# Robinson-Schensted-Knuth Superinsertion and Super Frobenius Formulae

Deke Zhao

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**Abstract**

In this paper we extend the Robinson-Schensted-Knuth (RSK) superinsertion algorithm to hook-multipartitions and derive the super Frobenius formula for the characters of cyclotomic Hecke algebras via the RSK superinsertion algorithm. As a corollary, we obtain a new proof of Mitsuhashi's super Frobenius formula for the characters of Iwahori-Hecke algebras.

**Mathematics Subject Classifications:** 05E10, 05E05, 20C99, 20C15

## 1 Introduction

In [8], Frobenius gave a formula for computing the characters of symmetric groups, which is often referred as the Frobenius formula. In his study of representations of the general linear groups, Schur [21,22] showed the Frobenius formula can be obtained by the classical Schur-Weyl reciprocity. Inspired by Schur's classical work, Ram [16] got the Frobenius formula for the characters of Iwahori-Hecke algebras of type $A$ by applying the Schur-Weyl reciprocity established by Jimbo [10]; Shoji [24] obtained the Frobenius formula for the characters of cyclotomic-Hecke algebras by making use of the Schur-Weyl reciprocity given by Sakamoto-Shoji [20]; Mitsuhashi [14] showed the super Frobenius formula for the characters of Iwahori-Hecke algebras of type $A$ by using the Schur-Weyl reciprocity between the Iwahori-Hecke algebras of type $A$ and the quantum superalgebra [13, 15]; Recently the author [26] obtained the super Frobenius formulas for the characters of cyclotomic Hecke algebras by employing the super Schur-Weyl reciprocity between the cyclotomic Hecke algebra and the quantum superalgebra proved in [25].

In [17], Ram observed that the Roichman formula [18] and his Frobenius formula can be acquired by applying the Robinson-Schensted-Knuth (RSK) insertion algorithm. Following Ram's *loc. cit.* work, Cantrell et. al. [5] provided a new proof of Shoji's Frobenius formula via the modified RSK insertion algorithm for multipartitions. Note

School of Applied Mathematics, Beijing Normal University at Zhuhai, Zhuhai, China (deke@amss.ac.cn).

that there are super-analogues of RSK insertion algorithm for hook partitions (see e.g. [3, §2.5], [7, §4]). It is natural to ask whether the super Frobenius formulae in [14,26] can be showed by adapting the arguments of [5,17].

The purpose of the paper is to derive the super Frobenius formulae in terms of weights on standard tableaux by applying the modified RSK superinsertion algorithm for hook (multi)partitions. A point should be note that our strategy is very similar to the one in [5,17], that is, let $\mathscr{P}_{m,n}$ be the set of multipartitions of $n$, for $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$, we first define the $\boldsymbol{\mu}$-weight of parity-integer sequences. Therefore we can rewrite the super Frobenius formula as a sum over $\boldsymbol{\mu}$-weighted parity-integer sequences (see Corollary 13). Secondly, we introduce the RSK superinsertion algorithm for hook multipartitions, which gives a bijection between pairs of specific tableaux and parity-integer sequences (see §??). We then define the $\boldsymbol{\mu}$-weight $\mathrm{wt}_{\boldsymbol{\mu}}$ of standard tableaux, which is exactly the $\boldsymbol{\mu}$-weight of parity-integer sequences up to the RSK superinsertion algorithm (see Lemma 23). Thus we arrive at the super Frobenius formula for the characters of cyclotomic Hecke algebras (see Theorem 24)

$$q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \left( \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) \right) S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}),$$

where $\mathrm{std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux and $S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y})$ is the supersymmetric Schur function associated to $\boldsymbol{\lambda}$. As a direct application, we derive a formula to compute the irreducible characters of cyclotomic Hecke algebras as a sum of $\boldsymbol{\mu}$-weights over standard tableaux (see Corollary 26).

Let us remark that if $m = 1$ then the super Frobenius formula for the characters of cyclotomic Hecke algebras is Mitsuhashi's super Frobenius formula for the characters of Iwahori-Hecke algebras of type $A$, therefore we obtain a new proof of Mitsuhashi's super Frobenius formula (see Remark 25), which is a super-extension of Ram's work [17]. On the other hand, if we let $q = 1$ and $\boldsymbol{Q} = (\varsigma, \varsigma^2, \ldots, \varsigma^m)$, where $\varsigma$ is a fixed primitive $m$-th root of unity, then the super Frobenius formula gives the super Frobenius formula for the complex reflection group $W_{m,n}$(see [26, Remark 4.12]). So we derive a formula to compute the irreducible characters of complex reflection group $W_{m,n}$ in term of $\boldsymbol{\mu}$-weights over standard tableaux (see Remark 27).

The layout of the paper is as follows. In Section 2 we review briefly the cyclotomic Hecke algebras and fix our notations on (multi)partitions. Section 3 devotes to introduce the supersymmetric Schur functions and power sum supersymmetric functions associated to multipartitions. In particular, we present the super Frobenius formula for the characters of cyclotomic Hecke algebras. We define the ($\boldsymbol{\mu}$-)weighted parity-integer sequences and give a combinatorial version of the Frobenius formula in Section 4. The last section deals with the RSK superinsertion for hook-(multi)partitions and presents the new proof of the super Frobenius formula via the RSK superinsertion algorithm for multipartitions.

Throughout the paper, $\mathbb{K} = \mathbb{C}(q, \boldsymbol{Q})$ is the field of rational function in indeterminates $q$ and $\boldsymbol{Q} = (Q_1, \ldots, Q_m)$; $k_1, \ldots, k_m$, $\ell_1, \ldots, \ell_m$ are non-negative integers with $\sum_{i=1}^m k_i = k > 0$, $\sum_{i=1}^m \ell_i = \ell \geqslant 0$. We denote by $\boldsymbol{k} = (k_1, \ldots, k_m)$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m)$ and let $d_i = d_{i-1} + k_i + \ell_i$ ($d_0 = 0$) for $i = 1, \ldots, m$.

For $i = 1, \dots, k+\ell$, we define the *parity function* $i \mapsto \bar{i}$ by setting

$$\bar{i} = \begin{cases} \bar{0}, & \text{if } d_{a-1} < i \leqslant d_{a-1} + k_a \text{ for some } 1 \leqslant a \leqslant m; \\ \bar{1}, & \text{if } d_a - \ell_a < i \leqslant d_a \text{ for some } 1 \leqslant a \leqslant m; \end{cases}$$

we define the *color* of $i$ is $a$ when $d_{a-1} < i \leqslant d_a$ $(1 \leqslant a \leqslant m)$ and write $c(i) = a$.

## 2 Cyclotomic Hecke algebras

In this section we briefly review some facts about cyclotomic Hecke algebras for latter using and fix our notations on (multi)partitions.

Let $W_{m,n}$ be the complex reflection group of type $G(m,1,n)$. According to [23], $W_{m,n}$ is generated by $s_0, s_1, \dots, s_{n-1}$ with relations

$$\begin{aligned} & s_0^m = 1, \quad s_1^2 = \cdots = s_{n-1}^2 = 1, && \\ & s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, && \\ & s_i s_j = s_j s_i, && \text{if } |i-j| > 1, \\ & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, && \text{for } 1 \leqslant i < n-1. \end{aligned}$$

It is well-known that $W_{m,n} \cong (\mathbb{Z}/m\boldsymbol{z})^n \rtimes \mathfrak{S}_n$, where $s_1, \dots, s_{n-1}$ are generators of the symmetric group $\mathfrak{S}_n$ of degree $n$ corresponding to transpositions $(1\,2)$, …, $(n-1\,n)$. For $a = 1, \dots, n$, let $t_a = s_{a-1} \cdots s_1 s_0 s_1 \cdots s_{a-1}$. Then $t_1, \cdots, t_n$ are generators of $(\mathbb{Z}/m\boldsymbol{z})^n$ and any element $w \in W_{m,n}$ can be written in a unique way as $w = t_1^{c_1} \cdots t_n^{c_n} \sigma$, where $\sigma \in \mathfrak{S}_n$ and $c_i$ are integers such that $0 \leqslant c_i < m$.

The *cyclotomic Hecke algebras* or *Ariki-Koike* algebras were introduced independently by Cherednik [6], Broué-Malle [4], and Ariki-Koike [1], it is the unitary associative $\mathbb{K}$-algebra $H = H_{m,n}(q, \boldsymbol{Q})$ generated by $T_0, T_1, \dots, T_n$ and subject to relations

$$\begin{aligned} & (T_0 - Q_1) \dots (T_0 - Q_m) = 0, && \\ & T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, && \\ & T_i^2 = (q - q^{-1}) T_i + 1, && \text{for } 1 \leqslant i < n, \\ & T_i T_j = T_j T_i, && \text{for } |i-j| > 2, \\ & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, && \text{for } 1 \leqslant i \leqslant n-2. \end{aligned}$$

Note that if $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for $\sigma \in \mathfrak{S}_n$, then $T_\sigma := T_{i_1} T_{i_2} \cdots T_{i_k}$ is independent of the choice of reduced expressions and $\{T_\sigma | \sigma \in \mathfrak{S}_n\}$ is a linear basis of the subalgebra $H_n(q)$ of $H$ generated by $T_1, \dots, T_{n-1}$, which is isomorphic to the Iwahori-Hecke algebra of type $A$.

Let $\phi : \mathbb{K} \to \mathbb{C}$ be the specialization homomorphism defined by $\phi(q) = 1$ and $\phi(Q_i) = \varsigma^i$ for each $i$, where $\varsigma$ is a fixed primitive $m$-th root of unity. By the specialization homomorphism $\phi$, one obtains $\mathbb{C} \otimes_{\mathbb{K}} H \cong \mathbb{C} W_{m,n}$, that is, $H$ can be thought as a deformation of the group algebra $\mathbb{C} W_{m,n}$.

We will need the following presentation of $H$ due to Shoji [24, Theorem 3.7]. Let $\Delta$ be the determinant of the Vandermonde matrix $V(\boldsymbol{Q})$ of degree $m$ with $(a,b)$-entry

$Q_b^a$ for $1 \leqslant b \leqslant m$, $0 \leqslant a < m$. Clearly, we can write $V(\boldsymbol{Q})^{-1} = \Delta^{-1}V^*(\boldsymbol{Q})$, where $V^*(\boldsymbol{Q}) = (v_{ba}(\boldsymbol{Q}))$ and $v_{ba}(\boldsymbol{Q})$ is a polynomial in $\mathbb{Z}[\boldsymbol{Q}]$. For $1 \leqslant c \leqslant m$, we define a polynomial $F_c(X)$ with a variable $X$ with coefficients in $\mathbb{Z}[\boldsymbol{Q}]$ by

$$F_c(X) = \sum_{0 \leqslant i < m} v_{ci}(\boldsymbol{Q})X^i.$$

Then $H$ is (isomorphic to) the associative $\mathbb{K}$-algebra generated by $T_1, \dots, T_{n-1}$ and $\xi_1, \dots, \xi_n$ subject to

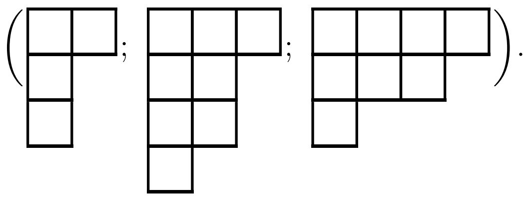
$$\begin{aligned}
&(T_i - q)(T_i + q^{-1}) = 0, && 1 \leqslant i < n,\\
&(\xi_i - Q_1)\cdots(\xi_i - Q_m) = 0, && 1 \leqslant i \leqslant n,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && 1 \leqslant i < n-1,\\
&T_iT_j = T_jT_i, && |i-j| \geqslant 2,\\
&\xi_i\xi_j = \xi_j\xi_i, && 1 \leqslant i, j \leqslant n,\\
&T_j\xi_i = \xi_iT_j, && i \neq j-1, j,\\
&T_j\xi_j = \xi_{j-1}T_j + \Delta^{-2}\sum_{a<b}(Q_a - Q_b)(q - q^{-1})F_a(\xi_{j-1})F_b(\xi_j), &&\\
&T_j\xi_{j-1} = \xi_jT_j - \Delta^{-2}\sum_{a<b}(Q_a - Q_b)(q - q^{-1})F_a(\xi_{j-1})F_b(\xi_j). &&
\end{aligned}$$

Recall that a composition (resp. partition) $\lambda = (\lambda_1, \lambda_2, \dots)$ of $n$, denoted $\lambda \models n$ (resp. $\lambda \vdash n$) is a sequence (resp. weakly decreasing sequence) of nonnegative integers such that $|\lambda| = \sum_{i \geqslant 1} \lambda_i = n$. For a composition or a partition $\lambda$, its length $\ell(\lambda)$ is the number of nonzero parts of $\lambda$. A *multicomposition* (resp. *multipartition*) of $n$ is an ordered tuple $\boldsymbol{\lambda} = (\lambda^{(1)}; \dots; \lambda^{(m)})$ of compositions (resp. partitions) $\lambda^{(i)}$ such that $|\boldsymbol{\lambda}| = \sum_{i=1}^m |\lambda^{(i)}| = n$. We refer to $\lambda^{(c)}$ as the $c$-component of $\boldsymbol{\lambda}$ and denote by $\mathscr{P}_{m,n}$ the set of all multipartitions of $n$ consisting of $m$ components.

