Hamilton Cycles in the Line Graph of a Random Hypergraph

Katarzyna Rybarczyk[a]

Submitted: Jun 30, 2022; Accepted: Jan 29, 2024; Published: Apr 19, 2024

Abstract

We establish the threshold function for the property that the line graph of a random r–uniform hypergraph has a Hamilton cycle. The main result gives also the threshold function for Hamiltonicity of uniform random intersection graphs with a bounded number of attributes assigned to each vertex. The problem is closely related to Berge Hamilton cycles in random r–uniform hypergraphs.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

A hypergraph is a pair $H = (V, E)$ where $V = V(H)$ is a non–empty vertex set and $E = E(H)$, called the edge set, is a collection of subsets of V. We will allow subsets to repeat in a collection E. A hypergraph is r–uniform (is an r–graph) if all of its edges are r–sets of V (subsets of V of cardinality r). If $r = 2$ then an r–graph is called simply a graph. We will study properties of a line graph of a random hypergraph. By $\mathcal{H}_r(n, M)$ we denote a random r–graph chosen uniformly at random from all r–graphs on n vertices ($V = \{1, 2, \ldots, n\}$) with exactly M distinct edges (chosen without replacement). Given a hypergraph $H = (V, E)$, a line graph of H (denoted by $L(H)$) is a graph with the vertex set E in which $\{e, e'\}$ is an edge whenever e and e' intersect in H. We will study Hamilton cycles in $L(\mathcal{H}_r(n, M))$. A cycle in a graph is an alternating sequence of distinct vertices and edges $(v_1, e_1, v_2, \ldots, v_n, e_n)$ in which $e_i = \{v_i, v_{i+1}\}$, for $i = 1, 2, \ldots, n$ (indices considered modulo n). A Hamilton cycle is a cycle which contains all vertices of the graph.

The problem of Hamilton cycles in random graphs has been one of the most inspiring problems in random graph theory. After a groundbreaking introduction of the rotation method by Pósa [15], numerous results concerning threshold functions for hamiltonicity of various random graphs have been obtained, starting with those by Ajtai, Komlós and Szemerédi [1], Bollobás [7], and Komlós and Szemerédi [12].

In this article we will be concentrating on the threshold function for Hamilton cycles in the line graph of random hypergraphs. This problem was studied before in the context

[a] Adam Mickiewicz University, Poznań, Poland (kryba@amu.edu.pl).

of random intersection graphs by Bloznelis and Radavičius [5] and Nikoletseas et al. [13]. Some partial results were obtained also by Rybarczyk in [17]. In the main theorem we generalise a result from [17] to the case of the line graph of a random hypergraph $\mathcal{H}_r(n, M)$ with $r \geqslant 3$. This result is particularly interesting, as the following theorem answers the questions raised in [5] and [13]. We explain it in more detail in Section 2.

Before formulating our result we mention that all limits in the paper are taken as $n \to \infty$. All inequalities are valid for n large enough. Throughout the paper we use standard asymptotic notation $a_n = o(b_n)$ for $a_n/b_n \to 0$ and $a_n = O(b_n)$ if $|a_n| < C|b_n|$ for all n and some constant $C > 0$. We also use a phrase *with high probability* to say that an event occurs with probability tending to 1 as $n \to \infty$.

Theorem 1. *Let $r \geqslant 2$ be an integer and $\mathcal{H}_r(n, M)$ be a random r-graph with*

$$M = \frac{n(\ln n + 2\ln\ln n - 2\ln r + c_n)}{r^2}, \qquad \text{and } c_n = o(\ln n). \tag{1}$$

Then

$$\Pr\{L(\mathcal{H}_r(n, M)) \text{ has a Hamilton cycle}\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ e^{-e^{-c}} & \text{for } c_n \to c; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

We should note here that Theorem 1 is not a hitting time result similar to those obtained in [2] or [7]. A hitting time result is not straight forwardly obtainable from the presented method of the proof, because hamiltonicity of the line graph is not a monotone property.

In the proof we will be studying subhypergraphs of $\mathcal{H}_r(n, M)$ and Hamilton cycles in them. A hypergraph $H' = (V', E')$ is a subhypergraph of $H = (V, E)$ (denoted $H' \subseteq H$) if $V' \subseteq V$ and $E' \subseteq E \cap V' := \{e \cap V' : e \in E\}$. Here E' is a collection of sets with possible repetitions. In what follows we will distinguish between the edges in H' which contain the same vertices, however are obtained from different edges of H. For any $V' \subseteq V$, $H' = (V', E')$ is the subhypergraph induced by V', if $E' = \{e \cap V' : e \in E, |e \cap V'| \geqslant 2\}$. $H' = (V', E')$ is a spanning subhypergraph of $H = (V, E)$ if $V' = V$.

There is a vast literature concerning threshold functions for Hamilton cycles in random hypergraphs. The subject is particularly intriguing due to the fact that there are various ways to define a cycle in a hypergraph: tight, loose, l-overlapping, Berge, weak Berge (see for example [2, 8, 9, 14]). We will be interested in Berge cycles. A Begre cycle is an alternating sequence of distinct vertices and distinct edges $(v_1, e_1, v_2 \ldots, v_k, e_k)$ in which $\{v_i, v_{i+1}\} \subseteq e_i$, for $i = 1, 2, \ldots, k$ (indices considered modulo k). A Berge cycle in a hypergraph is Hamilton if it contains all vertices of the hypergraph. A hypergraph with such a cycle is called Berge Hamiltonian. Berge Hamilton cycles in $\mathcal{H}_r(n, M)$ were studied by Bal, Berkowitz, Devlin, and Schacht [2]. In some parts of our reasoning we will rely on ideas from [2].

In the context of random graphs and hypergraphs, the minimum degree is crucial in considerations concerning Hamilton cycles. Let the degree of a vertex v, denoted by $\deg v$, be the number of edges containing v in a hypergraph. An obvious necessary condition for a hypergraph H to be Berge Hamiltonian is that each vertex has to have degree at least 2, i.e. minimum degree of H, $\delta(H) := \min_v \deg v$, should be at least 2.

The proof of Theorem 1 will proceed as follows. First for any H, instance of $\mathcal{H}_r(n, M)$, we will construct a subset of V denoted by V'. Then we will define H' to be the subhypergraph of $\mathcal{H}_r(n, M)$ induced by V'. We will show that with high probability $\mathcal{H}_r(n, M)$ is such that $\mathcal{H}_r(n, M)$ and H' have some anticipated properties.

