Flag-transitive, point-imprimitive 2-designs and direct products of symmetric groups

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Abstract

Consider the direct product of symmetric groups $S_c \times S_n$ and its natural action on $\mathcal{P} = C \times N$, where $|C| = c$ and $|N| = n$. We characterize the structure of 2-designs with point set $\mathcal{P}$ admitting flag-transitive, point-imprimitive automorphism groups $H \leqslant S_c \times S_n$. As an example of its applications, we show that H cannot be any subgroup of $D_{2c} \times S_n$ or $S_c \times D_{2n}$. Besides, some families of 2-designs admitting flag-transitive automorphism groups $S_c \times S_n$ are constructed by using complete bipartite graphs and cycles. Two families of these also admit flag-transitive, point-primitive automorphism groups $S_c \wr S_2$, a family of which attain the Cameron-Praeger upper bound $v = (k-2)^2$.

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1 Introduction

A 2-(v, k, λ) design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v elements (called points), $\mathcal{B}$ is a collection of b k-subsets of $\mathcal{P}$ (called blocks), such that any 2-subset of $\mathcal{P}$ is contained in precisely λ blocks. The number of blocks through a given point x is a constant, which is denoted by r. An automorphism group G of $\mathcal{D}$ is a permutation group on $\mathcal{P}$, preserving the block set $\mathcal{B}$. An automorphism group G is said to be point (or block)-transitive if G acts transitively on the point set $\mathcal{P}$ (or the block set $\mathcal{B}$), and is said to be flag-transitive if G acts transitively on the set of incident point-block pairs. The terminology point-imprimitive and point-primitive are defined similarly. It is well known that G is flag-transitive on $\mathcal{D}$ if and only if G is block-transitive and the block-stabilizer G_B is transitive on B for any block B.

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A 2-design $\mathcal{D}$ is said to be non-trivial if $2 < k < v - 1$. The parameters of $\mathcal{D}$ satisfy the two equations

$$\lambda(v-1) = r(k-1) \text{ and } bk = vr.$$

For further basic facts of 2-designs, refer to [8, Section 2.1] and [2, Chapter 3], for example.

The study of flag-transitive, point-imprimitive 2-designs is a long-term project. One of the most classic results would be due to Kantor in 1969. He gave some conditions in [10, Section 4], under which the flag-transitive automorphism groups of 2-designs are point-primitive. These results were also referred in [8, Section 2.3.7] by Dembowski. In 1987, Davies proved in [6] that there exist only finitely many flag-transitive, point-imprimitive 2-designs for fixed λ. In 1989, Delandtsheer and J. Doyen proved in [7, Theorem] that a block-transitive, point-imprimitive 2-design satisfies $v \leqslant (\binom{k}{2}-1)^2$. In 1993, Cameron and Praeger [4] proved that a flag-transitive, point-imprimitive 2-design satisfies $v \leqslant (k-2)^2$ (we call $v = (k-2)^2$ the Cameron-Praeger upper bound). In [4, Propositions 2.2, 3.6], they studied constructions of 2-designs admitting a block-transitive, point-imprimitive automorphism groups $S_c \wr S_n$ or $S_c \times S_n$ acting on the set $C \times N$, where C and N are sets of size c or n, respectively. In [5], they give a construction of a family of designs with a specified point partition, and give necessary and sufficient conditions for a design in the family to possess a flag-transitive automorphism group preserving the point partition.

Inspired by Cameron and Praeger's work in [4], we continue to consider flag-transitive 2-designs with the natural, imprimitive action of $S_c \times S_n$ on $C \times N$. In Section 2, by connecting with bipartite graphs, we characterize 2-designs admitting flag-transitive automorphism groups $S_c \times S_n$. In Section 3, we prove that if $H \leqslant S_c \times S_n$ and H acts as a flag-transitive automorphism group on a 2-design, then H cannot be any subgroup of $D_{2c} \times S_n$ or $S_c \times D_{2n}$, which are special subgroups of $S_c \times S_n$ with relatively small order. We tackle this by inspecting the projection maps ρ_1 and ρ_2 of H on S_c and S_n, respectively. It provides an example of tackling specific subgroups of $S_c \times S_n$ and we believe that the technique used in the proof in Section 3 could be applied to analyse other types of subgroups of G.

In the last section, some families of 2-designs admitting flag-transitive and point-imprimitive automorphism groups $S_c \times S_n$ are constructed using complete bipartite subgraphs and cycles. Two families of these also admit flag-transitive, point-primitive automorphism groups $S_c \wr S_2$ with the product action (Corollaries 23 and 28). Among them, a family of designs attain the Cameron-Praeger upper bound $v = (k-2)^2$(Corollary 28).

