] that for all $n \geqslant m \geqslant 1$, the metric dimension of a Cartesian product of two complete graphs is

$$\dim(K_m \square K_n) = \begin{cases} \lfloor \frac{2}{3}(n + m - 1) \rfloor & \text{if } m \leqslant n \leqslant 2m - 1 \\ n - 1 & \text{if } n \geqslant 2m - 1. \end{cases}$$

Moreover, [3, Lemma 6.2] provides necessary and sufficient conditions for a subset of vertices S to be resolving based on relationships between the elements of S. More recently, Kuziak et al. [10] determined an equivalent formula for the metric dimension of the direct product $K_m \times K_n$.

To solve the puzzle on an $n_1 \times n_2 \times n_3$ board, we want to find the metric dimension of $HG(n_1, n_2, n_3; 3) \cong K_{n_1} \times K_{n_2} \times K_{n_3}$ or $HG(n_1, n_2, n_3; 1, 2) \cong (K_{n_1} \square K_{n_2} \square K_{n_3})^{(2)}$. We are unaware of any results that give the metric dimension of these graphs, so that will be the focus of the remainder of the paper. In Section 2, we find a lower bound for the metric dimension. In Section 3, we focus our attention on the diagonal family $HG(n, n, n; 3) \cong K_n \times K_n \times K_n$ and develop a characterization of resolving sets in terms of forbidden edge-colored subgraphs of an auxiliary hypergraph. In Section 4, we apply the theorems from Section 3 to construct minimum resolving sets for diagonal Hamming graphs. We end with the comprehensive Theorem 17 for the metric dimension of the Hamming graphs $HG(n, n, n; 3) \cong K_n \times K_n \times K_n$ for $n \geqslant 3$ and indicate further research directions with connections to unitary Cayley graphs.

Throughout the paper, let $\mathcal{N} = \{(n_1, n_2, n_3) \in \mathbb{N}^3 \mid n_1, n_2, n_3 \geqslant 3\}$ and let the diagonal be $\text{diag}(\mathcal{N}) = \{(n, n, n) \mid n \geqslant 3\}$. We commonly denote an element of $\mathcal{N}$ as $\mathbf{n} = (n_1, n_2, n_3)$ and increment each coordinate to get $\mathbf{n} + \mathbf{1} = (n_1 + 1, n_2 + 1, n_3 + 1)$. For $n \in \mathbb{N}$, we let $[n] = \{1, 2, \ldots, n\}$.

2 Lower Bound

In this section, we prove a lower bound for the metric dimension of the Hamming graphs $HG(n_1, n_2, n_3; 3)$.

Given $\mathbf{n} \in \mathcal{N}$ and a subset of vertices W of the Hamming graph $HG(\mathbf{n}; 3)$, let $W_{i,a}$ be the set of landmarks in the plane $x_i = a$, i.e.,

$$W_{i,a} = \{(x_1, x_2, x_3) \in W \mid x_i = a\}$$

for $1 \leqslant i \leqslant 3$ and $a \in [n_i]$. We call the sets $W_{i,a}$ **blocks of color *i***.

We begin with a constraint on the minimum number of landmarks in the blocks of a given color. Namely, if W is a resolving set, then the absence of landmarks in a block forces the other blocks of that color to each contain at least three landmarks. Additionally, if a block has only one landmark in it, then the other blocks of that color are forced to each contain at least two landmarks.

Lemma 1. *If W is a resolving set for the Hamming graph $G = HG(n_1, n_2, n_3; 3)$ (each $n_i \geqslant 3$), then*

$$|W_{i,a}| + |W_{i,b}| \geqslant 3$$

for all $1 \leqslant i \leqslant 3$ and distinct $a, b \in [n_i]$.

Proof. Let W be a subset of the vertex set of G and let $i = 1$ (the proofs for $i = 2, 3$ are analogous). First suppose $|W_{1,a}| = 0$ and $|W_{1,b}| = 2$ for some $a, b \in [n_1]$. Say the two landmarks in $W_{1,b}$ are (b, x_1, y_1) and (b, x_2, y_2). Now, since $n_2, n_3 \geqslant 3$, we can choose $x_3 \in [n_2] - \{x_1, x_2\}$ and $y_3 \in [n_3] - \{y_1, y_2\}$ and define (x, y) as follows:

$$(x, y) = \begin{cases} (x_1, y_2) & \text{if } x_1 \neq x_2 \text{ and } y_1 \neq y_2 \\ (x_1, y_3) & \text{if } x_1 = x_2 \\ (x_3, y_1) & \text{if } y_1 = y_2. \end{cases}$$

To resolve the vertices (a, x, y) and (b, x, y), we would need a landmark that has a coordinate in common with one of the vertices but not the other, i.e., a landmark from $W_{1,a} \cup W_{1,b}$. But $W_{1,a}$ is empty, and, by construction, (x, y) has exactly one coordinate in common with (x_1, y_1) and exactly one coordinate in common with (x_2, y_2), which implies (a, x, y) and (b, x, y) are distance two from the elements of $W_{1,b}$. Hence (a, x, y) and (b, x, y) are unresolved.

Next suppose $|W_{1,a}| = 1$ and $|W_{1,b}| = 1$. Say the landmarks in $W_{1,a} \cup W_{1,b}$ are (a, x_1, y_1) and (b, x_2, y_2). Defining (x, y) as in the previous case, we can similarly show that (a, x, y) and (b, x, y) are distance two to the landmarks in $W_{1,a} \cup W_{1,b}$ and are hence unresolved.

Finally, if $|W_{1,a}| + |W_{1,b}| < 2$, the case is all the worse, for if W was resolving, then we could add another landmark to $W_{1,a} \cup W_{1,b}$ to create a resolving set W' with $|W'_{1,a}| + |W'_{1,b}| = 2$, a contradiction. □

We next use the pairwise bound from Lemma 1 to determine a lower bound on the total number of landmarks required to resolve Hamming graphs $HG(\mathbf{n}; 3)$.

Theorem 2. *For $n_3 \geqslant n_2 \geqslant n_1 \geqslant 3$, the Hamming graph $G = HG(n_1, n_2, n_3; 3)$ cannot be resolved with fewer than $2n_3 - 1$ landmarks.*

Proof. Suppose W resolves G. For $1 \leqslant i \leqslant 3$, the sum $|W_{i,1}| + |W_{i,2}| + \cdots + |W_{i,n_i}|$ is the total number of landmarks in W. If the minimum term in this sum is zero, Lemma 1 says the remaining terms are greater than or equal to three, giving at least $3(n_i - 1)$ landmarks. If the minimum term is one, Lemma 1 says the remaining terms are greater than or equal to two, giving at least $1 + 2(n_i - 1)$ landmarks. If the minimum term is two, we have at least $2n_i$ landmarks. Note that, under the assumption $n_i \geqslant 3$, we have

$$2n_i - 1 < 2n_i \leqslant 3(n_i - 1).$$

So, regardless of the minimum term in the sum, we have $2n_i - 1$ as a lower bound on the number of landmarks in W. In particular, $2n_3 - 1$ is a lower bound (and with the assumption $n_3 \geqslant n_2 \geqslant n_1$, it is the greatest of the lower bounds $2n_i - 1$). □

In Section 4, we will show the bound in Theorem 2 is sharp by resolving the Hamming graphs $HG(n, n, n; 3)$ for $n \geqslant 5$ in $2n - 1$ landmarks. In preparation, we develop an equivalent characterization of resolving sets.