In the next step we will use a modified technique from [2] to show that with high probability in $\mathcal{H}_r(n, M)$ either $\delta(L(\mathcal{H}_r(n, M))) < 2$ or H' (constructed based on $\mathcal{H}_r(n, M)$) has a Berge Hamilton cycle. Let H be an instance of $\mathcal{H}_r(n, M)$ with some anticipated properties. We will need to define some other subhypergraphs of H. First we will define Γ_0, a random subhypergraph of H. Then based on Γ_0 we will introduce Γ'_0, the subhypergraph of Γ_0 induced by V', and Γ^*_0, an auxiliary hypergraph. We will show that for H with some anticipated properties with high probability Γ'_0 has good expansion properties (for that we will introduce the notion of an almost expander in which we will use Γ^*_0). Moreover we will show that any subhypergraph of H' with good expansion properties is either Berge Hamiltonian itself or has a long path that may be lengthen by adding one edge from H'. By definition H' will have far more than n extra edges comparing to Γ'_0 and adding edges does not disrupt expansion properties. Therefore with high probability H' is Berge Hamiltonian for any instance H of $\mathcal{H}_r(n, M)$ with anticipated properties.

A Berge Hamilton cycle C in H' has its counterpart cycle C' in $L(\mathcal{H}_r(n, M))$. However C' is not necessarily a Hamilton cycle in $L(\mathcal{H}_r(n, M))$. The construction of H' will be such that with high probability each edge of $\mathcal{H}_r(n, M)$ intersects the vertex set of H'. All edges containing a common vertex in $\mathcal{H}_r(n, M)$ form a complete graph in $L(\mathcal{H}_r(n, M))$. Therefore an edge ee' in C' which is a counterpart of a fragment $\cdots eve' \cdots$ of C may be replaced in C' by any path $ee_1e_2 \cdots e_te'$ such that $v \in e_1, e_2, \ldots, e_t$. In this way, based on C and C', we may construct a Hamilton cycle in $L(\mathcal{H}_r(n, M))$.

Now it will be enough to determine the probability of the event $\{\delta(L(\mathcal{H}_r(n, M))) < 2\}$. For that we will use known results concerning random intersection graphs.

The article is organised as follows. In the following section we discuss the relation between Hamilton cycles in $L(\mathcal{H}_r(n, M))$ and Hamilton cycles in a random intersection graph. In particular, using known results concerning random intersection graphs, we state results concerning $\delta(L(\mathcal{H}_r(n, M)))$. Moreover in that section we present a theorem concerning Hamilton cycles in random intersection graphs that is derived from Theorem 1. Then, in Section 3, we present a lemma stating useful properties of $\mathcal{H}_r(n, M)$. Finally in Section 4 we show the constructions of subhypergraphs H', Γ, Γ' and hypergraph Γ^*. We show how to find a Hamilton Berge cycle in H'. Last but not least, in the last section, we present some technical details of the remaining proofs, omitted earlier from the main text for the clarity of presentation.

2 Random intersection graphs

Random intersection graphs have been attracting attention lately, mostly due to their numerous applications in theoretical computer science. For more on the subject of random intersection graphs we refer to survey papers [3, 4, 19] and references therein. The best studied random intersection graph model is binomial random intersection graph. Hamilton cycles in this model were studied by Efthymiou and Spirakis [10] and Rybarczyk [16, 18]. However in most cases those results are not transferable to other random intersection graph models.

Here we are discussing the problem of Hamilton cycles in uniform random intersection graph model $\mathcal{G}(M,n,r)$. In the uniform random intersection graph model $\mathcal{G}(M,n,r)$ there is a vertex set $\mathcal{V}$ ($|\mathcal{V}| = M$) and auxiliary attribute set $\mathcal{W}$ ($\mathcal{W} = \{1,2,\dots,n\}$). Each vertex $v \in \mathcal{V}$ is attributed an r–set $\mathcal{W}_v$, a random r–set of $\mathcal{W}$ chosen uniformly at random from all r–sets of $\mathcal{W}$, independently for each vertex from $\mathcal{V}$. Two vertices v, v' form an edge if their r–sets intersect, i.e. when $\mathcal{W}_v \cap \mathcal{W}_{v'} \neq \emptyset$.

The problem of Hamilton cycles in $\mathcal{G}(M,n,r)$ was first studied by Nikoletseas et. al. [13] however only very rough results were obtained. Already the result of Bloznelis and Radavičius [5] gave a far sharper result. They showed that if

$$M = \frac{1}{2}n\left(\ln n + \ln\ln n + \omega\right) \text{ and } \omega \to \infty,$$

then with high probability $\mathcal{G}(M,n,r)$ has a Hamilton cycle. Later Rybarczyk [17] managed to strengthen the result and showed that $\mathcal{G}(M,n,r)$ has Hamilton cycle with high probability if

$$M = \frac{1}{4}n\left(\ln n + 2\ln\ln n - 2\ln 2 + \omega\right) \text{ and } \omega \to \infty,$$

which was tight for $r = 2$. We will show that Theorem 1 with the result from [17] gives a sharp threshold function for the property of containing a Hamilton cycle in $\mathcal{G}(M,n,r)$ for $r = O(1)$.

Note that $\mathcal{G}(M,n,r)$ is in fact the line graph of an r-graph with n vertices and M edges chosen one by one with repetition from all r–sets of $\mathcal{W}$. We denote by $\mathcal{H}_r^{rep}(n,M)$ such a random r–graph with M edges chosen with repetition and we allow the edge set of $\mathcal{H}_r^{rep}(n,M)$ to be a collection of sets with possible repetitions. In the range of parameters considered in Theorem 1 the expected number of pairs of edges in $\mathcal{H}_r^{rep}(n,M)$ which have chosen the same r–set is

$$\binom{M}{2}\frac{1}{\binom{n}{r}} = o(1), \qquad \text{for } r \geqslant 3.$$

Moreover, given the fact that edges are distinct in $\mathcal{H}_r^{rep}(n,M)$, each choice of possible set of M edges is equiprobable in both models $\mathcal{H}_r(n,M)$ and $\mathcal{H}_r^{rep}(n,M)$. Therefore, in the case $r \geqslant 3$, the two models are asymptotically equivalent as far as $M = o(n^r)$ (here it is true on the threshold). Thus $\mathcal{G}(M,n,r)$ and $L(\mathcal{H}_r^{rep}(n,M))$ are the same models and, under assumptions of Theorem 1, $\mathcal{H}_r^{rep}(n,M)$ and $\mathcal{H}_r(n,M)$ have the same asymptotic properties. Therefore in the proofs we can use interchangeably $\mathcal{H}_r(n,M)$ and $\mathcal{H}_r^{rep}(n,M)$ if it facilitates the argument. Thus using Theorem 1 and the main result from [17] we get the following result.

Theorem 2. *Let $\mathcal{G}(M, n, r)$ be a uniform random intersection graph with $r \geqslant 2$, $r = O(1)$, and*

$$M = \frac{n(\ln n + 2\ln\ln n - 2\ln r + c_n)}{r^2}, \qquad \text{and } c_n = o(\ln n). \tag{2}$$

Then

$$\Pr\{\mathcal{G}(M, n, r) \text{ has a Hamilton cycle}\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ e^{-e^{-c}} & \text{for } c_n \to c; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

This gives the final answer to the problems considered in [5] and [13].

Now we may state the result concerning the necessary minimum degree condition for a Hamilton cycle in $L(\mathcal{H}_r(n, M))$. For that we use results concerning $\mathcal{G}(M, n, r)$. By Theorem 2 from [6] and the above discussion, we get the following lemma.