Recall that the *Young diagram* $Y(\boldsymbol{\lambda})$ of multipartition $\boldsymbol{\lambda}$ is the set

$$Y(\boldsymbol{\lambda}) = \{(i, j, c) \in \boldsymbol{z}_{>0} \times \boldsymbol{z}_{>0} \times \boldsymbol{m} | 1 \leqslant j \leqslant \lambda_i^c\},$$

where $\boldsymbol{m} = \{1, \dots, m\}$. The elements of $Y(\boldsymbol{\lambda})$ are the *nodes* of $\boldsymbol{\lambda}$. We will identify a (multi)partition with its Young diagram. For example, consider $\boldsymbol{\lambda} = ((2,1,1); (3,2,2,1); (4,3,1))$, then its Young diagram is

$\boldsymbol{\lambda} =$ 

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n$ is said to be a $(k,\ell)$-*hook partition* of $n$ if $\lambda_{k+1} \leqslant \ell$. We let $H(k|\ell; n)$ denote the set of all $(k,\ell)$-hook partitions of $n$, that is

$$H(k|\ell; n) = \{\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n \mid \lambda_{k+1} \leqslant \ell\}.$$

Further, we say a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}; \dots ; \lambda^{(m)})$ of $n$ is a $(\boldsymbol{k}, \boldsymbol{\ell})$-*hook multipartition* of $n$ if $\lambda^{(c)}$ is a $(k_c, \ell_c)$-hook partition for $c = 1, \dots , m$. We write $\boldsymbol{k}|\boldsymbol{\ell} = (k_1|\ell_1, \dots , k_m|\ell_m)$ and denote by $H(\boldsymbol{k}|\boldsymbol{\ell}; m, n)$ the set of all $(\boldsymbol{k}, \boldsymbol{\ell})$-hook multipartition of $n$. For example, $\boldsymbol{\lambda} = ((2, 1, 1); (3, 2, 2, 1); (4, 3, 1))$ is a $(1|1, 1|2, 1|3))$-hook 3-multipartition of 20.

Given a composition $\boldsymbol{c} = (c_1, \dots , c_b)$ of $n$, we denote by $\mathfrak{S}_n^{(i)}$ the subgroup of $\mathfrak{S}_n$ generated by $s_j$ such that $a_{i-1} + 1 < j \leqslant a_i$, where $a_0 = 0$ and $a_i = a_{i-1} + c_i$ for $1 \leqslant i \leqslant b$. Then $\mathfrak{S}_n^{(i)} \cong \mathfrak{S}_{c_i}$. Now we define a parabolic subgroup $\mathfrak{S}_{\boldsymbol{c}}$ of $\mathfrak{S}_n$ by

$$\mathfrak{S}_{\boldsymbol{c}} = \mathfrak{S}_n^{(1)} \times \mathfrak{S}_n^{(2)} \times \cdots \times \mathfrak{S}_n^{(b)}.$$

In other words, $\mathfrak{S}_{\boldsymbol{c}}$ is the Young subgroup of $\mathfrak{S}_n$ associated to $\boldsymbol{c}$.

Let $W_{\boldsymbol{c}}$ be the subgroup of $W_{m,n}$ obtained as the semidirect product of $\mathfrak{S}_{\boldsymbol{c}}$ with $(\mathbb{Z}/m\boldsymbol{z})^n$. Then $W_{\boldsymbol{c}}$ can be written as

$$W_{\boldsymbol{c}} = W^{(1)} \times W^{(2)} \times \cdots \times W^{(b)},$$

where $W^{(i)}$, $1 \leqslant i \leqslant b$, is the subgroup of $W_{m,n}$ generated by $\mathfrak{S}_n^{(i)}$ and $t_j$, $a_{i-1} < j \leqslant a_i$. Then $W^{(i)} \cong W_{m,c_i}$, which enables us to yield a natural embedding up to isomorphism

$$\theta_{\boldsymbol{c}} : W_{m,c_1} \times \cdots \times W_{m,c_b} \hookrightarrow W_{m,n}$$

for each composition $\boldsymbol{c}$ of $n$.

For positive integer $i$, we define

$$w(1, i) = t_1^i, \qquad w(a, i) = t_a^i s_{a-1} \cdots s_1, \quad 2 \leqslant a \leqslant n.$$

Let $\mu = (\mu_1, \mu_2, \cdots)$ be a partition of n. We define

$$w(\mu, i) \;=\; w(\mu_1, i) \times w(\mu_2, i) \times \cdots$$

with respect to the embedding $\theta_\mu$. More generally, for $\boldsymbol{\mu} = (\mu^{(1)}, \dots , \mu^{(m)}) \in \mathscr{P}_{m,n}$, we define

$$w(\boldsymbol{\mu}) \;=\; w(\mu^{(1)}, 1)w(\mu^{(2)}, 2) \cdots w(\mu^{(m)}, m).$$

Then $\{w(\boldsymbol{\mu})|\boldsymbol{\mu} \in \mathscr{P}_{m,n}\}$ is a set of conjugacy class representatives for $W_{m,n}$ (see [9, 4.2.8]).

Now we define the *standard element* $T(\boldsymbol{\mu}) \in H$ of type $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$ as follows. First for $a \geqslant 1$, we put $T(a, i) = \xi_a^i T_{a-1} \cdots T_1$. Then for a partition $\mu = (\mu_1, \mu_2, \dots) \vdash n$, we define

$$T(\mu, i) = T(\mu_1, i) \times T(\mu_2, i) \times \cdots .$$

Finally, for $\boldsymbol{\mu} = (\mu^{(1)}; \mu^{(2)}; \dots ; \mu^{(m)}) \in \mathscr{P}_{m,n}$, we define $T(\boldsymbol{\mu}) \in H$ by

$$T(\boldsymbol{\mu}) = T(\mu^{(1)}, 1) \times T(\mu^{(2)}, 2) \times \cdots \times T(\mu^{(m)}, m). \tag{1}$$

More generally, we define $T(w) = \xi_1^{c_1} \cdots \xi_n^{c_n} T_\sigma$ for $w = t_1^1 \cdots t_n^{c_n} \sigma \in W_{m,n}$.

It is known that the irreducible representations of $\mathbb{C}W_{m,n}$ and $H$ are indexed by $\mathscr{P}_{m,n}$ (see [1]). We denote by $S_1^{\boldsymbol{\lambda}}$ (resp. $S_q^{\boldsymbol{\lambda}}$) the irreducible representations of $\mathbb{C}W_{m,n}$ (resp. $H$) corresponding to $\boldsymbol{\lambda} \in \mathscr{P}_{m,n}$ and by $\chi_1^{\boldsymbol{\lambda}}$ (resp. $\chi_q^{\boldsymbol{\lambda}}$) its irreducible character. Furthermore, it is known that the characters $\chi_1^{\boldsymbol{\lambda}}$ (resp. $\chi_q^{\boldsymbol{\lambda}}$) are completely determined by their values on $w(\boldsymbol{\mu})$ (resp. $T(\boldsymbol{\mu})$) for all $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$ (see [24, Proposition 7.5]). For simplicity, we write $\chi_1^{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \chi_1^{\boldsymbol{\lambda}}(w(\boldsymbol{\mu}))$ and $\chi_q^{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \chi_q^{\boldsymbol{\lambda}}(T(\boldsymbol{\mu}))$ respectively.

## 3 The super Frobenius formula

In this section we first recall the definitions of the supersymmetric Schur functions and power sum supersymmetric functions indexed by multipartitions introduced in [26, §3], then we present the super Frobenius formula for the characters of cyclotomic Hecke algebras. Here we will follow [12] with respect to our notation about symmetric functions unless otherwise stated.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sets of $k$ and $\ell$ indeterminates, respectively, expressed as follow:

$$\begin{aligned} &\boldsymbol{x}^{(i)} = \left\{x_1^{(i)}, \cdots, x_{k_i}^{(i)}\right\}, \quad 1 \leqslant i \leqslant m; \\ &\boldsymbol{y}^{(i)} = \left\{y_1^{(i)}, \cdots, y_{\ell_i}^{(i)}\right\}, \quad 1 \leqslant i \leqslant m; \\ &\boldsymbol{x} = \boldsymbol{x}^{(1)} \cup \cdots \cup \boldsymbol{x}^{(m)}; \\ &\boldsymbol{y} = \boldsymbol{y}^{(1)} \cup \cdots \cup \boldsymbol{y}^{(m)}. \end{aligned}$$

We say that the variables in $\boldsymbol{x}^{(i)} \cup \boldsymbol{y}^{(i)}$ are of color $i$ and identify the variables $x_1^{(1)}, \ldots, y_{\ell_m}^{(m)}$ with the variables $x_1, \ldots, x_k, y_1, \cdots, y_\ell$, and the variables $z_1, \cdots, z_{k+\ell}$ as follows:

$$\begin{array}{ccccccccccccc} x_1^{(1)} & \cdots & x_{k_1}^{(1)} & y_1^{(1)} & \cdots & y_{\ell_1}^{(1)} & \cdots & x_1^{(m)} & \cdots & x_{k_m}^{(m)} & y_1^{(m)} & \cdots & y_{\ell_m}^{(m)} \\ \updownarrow & \vdots & \updownarrow & \updownarrow & \vdots & \updownarrow & \cdots & \updownarrow & \vdots & \updownarrow & \updownarrow & \vdots & \updownarrow \\ x_1 & \cdots & x_{k_1} & y_1 & \cdots & y_{\ell_1} & \cdots & x_{k-k_m+1} & \cdots & x_k & y_{\ell-\ell_m+1} & \cdots & y_\ell \\ \updownarrow & \vdots & \updownarrow & \updownarrow & \vdots & \updownarrow & \cdots & \updownarrow & \vdots & \updownarrow & \updownarrow & \vdots & \updownarrow \\ z_1 & \cdots & z_{k_1} & z_{k_1+1} & \cdots & z_{d_1} & \cdots & z_{d_{m-1}+1} & \cdots & z_{d_m-\ell_m} & z_{d_m-\ell_m+1} & \cdots & z_{k+\ell}. \end{array} \tag{2}$$

We define $\overline{z_i} = \bar{i}$ for $i = 1, \ldots, k+\ell$, that is, $\bar{x} = \bar{0}$ for $x \in \boldsymbol{x}$ and $\bar{y} = \bar{1}$ for $y \in \boldsymbol{y}$. Finally, we linearly order the variables $x_1^{(1)}, \cdots, x_{k_m}^{(m)}, y_1^{(1)}, \cdots, y_{\ell_m}^{(m)}$ by the rules:

$$\begin{aligned} x_*^{(i)} < y_*^{(i)} < x_*^{(i+1)}&, \\ x_a^{(i)} < x_b^{(j)} \text{ if and only if } i < j &\text{ or } i = j \text{ and } a < b, \\ y_a^{(i)} < y_b^{(j)} \text{ if and only if } i < j &\text{ or } i = j \text{ and } a < b, \end{aligned} \tag{3}$$

or equivalently $z_1 < z_2 < \cdots < z_{k+\ell}$.