3 Landmark Graphs and Forbidden Configurations

A priori, determining whether a subset of vertices W is a resolving set for a Hamming graph $HG(\mathbf{n};3)$ involves checking whether all pairs of vertices are resolved. Instead of doing this directly, we shift our focus to relationships between the elements of W. We define an edge-colored hypergraph based on W and then characterize whether W is resolving in terms of forbidden edge-colored subgraphs. This characterization will be used in Section 4 to construct minimum resolving sets for the diagonal family $HG(n,n,n;3)$. First, we establish terminology regarding hypergraphs.

3.1 Hypergraphs

A **hypergraph** consists of a nonempty set of vertices, say W, and a (multi)set of **hyperedges**, each of which is a nonempty subset of W. A **loop** is a hyperedge with only one vertex. A **plain edge** is a hyperedge with exactly two vertices. In this paper, we consider the collection of hyperedges as a multiset, so it is possible to have multiple hyperedges with the same set of vertices. In particular, we will encounter **triple loops** (three loops with the same vertex), **double loops** (two loops with the same vertex), and **double edges** (two plain edges with the same pair of vertices).

3.2 Landmark graphs and systems

As in Section 2, given a subset of vertices W in a Hamming graph $HG(\mathbf{n};3)$, let $W_{i,a}$ be the set of elements in W whose i-th coordinate is a. To highlight relationships between the coordinates of elements of W, we define the **landmark graph $\boldsymbol{\mathcal{G}(W)}$** to be the hypergraph with vertex set W and hyperedges those $W_{i,a}$ that are nonempty. We often refer to $\mathcal{G}(W)$ as the graph of W. The graph of W is 3-regular and has a proper 3-edge-coloring obtained by assigning color i to the sets $W_{i,a}$. Note that the hyperedges of color i form a partition of W. In practice, we will think of the first color as blue, second color as green, and third color as pink. As an example, Figure 1 shows the landmark graph of a resolving set for $HG(3,3,3;3)$.

The following lemma tells us a necessary condition on the structure of the landmark graph if a diagonal Hamming graph is to be resolved with the lower bound of $2n-1$ landmarks. We use the phrase "plain edge" to emphasize an edge with two endpoints (as opposed to a loop or larger hyperedge).

Lemma 3. *Let $n \geqslant 3$. If W is a resolving set of $HG(n,n,n;3)$ of size $2n-1$, then the landmark graph $\mathcal{G}(W)$ has exactly one loop of each color and exactly $(n-1)$ plain edges of each color.*

Proof. Suppose W is a resolving set of $HG(n,n,n;3)$ of size $2n-1$. First, note that if the n blocks of color i all have size 2 or greater, then $|W| \geqslant 2n$, a contradiction. Hence there must be at least one block of size 0 or 1. But if there is a block of color i that has size 0, then, by Lemma 1, the other blocks of color i all have size 3 or greater, which implies $|W| \geqslant 3(n-1) > 2n-1$, also a contradiction. Thus there must be a block $W_{i,a}$ of size 1.

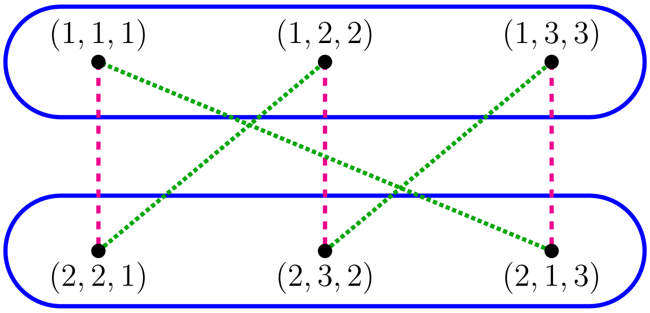


Figure 1: Landmark graph $\mathcal{G}(W)$ of a resolving set for $HG(3,3,3;3)$. The six landmarks in W are $(1,1,1)$, $(1,2,2)$, $(1,3,3)$, $(2,2,1)$, $(2,3,2)$, and $(2,1,3)$. Landmarks in the same blue (solid) hyperedge have the same first coordinate; landmarks connected by a green (dotted) edge have the same second coordinate; and landmarks connected by a pink (dashed) edge have the same third coordinate.

Then, by Lemma 1, all other blocks $W_{i,b}$ with $b \neq a$ have size at least 2. Moreover, to get exactly $2n-1$ elements in W, the sizes of $W_{i,b}$ for $b \neq a$ cannot exceed 2. □

To optimally resolve the graphs in the diagonal family, we are interested in subsets W that will produce a graph $\mathcal{G}(W)$ with the structure given in Lemma 3. Our strategy is to start with graphs that produce an equal number of plain edges of each color and then add on loops. Though not necessary, we attempt to put all loops at the same vertex to make the analysis and constructions easier. This leads us to the following definition.

Definition 4. Let $n \geqslant 3$ and $\mathbf{n} = (n,n,n)$. For W a subset of vertices of the graph $HG(\mathbf{n};3)$, we say W is a **2-basic landmark system** provided that:

1. $|W_{i,a}| = 2$ for all $1 \leqslant i \leqslant 3$ and $a \in [n]$; and
2. $|W_{i,a} \cap W_{j,b}| \leqslant 1$ for all $1 \leqslant i < j \leqslant 3$ and $a, b \in [n]$.

Letting $u = (n+1, n+1, n+1)$, we can extend a 2-basic landmark system for $HG(\mathbf{n};3)$ to a **triple-looped landmark system** $W \cup \{u\}$ for $HG(\mathbf{n}+\mathbf{1};3)$.

Condition (1) ensures that $\mathcal{G}(W)$ only has edges with two endpoints, and condition (2) ensures that $\mathcal{G}(W)$ has no double edges. Thus, when W is a 2-basic landmark system, $\mathcal{G}(W)$ is a simple graph with n edges of each color. Note that the definition of a landmark system makes no assertion as to whether W is a resolving set. Our goal is to find further conditions, based on subgraphs of $\mathcal{G}(W)$, which will allow us to determine if W is a resolving set.

3.3 Subgraphs

Before continuing, we establish notation and definitions for subgraphs. We denote a path with n vertices as P_n, a cycle with n vertices as C_n, and a complete bipartite graph with parts of size m and n as $K_{m,n}$. Some of our graphs will have loops and multiple edges, so

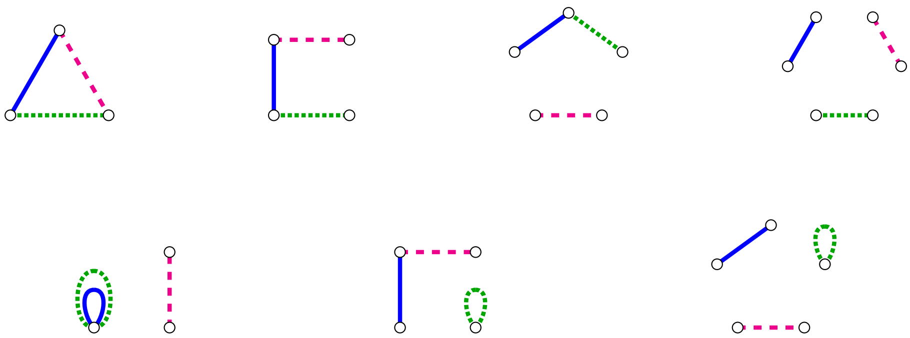


Figure 2: If W is a 2-basic landmark system, then there are four possible footprints for a non-landmark: rainbow C_3, P_4, $P_3 \cup P_2$, and $3P_2$. If W is a triple-looped landmark system, then there are three additional possible footprints: rainbow $L_2 \cup P_2$, $P_3 \cup L_1$, and $2P_2 \cup L_1$. Different line styles correspond to different edge colors.

we denote a vertex with n loops as L_n and a double edge between the same two vertices as D_2. We write $G \cup H$ for the disjoint union of graphs G and H.