Lemma 3. *Let $\mathcal{H}_r(n, M)$ be a random r–graph with M given by* (1)*. Then*

$$\Pr\{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\} = \begin{cases} 0 & \text{for } c_n \to -\infty; \\ e^{-e^{-c}} & \text{for } c_n \to c; \\ 1 & \text{for } c_n \to \infty. \end{cases}$$

3 Properties of $\mathcal{H}_r(n, M)$

Before we proceed with presenting the most important properties of $\mathcal{H}_r(n, M)$ we need to introduce some additional notations.

Let H be an instance of $\mathcal{H}_r(n, M)$. We call a vertex *small* if its degree is at most $\varepsilon_r \ln n$ in H (where $\varepsilon_r = \varepsilon/r$ for a small constant $\varepsilon > 0$ to be determined later), *tiny* if its degree is exactly 2 in H, and *insignificant* if its degree is at most 1 in H. All vertices which are not *small* we call *large*. For convenience we will denote by V_s the set of *small* vertices. We will say that sets (or edges) meet if they have a non–empty intersection. They meet t times if they intersect on at least t vertices. For any edge e, its degree $\deg e$ is the number of edges (except e) meeting e. We call an edge e in H *slim* if $\deg e \leqslant \varepsilon \ln n$. We call an edge *irrelevant* if it has at most one *large* vertex.

We define a path in a hypergraph to be a sequence of alternating distinct edges and distinct vertices

$$e_1, v_1, e_2, v_2, \ldots, e_{l-1}, v_{l-1}, e_l, \qquad \text{or} \qquad v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l,$$

such that $v_0 \in e_1$, $v_l \in e_l$, and $v_i \in e_i \cap e_{i+1}$, for all $i = 1, 2, ..., l-1$. A path is called an edge path (or a vertex path, resp.) if it starts and ends with an edge (or a vertex, resp). The length of the path is the number of edges contained in it. Two vertices or edges are at distance l if the shortest path which connects them is of length l.

For a hypergraph H we define the following properties

A1 $\max_{v \in V(H)} \deg v \leqslant 8 \ln n$.

A2 For some constant a_r, depending on r, $0 < a_r < 1$ we have $|V_s| \leqslant n^{a_r}$.

A3 No two *irrelevant* edges meet at a *large* vertex.

A4 No two *slim* edges are at distance at most 6 and there is no edge path $eve'v'e''$ such that e is *slim* and $|e' \cap e''| \geqslant 2$.

A5 If $U \subseteq V$ has size at most $|U| \leqslant n/\ln^{1/2} n$, then there are at most $0.9|U| \log^{3/4} n$ edges of H that meet U more than once.

A6 For every pair of disjoint vertex sets U of size $|U| \leqslant n/\ln^{1/2} n$ and W of size $0.9|U| \ln^{1/4} n \leqslant |W| \leqslant |U| \ln^{1/4} n$, there are at most $0.4\varepsilon \ln n |U|$ edges of H meeting U exactly once and also meeting W.

A7 For every pair of disjoint vertex sets U, W of sizes $|U| = 0.9n/\ln^{1/2} n$ and $|W| = n/5$, there are at least $n \ln^{1/3} n$ edges of H meeting U exactly once and meeting W exactly $r-1$ times.

Lemma 4. *Let $\mathcal{H}_r(n, M)$ be a random r–graph with M given by* (1). *Then with high probability $\mathcal{H}_r(n, M)$ has properties **A1**–**A7**.*

The proof of Lemma 4 is standard. It is very similar to the proof of Lemma 2.3 [2], therefore we defer it to the Appendix.

4 Subhypergraphs of $\mathcal{H}_r(n, M)$

Now we will define H' – an induced subhypergraph of $\mathcal{H}_r(n, M)$. If $H = \mathcal{H}_r(n, M)$ does not have properties **A1**–**A7** or has an edge of degree smaller than 2 then we set H' to be H.

Now let $H = (V, E)$ be an instance of $\mathcal{H}_r(n, M)$ with properties **A1**–**A7** and each edge with degree at least 2. We will construct H' based on H. Given H, we first define $V' \subseteq V$, the vertex set of H'. First we add to V' all *large* vertices from V. Now we consider all edges which are *irrelevant* and do not meet any *large* vertex. These edges consist only of *small* vertices and, by consequence, they are *slim*. For each such edge, if it contains a vertex which is neither *tiny* nor *insignificant*, we pick one such vertex and add it to V' (in order for H' to be uniquely defined by H we may choose based on an ordering of V). Note that by **A4** no *slim* edges meet, thus we will not add two *small* vertices from the same *slim* edge. If an edge contains only *tiny* and *insignificant* vertices then we add to V' two *tiny* vertices from this edge. Recall that each edge has degree at least two thus there are always such two *tiny* vertices in an edge with only *tiny* and *insignificant* vertices. Now we set H' to be an induced spanning subhypergraph of H, with the vertex set V' and edge collection

$$E' = \{e \cap V' : e \in E \text{ and } |e \cap V'| \geqslant 2\}.$$

We recall here that in H' we distinguish between the edges coming from different edges of H, even if they consist of the same vertices.

Note that, by the construction, for each $e \in E$ we have $|e \cap V'| \geqslant 1$. Moreover $|e \cap V'| = 1$ only for *irrelevant* edges e with at least one vertex which is neither *tiny* nor *insignificant*. Therefore by properties **A3** and **A4** in H' (compared to H) the degree of each vertex drops at most by one. Moreover each vertex has degree at least 2, because no *insignificant* vertex has been added and no *tiny* vertex from V' has changed its degree. We recall that the definitions of *small*, *tiny* and *large* refer to the degrees of these vertices in H (not in H').

Let H be an instance of $\mathcal{H}_r(n, M)$ such that H has properties **A1**–**A7** and $\delta(L(H)) \geqslant 2$. Now we will prove that H' based on H with high probability has a Berge Hamilton cycle. To this end we will use a similar technique to this utilised in [2]. First, for H we define Γ_0, a random spanning subhypergraph of H. In Γ_0 every $v \notin V_s$ chooses E_v, a subset of $\varepsilon_r \ln n$ many edges uniformly at random from the set of all edges incident to v. Moreover, for every $v \in V_s$, we set E_v to be the set of all edges incident to v. Then the edge set of Γ_0 is

$$E(\Gamma_0) = \bigcup_{v \in V} E_v.$$

In what follows we will consider Γ_0', which is the subhypergraph of Γ_0 induced by V'. By definition, Γ_0' is a random spanning subhypergraph of H' with the edge collection $\{e \cap V' : e \in E(\Gamma_0), |e \cap V'| \geqslant 2\}$.

We define expander and booster as they were defined in [2].

Definition 5. A hypergraph is a (k, α)–expander if and only if, for all disjoint sets of vertices X and Y, if $|Y| < \alpha|X|$ and $|X| \leqslant k$, then there is an edge, e, such that $|e \cap X| = 1$ and $e \cap Y = \emptyset$.