The first version of the current paper was finished and submitted in August, 2021. When the paper was under review, we were aware that some papers [1, 3, 12] concerning relevant topic appeared. In [1], the authors analyse block-transitive 2-designs and 3-designs with automorphism groups $S_c \times S_n$ or $S_c \wr S_2$. In [3], the authors observe the group $S_c \times S_n$ acting as a flag-transitive automorphism group of a block design with $4 \leqslant c \leqslant n \leqslant 70$ by developing and applying several algorithms. It is also worth noting that [12, Construction 3.2] presents a construction similar to our Construction 4.1 from different perspectives. Our work is independent of these papers. We deal with designs admitting flag-transitive automorphism groups $S_c \times S_n$ or their subgroups in the language of graphs and their automorphisms, and present some more general constructions.

2 Flag-transitive 2-designs with the action of $S_c \times S_n$

Throughout the paper we use conventional terminology of permutation groups and group actions, such as primitive groups, regular groups and induced action on a subset. For these basic definitions and facts, refer to [9], for example.

In this section, we characterize the flag-transitive 2-designs under the natural imprimitive action of $S_c \times S_n$. Throughout this paper, let $G = S_c \times S_n$ with $c, n \geqslant 2$, $H \leqslant G$ and $\mathcal{P} = C \times N$, where $C = \{1, 2, \ldots, c\}$ and $N = \{1, 2, \ldots, n\}$. In the following, we first present some necessary definitions.

Definition 1. Define the action φ of $G = S_c \times S_n$ on $\mathcal{P} = C \times N$:

$$\varphi : S_c \times S_n \longrightarrow Sym(\mathcal{P}) \text{ by } (g, h) \longrightarrow \varphi_{(g,h)}$$

with

$$(\alpha, \beta)^{\varphi_{(g,h)}} = (\alpha^g, \beta^h),\ (\alpha, \beta) \in \mathcal{P}$$

Clearly, the action φ described in Definition 1 is well-defined and faithful. The image is an imprimitive permutation group on $\mathcal{P}$ since the stabilizer $G_{(1,1)} \cong S_{c-1} \times S_{n-1}$ is not a maximal subgroup of G. In the current paper, the action of $G = S_c \times S_n$ on $\mathcal{P} = C \times N$ is always assumed to be the action defined in Definition 1.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-design admitting a flag-transitive automorphism group $H \leqslant G$. We can regard the points of a 2-design $\mathcal{D}$ as the edges of a complete bipartite graph $K_{c,n}$, whose bipartite blocks are C and N. Then regard a block of $\mathcal{D}$ as a set of k edges of $K_{c,n}$. Clearly, these $c + n$ vertices and k edges form a subgraph Γ of $K_{c,n}$. We denote by $E(\Gamma)$ the set of edges of Γ. Moreover, we have a natural permutation representation of H on the edges of $K_{c,n}$, which is equivalent to $\varphi|_H$ on $\mathcal{P}$. Clearly, we have the full automorphism group $Aut(K_{c,n}) = G$ if $c \neq n$, and $Aut(K_{c,n}) = G \wr S_2$ if $c = n$.

Definition 2. Let Γ be a bipartite graph with bipartite blocks C and N, and let $H \leqslant S_c \times S_n$. Then $\mathcal{D}(\Gamma, H)$ is defined to be the incidence structure whose points are all edges of the complete bipartite graph $K_{c,n}$, and block set is $\{E(\Gamma^h) : h \in H\}$.

Clearly, the group H acts block-transitively on the incidence structure $\mathcal{D}(\Gamma, H)$.

Definition 3. Let Γ be a bipartite graph with bipartite blocks C and N. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_c)$, where x_i is the number of vertices adjacent to vertex $i \in N$ and y_j is the number of vertices adjacent to vertex $j \in C$. The tuples $\mathbf{x}$, $\mathbf{y}$ are called the degree sequences of vertices of Γ in N and C, respectively.

In the remaining part of the paper, we always let Γ be a bipartite graph with bipartite blocks $C = \{1, 2, \ldots, c\}$ and $N = \{1, 2, \ldots, n\}$. Remove all isolated vertices of Γ and then the remaining vertices and edges of Γ form a subgraph of Γ. We denote this subgraph by $\overline{\Gamma}$. It is worth mentioning that some literature would use "$\overline{\Gamma}$" to denote the complement of the graph Γ but we still take the risk to use this notation. We use $\overline{\Gamma}_j (j = 1, 2, \ldots)$ to denote the connected components of $\overline{\Gamma}$. We always let $\mathbf{x}$ and $\mathbf{y}$ be the degree sequences of vertices in N and C, respectively. We say a group K is an edge-transitive automorphism group of Γ if $K \leqslant Aut(\Gamma)$ and K acts transitively on the set $E(\Gamma)$ of edges of Γ.