Let $\Lambda_k = \mathbb{Z}[x_1, \ldots, x_k]^{\mathfrak{S}_k}$ be the ring of symmetric polynomials of $k$ variables and $(\Lambda_k)_{\mathbb{Q}} = \Lambda_k \otimes_{\mathbb{Z}} \mathbb{Q}$. It is well-known that the *power sum symmetric polynomials* [12, §I.2]

$$p_0(\boldsymbol{x}) = 1 \text{ and } p_a(\boldsymbol{x}) = \sum_{i=1}^{k} x_i^a \text{ for } a = 1, 2, \cdots$$

generate $\Lambda_k$ under the power series multiplication and $\Lambda_k$ is a graded ring:

$$\Lambda_k = \oplus_{n \geqslant 0} \Lambda_k^n,$$

where $\Lambda_k^n$ consists of the homogeneous symmetric polynomials of degree $n$ together with zero polynomials. It is known that $\Lambda_k^n$ has a basis

$$p_\mu(\boldsymbol{x}) := p_{\mu_1}(\boldsymbol{x}) p_{\mu_2}(\boldsymbol{x}) \cdots \text{ with } \mu = (\mu_1, \mu_2, \cdots) \vdash n.$$

An alternative basis of $\Lambda_k^n$ is given by the *Schur functions* $S_\lambda(\boldsymbol{x})$, $\lambda \vdash n$, which can be defined as

$$S_\lambda(\boldsymbol{x}) = \sum_{\mu \vdash n} Z_\mu^{-1} \chi_1^\lambda(\mu) p_\mu(\boldsymbol{x}),$$

here $Z_\mu$ is the order of the centralizer in $\mathfrak{S}_n$ of an element of cycle type $\mu$. Let us remark that $S_\lambda(\boldsymbol{x}) = 0$ whenever $\ell(\lambda) > k$ (see e.g. [11, Equ. (3.8)]).

Now we denote by $\Lambda_{k,\ell}$ the ring of polynomials in $x_1, \ldots, x_k, y_1, \ldots, y_\ell$, which are separately symmetric in $\boldsymbol{x}$'s and $\boldsymbol{y}$'s, namely

$$\Lambda_{k,\ell} = \mathbb{Q}[x_1, \ldots, x_k]^{\mathfrak{S}_k} \otimes_{\mathbb{Q}} \mathbb{Q}[y_1, \ldots, y_\ell]^{\mathfrak{S}_\ell}.$$

Following [11, §4], we define the *power sum supersymmetric function* $p_i(\boldsymbol{x}/\boldsymbol{y})$ to be

$$\begin{aligned} &p_0(\boldsymbol{x}/\boldsymbol{y}) = 1, \\ &p_i(\boldsymbol{x}/\boldsymbol{y}) = p_i(\boldsymbol{x}) - p_i(\boldsymbol{y}) \text{ for } i \geqslant 1. \end{aligned}$$

Recall that $\varsigma$ is a fixed primitive $m$-th root of unity. For each integer $t \geqslant 1$ and $i$ such that $1 \leqslant i \leqslant m$, set

$$P_t^{(i)}(\boldsymbol{x}/\boldsymbol{y}) = \sum_{j=1}^{m} \varsigma^{-ij} p_t(\boldsymbol{x}^{(j)}/\boldsymbol{y}^{(j)}).$$

For $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)}) \in \mathscr{P}_{m,n}$, we define the power sum supersymmetric function $P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y})$ associated to $\boldsymbol{\mu}$ as follows. For a partition $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}, \cdots)$, we define a function $P_{\mu^{(i)}}(\boldsymbol{x}/\boldsymbol{y})$ by

$$P_{\mu^{(i)}}(\boldsymbol{x}/\boldsymbol{y}) = \prod_{j=1}^{\ell(\mu^{(i)})} P_{\mu_j^{(i)}}^{(i)}(\boldsymbol{x}/\boldsymbol{y}),$$

and define

$$P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}) = \prod_{i=1}^{m} P_{\mu^{(i)}}(\boldsymbol{x}/\boldsymbol{y}).$$

Following [3, §6] or [11, §4], the *supersymmetric Schur function* $S_{\lambda^{(i)}}(\boldsymbol{x}/\boldsymbol{y}) \in \Lambda_{k,\ell}$ corresponding to a partition $\lambda^{(i)} = \left(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots\right)$ is defined as

$$S_{\lambda^{(i)}}(\boldsymbol{x}^{(i)}/\boldsymbol{y}^{(i)}) := \sum_{\mu \subset \lambda} (-1)^{|\lambda-\mu|} S_\mu(\boldsymbol{x}^{(i)}) S_{\lambda'/\mu'}(\boldsymbol{y}^{(i)}),$$

where $S_{\lambda'/\mu'}(\boldsymbol{y}^{(i)})$ is the skew Schur function associated to the conjugate $\lambda'/\mu'$ of the skew partition $\lambda/\mu$. Note that the skew Schur function $S_{\eta/\theta}(\boldsymbol{y}^{(i)})$ can be calculated by

$$S_{\eta/\theta}(\boldsymbol{y}^{(i)}) = \sum_{\nu} c_{\theta\nu}^{\eta} S_\nu(\boldsymbol{y}^{(i)}),$$

where the coefficients $c_{\theta\nu}^{\eta}$ are determined by the Littlewood-Richardson rule in the product of Schur functions (see [12, §I.9]).

For $\boldsymbol{\lambda} = (\lambda^{(1)}; \dots ; \lambda^{(m)}) \in \mathscr{P}_{m,n}$, we define the supersymmetric Schur function associated with $\boldsymbol{\lambda}$ by

$$S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}) \;=\; \prod_{i=1}^{m} S_{\lambda^{(i)}}(\boldsymbol{x}^{(i)}/\boldsymbol{y}^{(i)}),$$

which is a super analogue of the Schur function associated to multipartitions defined in [24, (6.1.2)]. It should be noted that $S_{\lambda^{(i)}}(\boldsymbol{x}^{(i)}/\boldsymbol{y}^{(i)}) = 0$ unless $\lambda^{(i)}$ is a $(k,\ell)$-hook partition (see e.g. [11, Equ. (4.10)]), which implies that $S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}) = 0$ unless $\boldsymbol{\lambda}$ is a $(\boldsymbol{k},\boldsymbol{\ell})$-hook multipartition.

Now we are ready to give a combinatorial interpretation of the supersymmetric Schur functions in terms of $(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableau, which also shows $S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}) = 0$ unless $\boldsymbol{\lambda}$ is $(\boldsymbol{k},\boldsymbol{\ell})$-hook multipartition of $n$.

Let $\lambda$ be a partition of $n$. A *$(k,\ell)$-tableau* $\mathfrak{t}$ of shape $\lambda$ is obtained from the diagram of $\lambda$ by replacing each node by one of the variables $\boldsymbol{z}$, allowing repetitions. For a $(k,\ell)$-tableau $\mathfrak{t}$, we denote by $\mathfrak{t}_{\boldsymbol{x}}$ (resp. $\mathfrak{t}_{\boldsymbol{y}}$) the nodes filled with variables $\boldsymbol{x}$ (resp. $\boldsymbol{y}$). We say a $(k,\ell)$-tableau $\mathfrak{t}$ is *$(k,\ell)$-semistandard* if

(a) $\mathfrak{t}_{\boldsymbol{x}}$ (resp. $\mathfrak{t}_{\boldsymbol{y}}$) is a tableau (resp. skew tableau); and

(b) The entries of $\mathfrak{t}_{\boldsymbol{x}}$ are nondecreasing in rows, strictly increasing in columns, that is, $\mathfrak{t}_{\boldsymbol{x}}$ is a *row-semistandard tableau*; and

(c) The entries of $\mathfrak{t}_{\boldsymbol{y}}$ are nondecreasing in columns, strictly increasing in rows, that is, $\mathfrak{t}_{\boldsymbol{y}}$ is a *column-semistandard skew tableau.*

For $\boldsymbol{\lambda} \in \mathscr{P}_{m,n}$, the *$(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableau* $\mathfrak{t}$ of shape $\boldsymbol{\lambda}$ is a filling of boxes of $\boldsymbol{\lambda}$ with variables $\boldsymbol{z}$ such that its $i$-component $\mathfrak{t}^{(i)}$ is a $(k_i,\ell_i)$-semistandard tableau filled variables $\boldsymbol{x}^{(i)}$ and $\boldsymbol{y}^{(i)}$ for $i = 1,\dots,m$.

**Example 1.** For $\lambda = (3,3,2,2,1) \in H(2|2,11)$, we let

$$\mathfrak{t} = \begin{array}{|c|c|c|}\hline x_1 & x_1 & x_1 \\ \hline x_2 & y_1 & y_2 \\ \hline y_1 & y_2 \\ \cline{1-2} y_1 & y_2 \\ \cline{1-2} y_1 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \mathfrak{s} = \begin{array}{|c|c|c|}\hline x_1 & x_1 & x_1 \\ \hline x_2 & y_1 & x_2 \\ \hline y_1 & y_2 \\ \cline{1-2} y_1 & y_2 \\ \cline{1-2} y_1 \\ \cline{1-1}\end{array}$$

Then $\mathfrak{t}$ is a $(2,2)$-semistandard $\lambda$-tableau and $\mathfrak{s}$ is not a $(2,2)$-semistandard $\lambda$-tableau.

For $\boldsymbol{\lambda} = ((4,2),(3,1)) \in H((2|2,1|1);2,10)$,

$$\left( \begin{array}{|c|c|c|c|}\hline x_1^{(1)} & x_2^{(1)} & y_1^{(1)} & y_2^{(1)} \\ \hline x_2^{(1)} & y_1^{(1)} \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline x_1^{(2)} & x_1^{(2)} & y_1^{(2)} \\ \hline y_1^{(2)} \\ \cline{1-1}\end{array} \right)$$

is a $(2|2,1|1)$-semistandard $\boldsymbol{\lambda}$-tableau.

We denote by $\mathrm{sstd}_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda})$ the set of $(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableaux of shape $\boldsymbol{\lambda}$ and by $s_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda})$ its cardinality. Thanks to [3, § 2] and [2, Lemma 4.2], $s_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda}) \neq 0$ if and only if $\boldsymbol{\lambda} \in H(\boldsymbol{k}|\boldsymbol{\ell}; m, n)$.

For a $(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableau $\mathfrak{t}$ of shape $\boldsymbol{\lambda}$, we define

$$\mathfrak{t}(\boldsymbol{x}/\boldsymbol{y}) = \prod_{i=1}^{m} \prod_{(a,b)\in\lambda^{(i)}} (-1)^{\overline{z_{a,b}}} z_{a,b} \tag{4}$$

where $z_{a,b} \in \boldsymbol{z} = \{z_1, \ldots, z_{k+\ell}\}$ is the variable filled in the box $(a,b)$ of $\mathfrak{t}^{(i)}$. It follows from [3] and [11, Equ. (4.9)] that for $\boldsymbol{\lambda} \in H(\boldsymbol{k}|\boldsymbol{\ell}; m, n)$, we have

$$S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}) = \sum_{\mathfrak{t}\in\mathrm{sstd}_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda})} \mathfrak{t}(\boldsymbol{x}/\boldsymbol{y}).$$

The following fact clarifies the relationship between supersymmetric Schur functions and power sum supersymmetric functions indexed by multipartitions, which is a cyclotomic version of [11, (4.4)] and may serve as a definition of supersymmetric Schur functions associated to (hook)mutlipartitions.

**Proposition 2** ( [26, Proposition 3.8]). *Let $Z_{\boldsymbol{\mu}}$ be the order of the centralizer in $W_{m,n}$ of an element of cycle type $\boldsymbol{\mu}$. Then*

$$S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}) = \sum_{\boldsymbol{\mu}\in\mathscr{P}_{m,n}} Z_{\boldsymbol{\mu}}^{-1} \chi_1^{\boldsymbol{\lambda}}(\boldsymbol{\mu}) P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}).$$

Now we are going to define a deformation of power sum supersymmetric functions indexed by multipartitions. We need the following notations.

For positive integer $t$, we let

$$\begin{aligned}
\mathscr{I}(t; k|\ell) &= \{\boldsymbol{i} = (i_1, \ldots, i_t) | 1 \leqslant i_a \leqslant k+\ell, a = 1, \ldots, t\},\\
\mathscr{I}_{\leqslant}(t; k|\ell) &= \{\boldsymbol{i} = (i_1, \ldots, i_t) | 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t \leqslant k+\ell\},
\end{aligned}$$

and let $\mathscr{C}(t; k+\ell)$ be the set of compositions of $t$ with $(k+\ell)$ parts, that is,

$$\mathscr{C}(t; k+\ell) = \left\{(\alpha;\beta) = \left((\alpha^{(1)}, \ldots, \alpha^{(m)}); (\beta^{(1)}, \cdots, \beta^{(m)})\right) \big| \, |\alpha| + |\beta| = t\right\}, \tag{5}$$

where $\alpha^{(i)} = (\alpha_1^{(i)}, \ldots, \alpha_{k_i}^{(i)})$ and $\beta^{(i)} = (\beta_1^{(i)}, \ldots, \beta_{\ell_i}^{(i)})$ for $1 \leqslant i \leqslant m$.