Definition 6. For a hypergraph H, a booster is a non–edge of H such that either $H \cup e$ has a longer path than H or $H \cup e$ is Berge Hamiltonian.

We note that Γ_0' is not $(k, 2)$–expander for any $k \geqslant 2$. Let $X = \{v_1, v_2\}$ be the set of two *tiny* vertices contained in the same edge in Γ_0'. Let e_1 (resp. e_2) by the edge in Γ_0' containing v_1 (resp. v_2) different from edge $\{v_1, v_2\}$. Define a set Y that contains one vertex from $e_1 \setminus X$ and one vertex from $e_2 \setminus X$. Obviously $|Y| < 4$ and there is no edge e in Γ_0' such that $|e \cap X| = 1$ and $e \cap Y = \emptyset$. Therefore with high probability Γ_0' does not fit the definition of $(k, 2)$–expander for any $k \geqslant 2$. However we may use other properties of Γ_0' to show that with high probability Γ_0' either is Berge Hamiltonian or has many boosters.

Recall that we consider the case of H with properties **A1**–**A7** and the minimum edge degree at least 2. Then the edges constituted of two vertices of degree 2 in Γ_0' and in H' are at distance at least 6. This follows by **A4** and the fact that these edges were obtained from a *slim* edge by adding two *tiny* vertices from it to V'. For any spanning subhypergraph Γ' of H', $\Gamma' \subseteq H'$, we define $\Gamma^* = \Gamma^*(\Gamma')$ which is obtained in the following manner: any edge consisting of two vertices of degree 2 (in this case *tiny*) is contracted into one vertex. We denote by $V_s(\Gamma^*)$ the set of all vertices from Γ^* which are *small* in H plus all contracted ones.

Now we are ready to define a notion which will be helpful in describing hypergraphs with many boosters.

Definition 7. A hypergraph Γ' on n' vertices with no trivial edges we call an *almost expander* if

(i) there are at most $o(n')$ edges consisting of pairs of adjacent vertices of degree 2 in Γ' and they are at distance at least 6 from each other;

(ii) Γ^* obtained from Γ' is $(n_*/4, 2)$–expander, where n_* is the cardinality of the vertex set of Γ^*.

Now we may state the main lemmas which will lead us to the result concerning the Berge Hamiltonicity of H'.

Lemma 8. *For $\mathcal{H}_r(n, M)$ with parameters given by (1), with high probability either $\delta(L(\mathcal{H}_r(n, M))) < 2$ or Γ_0' is a connected almost expander.*

Lemma 9. *Let H' be constructed based on $\mathcal{H}_r(n, M)$. With high probability, for any spanning subhypergraph Γ', $\Gamma' \subseteq H' = H'$, which is a connected almost expander with at most $\varepsilon_r n \ln n + n$ edges, either Γ' is Berge Hamiltonian or it has at least one booster edge in H'.*

We defer the proofs of Lemmas 8 and 9 to the next section. Now we will show how Theorem 1 is derived from the lemmas and previously presented results.

Let B be the event that Γ_0', obtained from $\mathcal{H}_r(n, M)$, is a connected almost expander and H' satisfies the conclusions of Lemma 9. Note that, by Lemmas 8 and 9

$$\Pr\{B \cap \{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\}\} = \Pr\{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\} - o(1)$$

Now we show that, if for an instance H of $\mathcal{H}_r(n, M)$ event $B \cap \{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\}$ is satisfied, then H' (an instance of H' derived from H) has a Berge Hamilton cycle.

First consider any Γ which is obtained from Γ_0 by adding some edges from H, i.e. $\Gamma_0 \subseteq \Gamma \subseteq H$. Note that, by definition of Γ_0, none of the added edges contains any *tiny* (or *small*) vertex since these edges were added by default to Γ_0. Moreover they have the same edges consisting of two *tiny* vertices. Thus Γ_0^* has the same vertex set as Γ^* (related to Γ' – the subhypergraph of Γ induced by V') and Γ_0^* is a spanning subhypergraph of Γ^*. Therefore if Γ_0^* is $(n_*/4, 2)$–expander then Γ^* is also $(n_*/4, 2)$–expander. Hence, if Γ_0' is a connected almost expander then Γ' (a subhypergraph of Γ induced by V') is also a connected almost expander. Now we start with Γ_0' and iteratively step by step, if it is possible, add booster edges from H' to almost expanders obtained in this way. Note that we cannot add more than n edges without obtaining a Berge Hamilton cycle. Γ_0' did not have more that $\varepsilon_r n \ln n$ edges, therefore obtained almost expanders do not have more that $\varepsilon_r n \ln n + n$ edges. Therefore, as event B is satisfied, at some point of adding boosters we will get an almost expander Γ', $\Gamma_0' \subseteq \Gamma' \subseteq H'$, which is Berge Hamiltonian.

In conclusion, the probability that $L(\mathcal{H}_r(n, M))$ is Hamiltonian is at least the probability that H' is Berge Hamiltonian, which is at least $\Pr\{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\} - o(1)$. Recall that $\{\delta(L(\mathcal{H}_r(n, M))) \geqslant 2\}$ is a necessary condition for $L(\mathcal{H}_r(n, M))$ to be Hamiltonian. Therefore Theorem 1 follows by applying Lemma 3.

5 Proofs of Lemmas 8 and 9

In view of Lemma 4 and by the construction of H', in order to prove Lemma 8, it is enough to prove the following lemma.

Lemma 10. *If H has properties **A1**–**A7** and each edge in H has degree at least 2 then with high probability Γ_0^* is $(n_*/4, 2)$–expander, where n_* is the cardinality of the vertex set of Γ_0^*.*

Proof of Lemma 10. We will use the following lemma from [2](we have kept the numbering in line with [2]).

Lemma 11 (Lemma 2.4 from [2])**.** *If Γ^* on n_* vertices has the following properties*

P0 *All vertices in Γ^* have degree at least 2.*

P3 *Let $N = \{v \in V^* : \exists_{e\in E(\Gamma^*), v\in e} V_s(\Gamma^*) \cap e \neq \emptyset\}$. No edge meets $V_s(\Gamma^*)$ more than once, and no $u \notin V_s(\Gamma^*)$ lies in more than one edge meeting $N \setminus \{u\}$.*

P4 *If $U \subseteq V^*$ has size at most $|U| \leqslant n_*/\ln^{1/2} n_*$, then there are at most $|U|\log^{3/4} n_*$ edges of Γ^* that meet U more than once.*

P5 *For every pair of disjoint vertex sets U, W of sizes $|U| \leqslant n_*/\ln^{1/2} n_*$ and $|W| \leqslant |U|\ln^{1/4} n_*$, there are at most $\varepsilon \ln n_* |U|/2$ edges of Γ^* meeting U exactly once and also meeting W.*

P7 *For every pair of disjoint vertex sets U, W of sizes $|U| = n_*/\ln^{1/2} n_*$ and $|W| = n_*/4$, there is at least one edge of Γ^* meeting U exactly once and not meeting $V^* \setminus (W \cup U)$.*

then Γ^* is a connected $(n^*/4, 2)$–expander.