We say edge e **covers** vertex u if $u \in e$. A subset of edges E' determines an **edge-induced subgraph** consisting of the edges in E' and the vertices they cover. In a graph with an edge-coloring, a **rainbow subgraph** is a subgraph whose edges are all different colors. We can identify (almost) every vertex in a Hamming graph with an edge-induced rainbow subgraph of a landmark graph.

Definition 5. Let $\mathbf{n} \in \mathcal{N}$ and let W be a subset of vertices of the Hamming graph $HG(\mathbf{n}; 3)$. Given a vertex $\alpha = (a_1, a_2, a_3)$ of the Hamming graph, define the **footprint of** $\boldsymbol{\alpha}$ (relative to W) to be the subgraph of $\mathcal{G}(W)$ induced by the edges W_{i,a_i}. If $W_{i,a_i} = \varnothing$ for all $1 \leqslant i \leqslant 3$, then there is no edge-induced subgraph and α has no footprint.

In Figure 2, we show the possible footprints of a non-landmark vertex in a Hamming graph $HG(\mathbf{n}; 3)$. There are four possible footprints assuming that W is a 2-basic landmark system, and three additional footprints assuming W is a triple-looped landmark system. The vertices of the footprint of α are the landmarks that have a coordinate in common with α, i.e., the vertices in W that have distance 2 from α in $HG(\mathbf{n}; 3)$. Notice that the footprint of a landmark has three edges/loops that share a common vertex, which in the case of a triple-looped landmark system can either be a $K_{1,3}$ or an L_3. We omit these from Figure 2 since landmarks are automatically resolved.

Because we are working with Hamming graphs of diameter two, the following straightforward but important observation allows us to determine whether W is resolving by examining how footprints overlap.

Remark 6. Let W be a subset of vertices of a Hamming graph $HG(\mathbf{n}; 3)$. Note that in the Hamming graph, a vertex $(a_1, a_2, a_3) \notin W$ has distance two to the landmarks in $W_{1,a_1} \cup W_{2,a_2} \cup W_{3,a_3}$ and distance one to the remaining landmarks in W. It follows that distinct vertices $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ not in W are resolved if and only if

$$W_{1,a_1} \cup W_{2,a_2} \cup W_{3,a_3} \neq W_{1,b_1} \cup W_{2,b_2} \cup W_{3,b_3},$$

i.e., the footprints of α and β cover different sets of vertices.

3.4 Forbidden configurations for two-basic landmark systems

We are now ready to characterize whether W is resolving based on properties of the landmark graph of W. In particular, we will find necessary and sufficient conditions in terms of forbidden edge-colored subgraphs. We begin with two examples of configurations that cannot occur in the landmark graph of a resolving set.

Lemma 7 (Forbidden **4**-cycle)**.** *Let* $\mathbf{n} \in \mathcal{N}$ *and let* W *be a subset of vertices of the graph* $HG(\mathbf{n}; 3)$*. If the hypergraph* $\mathcal{G}(W)$ *contains a cycle* C_4 *with edge color pattern blue-green-blue-pink (or colors permuted), then* W *is not a resolving set.*

Proof. Suppose the graph of W contains a 4-cycle, as illustrated in Figure 3, with opposite blue edges $W_{1,a_1} = \{w_1, w_2\}$ and $W_{1,b_1} = \{w_3, w_4\}$, a green edge $W_{2,a_2} = \{w_2, w_3\}$, and a pink edge $W_{3,a_3} = \{w_1, w_4\}$. Then the vertices (a_1, a_2, a_3) and (b_1, a_2, a_3) are unresolved in $HG(\mathbf{n}; 3)$ since neither is a landmark and

$$W_{1,a_1} \cup W_{2,a_2} \cup W_{3,a_3} = \{w_1, w_2, w_3, w_4\} = W_{1,b_1} \cup W_{2,a_2} \cup W_{3,a_3}.$$

Similarly, if edge colors are permuted, there will be an unresolved pair of vertices. □

Lemma 8 (Forbidden **6**-cycle)**.** *Let* $\mathbf{n} \in \mathcal{N}$ *and let* W *be a subset of vertices of the graph* $HG(\mathbf{n}; 3)$*. If the hypergraph* $\mathcal{G}(W)$ *contains a cycle* C_6 *with edge color pattern blue-green-pink-blue-green-pink, then* W *is not a resolving set.*

Proof. Suppose the graph of W contains a 6-cycle, as illustrated in Figure 3, with opposite blue edges $W_{1,a_1} = \{w_1, w_2\}$ and $W_{1,b_1} = \{w_4, w_5\}$; opposite green edges $W_{2,a_2} = \{w_5, w_6\}$ and $W_{2,b_2} = \{w_2, w_3\}$; and opposite pink edges $W_{3,a_3} = \{w_3, w_4\}$ and $W_{3,b_3} = \{w_1, w_6\}$. Then the vertices (a_1, a_2, a_3) and (b_1, b_2, b_3) are unresolved in $HG(\mathbf{n}; 3)$ since neither is a landmark and

$$W_{1,a_1} \cup W_{2,a_2} \cup W_{3,a_3} = \{w_1, w_2, w_3, w_4, w_5, w_6\} = W_{1,b_1} \cup W_{2,b_2} \cup W_{b_3}.$$

Hence W is not a resolving set. □

When either the 4-cycle from Lemma 7 (possibly with colors permuted) or the 6-cycle from Lemma 8 appear in a landmark graph, the Hamming graph $HG(\mathbf{n}; 3)$ is unresolved. We next show that when W is a 2-basic landmark system, these are in fact the only configurations we must avoid to guarantee W is a resolving set.

Theorem 9. *Let* $\mathbf{n} \in \operatorname{diag}(\mathcal{N})$*. If* W *is a* 2*-basic landmark system for the Hamming graph* $HG(\mathbf{n}; 3)$*, then* W *is a resolving set if and only if the landmark graph* $\mathcal{G}(W)$ *avoids the following forbidden subgraphs:*

1. *a* 4*-cycle that contains all three colors of edges*
2. *a* 6*-cycle with opposite edges of the same color.*

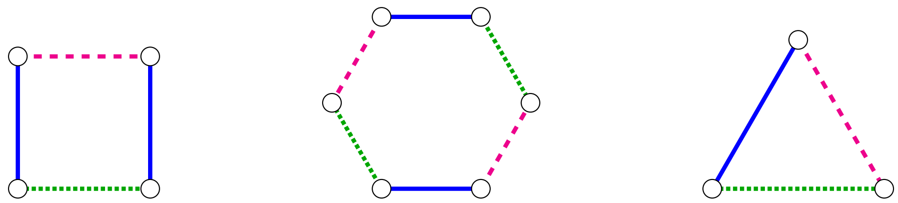


Figure 3: The landmark graph of a resolving set must avoid forbidden 4-cycles and forbidden 6-cycles. The landmark graph of a resolving triple-looped landmark system must also avoid rainbow triangles. Solid edges are blue, dashed edges are pink, and dotted edges are green.