For integers $a \geqslant 1$ and $b \geqslant 0$, we denote by $(a^b)$ the composition $(a, \ldots, a)$ consisting of $b$ parts. Note that $\boldsymbol{i} = (i_1, \ldots, i_t) \in \mathscr{I}_{\leqslant}(t; k|\ell)$ may be written uniquely as follows:

$$\begin{aligned}
\boldsymbol{i}(\alpha;\beta) \;=\; \Big( & 1^{\alpha_1^{(1)}}, \cdots, k_1^{\alpha_{k_1}^{(1)}}, (k_1{+}1)^{\beta_1^{(1)}}, \cdots, d_1^{\beta_{\ell_1}^{(1)}}; \cdots, \cdots;\\
& (d_{m-1}{+}1)^{\alpha_1^{(m)}}, \cdots, (d_m{-}\ell_m)^{\alpha_{k_m}^{(m)}}, (d_{m-1}{+}1)^{\beta_1^{(m)}}, \cdots, d_m^{\beta_{\ell_m}^{(m)}} \Big)
\end{aligned}$$

for some $(\alpha;\beta) \in \mathscr{C}(t;k+\ell)$, where $\alpha = (\alpha^{(1)};\ldots;\alpha^{(m)})$, $\beta = (\beta^{(1)};\ldots;\beta^{(m)})$. Thus we may and will identify $\mathscr{I}_{\leqslant}(t;k|\ell)$ with $\mathscr{C}(t;k+\ell)$ and write $\boldsymbol{i} = (\alpha;\beta)$ when $\boldsymbol{i} = \boldsymbol{i}(\alpha;\beta)$.

Finally, for $\boldsymbol{i} = (i_1,\ldots,i_t) \in \mathscr{I}(t;k|\ell)$, we define its length $\ell(\boldsymbol{i}) = t$, its color $c(\boldsymbol{i})$ to be the color of the maximal integer in $\{i_1,\ldots,i_t\}$, and write $\boldsymbol{z_i} = \prod_{s=1}^{t}(-1)^{\overline{z_{i_s}}} z_{i_s}$. Clearly, if $\boldsymbol{i} = (\alpha;\beta) \in \mathscr{I}_{\leqslant}(t;k|\ell)$, then $\ell(\boldsymbol{i}) = \ell(\alpha)+\ell(\beta)$, $c(\boldsymbol{i})$ is the maximal $b$ such that $(\alpha^{(b)},\beta^{(b)}) \neq \emptyset$ (For simplicity, we write $Q_{c(\boldsymbol{i})} = Q_{(\alpha;\beta)}$), and $\boldsymbol{z_i} = \boldsymbol{x}^{\alpha}(-\boldsymbol{y})^{\beta}$.

For integer $t \geqslant 1$ and $a = 1,\ldots,m$, we define

$$q_t^{(a)}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q}) = \sum_{(\alpha;\beta)\in\mathscr{C}(t;k+\ell)} Q^a_{(\alpha;\beta)}\tilde{q}_{(\alpha;\beta)}(\boldsymbol{x}/\boldsymbol{y};q), \tag{6}$$

where

$$\tilde{q}_{(\alpha;\beta)}(\boldsymbol{x}/\boldsymbol{y};q) = (-q^{-1})^{|\beta|-\ell(\beta)}q^{|\alpha|-\ell(\alpha)}(q-q^{-1})^{\ell(\alpha;\beta)-1}\boldsymbol{x}^{\alpha}(-\boldsymbol{y})^{\beta}. \tag{7}$$

For $\boldsymbol{\mu} = (\mu^{(1)},\ldots,\mu^{(m)}) \in \mathscr{P}_{m,n}$, we define a deformation of $P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y})$ as follows

$$q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q}) = \prod_{i=1}^{m}\prod_{j=1}^{\ell(\mu^{(i)})} q^{(i)}_{\mu^{(i)}_j}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q}). \tag{8}$$

Note that $q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y};1,\boldsymbol{\varsigma}) = P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y})$ with $\boldsymbol{\varsigma} = (\varsigma,\varsigma^2,\ldots,\varsigma^m)$.

In [26, Theorem 4.11], we obtain the following super Frobenius formula for characters of $H$,

$$q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q}) = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{m,n}} \chi_q^{\boldsymbol{\lambda}}(\boldsymbol{\mu})S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}), \tag{9}$$

for each $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$.

## 4 The combinatorial formula of $q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q})$

In this section, we define the $\boldsymbol{\mu}$-weight of parity-integer sequences and interpret the function $q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y};q,\boldsymbol{Q})$ as a sum over $\boldsymbol{\mu}$-weighted parity-integer sequences.

**Definition 3.** For $\boldsymbol{i} = (i_1,\ldots,i_t) \in \mathscr{I}(t;k|\ell)$, we say that $\boldsymbol{i}$ is *up-down* if there exists a nonnegative integer $p$ with $0 \leqslant p < t$ such that

$$i_1 < \cdots < i_p < i_{p+1} \geqslant \cdots \geqslant i_t,$$

and we say that $i_{p+1}$ is the *peak* of the up-down sequence $\boldsymbol{i}$.

For $\boldsymbol{i} = (i_1,\ldots,i_t) \in \mathscr{I}(t;k|\ell)$ and $a = 0,1$, we denote by $\ell_a(\boldsymbol{i})$ the number of the integers in $\boldsymbol{i}$ with parity $\bar{a}$ and by $\ell_a(\boldsymbol{i}_<)$ (resp. $\ell_a(\boldsymbol{i}_=)$) the number of the integers $i_j$ in $\boldsymbol{i}$ such that $\overline{i_j} = \bar{a}$ and $i_j < i_{j+1}$ (resp. $i_j = i_{j+1}$). In particular, for an up-down sequence $\boldsymbol{i} = (i_1,\ldots,i_t)$ with peak $i_{p+1}$, we set $\ell_a(\boldsymbol{i}_{<p})$ the number of the integers $i_j$ in $\{i_1,\ldots,i_p\}$ with parity $\bar{a}$. Further, we let $\ell(\boldsymbol{i}_<)$ be the number of the integers $i_j$ in $\boldsymbol{i}$ such that $i_j < i_{j+1}$. Clearly $\ell(\boldsymbol{i}_<) = \ell_0(\boldsymbol{i}_<) + \ell_1(\boldsymbol{i}_<)$.

**Definition 4.** For a sequence $\boldsymbol{i} = (i_1, \ldots, i_t) \in \mathscr{I}(t; k|\ell)$, we define its *weight*

$$\mathrm{wt}(\boldsymbol{i}) = \begin{cases} 0, & \text{if } \boldsymbol{i} \text{ is not up-down;} \\ \dfrac{q^{\ell_0(\boldsymbol{i})+\ell_1(\boldsymbol{i}_{<p})-\ell_0(\boldsymbol{i}_{<p})-1}}{(-q)^{\ell_1(\boldsymbol{i})+\ell_0(\boldsymbol{i}_{<p})-\ell_1(\boldsymbol{i}_{<p})}}, & \text{if } \boldsymbol{i} \text{ is up-down with peak } i_{p+1} \text{ and } \overline{i_{p+1}} = \bar{0}; \\ \dfrac{q^{\ell_0(\boldsymbol{i})+\ell_1(\boldsymbol{i}_{<p})-\ell_0(\boldsymbol{i}_{<p})}}{(-q)^{\ell_1(\boldsymbol{i})+\ell_0(\boldsymbol{i}_{<p})-\ell_1(\boldsymbol{i}_{<p})-1}}, & \text{if } \boldsymbol{i} \text{ is up-down with peak } i_{p+1} \text{ and } \overline{i_{p+1}} = \bar{1}. \end{cases}$$

*Remark* 5. If $\ell(\boldsymbol{i}) = 1$ then $\mathrm{wt}(\boldsymbol{i}) = 1$. If $\ell = 0$ then Definition 4 reduces to the one defined in Ram's work [17, Lemma 1.5] (see also [5, Equ. (2.13)]).

**Lemma 6.** *Given $\boldsymbol{i} = (i_1, \ldots, i_t) \in \mathscr{I}_{\leqslant}(t; k|\ell)$ and let $\mathfrak{S}_{\boldsymbol{i}}$ denote the set of all distinct permutations of $\boldsymbol{i}$. Then*

$$\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) = (-q^{-1})^{\ell_1(\boldsymbol{i}_=)} q^{\ell_0(\boldsymbol{i}_=)} (q - q^{-1})^{\ell(\boldsymbol{i}_<)}.$$

*Proof.* We adapt the argument of the proof of [17, Lemma 1.5]. Let $\tilde{D}$ be the set of distinct elements in the sequence $\boldsymbol{i}$ and let $D = \tilde{D} - \{i_t\}$. For $s = 0, 1$, let $D_s$ be the subset of $D$ consisting of elements with parity $\bar{s}$ and let $d_s = |D_s|$. For each subset $A$ of $D$, there is a disjoint decomposition $A = A_0 \cup A_1$ with $A_s \subseteq D_s$ for $s = 0, 1$. We let $a_s = |A_s|$, $a = |A|$, $j_1 < j_1 < \cdots < j_a$ the elements of $A$ in increasing order, $j_{a+1} = i_t$, and let $j_{a+2} \geqslant j_{a+3} \geqslant \cdots \geqslant j_t$ be the remainder of the elements of the sequence $\boldsymbol{i}$ arranged in decreasing order. In this way, we obtain an up-down sequence

$$\boldsymbol{j} = (j_1 < \cdots < j_a < j_{a+1} \geqslant j_{a+2} \geqslant j_{a+3} \geqslant \cdots \geqslant j_t)$$

with peak $j_{a+1}$. According to Definition 4,

$$\mathrm{wt}(\boldsymbol{j}) = \begin{cases} (-q^{-1})^{\ell_1(\boldsymbol{i})+a_0-a_1} q^{\ell_0(\boldsymbol{i})+a_1-a_0-1}, & \text{if } \overline{i_t} = \bar{0}; \\ (-q^{-1})^{\ell_1(\boldsymbol{i})+a_0-a_1-1} q^{\ell_0(\boldsymbol{i})+a_1-a_0}, & \text{if } \overline{i_t} = \bar{1}. \end{cases}$$

Note that for $\sigma \in \mathfrak{S}_{\boldsymbol{i}}$, $\mathrm{wt}(\sigma(\boldsymbol{i})) \neq 0$ if and only if

$$\sigma(\boldsymbol{i}) = \big(i_{\sigma(1)}, \ldots, i_{\sigma(a)}, i_{\sigma(a+1)}, i_{\sigma(a+2)}, \ldots, i_{\sigma(t)}\big)$$

is an up-down sequence with peak $i_{\sigma(a+1)} = i_t$. It follows from this that every $\sigma \in \mathfrak{S}_{\boldsymbol{i}}$ such that $\mathrm{wt}(\sigma(\boldsymbol{i})) \neq 0$ is given by a unique subset $A$ of $D$ and the parity of its peak is $\mathfrak{S}_{\boldsymbol{i}}$-stable.

Now we assume that $\overline{i_t} = \bar{0}$. Then $\ell_0(\boldsymbol{i}_=) = \ell_0(\boldsymbol{i}) - \ell_0(\boldsymbol{i}_<) - 1$ and

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) &= \sum_{A_0 \subseteq D_0, A_1 \subseteq D_1} (-q^{-1})^{\ell_1(\boldsymbol{i})+|A_0|-|A_1|} q^{\ell_0(\boldsymbol{i})+|A_1|-|A_0|-1} \\ &= \sum_{a_0=0}^{d_0} \sum_{a_1=0}^{d_1} \binom{d_0}{a_0} \binom{d_1}{a_1} (-q^{-1})^{\ell_1(\boldsymbol{i})+a_0-a_1} q^{\ell_0(\boldsymbol{i})+a_1-a_0-1} \end{aligned}$$

$$
\begin{aligned}
&= (-q^{-1})^{\ell_1(\boldsymbol{i})-d_1} q^{\ell_0(\boldsymbol{i})-d_0-1}(q-q^{-1})^{d_0+d_1} \\
&= (-q^{-1})^{\ell_1(\boldsymbol{i})-\ell_1(\boldsymbol{i}_<)} q^{\ell_0(\boldsymbol{i})-\ell_0(\boldsymbol{i}_<)-1}(q-q^{-1})^{d_0+d_1} \\
&= (-q^{-1})^{\ell_1(\boldsymbol{i}_=)} q^{\ell_0(\boldsymbol{i}_=)}(q-q^{-1})^{\ell(\boldsymbol{i}_<)},
\end{aligned}
$$

where the last equality follows by that $\ell_1(\boldsymbol{i}_=) = \ell_1(\boldsymbol{i}) - \ell_1(\boldsymbol{i}_<)$ and $d_0 + d_1 = \ell(\boldsymbol{i}_<)$.