The lemma in [2] concerns r–graphs. However, after rephrasing **P7**, the lemma and the proof from [2] stay valid for non–uniform hypergraphs.

Assume that H has properties **A1**–**A7** and each edge in H has degree at least 2. We will show that then with high probability Γ_0^* has properties listed in Lemma 11. First of all, vertices from V which were not included in V' or were contracted in V^* are *small* in H. Therefore by **A2**

$$n \geqslant n_* \geqslant n - n^{a_r} = (1 + o(1))n.$$

P0 Follows by the following facts:
– the minimum edge degree in H is 2;
– no *insignificant* vertex has been added to H';
– all *tiny* vertices in H' and contracted *tiny* vertices in Γ_0^* have degree 2 in H', Γ_0' and, by consequence, in Γ_0^*;
– the degree of any vertex, that is not tiny, have decreased in H' (Γ_0', Γ_0^*) by at most one compared to H (by **A3** and **A4**).

P3 Recall that *small* vertices were included in H' only if they were in *slim* edges. From each *slim* edge we have included in H' either only *large* vertices or exactly one *small* vertex or exactly two *tiny* ones which were contracted into one in Γ_0^*. No two *slim* edges in H were at distance at most 6 (**A4**) therefore no two vertices from $V_s(\Gamma_0^*)$ in Γ_0^* are at distance at most 4. Thus in Γ_0^* no edge meets two vertices from $V_s(\Gamma_0^*)$ and no $u \notin N$ lies in more than one edge meeting N (otherwise there would be a path of length at most 4 between two vertices). Similarly, for any $u \in N \setminus V_s(\Gamma_0^*)$, if there were two edges meeting u and $N \setminus \{u\}$ then it would contradict either the first or the second part of **A4**.

P4 and **P5** follow easily by **A5** and **A6** and the fact that $n_* = (1 + o(1))n$.

P7 Let $U, V \subseteq V^*$ be disjoint sets in Γ_0^*. Sets U and W have been obtained from $U', W' \subseteq V'$ in Γ_0'. We note that $|U'| \geqslant n_*/\ln^{1/2} n^* > 0.9n/\ln^{1/2} n$ and $|W'| \geqslant n_*/4 > n/5$. Therefore by **A7** there are at least $n \ln^{1/3} n$ edges in H' (and also in H) meeting U' once and W' exactly $r-1$ times (i.e. not meeting $V' \setminus (U' \cup W')$). Therefore it is enough to prove that with high probability for every such U' and W' at least one of these edges remains in Γ_0' (i.e. also for U and W in Γ_0^*). This can be proved using exactly the same calculations as in the proof of **P7** in [2] (in these calculations we use **A1**). □

Now we are going towards proving Lemma 9.

Let H' be constructed based on $\mathcal{H}_r(n, M)$. Let $\Gamma' \subseteq H'$ be spanning subhypergraph of H' which is a connected almost expander. Let n' be the number of vertices of H'. Let P be the longest path in Γ' and let P^* be its counterpart in Γ^*. Note that if P is a longest path, then by properties of *tiny* vertices in a connected almost expander and definition of Γ^*, there is no longer path containing $V(P^*)$ than P^* in Γ^* (otherwise there would be a longer path in Γ').

For P^* we will use a consequence of a part of the proof of Lemma 2.1 [2] which we state as the following lemma.

Lemma 12. *Let G be a connected $(k, 2)$–expander and P be a path in G such that there is no path containing $V(P)$ longer than P. Then G is Berge Hamiltonian or there are at least k^2 distinct pairs of vertices (v, w) which are the ends of the paths (in G) with the vertex set $V(P)$.*

We will use Lemma 12 in the proof of the following Lemma.

Lemma 13. *Let Γ' be a subhypergraph of a complete hypergraph $K_r(n)$ with n vertices and all its r-sets as edges. If Γ' is a connected almost expander then either it is Berge Hamiltonian or it has at least $c_r(n')^2 n^{r-1}$ booster edges in $K_r(n)$, where n' is the number of vertices in Γ'.*

Proof. Assume that Γ' is not Hamiltonian. Let P be a longest path in Γ' and let P^* be the path corresponding to P in Γ^*. Γ' is a connected almost expander therefore Γ^* has $n^* = n'(1 + o(1))$ vertices and is $(n_*/4, 2)$–expander. By Lemma 12 there are (in Γ^*) at least $n_*^2/16 = (1 + o(1))(n')^2/16$ pairs of ends of paths with the vertex set $V(P^*)$. Thus there are at least $(1 + o(1))(n')^2/16$ pairs of ends of paths with the vertex set $V(P')$ in

Γ' (if a contracted vertex is the end in a path in Γ^* then exactly one vertex from the contracted vertices of degree two is the end in a corresponding path in Γ'). Now we will show that these paths give rise to many boosters.

Let (u, v) be a pair of vertices which are ends of a longest path P in $\Gamma' \subseteq H'$. Let e be an r–set in V such that $\{u, v\} \subseteq e$. If $e' = e \cap V'$ is contained in Γ' then it must be a part of P. Otherwise it would be closing a cycle $P \cup e'$. If P contains all vertices of Γ' then this cycle is a Hamilton cycle. If not, then there is a vertex w outside P connected by a path with the cycle (as Γ' is connected). Therefore we may construct a path longer than P using: the second last vertex of this path (i.e. the first one outside the cycle), an edge between it and the cycle, and the edges and vertices of the cycle. Therefore either Γ' has a Hamilton cycle or $e' = e \cap V'$ is a booster edge. Note that $e \subseteq V$ and Γ' has vertex set V'. However while constructing H' we have distinguished between the edges if they came from different r–sets of V. Therefore for any pair of ends (u, v) of a longest path P in Γ', each r–set e of V, such that $\{u, v\} \subseteq e$ and $e \cap V'$ is not an edge in P, is a booster edge in $K_r(n)$. Therefore there are at least $\binom{n-1}{r-1} - (n-1)$ booster edges in V containing (u, v) (and at most $\binom{n-1}{r-1} - 2$ booster edges for $r = 3$, as there might be at most 2 edges on the path that also contain $\{u, v\}$). Therefore, by Lemma 12 the number of boosters of a connected almost expander with no Berge Hamilton cycle is at least

$$\frac{1}{r(r-1)} \frac{(1 + o(1))(n')^2}{16} \left(\binom{n-2}{r-2} - (n-1) \right) \geqslant c'_r (n')^2 n^{r-2}, \text{ for } r > 3$$

and

$$\frac{1}{r(r-1)} \frac{(1 + o(1))(n')^2}{16} \left(\binom{n-2}{r-2} - 2 \right) \geqslant c'_r (n')^2 n^{r-2}, \text{ for } r = 3$$

for some constant $c'_r > 0$ depending only on r. □

Now we can conclude the proof of Lemma 9 using the following auxiliary result.