Proof. Using Lemma 7 and Lemma 8, we can see that W is not resolving when the landmark graph $\mathcal{G}(W)$ contains a 4-cycle that contains all three colors of edges or a 6-cycle with opposite edges of the same color.

Conversely, suppose W is a 2-basic landmark system whose graph $\mathcal{G}(W)$ avoids forbidden 4-cycles and 6-cycles. We will show that W is a resolving set. Suppose $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ are distinct vertices in $HG(\mathbf{n}; 3)$ and that neither is a landmark. We will show the edges W_{i,a_i} cannot cover the same set of vertices as the edges W_{i,b_i}. We consider cases based on the number of vertices covered. Throughout, we use the fact that $\mathcal{G}(W)$ is a simple graph with a proper 3-edge coloring.

Case 1 (3 vertices). If the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of three vertices, then we must have $W_{i,a_i} = W_{i,b_i}$ for $i = 1, 2, 3$ since it is not possible to have two different edges of the same color in a subgraph with only three vertices. But this contradicts our assumption that α and β are distinct.

Case 2 (4 vertices). Suppose the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of four vertices. Since α is not a landmark, the edges W_{i,a_i} cannot induce a $K_{1,3}$ and hence must induce a rainbow path P_4. Likewise, the edges W_{i,b_i} induce a rainbow path P_4. The only way for two different rainbow paths to cover all four vertices is to make a 4-cycle with one set of opposite edges the same color and the other two edges different colors, but we are assuming that $\mathcal{G}(W)$ avoids this forbidden configuration.

Case 3 (5 vertices). Suppose the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of five vertices. Then each set of edges induces a rainbow $P_2 \cup P_3$. Up to permuting colors, suppose the first $P_2 \cup P_3$ has blue edge $W_{1,a_1} = \{x, y\}$, green edge $W_{2,a_2} = \{u, v\}$, and pink edge $W_{3,a_3} = \{u, w\}$, as in Figure 4[A]. If the second rainbow $P_2 \cup P_3$ also has blue edge $\{x, y\}$, then since there are no double edges, the green and pink edges must also coincide, i.e., $W_{2,a_2} = W_{2,b_2}$ and $W_{3,a_3} = W_{3,b_3}$, contradicting α and β being distinct. The only other option is for the second rainbow $P_2 \cup P_3$ to have blue edge $W_{1,b_1} = \{v, w\}$ as in Figure 4[B]. If the blue edge vw is the P_2, then the edges W_{2,b_2} and W_{3,b_3} would have to create a green-pink path xuy or yux, but this is not possible with a proper edge coloring. If the blue edge vw is part of the P_3, then the P_2 would have to create a double edge between x and y, which is not allowed.

Case 4 (6 vertices). Suppose the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of six vertices. Then each set of edges creates a rainbow matching $3P_2$. We will show

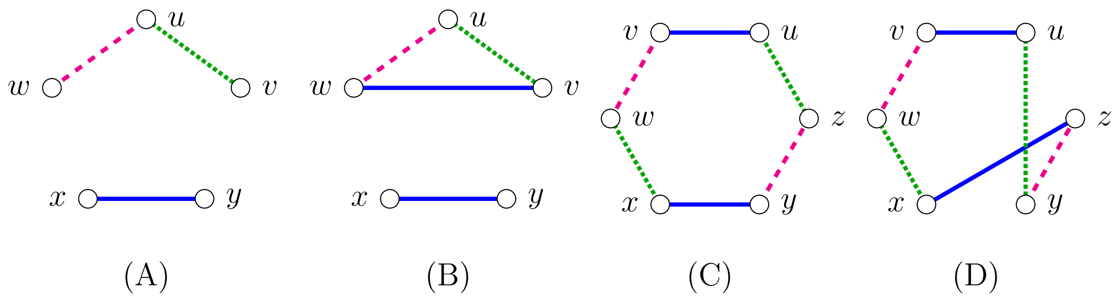


Figure 4: Illustrations for Cases 3 and 4 in the proof of Theorem 9. Line styles correspond to the edge colors blue (solid), green (dotted), and pink (dashed). (A) and (B) It is not possible for two different rainbow $P_2 \cup P_3$ graphs to cover the same set of five vertices. (C) and (D) If two different rainbow matchings cover the same set of six vertices, the edges together form a forbidden 6-cycle.

the only way this can happen is if the edges of the matchings together form a forbidden 6-cycle. Suppose the first rainbow matching has blue edge $W_{1,a_1} = \{u, v\}$, green edge $W_{2,a_2} = \{w, x\}$, and pink edge $W_{3,a_3} = \{y, z\}$, as in Figure 4[C and D]. The two matchings must differ by at least one edge, so up to permuting colors, suppose they have different pink edges. Then W_{3,b_3} must connect a vertex from the blue edge W_{1,a_1} to a vertex from the green edge W_{2,a_2}. Without loss of generality, say $W_{3,b_3} = \{v, w\}$. Now, the blue edge W_{1,b_1} and green edge W_{2,b_2} must cover the vertices u, x, y, and z. There are two ways to do this, leading to either the 6-cycle $uvwxyzu$ or $uvwxzyu$. Either way, the pattern of edge colors around the cycle is blue, pink, green, blue, pink, green. But this is a contradiction since we assumed that the graph of W has no forbidden 6-cycles.

Thus, in all cases, the edges W_{i,a_i} and the edges W_{i,b_i} cannot cover the same set of vertices and so α and β are resolved. Hence W is a resolving set. □

3.5 Forbidden configurations for triple-looped landmark systems

We next investigate conditions for when a triple-looped landmark system will resolve a Hamming graph. We will use these conditions in Section 4 to construct resolving sets for most of the Hamming graphs $HG(n, n, n; 3)$ in $2n - 1$ landmarks.

Recall that the graph of a triple-looped landmark system is obtained from the graph of a 2-basic landmark system by adding a vertex with a triple loop. The next lemma shows that if there is a rainbow triangle in the graph of a 2-basic landmark system, then the extension to a triple-looped landmark system will not be a resolving set.

Lemma 10 (Triangle union triple loop)**.** *Let* $\mathbf{n} \in \mathcal{N}$ *and let* W *be a subset of vertices of the graph* $HG(\mathbf{n}; 3)$*. If the hypergraph* $\mathcal{G}(W)$ *contains a rainbow triangle and a triple loop, then* W *is not a resolving set.*

Proof. Suppose the graph of W contains a rainbow triangle with blue edge $W_{1,a_1} = \{w_1, w_2\}$, green edge $W_{2,a_2} = \{w_2, w_3\}$, and pink edge $W_{3,a_3} = \{w_3, w_1\}$, as in Figure 3.