If $\overline{i_t} = \bar{1}$ then $\ell_1(\boldsymbol{i}_=) = \ell_1(\boldsymbol{i}) - \ell_1(\boldsymbol{i}_<) - 1$ and, by applying a similar argument, we obtain that

$$
\begin{aligned}
\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) &= (-q^{-1})^{\ell_1(\boldsymbol{i})-\ell_1(\boldsymbol{i}_<)-1} q^{\ell_0(\boldsymbol{i})-\ell_0(\boldsymbol{i}_<)}(q-q^{-1})^{d_0+d_1} \\
&= (-q^{-1})^{\ell_1(\boldsymbol{i}_=)} q^{\ell_0(\boldsymbol{i}_=)}(q-q^{-1})^{\ell(\boldsymbol{i}_<)},
\end{aligned}
$$

where the last equality follows by noting that $\ell_0(\boldsymbol{i}_=) = \ell_0(\boldsymbol{i}) - \ell_0(\boldsymbol{i}_<)$ and $d_0 + d_1 = \ell(\boldsymbol{i}_<)$. Thus we complete the proof. ☐

Now assume that $\boldsymbol{i} = (\alpha;\beta)$ with $(\alpha;\beta) \in \mathscr{C}(t; k+\ell)$ for some $t$. Then $\ell_0(\boldsymbol{i}) = |\alpha|$, $\ell_1(\boldsymbol{i}) = |\beta|$, $\ell(\boldsymbol{i}_<) = \ell(\alpha;\beta) - 1$, and

$$
\begin{aligned}
\ell_0(\boldsymbol{i}_<) &= \begin{cases} \ell(\alpha) - 1, & \text{if the maximal element in } \boldsymbol{i} \text{ has parity } \bar{0}; \\ \ell(\alpha), & \text{otherwise;} \end{cases} \\
\ell_1(\boldsymbol{i}_<) &= \begin{cases} \ell(\beta), & \text{if the maximal element in } \boldsymbol{i} \text{ has parity } \bar{0}; \\ \ell(\beta) - 1, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Thus $\ell_1(\boldsymbol{i}_=) = |\beta| - \ell(\beta)$, $\ell_0(\boldsymbol{i}) = |\alpha| - \ell(\alpha)$.

According to Equ. (7), Lemma 6 can be rephrased as follows:

**Corollary 7.** *Given $\boldsymbol{i} = (\alpha;\beta) \in \mathscr{C}(t; k+\ell)$ and let $\mathfrak{S}_{\boldsymbol{i}}$ denote the set of all distinct permutations of $\boldsymbol{i}$. Then*

$$
\mathrm{wt}(\alpha;\beta) := \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) = (-q^{-1})^{|\beta|-\ell(\beta)} q^{|\alpha|-\ell(\alpha)} (q-q^{-1})^{\ell(\alpha;\beta)-1},
$$

*or equivalently,*

$$
\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) \boldsymbol{z}_{\sigma(\boldsymbol{i})} = \tilde{q}_{(\alpha;\beta)}(\boldsymbol{x}/\boldsymbol{y}; q),
$$

Now we can rephrase the function $q_t^{(a)}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q})$ in terms of the weight of sequence.

**Corollary 8.** *For integers $t \geqslant 1$ and $1 \leqslant a \leqslant m$, we have*

$$
q_t^{(a)}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) = \sum_{\boldsymbol{i} \in \mathscr{I}(t;k+\ell)} \mathrm{wt}(\boldsymbol{i}) Q^a_{c(\boldsymbol{i})} \boldsymbol{z}_{\boldsymbol{i}}.
$$

*Proof.* First note that for $\boldsymbol{i} = (i_1, \ldots, i_t)$, we have $c(\sigma(\boldsymbol{i})) = c(\boldsymbol{i})$,

$$
\boldsymbol{z}_{\sigma(\boldsymbol{i})} = \prod_{j=1}^{t} (-1)^{\overline{z_{\sigma(i_j)}}} z_{\sigma(i_j)} = \prod_{j=1}^{t} (-1)^{\overline{z_{i_j}}} z_{i_j} = \boldsymbol{z}_{\boldsymbol{i}}
$$

for all $\sigma \in \mathfrak{S}_{\boldsymbol{i}}$, and $\mathrm{wt}(i_1, \ldots, i_t) = 0$ unless it is up-down.

As a consequence, we get

$$\begin{aligned}
\sum_{\boldsymbol{i}\in\mathscr{I}(t;k|\ell)} \mathrm{wt}(\boldsymbol{i}) Q^a_{c(\boldsymbol{i})} \boldsymbol{z}_{\boldsymbol{i}} &= \sum_{\boldsymbol{i}\in\mathscr{I}_{\leqslant}(t;k|\ell)} \sum_{\sigma\in\mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) Q^a_{c(\sigma(\boldsymbol{i}))} \boldsymbol{z}_{\sigma(\boldsymbol{i})} \\
&= \sum_{\boldsymbol{i}\in\mathscr{I}_{\leqslant}(t;k|\ell)} Q^a_{c(\boldsymbol{i})} \left( \sum_{\sigma\in\mathfrak{S}_{\boldsymbol{i}}} \mathrm{wt}(\sigma(\boldsymbol{i})) \right) \boldsymbol{z}_{\sigma(\boldsymbol{i})} \\
&= \sum_{(\alpha;\beta)\in\mathscr{C}(t;k+\ell)} Q^a_{(\alpha;\beta)} \tilde{q}_{(\alpha;\beta)}(\boldsymbol{x}/\boldsymbol{y}; q).
\end{aligned}$$

It completes the proof owing to Equ. (6). □

For $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$, we denote by $\mathfrak{t}^{\boldsymbol{\mu}}$ the standard $\boldsymbol{\mu}$-tableau with the boxes filled in with the numbers $1, 2, \ldots, n$ so that $\mu^{(1)}$ contains the numbers $1, \ldots, |\mu^{(1)}|$ in order from left to right and top to bottom, $\mu^{(2)}$ contains the numbers $|\mu^{(1)}|+1, \ldots, |\mu^{(1)}|+|\mu^{(2)}|$ in order from left to right and top to bottom, and so on. For $i = 1, \ldots, n$, we will write $c_{\mathfrak{t}^{\boldsymbol{\mu}}}(i) = c$ if the number $i$ is entered in the $c$-component of $\mathfrak{t}^{\boldsymbol{\mu}}$. More generally, given a standard $\boldsymbol{\lambda}$-tableau $T = (T^{(1)}; \ldots, T^{(m)})$ filling the numbers $1, 2, \ldots, n$, we write $c_T(i) = c$ when the number $i$ is entered in the $c$-component of $T$.

**Definition 9.** We say that $\boldsymbol{i} = (i_1, \ldots, i_n)$ is $\boldsymbol{\mu}$*-up-down* if it satisfies the following property:

> if $a, a+1, \ldots, a+r$ is a row of $\mathfrak{t}^{\boldsymbol{\mu}}$ then the subsequence $(i_a, i_{a+1}, \ldots, i_{a+r})$ is an up-down sequence, i.e., $i_a < i_{a+1} < \cdots < i_p \geqslant \cdots \geqslant i_{a+r}$ for some $p \geqslant a$.

The index $i_p$ is the peak of the row.

For a $\boldsymbol{\mu}$-up-down sequence $\boldsymbol{i}$, we denote by $P^{\boldsymbol{\mu}}_{\boldsymbol{i}}$ the set of peaks in $\boldsymbol{i}$, one for each row of $\mathfrak{t}^{\boldsymbol{\mu}}$.

**Definition 10.** For a sequence $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathscr{I}(n; k|\ell)$, we define its $\boldsymbol{\mu}$-weight

$$\mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) = \begin{cases} 0, & \text{if } \boldsymbol{i} \text{ is not } \boldsymbol{\mu}\text{-up-down;} \\ (-q^{-1})^{\ell_1(\boldsymbol{i}_{\geqslant})+\ell_0(\boldsymbol{i}_{<})} q^{\ell_0(\boldsymbol{i}_{\geqslant})+\ell_1(\boldsymbol{i}_{<})} \displaystyle\prod_{i_p\in P^{\boldsymbol{\mu}}_{\boldsymbol{i}}} Q^{c_{\mathfrak{t}^{\boldsymbol{\mu}}}(i_p)}_{c(i_p)}, & \text{if } \boldsymbol{i} \text{ is } \boldsymbol{\mu}\text{-up-down,} \end{cases}$$

where $\ell_a(\boldsymbol{i}_{\geqslant})$ (resp. $\ell_a(\boldsymbol{i}_{<})$) is the number of $i_j$ with parity $a$ ($a = 0, 1$) such that $i_j \geqslant i_{j+1}$ (resp. $i_j < i_{j+1}$) and $j$, $j+1$ in the same row of $\mathfrak{t}^{\boldsymbol{\mu}}$.

Note that we may index the sequence $\boldsymbol{i} = (i_1, \ldots, i_n)$ by (row reading sequence of) the nodes of $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$, namely for $t = 1, \ldots, n$, we write $i_t = i_{a,b,c}$ when the number $t$ is entered in the $a$-th row and the $b$-column of the $c$-component of $\mathfrak{t}^{\boldsymbol{\mu}}$.

Now we assume that $\boldsymbol{i} = (i_{a,b,c})_{(a,b,c)\in\boldsymbol{\mu}}$ is $\boldsymbol{\mu}$-up-down and define the weight of $i_{a,b,c}$ as follows:

$$\mathrm{wt}_{\boldsymbol{\mu}}(i_{a,b,c}) = \begin{cases} Q^c_{c(i_{a,b,c})}, & \text{if } i_{a,b,c} \text{ is a peak and } b = \mu_a^{(c)}; \\ (-1)^s q^{(-1)^s} Q^c_{c(i_{a,b,c})}, & \text{if } i_{a,b,c} \text{ is a peak with parity } s \text{ and } i_{a,b,c} \geqslant i_{a,b+1,c}; \\ (-1)^s q^{(-1)^s}, & \text{if } i_{a,b,c} \text{ is a non-peak with parity } s \text{ and } i_{a,b,c} \geqslant i_{a,b+1,c}; \\ (-1)^{s+1} q^{(-1)^{s+1}}, & \text{if } i_{a,b,c} \text{ is a non-peak with parity } s \text{ and } i_{a,b,c} < i_{a,b+1,c}; \\ 1, & \text{otherwise.} \end{cases}$$

Further, we set $\mathrm{wt}(i_{a,b,c}) = 0$ for all $(a,b,c) \in \boldsymbol{\mu}$ when $\boldsymbol{i}$ is not $\boldsymbol{\mu}$-up-down. It is easy to see that for any (row-reading $\boldsymbol{\mu}$-sequence) $\boldsymbol{i}$, we have

$$\mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) \;=\; \prod_{(a,b,c)\in\boldsymbol{\mu}} \mathrm{wt}_{\boldsymbol{\mu}}(i_{a,b,c}). \tag{10}$$

**Example 11.** Let $n = 20$, $m = 3$,

$$\boldsymbol{k}|\boldsymbol{\ell} = (1|1, 1|2, 1|3)$$

and

$$\boldsymbol{\mu} = ((2,1,1);(3,2,2,1);(4,3,1)).$$

Then

$$\mathfrak{t}^{\boldsymbol{\mu}} = \left( \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|} \hline 5 & 6 & 7 \\ \hline 8 & 9 \\ \cline{1-2} 10 & 11 \\ \cline{1-2} 12 \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|c|} \hline 13 & 14 & 15 & 16 \\ \hline 17 & 18 & 19 \\ \cline{1-3} 20 \\ \cline{1-1} \end{array} \right),$$

the (row-reading) sequence

$$\boldsymbol{i} = \left( \begin{array}{|c|c|} \hline 1 & \underline{3} \\ \hline \underline{2} \\ \cline{1-1} \underline{4} \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|} \hline 6 & 7 & \underline{9} \\ \hline \underline{2} & 2 \\ \cline{1-2} \underline{5} & 4 \\ \cline{1-2} \underline{7} \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|c|} \hline 8 & \underline{9} & 5 & 1 \\ \hline 3 & \underline{6} & 1 \\ \cline{1-3} \underline{8} \\ \cline{1-1} \end{array} \right).$$

is $\boldsymbol{\mu}$-up-down, its peaks are underlined numbers, and its weight $\mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i})$ is the product of the entries of the following diagram

$$\left( \begin{array}{|c|c|} \hline -q^{-1} & Q_2 \\ \hline Q_1 \\ \cline{1-1} Q_2 \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|} \hline -q^{-1} & q & Q_3^2 \\ \hline -q^{-1}Q_1^2 & 1 \\ \cline{1-2} -q^{-1}Q_2^2 & 1 \\ \cline{1-2} Q_3^2 \\ \cline{1-1} \end{array} ; \begin{array}{|c|c|c|c|} \hline q & -q^{-1}Q_3^3 & -q^{-1} & 1 \\ \hline -q^{-1} & qQ_3^3 & 1 \\ \cline{1-3} Q_3^3 \\ \cline{1-1} \end{array} \right),$$

where the entry in the node $(a,b,c)$ is the weight $\mathrm{wt}(i_{a,b,c})$ of $i_{a,b,c}$, that is,

$$\mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) = (-q^{-1})^7 q^3 Q_1^3 Q_2^4 Q_3^{13}.$$

The following easily verified fact shows Definitions 4 and 10 are compatible, which enable us to view the function $q_{\boldsymbol{\mu}}$ (see Equ. (8)) as the product of the function $q_t^{(i)}$ (see Equ. (6)) over the rows of $\mathfrak{t}^{\boldsymbol{\mu}}$ where $t$ are the lengths of the rows in the $i$-component $\mathfrak{t}^{\boldsymbol{\mu}}$.