Lemma 14. *Let M be given by* (1) *and let c_r be a constant depending only on r. Then with high probability for any subhypergraph $\Gamma' \subseteq H' = H'$ with at least $c_r n^r$ boosters and at most $\varepsilon_r n \ln n + n$ edges, H' contains at least one of the booster edges of Γ'.*

The proof of Lemma 14 is analogous to that of Lemma 2.2 from [2]. We give it for completeness in the Appendix.

By definition of H', Lemma 4 and **A2**, with high probability H' has $(1 + o(1))n$ vertices. Therefore by Lemma 13 with high probability any spanning subhypergraph $\Gamma' \subseteq H'$ which is a connected almost expander has at least $c_r n^r$ boosters for some $c_r > 0$. This and Lemma 14 imply Lemma 9.

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi. First occurrence of Hamilton cycles in random graphs. In B.R. Alspach and C.D. Godsil, editors, *Annals of Discrete Mathematics (27): Cycles in Graphs*, volume 115 of *North-Holland Mathematics Studies*, pages 173–178. North-Holland, 1985.

[2] D. Bal, R. Berkowitz, P. Devlin, and M. Schacht. Hamiltonian Berge cycles in random hypergraphs. *Combinatorics, Probability and Computing*, 30(2):228-238, 2021.

[3] M. Bloznelis, E. Godehardt, J. Jaworski, V. Kurauskas, and K. Rybarczyk. Recent progress in complex network analysis: Models of random intersection graphs. In Berthold Lausen, Sabine Krolak-Schwerdt, and Matthias Böhmer, editors, *Data Science, Learning by Latent Structures, and Knowledge Discovery*, pages 69–78, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[4] M. Bloznelis, E. Godehardt, J. Jaworski, V. Kurauskas, and K. Rybarczyk. Recent progress in complex network analysis: Properties of random intersection graphs. In Berthold Lausen, Sabine Krolak-Schwerdt, and Matthias Böhmer, editors, *Data Science, Learning by Latent Structures, and Knowledge Discovery*, pages 79–88, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[5] M. Bloznelis and I. Radavičius. A note on Hamiltonicity of uniform random intersection graphs. *Lithuanian Mathematical Journal*, 51(2):155, 2011.

[6] M. Bloznelis and K. Rybarczyk. k-connectivity of uniform s-intersection graphs. *Discrete Mathematics*, 333:94–100, 2014.

[7] B. Bollobás. *Graph theory and combinatorics (Cambridge,1983)*, chapter The evolution of sparse graphs, pages 35–57. Academic Press, London, 1984.

[8] A. Dudek and A. Frieze. Loose Hamilton cycles in random uniform hypergraphs. *The Electronic Journal of Combinatorics*, 18(1):#P48, 2011.

[9] A. Dudek and A. Frieze. Tight Hamilton cycles in random uniform hypergraphs. *Random Structures & Algorithms*, 42(3):374–385, 2013.

[10] C. Efthymiou and P. G. Spirakis. Sharp thresholds for hamiltonicity in random intersection graphs. *Theoretical Computer Science*, 411(40):3714–3730, 2010.

[11] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs*. Wiley, 2001.

[12] J. Komlós and E. Szemerédi. Limit distribution for the existence of hamiltonian cycles in a random graph. *Discrete Mathematics*, 43(1):55–63, 1983.

[13] S. Nikoletseas, C. Raptopoulos, and P.G. Spirakis. On the independence number and Hamiltonicity of uniform random intersection graphs. *Theoretical Computer Science*, 412(48):6750–6760, 2011.

[14] O. Parczyk and Y. Person. Spanning structures and universality in sparse hypergraphs. *Random Structures & Algorithms*, 49(4):819–844, 2016.

[15] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Mathematics*, 14(4):359–364, 1976.

[16] K. Rybarczyk. The coupling method for inhomogeneous random intersection graphs. *The Electronic Journal of Combinatorics*, 24(2):#P2.10, 2017.

[17] K. Rybarczyk. Hamilton cycles in the line graph of a random graph. *Electronic Notes in Discrete Mathematics*, 61:1027–1031, 2017. The European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB'17).

[18] Katarzyna Rybarczyk. Finding Hamilton cycles in random intersection graphs. *Discrete Mathematics & Theoretical Computer Science*, Vol. 20 no. 1, March 2018.

[19] P. Spirakis, S. Nikoletseas, and Ch. Raptopoulos. A guided tour in random intersection graphs. In F.V. Fomin, R. Freivalds, M. Kwiatkowska, and D. Peleg, editors, *Automata, Languages, and Programming. ICALP 2013.*, volume 7966 of *Lecture Notes in Computer Science*, pages 29–35, Springer, Berlin, Heidelberg, 2013.

Appendix

A Proof of Lemma 4

In the proofs we will use Chernoff's bound (see for example Theorem 2.1 in [11]). Let X be a random variable with a binomial distribution and let $\varphi(x) = x\ln x + 1 - x$. Then

$$\Pr\{X \leqslant x\mathbb{E}X\} \leqslant \exp(-\varphi(x)\mathbb{E}X), \text{ for } x < 1$$

and

$$\Pr\{X \geqslant x\mathbb{E}X\} \leqslant \exp(-\varphi(x)\mathbb{E}X) \text{ for } x > 1.$$

Note that $\varphi(x_n) = (1+o(1))x_n \ln x_n$ for $x_n \to \infty$ and $\varphi(x_n) \to 1$ if $x_n \to 0$.

We will set $0 < \varepsilon < 1$ to be a constant such such that $\varphi(\varepsilon) > 0.8$ and $\varepsilon < 1/(2r)$.

A1 In $\mathcal{H}_r^{rep}(n, M)$ $\deg v$ has the binomial distribution with parameters M and $\binom{n-1}{r-1}/\binom{n}{r} = r/n$. Therefore

$$\Pr\{\exists_v \deg v > 8\ln n\} \leqslant n\binom{M}{8\ln n}\left(\frac{r}{n}\right)^{8\ln n} \leqslant n\left(\frac{eMr}{8n\ln n}\right)^{8\ln n} < n\left(\frac{e}{7r}\right)^{8\ln n} = o(1).$$

A2 We recall that, in $\mathcal{H}_r^{rep}(n, M)$, $\deg v$ has binomial distribution with the expected value

$$M\frac{r}{n} = (1+o(1))\frac{\ln n}{r}.$$

Therefore, by Chernoff's bound, the expected value of the number of *small* vertices is (recall that $\varphi(\varepsilon) > 0.8$ is a constant)

$$n\Pr\{\deg v \leqslant \varepsilon \ln n/r\} \leqslant n\exp(-0.8\ln n/r) = n^{1-0.8r^{-1}}. \tag{3}$$

Therefore with high probability $|V_s| \leqslant n^{a_r}$, for some constant $0 < a_r < 1$ depending on r.