Suppose the graph of W also contains a triple loop, say $W_{1,b_1} = W_{2,b_2} = W_{3,b_3} = \{w_4\}$. Note that since the graph of W is properly colored, $w_4 \notin \{w_1, w_2, w_3\}$. Now, the vertices (a_1, a_2, b_3) and (a_1, b_2, a_3) in $HG(\mathbf{n}; 3)$ are unresolved since neither is a landmark and

$$W_{1,a_1} \cup W_{2,a_2} \cup W_{3,b_3} = \{w_1, w_2, w_3, w_4\} = W_{1,a_1} \cup W_{2,b_2} \cup W_{3,a_3}.$$

Hence W is not a resolving set. □

The next theorem is analogous to Theorem 9, but allows for a triple loop at a single vertex. Furthermore this theorem allows us to construct minimum resolving sets in the next section.

Theorem 11. *Let $\mathbf{n} \in \operatorname{diag}(\mathcal{N})$, and suppose W is a 2-basic landmark system for $HG(\mathbf{n}; 3)$. Let $u = (n+1, n+1, n+1)$. Then the triple-looped landmark system $W \cup \{u\}$ resolves $HG(\mathbf{n}+\mathbf{1}; 3)$ if and only if the landmark graph $\mathcal{G}(W)$ avoids the following forbidden subgraphs:*

1. *a 4-cycle that contains all three colors of edges,*
2. *a 6-cycle with opposite edges of the same color, and*
3. *a rainbow triangle.*

Proof. Let V_n and V_{n+1} be the vertex sets of the Hamming graphs $HG(\mathbf{n}; 3)$ and $HG(\mathbf{n}+\mathbf{1}; 3)$, respectively. If a forbidden 4-cycle or 6-cycle exists in the graph of W, then W does not resolve $HG(\mathbf{n}; 3)$, and so $W \cup \{u\}$ does not resolve $HG(\mathbf{n} + \mathbf{1}; 3)$. This is because the new landmark u is adjacent to all of the vertices in V_n, so vertices in V_n can only be resolved using a landmark from W. If a rainbow triangle exists in the graph of W, then a rainbow triangle and triple loop exist in the graph of $W \cup \{u\}$, so by Lemma 10, $W \cup \{u\}$ does not resolve $HG(\mathbf{n} + \mathbf{1}; 3)$.

Conversely, suppose W is a 2-basic landmark system such that $\mathcal{G}(W)$ avoids forbidden 4-cycles, forbidden 6-cycles, and rainbow triangles. We will show that $W \cup \{u\}$ is a resolving set for $HG(\mathbf{n} + \mathbf{1}; 3)$.

Suppose $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ are distinct vertices in $HG(\mathbf{n} + \mathbf{1}; 3)$ and that neither is a landmark. Note that, by Theorem 9, W is a resolving set for $HG(\mathbf{n}; 3)$, so if α and β are both in V_n, then they are resolved by some landmark in W. If one vertex, say α, is in V_n and the other vertex, β, is in $V_{n+1} - V_n$, then u resolves α and β, as u sees β but not α. The remaining case, where α and β are both in $V_{n+1} - V_n$, requires the most work. We will show the edges W_{i,a_i} cannot cover the same set of vertices as the edges W_{i,b_i}. We consider cases based on the number of vertices covered.

Note that the edges W_{i,a_i} (likewise, W_{i,b_i}) must consist of either a loop and two plain edges, or a double loop and a plain edge. Thus the number of vertices covered ranges between three and five.

Case 1 (3 vertices). If the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of three vertices, then each set of edges induces a rainbow $P_2 \cup L_2$, where the L_2 must be at the same vertex. And then we must have $W_{i,a_i} = W_{i,b_i}$ for $i = 1, 2, 3$ since there are no double edges. But this contradicts our assumption that α and β are distinct.

Case 2 (4 vertices). Suppose the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of four vertices. Both edge sets induce a rainbow $P_3 \cup L_1$. Since the loops occur at the same vertex, the only way for two different rainbow $P_3 \cup L_1$ subgraphs to cover all four vertices is if they together make a $C_3 \cup L_2$, but we are assuming that $\mathcal{G}(W)$ avoids rainbow triangles.

Case 3 (5 vertices). Suppose the edges W_{i,a_i} and the edges W_{i,b_i} cover the same set of five vertices. Both edge sets induce a rainbow $P_2 \cup P_2 \cup L_1$. Both sets must have their L_1 at the same vertex (although possibly different colors). There is no way to make two different rainbow $2P_2$'s cover the same set of 4 vertices since once the first rainbow $2P_2$ is placed, the only other possible $2P_2$ would have edges of the same color instead of different colors. Hence we must have $W_{i,a_i} = W_{i,b_i}$ for $i = 1, 2, 3$, which contradicts α and β being distinct.

Thus in all cases, the edges W_{i,a_i} and the edges W_{i,b_i} cannot cover the same set of vertices and so α and β are resolved. Hence $W \cup \{u\}$ is a resolving set. $\square$

4 Metric Bases for the Diagonal Family

Now that we know which configurations must be avoided in the landmark graph, we are able to construct minimum resolving sets for the diagonal family of Hamming graphs $HG(n, n, n; 3)$. The graphs with $n \geqslant 5$ achieve the metric dimension lower bound of $2n - 1$, while the graphs for $n = 3$ and $n = 4$ require $2n$ landmarks.

4.1 Construction of metric bases

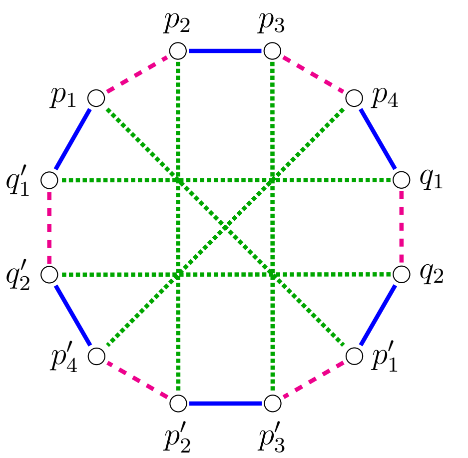
For the main case of $n \geqslant 5$, our strategy will be to construct an order $2(n-1)$ graph G that avoids the configurations from Theorem 11 and define a 2-basic landmark system W with $\mathcal{G}(W) = G$. We then extend W to a triple-looped landmark system that resolves $HG(n, n, n; 3)$ in $2n - 1$ landmarks.