**Lemma 12.** *Given an up-down sequence $\boldsymbol{i} = (i_1, \ldots, i_t)$ and denote by $\ell_a(\boldsymbol{i}_{\geqslant})$ is the number $i_j$ with parity $a$ ($a = 0, 1$) such that $i_j \geqslant i_{j+1}$. Then*

$$\mathrm{wt}(\boldsymbol{i}) = (-q^{-1})^{\ell_1(\boldsymbol{i}_{\geqslant})+\ell_0(\boldsymbol{i}_{<})} q^{\ell_0(\boldsymbol{i}_{\geqslant})+\ell_1(\boldsymbol{i}_{<})}.$$

*Proof.* Assume that $\boldsymbol{i} = (i_1, \ldots, i_t)$ such that $i_1 < \cdots < i_p < i_{p+1} \geqslant \cdots \geqslant i_t$. Clearly $\ell_0(\boldsymbol{i}_{<p}) = \ell_0(\boldsymbol{i}_{<})$, $\ell_0(\boldsymbol{i}_{<p}) = \ell_0(\boldsymbol{i}_{<})$, $\ell_1(\boldsymbol{i}_{<p}) = \ell_1(\boldsymbol{i}_{<})$. Further $\ell_1(\boldsymbol{i}) - \ell_1(\boldsymbol{i}_{<p}) = \ell_1(\boldsymbol{i}_{\geqslant})$ when $\overline{i_{p+1}} = \overline{0}$; otherwise $\ell_0(\boldsymbol{i}) - \ell_0(\boldsymbol{i}_{<p}) = \ell_0(\boldsymbol{i}_{\geqslant})$ and $\ell_1(\boldsymbol{i}) - \ell_1(\boldsymbol{i}_{<p}) - 1 = \ell_1(\boldsymbol{i}_{\geqslant})$. As a consequence, we prove the lemma. □

Now we can rewrite the super Frobenius formula Equ. (9) for the characters of $H$ as a sum over $\boldsymbol{\mu}$-weighted parity-integer sequences.

**Corollary 13.** *For $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$,*

$$q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) \;=\; \sum_{\boldsymbol{i}\in\mathscr{I}(n;k|\ell)} \mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i})\boldsymbol{z}_{\boldsymbol{i}} = \sum_{\boldsymbol{i} \text{ is } \boldsymbol{\mu}\text{-up-down}} \mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i})\boldsymbol{z}_{\boldsymbol{i}}.$$

*Proof.* Assume that $\boldsymbol{\mu} = (\mu^{(1)}; \ldots; \mu^{(m)})$ and $\mu^{(c)} = (\mu_1^{(c)}, \ldots, \mu_{\ell_c}^{(c)})$ where $\ell_c = \ell(\mu^{(c)})$ for $= 1, \ldots, m$. Given any (row-reading $\boldsymbol{\mu}$-sequence) $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathscr{I}(n; k|\ell)$, we rewrite it as the following form

$$\boldsymbol{i} = (\boldsymbol{i}_1^{(1)}, \ldots, \boldsymbol{i}_{\ell_1}^{(1)}, \ldots, \boldsymbol{i}_1^{(m)}, \ldots, \boldsymbol{i}_{\ell_m}^{(m)}),$$

where $\boldsymbol{i}_a^{(c)}$ is the subsequence of $\boldsymbol{i}$ indexed by the nodes lying in the $a$-th row of the $c$-component of $\boldsymbol{\mu}$ for $a = 1, \ldots, \ell_c$ and $c = 1, \ldots, m$. Furthermore, we let $P_{\boldsymbol{i}}^c$ be set of peaks in $\boldsymbol{i}^{(c)} = (\boldsymbol{i}_1^{(c)}, \ldots, \boldsymbol{i}_{\ell_c}^{(c)})$ for $c = 1, \ldots, m$. Clearly Definition 10 shows $\boldsymbol{i}$ is $\boldsymbol{\mu}$-up-down if and only if $\boldsymbol{i}_a^{(c)}$ is up-down for all $a = 1, \ldots, \ell_c$ and $c = 1, \ldots, m$; and $P_{\boldsymbol{i}}^{\boldsymbol{\mu}} = P_{\boldsymbol{i}}^1 \cup \cdots P_{\boldsymbol{i}}^m$.

Now combining Equ. (8) and Corollary 8, we yield

$$\begin{aligned}
q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) &= \prod_{c=1}^{m}\prod_{a=1}^{\ell_c} \sum_{\boldsymbol{i}_a^{(c)}\in\mathscr{I}(\mu_a^{(c)};k+\ell)} \mathrm{wt}(\boldsymbol{i}_a^{(c)}) Q^c_{c(\boldsymbol{i}_a^{(c)})} \boldsymbol{z}_{\boldsymbol{i}_a^{(c)}} \\
&= \prod_{c=1}^{m} \sum_{\substack{\boldsymbol{i}_a^{(c)}\in\mathscr{I}(\mu_a^{(c)};k+\ell)\\ a=1,\ldots,\ell_c}} \mathrm{wt}(\boldsymbol{i}_1^{(c)}) \cdots \mathrm{wt}(\boldsymbol{i}_{\ell_c}^{(c)}) Q^c_{c(\boldsymbol{i}_1^{(c)})} \cdots Q^c_{c(\boldsymbol{i}_{\ell_c}^{(c)})} \boldsymbol{z}_{\boldsymbol{i}_1^{(c)}} \cdots \boldsymbol{z}_{\boldsymbol{i}_{\ell_c}^{(c)}} \\
&= \sum_{\substack{\boldsymbol{i}_a^{(c)}\in\mathscr{I}(\mu_a^{(c)};k+\ell)\\ a=1,\ldots,\ell_c\\ c=1,\ldots,m}} \mathrm{wt}(\boldsymbol{i}_1^{(1)}) \cdots \mathrm{wt}(\boldsymbol{i}_{\ell_m}^{(m)}) Q^1_{c(\boldsymbol{i}_1^{(1)})} \cdots Q^m_{c(\boldsymbol{i}_{\ell_m}^{(m)})} \boldsymbol{z}_{\boldsymbol{i}_1^{(1)}} \cdots \boldsymbol{z}_{\boldsymbol{i}_{\ell_m}^{(m)}} \\
&= \sum_{(\boldsymbol{i}^{(1)},\ldots,\boldsymbol{i}^{(m)})\in\mathscr{I}(n;k+\ell)} \mathrm{wt}(\boldsymbol{i}_1^{(1)}) \cdots \mathrm{wt}(\boldsymbol{i}_{\ell_m}^{(m)}) Q^1_{c(\boldsymbol{i}_1^{(1)})} \cdots Q^m_{c(\boldsymbol{i}_{\ell_m}^{(m)})} \boldsymbol{z}_{\boldsymbol{i}^{(1)}} \cdots \boldsymbol{z}_{\boldsymbol{i}^{(m)}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(\boldsymbol{i}^{(1)},\ldots,\boldsymbol{i}^{(m)})\in\mathscr{I}(n;k+\ell)} \mathrm{wt}(\boldsymbol{i}_1^{(1)})\cdots\mathrm{wt}(\boldsymbol{i}_{\ell_m}^{(m)})\boldsymbol{z}_{\boldsymbol{i}^{(1)}}\cdots\boldsymbol{z}_{\boldsymbol{i}^{(m)}}\prod_{c=1}^{m}\prod_{x\in P_{\boldsymbol{i}}^{c}} Q_{c(x)}^{c} \\
&= \sum_{(\boldsymbol{i}^{(1)},\ldots,\boldsymbol{i}^{(m)})\in\mathscr{I}(n;k+\ell)} \mathrm{wt}(\boldsymbol{i}_1^{(1)})\cdots\mathrm{wt}(\boldsymbol{i}_{\ell_m}^{(m)})\boldsymbol{z}_{\boldsymbol{i}^{(1)}}\cdots\boldsymbol{z}_{\boldsymbol{i}^{(m)}}\prod_{x\in P_{\boldsymbol{i}}^{\boldsymbol{\mu}}} Q_{c(x)}^{c_{\mathfrak{t}^{\boldsymbol{\mu}}}(x)} \\
&= \sum_{\boldsymbol{i}\in\mathscr{I}(n;k|\ell)} \mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i})\boldsymbol{z}_{\boldsymbol{i}} \\
&= \sum_{\boldsymbol{i}\text{ is }\boldsymbol{\mu}\text{-up-down}} \mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i})\boldsymbol{z}_{\boldsymbol{i}},
\end{aligned}
$$

where the last third equality follows by applying Equ. (10) and the last second equality follows by applying Definition 10 and Lemma 12. □

## 5 RSK superinsertion

In the section, we first review the RSK superinsertion algorithm for partitions, then extend it to work for multipartitions and define the $\boldsymbol{\mu}$-weight of standard tableaux, which leads to a new proof of the super Frobenius formula.

Let

$$
\mathcal{Z}_{k,\ell}(n) = \left\{ z_{\boldsymbol{i}} = \begin{pmatrix} 1 & \cdots & n \\ z_{i_1} & \cdots & z_{i_n} \end{pmatrix} \middle| \, 1 \leqslant i_1,\ldots,i_n \leqslant k+\ell \right\}.
$$

Abusing notation, we write $z_{\boldsymbol{i}} = z_{i_1},\ldots,z_{i_n}$. Following Berele-Regev [3, §2] (see also [7, §4]), the *RSK superinsertion* is a bijection from $\mathcal{Z}_{k,\ell}(n)$ to pairs of tableaux $(S,T)$, where $S$ is a $(k,\ell)$-semistandard tableau, $T$ is a standard tableau (which is called the *recording tableau*), and $\mathrm{shape}(S) = \mathrm{shape}(T) = \lambda$ for some partition $\lambda$ of $n$. More precisely, for any $z_{\boldsymbol{i}} \in \mathcal{Z}_{k,\ell}(n)$, the RSK superinsertion constructs the pairs of tableaux $(S,T) = (S(z_{\boldsymbol{i}}), T(z_{\boldsymbol{i}}))$ iteratively:

$$
(\emptyset,\emptyset) = (S_0,T_0), (S_1,T_1),\ldots,(S_n,T_n) = (S,T)
$$

by applying the following rules:

(1) $S_j$ is a $(k,\ell)$-semistandard tableau containing $j$ nodes and $T_j$ is a standard tableau with $\mathrm{shape}(T_j) = \mathrm{shape}(S_j)$.

(2) $S_j$ is obtained from $S_{j-1}$ by inserting $z_{i_j}$ into $S_{j-1}$ as follows:

  (i) If $z_{i_j} \in \boldsymbol{x}$ then $S_j$ is obtained from $S_{j-1}$ by column inserting $z_{i_j}$ into $S_{j-1}$ as follows:

    (a) Insert $z_{j_i}$ into the first column of $S_{j-1}$ by displacing the smallest variables $\geqslant z_{i_j}$; if every variable is $< z_{i_j}$, add $z_{i_j}$ to the bottom of the first column.