A3 Let e and e' be two *irrelevant* edges such that $|e \cap e'| = i \geqslant 1$ which meet at a *large* vertex. There are $2r - i - 1$ *small* vertices in $e \cup e'$, i.e. there are at most

$$\varepsilon_r(2r - i - 1)\ln n = \varepsilon \cdot 2\left(1 - \frac{i+1}{2r}\right)\ln n$$

edges meeting these $2r - i - 1$ *small* vertices from $(e \cup e') \cap V_s$. Moreover the number of edges meeting these vertices (not counting e and e') in $\mathcal{H}_r^{rep}(n, M)$ is binomial with the expected value

$$\begin{aligned}(M-2)\left(1 - \binom{n-2r+i+1}{r}/\binom{n}{r}\right) &= (1+o(1))M\frac{r(2r-i-1)}{n} \\ &= (1+o(1))2\left(1 - \frac{i+1}{2r}\right)\ln n.\end{aligned}$$

Using Chernoff's bound (recall that $\varphi(\varepsilon) > 0.8$ is a constant) we get that probability that such a configuration of *irrelevant* edges exists in $\mathcal{H}_r^{rep}(n, M)$ (for $r \geqslant 3$) is at most

$$
\begin{aligned}
&M^2 \sum_{i=1}^{r} \frac{\binom{r}{i} i \binom{n-r}{r-i}}{\binom{n}{r}} \exp\left(-(1+o(1))2\varphi(\varepsilon)\left(1 - \frac{i+1}{2r}\right)\ln n\right) \\
&\leqslant \sum_{i=1}^{r} M^2 \frac{ir^{2i}}{n^i} n^{-1.6(1-(i+1)/(2r))} \\
&\leqslant \sum_{i=1}^{r} M^2 \frac{r^{2i+1}}{n^i} n^{-1.6(1-(2r)^{-1})} n^{i \cdot 0.8 r^{-1}} \\
&\leqslant \sum_{i=1}^{r} r \frac{M^2}{n^{1.33}} \left(\frac{r^2}{n^{0.73}}\right)^i = o(1).
\end{aligned}
$$

A4 First we will prove that no two *slim* edges meet. The proof is analogous to this of **A3**. Let e and e' be *slim* and $|e \cap e'| = i \geqslant 1$. Then the number of edges (not counting e and e') meeting $e \cup e'$ is at most $2\varepsilon \ln n$. Moreover in $\mathcal{H}_r^{rep}(n, M)$ the number of edges (not counting e and e') meeting $e \cup e'$ has binomial distribution with the expected value

$$
(M-2)\left(1 - \frac{\binom{n-|e\cup e'|}{r}}{\binom{n}{r}}\right) = (1+o(1))M\frac{(2r-i)\,r}{n} = (1+o(1))2\left(1 - \frac{i}{2r}\right)\ln n. \tag{4}
$$

Using Chernoff's bound we get that the probability that there are two *slim* edges that meet is at most (as in **A3**)

$$
M^2 \sum_{i=1}^{r} \frac{\binom{r}{i}\binom{n-r}{r-i}}{\binom{n}{r}} \exp\left(-(1+o(1))2\varphi(\varepsilon)\left(1 - \frac{i}{2r}\right)\ln n\right) = o(1)
$$

Now we will determine the expected number of edge paths

$$
e_1, v_1, e_2, \ldots v_{l-1}, e_l \quad \text{with } l = 3,4,5,6;\; e_1 \cap e_l = \emptyset;\; e_1, e_l \in V_s.
$$

The number of edges meeting both e_1 and e_l (not counting $e_1, e_2, \ldots, e_l$) is binomially distributed with the expected value $(1+o(1))2\ln n$ (similarly as in (4))). Moreover if $e_1, e_l \in V_s$ then the number of edges meeting both e_1 and e_l (not counting $e_1, e_2, \ldots, e_l$) is at most $2\varepsilon \ln n$. Therefore, using Chernoff's bound as before, the expected number of considered paths in $\mathcal{H}_r^{rep}(n, M)$ is at most

$$
\begin{aligned}
&\sum_{l=2}^{6} M^l n^{l-1} \left(\frac{\binom{n}{r-1}}{\binom{n}{r}}\right)^2 \left(\frac{\binom{n}{r-2}}{\binom{n}{r}}\right)^{l-2} \exp\left(-\varphi(\varepsilon)(1+o(1))2\ln n\right) \\
&\leqslant \sum_{l=2}^{6} (n \ln n)^l n^{l-1} \frac{1}{n^{2l-2}} n^{-1.6} = \sum_{l=2}^{6} \frac{\ln^l n}{n^{0.6}} = o(1).
\end{aligned}
$$

Now we prove the second part of **A4**. Note that in the path $eve'v'e''$ considered in **A4**, edge e is *slim*, thus it meets at most $\varepsilon \ln n$ other edges. In $\mathcal{H}_r^{rep}(n, M)$ the number of edges (not counting e,e', and e'') meeting e is binomial with the expected value

$$(M-3)\left(1-\frac{\binom{n-r}{r}}{\binom{n}{r}}\right) = (1+o(1))\frac{Mr^2}{n} = (1+o(1))\ln M.$$

Therefore, using standard counting and Chernoff's bound (recall that $\varphi(\varepsilon) > 0.8$ is a constant), we get that in $\mathcal{H}_r^{rep}(n, M)$ the expected number of edge paths $eve'v'e''$ with $e \in V_s$ and $|e' \cap e''| \geqslant 2$ is at most

$$M^3 e^{-0.8 \ln M} r \frac{\binom{n-1}{r-1}}{\binom{n}{r}} \binom{r}{2} \frac{\binom{n-2}{r-2}}{\binom{n}{r}} \leqslant M^{2.2}\frac{r^6}{n^3} = o(1).$$

A5 Given U, $|U| = u$, the number of edges in $\mathcal{H}_r^{rep}(n, M)$ which meet U at least two times is dominated by the binomial random variable Y_u with the expected value

$$\mathbb{E}Y_u = M\frac{\binom{u}{2}\binom{n}{r-2}}{\binom{n}{r}} = (1+o(1))M\frac{u^2}{2}\frac{r(r-1)}{n^2} = (1+o(1))\frac{u^2}{2n}\frac{r-1}{r}\frac{Mr^2}{n} = C_r\frac{u^2\ln n}{n},$$

for some bounded C_r.
We will use the fact that for $u \leqslant n/\ln^{1/2} n$ we have $\ln(n/(u\ln^{1/4} n)) > (\ln(n/u))/2$. Thus by Chernoff's bound

$$\begin{aligned}\Pr\left\{Y_u \geqslant 0.9u\ln^{3/4} n\right\} &\leqslant \exp\left(-\varphi\left(\frac{0.9n}{C_r u \ln^{1/4} n}\right) C_r \frac{u^2 \ln n}{n}\right)\\ &\leqslant \exp\left(-(1+o(1))0.9u\ln^{3/4} n \ln\left(0.9C_r^{-1}\frac{n}{u\ln^{1/4} n}\right)\right)\\ &\leqslant \exp\left(-(1+o(1))0.9u\ln^{3/4} n \frac{\ln\frac{n}{u}}{2}\right)\\ &\leqslant \exp\left(-0.4u\ln^{3/4} n \ln\frac{n}{u}\right).\end{aligned}$$