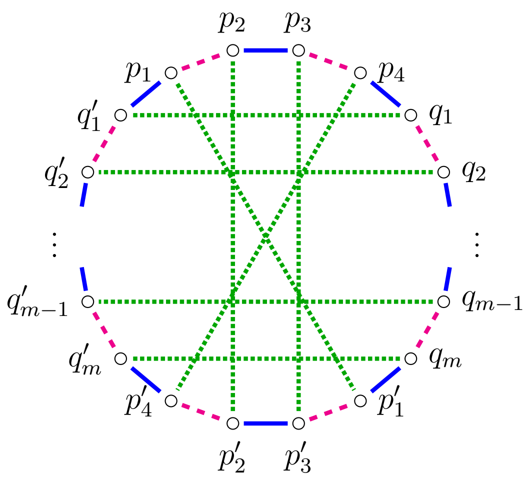
Lemma 12. *For $k \geqslant 6$, there exists a cubic graph on $2k$ vertices with a proper 3-edge-coloring that avoids the forbidden edge-colored subgraphs from Theorem 11:*

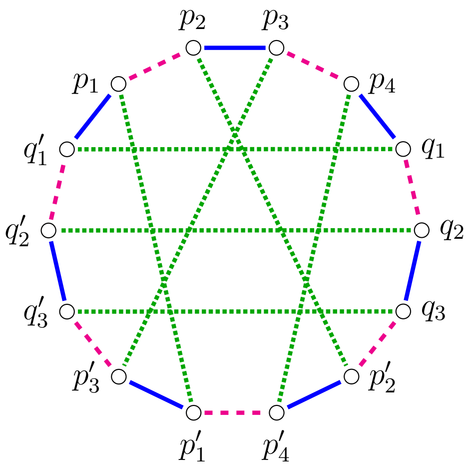
1. *a 4-cycle that contains all three colors of edges,*
2. *a 6-cycle with opposite edges of the same color, and*
3. *a rainbow triangle.*

Proof. To construct a graph with the desired properties (see Figure 5), we let $k \geqslant 6$ and start with an order $2k$ Hamiltonian cycle whose edge colors alternate, say blue and pink. Next, we label the vertices of the cycle in a way that helps us define green edges and avoid the forbidden configurations. Let $m = k - 4$. If k is even, label the vertices of the cycle in the order

$$(p_1, p_2, p_3, p_4, q_1, \ldots, q_m, p_1', p_3', p_2', p_4', q_m', \ldots, q_1'),$$







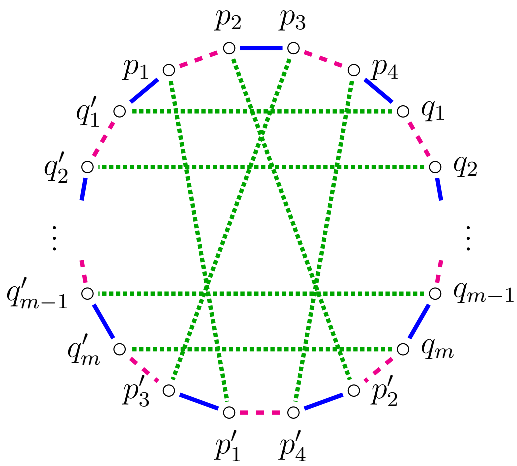


Figure 5: Cubic graph on $2k$ vertices with a proper 3-edge-coloring for even $k \geqslant 6$ (top row images) and odd $k \geqslant 7$ (bottom row images). Edges of outer Hamiltonian cycle alternate blue (solid) and pink (dashed). All other edges are green (dotted). Forbidden 4-cycles, forbidden 6-cycles, and rainbow triangles are avoided.

and if k is odd, label the vertices in the order

$$(p_1, p_2, p_3, p_4, q_1, \ldots, q_m, p'_2, p'_4, p'_1, p'_3, q'_m, \ldots, q'_1).$$

To finish the graph, add green edges $p_ip'_i$ (for $i = 1, 2, 3, 4$) and $q_iq'_i$ (for $i = 1, \ldots, m$). For notational convenience, we let $(p'_i)' = p_i$ and $(q'_i)' = q_i$ so that for any vertex u in the graph, the vertex u' is the other endpoint of the green edge containing u. We claim these graphs avoid the forbidden triangles, 4-cycles, and 6-cycles of Theorem 11.

If there was a rainbow triangle, the outer Hamiltonian cycle would contain a sequence of vertices u, v, u', which we don't have.

If there was a forbidden 4-cycle with two blue edges or two pink edges, then we would have a path $uvwu'$ along the outer Hamiltonian cycle, but this does not occur (u and u' are always at least five edges apart). If there was a forbidden 4-cycle with two green edges, then there would exist edges uv and $u'v'$ of different colors, but this does not occur in our construction.

Lastly, if there was a forbidden 6-cycle, then it would include a path uvw on the outer Hamiltonian cycle such that u' and w' are distance two apart on the Hamiltonian cycle. This would imply u and w are both q's, but then we see that the coloring of the 6-cycle is not a forbidden coloring. □

Remark 13. Note that Lemma 12 is not true for $k = 5$. Using `GraphData` in Mathematica [7], we find that, of the 21 cubic graphs with 10 vertices (19 of which are connected [2]), there are only five that are triangle-free and have edge-chromatic number 3. Moreover, it can be verified that all possible proper edge-colorings of these graphs lead to a forbidden configuration.

In Proposition 14, we show that for $n \geqslant 5$, the diagonal family $HG(n, n, n; 3)$ achieves the metric dimension lower bound $2n - 1$. To prove the cases $n \geqslant 7$, we make use of the graphs from Lemma 12, and to prove the cases $n = 5$ and $n = 6$ we use ad hoc constructions.

Proposition 14. *For $n \geqslant 5$, the Hamming graph $HG(n, n, n; 3)$ has metric dimension $2n - 1$.*

Proof. For $n \geqslant 7$, let $k = n - 1$ and let G be the graph constructed as in Lemma 12. Label the blue edges $1, \ldots, n - 1$ and likewise for the green and pink edges. If the blue, green, and pink edges incident to a vertex are labeled a_1, a_2, and a_3, respectively, then we create a landmark (a_1, a_2, a_3). Do this for each vertex in G to get a landmark set W with graph $\mathcal{G}(W) = G$. The order $2(n-1)$ graph G avoids forbidden 4-cycles and 6-cycles and rainbow triangles, so by Theorem 11, if we let $u = (n, n, n)$, then the set $W \cup \{u\}$ is a resolving set of size $2n-1$ for $HG(n, n, n; 3)$. By the lower bound in Theorem 2, $W \cup \{u\}$ is a minimum resolving set.

For $n = 5$, note that Lemma 12 is also true for $k = 4$. As illustrated in Figure 6, there is a proper 3-edge coloring of the order 8 Möbius ladder that avoids rainbow triangles, forbidden 4-cycles, and forbidden 6-cycles. Thus we can resolve $HG(5, 5, 5; 3)$ in the style of the previous paragraph, taking G to be the colored Möbius ladder.

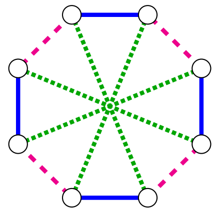


Figure 6: Proper 3-edge coloring of the order eight Möbius ladder M_4. Edges of outer Hamiltonian cycle alternate blue (solid) and pink (dashed), while green (dotted) edges join antipodal points on the cycle. All forbidden configurations are avoided.

For $n = 6$, we take a different approach, as Lemma 12 is not true for $k = 5$ (see Remark 13). In this case, rather than concentrating all loops at the same vertex, we spread them throughout the landmark graph. Letting symbol k in row i, column j correspond to a landmark (i, j, k), we can verify by computer that the partial Latin square

	1				
1	2				
		2	3		
		3	4		
				4	5
				5	6

represents a resolving set of size 11 for $HG(6, 6, 6; 3)$.[1] □

4.2 Small exceptions

In contrast to the diagonal cases $n \geqslant 5$, which can be resolved in $2n - 1$ landmarks, we will show that for $n = 3$ and $n = 4$, the metric dimension of $HG(n, n, n; 3)$ is $2n$.