    (b) If $z_{i_j}$ displace $z_?$ from the first column, insert $z_?$ into the second column using the above rule whenever $z_? \in \boldsymbol{x}$, otherwise insert $z_?$ to the first row using the rule (ii).

(c) Repeat for each subsequent column, until a variable in $\boldsymbol{x}$'s is added to the bottom of some (possibly empty) column.

(ii) If $z_{i_j} \in \boldsymbol{y}$ then $S_j$ is obtained from $S_{j-1}$ by row inserting $z_{i_j}$ into $S_{j-1}$ as follows:

(a) Insert $z_{i_j}$ into the first row of $S_{j-1}$ by displacing the smallest variable $\geqslant z_{i_j}$; if every variable is $< z_{i_j}$, add $z_{i_j}$ to the right-end of the first row.

(b) If $z_{i_j}$ displacing $z_?$ from the first row, insert $z_?$ into the second row using the above rules (note that here $z_?$ must be in $\boldsymbol{y}$).

(c) Repeat for each subsequent row, until a variable in $\boldsymbol{y}$'s is added to the right-end of some (possibly empty) row.

(3) $T_j$ is obtained from $T_{j-1}$ by inserting number $j$ in the newly added box.

**Example 14.** The result of inserting of $\boldsymbol{z_i} = y_1, x_2, x_2, x_1, x_3, y_1, y_1, y_3, y_2$ is

$$S: \quad \begin{array}{|c|}\hline y_1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline x_2 & y_1\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline x_2 & x_2 & y_1\\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline x_1 & x_2 & x_2 & y_1\\\hline\end{array} \quad \begin{array}{llll} x_1 & x_2 & x_2 & y_1\\ x_3 \end{array}$$

$$T: \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\\hline\end{array} \quad \begin{array}{llll} 1 & 2 & 3 & 4\\ 5 \end{array}$$

$$S: \quad \begin{array}{llll} x_1 & x_2 & x_2 & y_1\\ x_3 & y_1 \end{array} \quad \begin{array}{llll} x_1 & x_2 & x_2 & y_1\\ x_3 & y_1\\ y_1 \end{array} \quad \begin{array}{lllll} x_1 & x_2 & x_2 & y_1 & y_3\\ x_3 & y_1\\ y_1 \end{array} \quad \begin{array}{lllll} x_1 & x_2 & x_2 & y_1 & y_2\\ x_3 & y_1 & y_3\\ y_1 \end{array}$$

$$T: \quad \begin{array}{llll} 1 & 2 & 3 & 4\\ 5 & 6 \end{array} \quad \begin{array}{llll} 1 & 2 & 3 & 4\\ 5 & 6\\ 7 \end{array} \quad \begin{array}{lllll} 1 & 2 & 3 & 4 & 8\\ 5 & 6\\ 7 \end{array} \quad \begin{array}{lllll} 1 & 2 & 3 & 4 & 8\\ 5 & 6 & 9\\ 7 \end{array}$$

Now we generalize the RSK superinsertion algorithm to work for multipartitions, that is, a map from $\mathscr{Z}_{k,\ell}(n)$ to pairs of tableaux $(S,T)$, where $S$ is a $(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableau, the *recording tableau* $T$ is a standard tableau, and $\text{shape}(S) = \text{shape}(T) = \boldsymbol{\lambda}$ for some multipartition $\boldsymbol{\lambda} \in \mathscr{P}_{m,n}$. For any $z_{\boldsymbol{i}} \in \mathscr{Z}_{k,\ell}(n)$, the RSK superinsertion algorithm constructs the pairs of tableaux $(S,T) = (S(z_{\boldsymbol{i}}), T(z_{\boldsymbol{i}}))$ iteratively:

$$(\emptyset, \emptyset) = (S_0, T_0), (S_1, T_1), \ldots, (S_n, T_n) = (S, T)$$

by applying the following rules:

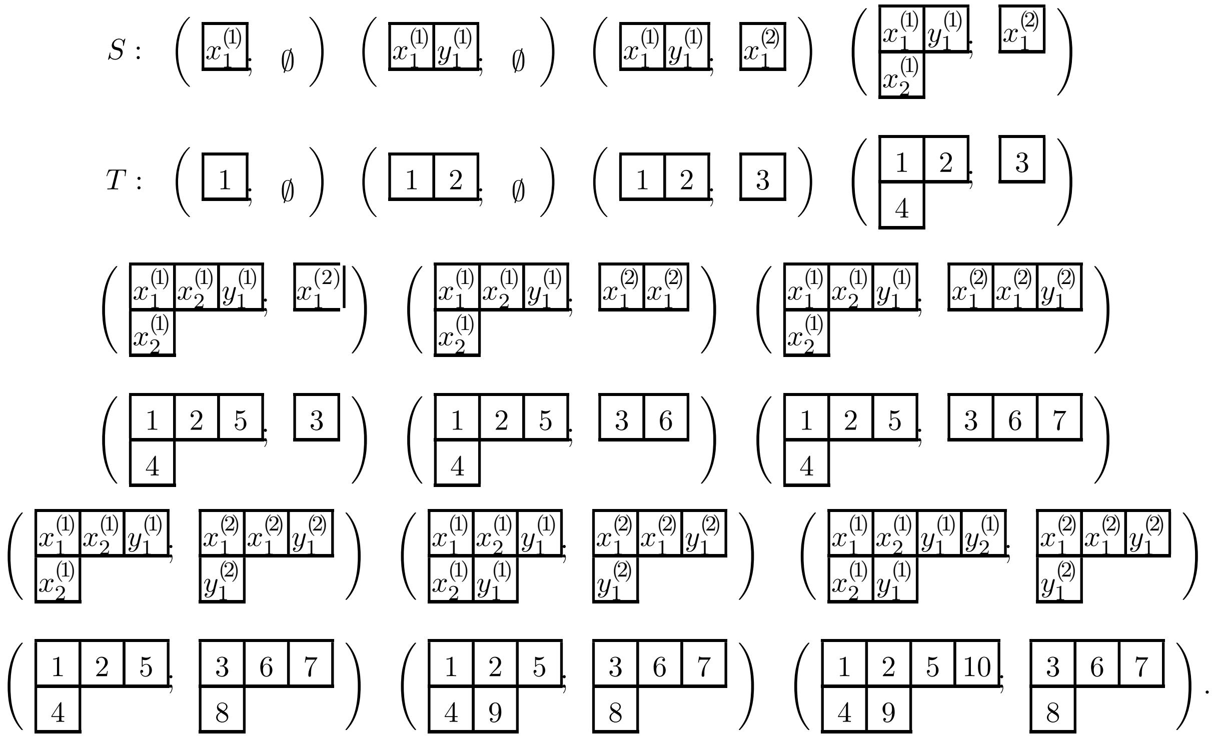
(1) $S_j$ is a $(\boldsymbol{k},\boldsymbol{\ell})$-semistandard tableau containing $j$ nodes and $T_j$ is a standard tableau with $\text{shape}(T_j) = \text{shape}(S_j)$;

(2) $S_j$ is obtained from $S_{j-1}$ by inserting $z_{i_j}$ into $S^{(c)}_{j-1}$, write as $S_j = S_{j-1} \leftarrow z_{i_j}$, by applying the RSK superinsertion algorithms when $z_{i_j}$ is of color $c$;

(3) $T_j$ is obtained from $T_{j-1}$ by inserting number $j$ in the newly added box.

*Remark* 15. If $\ell = 0$ (resp. $\boldsymbol{\ell} = \mathbf{0}$) then the RSK superinsertion reduces to the classical RSK column insertions for partitions (resp. multipartitions) (see e.g. [19, Chapter 3.1] and [5, § 3]). The RSK superinsertion here is different from the $(k, \ell)$-RoSch insertion given in [3, §2.5].

**Example 16.** The inserting $z_{\boldsymbol{i}} = x_1^{(1)}, y_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(1)}, x_1^{(2)}, y_1^{(2)}, y_1^{(2)}, y_1^{(1)}, y_2^{(1)}$ is



To show that this RSK superinsertion is a bijection, we can construct the inverse algorithm by using the RSK superinsertion for partitions in the reverse order of the entries of $T$, and using the component of $S$ to determine the uninserted variable. We denote this bijection by

$$(S, T) \xleftrightarrow{\text{RSK}} z_{i_1}, \ldots, z_{i_n}.$$

The following definition devotes to describe the linear order defined in Equ. (3) via the relative positions of nodes in the recoding tableau.

**Definition 17.** Assume that $(S, T) \xleftrightarrow{\text{RSK}} z_{i_1}, \ldots, z_{i_n}$ and put $\bar{a} = \overline{z_{i_a}}$ for $a = 1, \ldots, n$. For $a, b \in T$, we define

$$a \xrightarrow{\text{SW}} b \text{ if } \begin{cases} \text{either } a \in T^{(i)}, b \in T^{(j)} \text{ and } i < j; \text{ or } a, b \in T^{(j)} \text{ with } (\bar{a}, \bar{b}) = (\bar{0}, \bar{1}); \\ \text{or } b \text{ is south (below) or west (left) of } a \text{ in } T^{(j)} \text{ with } \bar{a} = \bar{b} = \bar{0}; \\ \text{or } b \text{ is north (above) or east (right) of } a \text{ in } T^{(j)} \text{ with } \bar{a} = \bar{b} = \bar{1}. \end{cases}$$

$$a \xrightarrow{\text{NE}} b \text{ if } \begin{cases} \text{either } a \in T^{(i)}, b \in T^{(j)} \text{ and } j < i; \text{ or } a, b \in T^{(j)} \text{ with } (\bar{a}, \bar{b}) = (\bar{1}, \bar{0}); \\ \text{or } b \text{ is north (above) or east (right) of } a \text{ in } T^{(j)} \text{ with } \bar{a} = \bar{b} = \bar{0}; \\ \text{or } b \text{ is south (below) or west (left) of } a \text{ in } T^{(j)} \text{ with } \bar{a} = \bar{b} = \bar{1}. \end{cases}$$

**Example 18.** Let $\boldsymbol{z_i}$ be the one in Examples 14 and 16 respectively. Then we have

$$1 \xrightarrow{\mathrm{NE}} 2 \xrightarrow{\mathrm{NE}} 3 \xrightarrow{\mathrm{NE}} 4 \xrightarrow{\mathrm{SW}} 5 \xrightarrow{\mathrm{SW}} 6 \xrightarrow{\mathrm{NE}} 7 \xrightarrow{\mathrm{SW}} 8 \xrightarrow{\mathrm{NE}} 9 \text{ and}$$
$$1 \xrightarrow{\mathrm{SW}} 2 \xrightarrow{\mathrm{SW}} 3 \xrightarrow{\mathrm{NE}} 4 \xrightarrow{\mathrm{NE}} 5 \xrightarrow{\mathrm{SW}} 6 \xrightarrow{\mathrm{SW}} 7 \xrightarrow{\mathrm{NE}} 8 \xrightarrow{\mathrm{NE}} 9 \xrightarrow{\mathrm{SW}} 10.$$

Recall that the linear order of the variables defined in Equ. (3). Then we have the following fact, which is a super-version of the ones about RSK insertion (see e.g. [17, Proposition 2.1] or [5, Proposition 3.1]).

**Proposition 19.** *Let* $S_{j+1} = (S_{j-1} \leftarrow z_{i_j}) \leftarrow z_{i_{j+1}}$ *with* $S_{j-1}$ *is* $(\boldsymbol{k}, \boldsymbol{\ell})$*-semistandard and let* $T_{j+1}$ *be the associated recording tableau. The following assertions hold:*

*(1)* $z_{i_j} < z_{i_{j+1}}$ *if and only if* $j \xrightarrow{\mathrm{SW}} j+1$ *in* $T_{j+1}$*.*

*(2)* $z_{i_j} \geqslant z_{i_{j+1}}$ *if and only if* $j \xrightarrow{\mathrm{NE}} j+1$ *in* $T_{j+1}$*.*

*Proof.* Note that the RSK superinsertion algorithm inserts the variables in $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) by applying classical column (resp. row) RSK insertion algorithm and $\boldsymbol{x}^{(a)} < \boldsymbol{y}^{(a)} < \boldsymbol{x}^{(a+1)}$ for $a = 1, \ldots, m-1$. The two follows directly by the linearly ordering on variables and well-known facts about the classical column and row RSK insertion algorithms. □

**Definition 20.** For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{m,n}$, we say that $T \in \mathrm{std}(\boldsymbol{\lambda})$ is $\boldsymbol{\mu}$*-SW-NE* if it satisfies the property: if $i, i+1, \ldots, i+j$ is in a row of $\mathfrak{t}^{\boldsymbol{\mu}}$ then

$$i \xrightarrow{\mathrm{SW}} i+1 \xrightarrow{\mathrm{SW}} \cdots \xrightarrow{\mathrm{SW}} p \xrightarrow{\mathrm{NE}} \cdots \xrightarrow{\mathrm{NE}} i+j \text{ in } T.$$

The number $p$ here is called the *peak* of the row.