Thus the expected number of sets U such that many edges meet more than two vertices from U is at most

$$\begin{aligned}&\sum_{u=2}^{n/\ln^{1/2} n}\binom{n}{u}\exp\left(-0.4u\ln^{3/4} n\ln\frac{n}{u}\right)\\ \leqslant &\sum_{u=2}^{n/\ln^{1/2} n}\left(\frac{en}{u}\right)^u \exp\left(-0.4u\ln^{3/4} n\ln\frac{n}{u}\right)\end{aligned}$$

$$\leqslant \sum_{u=2}^{n/\ln^{1/2} n} \exp\left(u\left(-0.4\ln^{3/4} n \ln\frac{n}{u} + 1 + \ln\frac{n}{u}\right)\right)$$
$$\leqslant \sum_{u=2}^{n/\ln^{1/2} n} \exp\left(u\left(-0.3\ln^{3/4} n \ln\frac{n}{u}\right)\right) = o(1).$$

A6 This proof is almost identical to the proof of **A5**. Given disjoint U and W ($|U| = u$ and $|W| = w$), the number of edges in $\mathcal{H}_r^{rep}(n, M)$ which meet U exactly once and meet W is dominated by the binomial random variable Y_{uw} with the expected value

$$\mathbb{E}Y_{uw} = M\frac{uw\binom{n}{r-2}}{\binom{n}{r}} = C_r uw\frac{\ln n}{n}.$$

Thus for $u \leqslant n/\ln^{1/2} n$, $0.9u\ln^{1/4} n \leqslant w \leqslant u\ln^{1/4} n$, we get

$$\frac{0.4\varepsilon u\ln n}{\mathbb{E}Y_{uw}} = C_{r,\varepsilon}\frac{n}{w}, \quad \text{for some boundend } C_{r,\varepsilon} > 0.$$

Moreover, similarly as in **A5**, we have $\ln(C_{r,\varepsilon}\frac{n}{w}) > (1 + o(1))\frac{1}{2}\ln\frac{n}{u}$ for any w such that $0.9u\ln^{1/4} n \leqslant w \leqslant u\ln^{1/4} n$. Therefore by Chernoff's bound we have

$$\begin{aligned}
\Pr\{Y_{uw} \geqslant 0.4\varepsilon u\ln n\} &\leqslant \exp\left(-\varphi\left(C_{r,\varepsilon}\frac{n}{w}\right)C_r uw\frac{\ln n}{n}\right)\\
&\leqslant \exp\left(-(1+o(1))C_r\cdot C_{r,\varepsilon}u\ln n\ln\left(C_{r,\varepsilon}\frac{n}{w}\right)\right)\\
&\leqslant \exp\left(-A_{r,\varepsilon}u\ln n\ln\left(\frac{n}{u}\right)\right),
\end{aligned}$$

for some constat $A_{r,\varepsilon} > 0$. Thus the expected number of disjoint sets U and W ($|U| = u$ and $|W| = w$) with the small number of edges meeting both of them is at most (note that $|W| < n/2$)

$$\sum_{u=1}^{n/\ln^{1/2} n}\sum_{w=0.9u\ln^{1/4}}^{u\ln^{1/4}}\binom{n}{u}\binom{n}{w}\exp\left(-A_{r,\varepsilon}u\ln n\ln\left(\frac{n}{u}\right)\right)$$
$$\sum_{u=1}^{n/\ln^{1/2} n}\sum_{w=0.9u\ln^{1/4}}^{u\ln^{1/4}}\binom{n}{u\ln^{1/4} n}^2\exp\left(-A_{r,\varepsilon}u\ln n\ln\left(\frac{n}{u}\right)\right)$$
$$\leqslant \sum_{u=1}^{n/\ln^{1/2} n} n\exp\left(u\left(-A_{r,\varepsilon}\ln n\ln\left(\frac{n}{u}\right) - (1/2)\ln\ln n + 2\ln\frac{n}{u}\right)\right)$$
$$\leqslant \sum_{u=1}^{n/\ln^{1/2} n} n\exp\left(u\left(-0.9A_{r,\varepsilon}\ln n\ln\left(\frac{n}{u}\right)\right)\right) = o(1).$$

A7 Let Y_{uw} be the number of edges in $\mathcal{H}_r^{rep}(n, M)$ meeting U at one vertex and W at $r-1$ vertices (U and W disjoint, $|U| = u := 0.9n/\ln^{1/2} n$ and $|W| = w := n/5$). Y_{uw} is a binomial random variable with

$$\mathbb{E}Y_{uw} = M\frac{u\binom{w}{r-1}}{\binom{n}{r}} = (1+o(1))M\frac{ruw^{r-1}}{n^r} = C_r n \ln^{1/2} n.$$

Therefore

$$\Pr\left\{\mathbb{E}Y_{uw} \leqslant n\ln^{1/3} n\right\} \leqslant \exp\left(-\varphi((C_r \ln^{1/6} n)^{-1})C_r n \ln^{1/2} n\right) \leqslant \exp\left(-C_r'' n \ln^{1/2} n\right)$$

and the expected number of pairs of sets U and W with small number of intersecting edges is at most

$$\begin{aligned}\binom{n}{u}\binom{n}{w}\exp\left(-C_r'' n\ln^{1/2} n\right) &\leqslant \left(0.9^{-1} e \ln^{1/2} n\right)^{0.9n/\ln^{1/2} n} (5e)^{n/5} \exp\left(-C_r'' n \ln^{1/2} n\right)\\ &\leqslant \exp\left(An\ln\ln n \ln^{-1/2} n + A'n - C_r'' n \ln^{1/2} n\right)\\ &= o(1).\end{aligned}$$

B Proof of Lemma 14

Let $N = \binom{n}{r}$ and $\gamma = \varepsilon_r n \ln n + n$. Recall that ε is such that $\varepsilon_r < 1/(2r^2)$, thus $M - \gamma > M/3$.

Probability that there is a subhypergraph $\Gamma' \subseteq H'$ with γ edges and such that none of its $c_r n^r = c_r' N$ boosters is present in H' is at most

$$\begin{aligned}\sum_{i\leqslant\gamma}\frac{\binom{N}{i}\binom{N-i-c_r'N}{M-i}}{\binom{N}{M}} &\leqslant \sum_{i\leqslant\gamma}\exp\left(\frac{-c_r' N(M-i)}{(N-i)}\right)\frac{\binom{N}{i}\binom{N-i}{M-i}}{\binom{N}{M}}\\ &\leqslant \sum_{i\leqslant\gamma}\exp\left(-c_r'(M-\gamma)\right)\binom{N}{i}\left(\frac{M}{N}\right)^i\\ &\leqslant \sum_{i\leqslant\gamma}\exp\left(-c_r' M/3\right)\left(\frac{eM}{i}\right)^i\\ &\leqslant \gamma\exp\left(-c_r' M/3\right)\left(\frac{eM}{\gamma}\right)^\gamma = o(1).\end{aligned}$$