To show that $HG(n, n, n; 3)$ cannot be resolved in $2n - 1$ landmarks for $n = 3, 4$, we need to consider all possible subsets of $2n - 1$ vertices, not only triple-looped landmark systems. With fewer constraints on W, there are more possible footprints than shown in Figure 2—in general, a footprint may have double edges, and loops do not all have to be at the same vertex. In turn, there are more ways for two different footprints to cover the same set of vertices in the landmark graph, and such configurations indicate pairs of unresolved vertices in the Hamming graph.

In addition to the forbidden 4-cycles and 6-cycles in Figure 3, the landmark graph of a resolving set has to avoid the configurations in Table 1. If any of the graphs in the "Forbidden Subgraph" column of Table 1 appear as subgraphs of the landmark graph $\mathcal{G}(W)$, then we can find two footprints that cover the same set of vertices, as illustrated

[1] Incidentally, our approach to constructing this resolving set of size 11 was to attempt to prove there was none. We failed and found this one! Our code for checking resolving sets with SageMath [17] is available at https://github.com/fostergreenwood/metric-dimension.

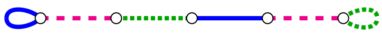
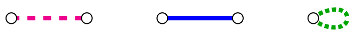
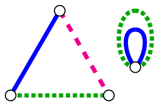
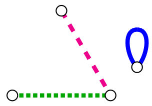
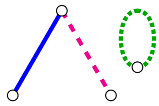
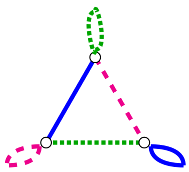
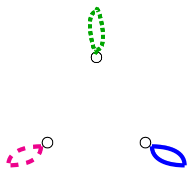
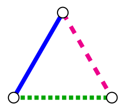
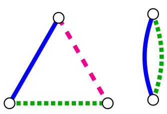
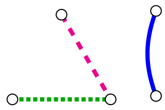
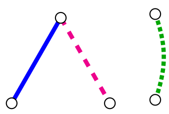
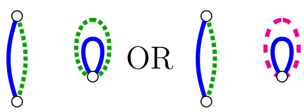
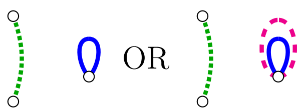
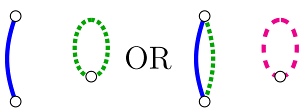
Forbidden Subgraph	**Footprint 1**	**Footprint 2**
Snake 		
Short Two-Headed Snake 		
Long Two-Headed Snake 		
Triangle ∪ Double Loop $C_3 \cup L_2$ 		
Loopy Triangle 		
Two Double Edges $2D_2$ 		
Triangle ∪ Double Edge $C_3 \cup D_2$ 		
Double Edge ∪ Double Loop $D_2 \cup L_2$ 		

Table 1: Forbidden subgraphs and two different footprints covering the same set of vertices. Different line styles correspond to different edge colors. Configurations with only two colors imply the addition of any edge of the third color will cause an issue.

in the “Footprint” columns of Table 1. In view of Remark 6, this would mean there is an unresolved pair of vertices in the graph $HG(n,n,n;3)$.

Note that the table is not exhaustive but includes what we need in order to prove Lemma 15 and Lemma 16. Of the configurations we require, we only show one representative coloring, with the understanding that the colors may be permuted. Note that which edges are the same color is significant; for example, in the long two-headed snake, it is important that the middle two colors match the loop colors on the end, but in the opposite order.

Lemma 15. *The Hamming graph $HG(3,3,3;3)$ has metric dimension 6.*

Proof. Letting symbol k in row i, column j correspond to a landmark (i,j,k), we can verify by computer that the partial Latin square

1	2	3
3	1	2

represents a resolving set of size 6 (whose landmark graph is in Figure 1).

To show there is no smaller resolving set, suppose, towards a contradiction, that we have a resolving set W with only 5 vertices. Then by Lemma 3, the graph of W has 2 plain edges and a loop for each of three colors (say blue, green, pink). We will show that all possible placements of the edges and loops lead to a forbidden configuration.

Case 1 (no double edges). First suppose the graph of W has no double edges.

Case 1.1 (blue and green loop on same vertex). If the blue and green loops are on the same vertex, say u, then the remaining two blue edges and two green edges are forced to create an alternating 4-cycle. Note that distinct vertices in the 4-cycle cannot be joined by a pink edge since that would create either a double edge or a forbidden $C_3 \cup L_2$. Thus every pink plain edge must have u as one of its endpoints. But this means we can only have one pink plain edge instead of the two required.

Case 1.2 (blue and green loop on different vertices). Suppose the graph of W has a blue loop at vertex b and a green loop at vertex g. Now there must be a green edge, say bu, and a blue edge, say gv. Note that if we had $u = v$, then the remaining blue and green edges would create a double edge. Thus we have $u \neq v$. Now, if the remaining vertex is labeled w, we are forced to have a blue edge wu and a green edge wv. We claim there is no way to place a pink at w. A loop at w gives short two-headed snakes (buw and gvw). A plain pink edge from w causes a double edge (uw or vw) or a snake ($buwg$ or $gvwb$).

Case 2 (double edge exists). Next, suppose the graph of W has a double edge. Without loss of generality, say it is a blue-green double edge between vertices u and v. Then, to avoid a forbidden $D_2 \cup L_2$, the blue and green loops must be on different vertices, say b and g. If we label the final vertex as w, the only way to place the remaining blue and green edges is with a green edge bw and a blue edge gw. We claim there is no way to place a pink at b. A plain pink edge bu or bv give us a snake with head g. A pink edge bw creates a second double edge. A pink loop at b creates a $D_2 \cup L_2$. A pink edge bg creates a forbidden $D_2 \cup C_3$.

Since all possibilities lead to forbidden configurations, there is no way to resolve the Hamming graph $HG(3,3,3;3)$ using only five landmarks. □

Lemma 16. *The Hamming graph* $HG(4,4,4;3)$ *has metric dimension* 8.

Proof. To construct a resolving set, let M_4 be the order 8 edge-colored Möbius ladder in Figure 6. Number the blue (likewise, green and pink) edges $1,\ldots,4$. If the blue, green, and pink edges incident to a vertex are labeled a_1, a_2, and a_3, respectively, then we create a landmark (a_1,a_2,a_3). Do this for each vertex in M_4 to get a landmark set W with graph $\mathcal{G}(W)=M_4$. By Theorem 9, the set W is a resolving set of size 8 for $HG(4,4,4;3)$.

To show there is no smaller resolving set, suppose, towards a contradiction, that we have a resolving set W with only 7 vertices. Then by Lemma 3, the graph of W has for each color three plain edges and a loop.

Case 1 (no double edges). First suppose the graph of W has no double edges.

Case 1.1 (blue and green loop on same vertex). Suppose the blue and green loops are on the same vertex, say u. Covering the six remaining vertices, we have three blue edges that make a perfect matching and three green edges that make a perfect matching. Since we are in the case of no double edges, it is straightforward to verify that the edges of the two matchings must combine to create an alternating blue-green 6-cycle.

Now we try to determine the pink edges. Note that distinct vertices in the 6-cycle cannot be joined by a pink edge since that would create either a double edge, a $C_3 \cup L_2$, or a forbidden 4-cycle. Thus every pink plain edge must have u as one of its endpoints. But this means we can only have one pink plain edge instead of the three required.