For a $\boldsymbol{\mu}$-SW-NE tableau $T$, we denote by $P_T^{\boldsymbol{\mu}}$ the set of all peaks in $T$ and let $T_{\mathrm{SW}}$ (resp. $T_{\mathrm{NE}}$) be the set of numbers $j$ such that $j \xrightarrow{\mathrm{SW}} j+1$ (resp. $j \xrightarrow{\mathrm{NE}} j+1$) in $T$, one for each row of $\mathfrak{t}^{\boldsymbol{\mu}}$. Further, for $a = 0, 1$, we let $\ell_a(T_{\mathrm{SW}})$ (resp. $\ell_a(T_{\mathrm{NE}})$) be the number of elements in $T_{\mathrm{SW}}$ (resp. $T_{\mathrm{NE}}$) with parity $\bar{a}$.

**Definition 21.** Let $T$ be standard $\boldsymbol{\lambda}$-tableau. We define its $\boldsymbol{\mu}$-weight

$$\mathrm{wt}_{\boldsymbol{\mu}}(T) = \begin{cases} 0, & \text{if } T \text{ is not } \boldsymbol{\mu}\text{-SW-NE;} \\ (-q^{-1})^{\ell_0(T_{\mathrm{SW}})+\ell_1(T_{\mathrm{NE}})} q^{\ell_0(T_{\mathrm{NE}})+\ell_1(T_{\mathrm{SW}})} \prod\limits_{i \in P_T^{\boldsymbol{\mu}}} Q_{c_T(i)}^{c_{\mathfrak{t}^{\boldsymbol{\mu}}}(i)}, & \text{if } T \text{ is } \boldsymbol{\mu}\text{-SW-NE.} \end{cases}$$

**Example 22.** Let $n = 20$, $m = 3$,

$$\boldsymbol{k}|\boldsymbol{\ell} = (1|1, 1|2, 1|3)$$

and

$$\boldsymbol{\mu} = ((2,1,1); (3,2,2,1); (4,3,1)).$$

Then

$$\mathfrak{t}^{\boldsymbol{\mu}} = \left( \begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \end{array} ; \begin{array}{lll} 5 & 6 & 7 \\ 8 & 9 & \\ 10 & 11 & \\ 12 & & \end{array} ; \begin{array}{llll} 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & \\ 20 & & & \end{array} \right)$$

Now we insert the sequence $\boldsymbol{i}$ of Example 11 via the RSK superinsertion algorithm. Thanks to the bijection in Equ. (2), the sequence $\boldsymbol{i}$ is the row reading sequence of the following diagram

$$\boldsymbol{i} = \left( \begin{array}{ll} x_1^{(1)} & x_1^{(2)} \\ y_1^{(1)} & \\ y_1^{(2)} & \end{array} ; \begin{array}{lll} x_1^{(3)} & y_1^{(3)} & y_3^{(3)} \\ y_1^{(1)} & y_1^{(1)} & \\ y_2^{(2)} & y_1^{(2)} & \\ y_1^{(3)} & & \end{array} ; \begin{array}{llll} y_2^{(3)} & y_3^{(3)} & y_2^{(2)} & x_1^{(1)} \\ x_1^{(2)} & x_1^{(3)} & x_1^{(1)} & \\ y_2^{(3)} & & & \end{array} \right).$$

Applying the RSK superinsertion algorithm, we obtain

$$S = \left( \begin{array}{llll} x_1^{(1)} & x_1^{(1)} & x_1^{(1)} & y_1^{(1)} \\ y_1^{(1)} & & & \\ y_1^{(1)} & & & \end{array} ; \begin{array}{lll} x_1^{(2)} & x_1^{(2)} & y_1^{(2)} \\ y_1^{(2)} & y_2^{(2)} & \\ y_2^{(2)} & & \end{array} ; \begin{array}{lllll} x_1^{(3)} & x_1^{(3)} & y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} & & \\ y_3^{(3)} & & & & \end{array} \right),$$

$$T = \left( \begin{array}{llll} 1 & 3 & 16 & 19 \\ 8 & & & \\ 9 & & & \end{array} ; \begin{array}{lll} 2 & 4 & 10 \\ 11 & 15 & \\ 17 & & \end{array} ; \begin{array}{llll} 5 & 6 & 7 & 14 \\ 12 & 13 & 20 & \\ 18 & & & \end{array} \right).$$

By Definition 17, we yield

$$1 \xrightarrow{\text{SW}} 2, \quad 3, \quad 4;$$
$$5 \xrightarrow{\text{SW}} 6 \xrightarrow{\text{SW}} 7, \quad 8 \xrightarrow{\text{NE}} 9, \quad 10 \xrightarrow{\text{NE}} 11, \quad 12;$$
$$13 \xrightarrow{\text{SW}} 14 \xrightarrow{\text{NE}} 15 \xrightarrow{\text{NE}} 16, \quad 17 \xrightarrow{\text{SW}} 18 \xrightarrow{\text{NE}} 19, \quad 20.$$

Therefore

$$T_{\text{SW}} = \{1, 5, 6, 13, 17\},$$
$$T_{\text{NE}} = \{8, 10, 14, 15, 18\},$$
$$P_T^{\boldsymbol{\mu}} = \{2, 3, 4, 7, 8, 10, 12, 14, 18, 20\}.$$

Clearly $\ell_0(T_{\text{SW}}) = 3$, $\ell_1(T_{\text{SW}}) = 2$, $\ell_0(T_{\text{NE}}) = 1$, $\ell_1(T_{\text{NE}}) = 4$.

Finally, by Definition 21, we get

$$\text{wt}_{\boldsymbol{\mu}}(T) = \text{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) = (-q^{-1})^7 q^3 Q_1^3 Q_2^4 Q_3^{13},$$

which coincides with that of Example 11.

The following fact clarify the relationship between Definitions 10 and 21.

**Lemma 23.** *If* $(S,T) \stackrel{\mathrm{RSK}}{\longleftrightarrow} z_{\boldsymbol{i}} = z_{i_1}, z_{i_2}, \ldots, z_{i_n}$, *then* $\mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) = \mathrm{wt}_{\boldsymbol{\mu}}(T)$.

*Proof.* Proposition 19 shows $\boldsymbol{i} = (i_1, \ldots, i_n)$ is (not) $\boldsymbol{\mu}$-up-down if and only if $T$ is (not) $\boldsymbol{\mu}$-SW-NE, and $i_p$ is a peak of $\boldsymbol{i}$ if and only if $p$ is a peak of $T$. According to Definitions 10 and 21, we prove the lemma by noting that $\ell_a(\boldsymbol{i}_<) = \ell_a(T_{\mathrm{SW}})$ and $\ell_a(\boldsymbol{i}_\geqslant) = \ell_a(T_{\mathrm{NE}})$ for $a = 0, 1$. □

**Theorem 24.** *For* $\boldsymbol{\mu} \in \mathscr{P}_{m,n}$, *we have*

$$q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \Bigg( \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) \Bigg) S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}).$$

*Proof.* Recall that for $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathscr{I}(n; k|\ell)$, we write $\boldsymbol{z_i} = \prod_{j=1}^n (-1)^{\overline{z_{i_j}}} z_{i_j}$. Thanks to Corollary 13 and Definition 10, we yield

$$\begin{aligned} q_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}; q, \boldsymbol{Q}) &= \sum_{\boldsymbol{i} \in \mathscr{I}(n; k+\ell)} \mathrm{wt}_{\boldsymbol{\mu}}(\boldsymbol{i}) \boldsymbol{z_i} \\ &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \sum_{(S,T) \in \mathrm{sstd}_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda}) \times \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) S(\boldsymbol{x}/\boldsymbol{y}) \\ &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) \sum_{S \in \mathrm{sstd}_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda})} S(\boldsymbol{x}/\boldsymbol{y}) \\ &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y}), \end{aligned}$$

where the second and the last qualities follow by using the RSK superinsertion algorithm and Equ. (4) respectively. □

*Remark* 25. If $m = 1$ then the super Frobenius formula Equ. (9) is Mitsuhashi's super Frobenius formula for the characters of Iwahaori-Hecke algebra $H_n(q)$ (cf. [14, Theorem 5.5]), that is, for $\mu \vdash n$,

$$q_\mu(\boldsymbol{x}/\boldsymbol{y}; q) = \sum_{\lambda \vdash n} \chi_q^\lambda(\mu) S_\lambda(\boldsymbol{x}/\boldsymbol{y}).$$

Now Theorem 24 enables us to obtain a super-version of Ram's formula in [16, 17]: for $\mu \vdash n$,

$$q_\mu(\boldsymbol{x}/\boldsymbol{y}; q) = \sum_{\lambda \vdash n} \Bigg( \sum_{T \in \mathrm{std}(\lambda)} \mathrm{wt}_\mu(T) \Bigg) S_\lambda(\boldsymbol{x}/\boldsymbol{y}),$$

which provides a new proof of Mitsuhashi's formula.

Note that the super Schur functions $S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y})$ are linearly independent [3, Lemma 6.4]. Equ. (9) and Theorem 24 implies

**Corollary 26.** *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{m,n}$, we have*

$$\chi_q^{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T).$$

*Remark* 27. Putting $q = 1$ and $Q_i = \varsigma^i$, Theorem 24 enables us to obtain the following super-symmetric function identity

$$P_{\boldsymbol{\mu}}(\boldsymbol{x}/\boldsymbol{y}) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{m,n}} \left( \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) \Big|_{\substack{q=1 \\ Q_i = \varsigma^i}} \right) S_{\boldsymbol{\lambda}}(\boldsymbol{x}/\boldsymbol{y})$$

and a character formula

$$\chi_1^{\boldsymbol{\lambda}}(w_{\boldsymbol{\mu}}) = \sum_{T \in \mathrm{std}(\boldsymbol{\lambda})} \mathrm{wt}_{\boldsymbol{\mu}}(T) \Big|_{\substack{q=1 \\ Q_i = \varsigma^i}}$$

for complex reflection group $W_{m,n}$. Note that Cantrell et. al. [5, Remark 3.5] give a character formula for $W_{m,n}$.

*Remark* 28. For $\boldsymbol{\lambda} \in \mathscr{P}_{m,n}$, we denote by $d_{\boldsymbol{\lambda}}$ the number of standard $\boldsymbol{\lambda}$-tableaux, that is, $d_{\boldsymbol{\lambda}}$ is the dimension of the irreducible representation of $H$ or $W_{m,n}$ corresponding to $\boldsymbol{\lambda}$. As a special case of the RSK superinsertion, we can restrict to sequences $\boldsymbol{z} = z_{i_1}^{(j_1)} \cdots z_{i_n}^{(j_n)}$ where $i_1, \ldots, i_n$ is a permutation of $n$ and $1 \leqslant j_r \leqslant m$, that is, $k_i + \ell_i = n$ for $i = 1, \ldots, m$. There are $n!m^n$ such sequences. Further, when we insert these special sequences, we get a pair $(S, T)$-standard tableaux (the $(\boldsymbol{k}, \boldsymbol{\ell})$-semistandard tableau $S$ is standard because its entries are different). Thus the RSK superinsertion algorithm gives the following equality

$$n!m^n = \sum_{\boldsymbol{\lambda} \in H(\boldsymbol{k}|\boldsymbol{\ell}; m,n)} d_{\boldsymbol{\lambda}}^2,$$

where $\boldsymbol{k}$, $\boldsymbol{\ell}$ satisfies $k_i + \ell_i = n$ for $i = 1, \ldots, m$.

We remark in closing that if $\ell = 0$, i.e. $\boldsymbol{\ell} = 0$, then $\mathrm{sstd}_{\boldsymbol{k}|\boldsymbol{\ell}}(\boldsymbol{\lambda})$ is the set of row-semistandard tableaux of shape $\boldsymbol{\lambda} \in \mathscr{P}_{m,n}$, the RSK superinsertion is the RSK insertion in [5, §5]. Thus Corollary 26 is Cantrell, Halverson and Miller's combinatorial formula [5, Corollary 3.4]. Both formulas compute the irreducible characters of $H$ via the weights of standard tableaux, while the definitions of weights are different.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