Case 1.2 (blue and green loops on different vertices). Suppose the graph of W has a blue loop at vertex b and a green loop at vertex g. Now there must be a green edge, say bu, and a blue edge, say gv.

Case 1.2.1 ($u=v$). If $u=v$, then the remaining blue and green edges create an alternating 4-cycle. Note that we cannot have a pink edge from b or g to a vertex on the 4-cycle, since that would create a snake with head g or b. We also cannot have a pink loop at b or g since that creates a short two-headed snake with underlying path bug. To avoid double edges, the only remaining option is a pink edge bg. Now, we cannot have a pink loop at u (since it would create a triangle with a loop at each vertex), so there must be a pink edge from u to a vertex, say w, on the blue-green 4-cycle. Since we can't have a double edge, the pink loop is forced to be at the vertex w' on the 4-cycle that is nonadjacent to w. But now there is a long two-headed snake (with heads b and w').

Case 1.2.2 ($u \neq v$). If $u \neq v$, then we have a blue edge, say uu' and a green edge vv'. Note that $u' \neq v'$ since $u'=v'$ would force a double edge with the remaining blue and green edges. There is only one vertex remaining, call it w, and we are forced to have a green edge wu' and a blue edge wv'.

Now we consider pink edges at u' and v'. Due to the path buu', we cannot have a pink loop at u' (would create a short two-headed snake), and the only way to avoid a double edge or a snake with head b is to put a pink edge $u'b$. Similarly, we must have a pink edge $v'g$. But now we have a long two-headed snake $bu'wv'g$.

Case 2 (there exists a double edge). By symmetry of permuting colors, it suffices to consider the case of a blue-green double edge, say between vertices u and v. Then the blue and green loops must be at distinct vertices, say b and g (otherwise we obtain $D_2 \cup L_2$). Now, there is a green edge from b to some vertex u' and a blue edge from g to some vertex v'. If $u' = v'$, we would be forced to make a blue-green double edge on the remaining two vertices which gives two double edges which is forbidden, so it must be that $u' \neq v'$. Similar to Case 1.2.2, there is only one vertex remaining, call it w, and we are forced to create a blue edge $u'w$ and a green edge $v'w$.

Now we consider pink edges. Note that at w, we cannot have a pink loop since we would get a short two-headed snake $bu'w$. We also cannot have a pink edge wu or wv since that would create a snake with head b or g. A pink edge wu' or wv' would create two double edges. Lastly, a pink edge wb or wg would create a $C_3 \cup D_2$. Thus there is no way to place a pink at w. This completes the case where the graph of W has a double edge.

Since all possibilities lead to forbidden configurations, there is no way to resolve the Hamming graph $HG(4, 4, 4; 3)$ using only seven landmarks. □

4.3 Conclusion and further directions

Compiling the results of the previous two subsections, we now have the metric dimension for all graphs in the diagonal family $HG(n, n, n; 3)$ for $n \geqslant 3$.

Theorem 17. *For $n \geqslant 3$, the metric dimension of the Hamming graph $HG(n, n, n; 3)$ is*

$$\dim HG(n, n, n; 3) = \begin{cases} 2n & \text{if } n \in \{3, 4\} \\ 2n - 1 & \text{if } n \geqslant 5. \end{cases}$$

Proof. This is a corollary of Proposition 14, Lemma 15, and Lemma 16. □

We are currently working to generalize the results of Section 3 and construct minimum resolving sets for non-diagonal graphs $HG(n_1, n_2, n_3; 3)$. These graphs are of special interest because when the n_i's are distinct odd primes, the graphs are isomorphic to unitary Cayley graphs [15]. We know that at least some of these graphs also achieve the lower bound of Theorem 2, as illustrated in our final example.

Example 18. Letting symbol k in row i, column j correspond to a landmark (i, j, k), we can check by computer that the partial Latin rectangle

1	2	3	10			
4	5	6	1	2	3	
7	8	9	4	5	6	
10			7	8	9	
						11

represents a minimum resolving set of size 21 for the graph $HG(5, 7, 11; 3)$.

Acknowledgements

We thank Ismael G. Yero for pointing us to related references. We thank an anonymous referee for suggestions that improved the clarity of our paper.

References

[1] Boštjan Brešar, Nicolas Gastineau, Sandi Klavžar, and Olivier Togni. Exact distance graphs of product graphs. *Graphs Combin.*, 35(6):1555–1569, 2019.

[2] F. C. Bussemaker, S. Čobeljić, D. M. Cvetković, and J. J. Seidel. Cubic graphs on $\leqslant 14$ vertices. *J. Combinatorial Theory Ser. B*, 23(2-3):234–235, 1977.

[3] José Cáceres, Carmen Hernando, Mercè Mora, Ignacio M. Pelayo, María L. Puertas, Carlos Seara, and David R. Wood. On the metric dimension of Cartesian products of graphs. *SIAM J. Discrete Math.*, 21(2):423–441, 2007.

[4] David G. Cantor and W. H. Mills. Determination of a subset from certain combinatorial properties. *Canadian J. Math.*, 18:42–48, 1966.

[5] Paul Erdős and Alfréd Rényi. On two problems of information theory. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 8:229–243, 1963.

[6] Frank Harary and Robert A. Melter. On the metric dimension of a graph. *Ars Combin.*, 2:191–195, 1976.

[7] Wolfram Research, Inc. Mathematica, Version 12.0. Champaign, IL, 2019.

[8] Zilin Jiang and Nikita Polyanskii. On the metric dimension of cartesian powers of a graph. *Journal of Combinatorial Theory, Series A*, 165:1–14, 2019.

[9] Ville Junnila, Tero Laihonen, and Tuomo Lehtilä. On a conjecture regarding identification in Hamming graphs. *Electron. J. Combin.*, 26(2):#P 2.45, 2019.

[10] Dorota Kuziak, Iztok Peterin, and Ismael G. Yero. Resolvability and strong resolvability in the direct product of graphs. *Results Math.*, 71(1-2):509–526, 2017.

[11] Dorota Kuziak and Ismael G. Yero. Metric dimension related parameters in graphs: A survey on combinatorial, computational and applied results, 2021. arXiv:2107.04877.

[12] Lucas Laird, Richard C. Tillquist, Stephen Becker, and Manuel E. Lladser. Resolvability of Hamming graphs. *SIAM J. Discrete Math.*, 34(4):2063–2081, 2020.

[13] Bernt Lindström. On a combinatory detection problem. I. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:195–207, 1964.

[14] Changhong Lu and Qingjie Ye. A bridge between the minimal doubly resolving set problem in (folded) hypercubes and the coin weighing problem. *Discrete Applied Mathematics*, 309:147–159, 2022.

[15] Torsten Sander. Eigenspaces of Hamming graphs and unitary Cayley graphs. *Ars Math. Contemp.*, 3(1):13–19, 2010.

[16] Peter J. Slater. Leaves of trees. In *Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975)*, pages 549–559. Congressus Numerantium, No. XIV, 1975.

[17] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.6)*, 2022. https://www.sagemath.org.

[18] Richard C. Tillquist, Rafael M. Frongillo, and Manuel E. Lladser. Getting the lay of the land in discrete space: a survey of metric dimension and its applications. *SIAM Rev.*, 65(4):919–962, 2023.