It is clear that any incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with a block-transitive automorphism group H is isomorphic to $\mathcal{D}(\Gamma, H)$ for some bipartite graph Γ when we regard the set $\mathcal{P}$ as the edges of $K_{c,n}$ and a block B as the set of edges of Γ. Clearly, H has equivalent representations on $\mathcal{D}$ and $\mathcal{D}(\Gamma, H)$.

If Γ^* is a subgraph of $K_{c,n}$, then we write H_{Γ^*} as the subgroup of H stabilizing the set of edges and the set of vertices of Γ^*. Clearly, the induced permutation group $H_{\Gamma^*}^{\Gamma^*} \leqslant Aut(\Gamma^*)$. For convenience, we write H^{Γ^*} instead of $H_{\Gamma^*}^{\Gamma^*}$.

Lemma 4. *If a 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with $\mathcal{P} = C \times N$ admits a flag-transitive automorphism group $H \leqslant S_c \times S_n$, then there exist Γ such that $\mathcal{D} \cong \mathcal{D}(\Gamma, H)$, and, for any two components $\overline{\Gamma}_{j_1}$, $\overline{\Gamma}_{j_2}$ of $\overline{\Gamma}$, there exists $h \in H$ such that $(\overline{\Gamma}_{j_1})^h = \overline{\Gamma}_{j_2}$. Moreover, H is edge-transitive on $K_{c,n}$, and all $H^{\overline{\Gamma}_j}$ are edge-transitive and permutationly isomorphic.*

Proof. For any such 2-design $\mathcal{D}$, we know that there exists a bipartite graph Γ such that $\mathcal{D} \cong \mathcal{D}(\Gamma, H)$. The flag-transitivity of H on $\mathcal{D}$ implies that H is transitive on $\mathcal{P}$, and H_B is transitive on B for any block B. Thus H is edge-transitive on $K_{c,n}$. Moreover, since the set of edges of $\overline{\Gamma}$ is exactly a block of $\mathcal{D}$, equivalently we have $H^{\overline{\Gamma}}$ is an edge-transitive automorphism group of $\overline{\Gamma}$.

If $\overline{\Gamma}$ is not connected, then $\overline{\Gamma}$ is a union of connected components $\overline{\Gamma}_j$. Let $\ell_1, \ell_2 \in E(\Gamma)$. There exists $h \in H_\Gamma$ such that $\ell_1^h = \ell_2$. If ℓ_1 and ℓ_2 are edges of the same component $\overline{\Gamma}_j$, then h induces an automorphism of $\overline{\Gamma}_j$, i.e., $h \in H_{\overline{\Gamma}_j}$. Thus $H^{\overline{\Gamma}_j}$ is edge-transitive. If $\ell_1 \in E(\overline{\Gamma}_{m_1})$ and $\ell_2 \in E(\overline{\Gamma}_{m_2})$ with $m_1 \neq m_2$, by the connectivity of $\overline{\Gamma}_{m_1}$ and $\overline{\Gamma}_{m_2}$, we have h maps all vertices and edges of $\overline{\Gamma}_{m_1}$ onto $\overline{\Gamma}_{m_2}$, and h^{-1} maps all vertices and edges of $\overline{\Gamma}_{m_2}$ onto $\overline{\Gamma}_{m_1}$. So h induces an isomorphism from $\overline{\Gamma}_{m_1}$ to $\overline{\Gamma}_{m_2}$. Since $h \in H$, we have $(H_{\overline{\Gamma}_{m_1}})^h = H_{(\overline{\Gamma}_{m_1})^h} = H_{\overline{\Gamma}_{m_2}}$. Thus $H^{\overline{\Gamma}_{m_1}}$ is permutationly isomorphic to $H^{\overline{\Gamma}_{m_2}}$. The arbitrariness of ℓ_1 and ℓ_2 yields the result. □

Lemma 5. *Suppose that $\overline{\Gamma}$ is a union of isomorphic components with bipartite blocks of size d and i, and for any two components $\overline{\Gamma}_{j_1}$, $\overline{\Gamma}_{j_2}$ there exists $g \in G$ such that $(\overline{\Gamma}_{j_1})^g = \overline{\Gamma}_{j_2}$. If one of such components admits a subgroup of $S_d \times S_i$ as an edge-transitive automorphism group, then there exists a group $H \leqslant G$ such that $\mathcal{D}(\Gamma, H)$ admits a flag-transitive automorphism group H. In particular, G is a flag-transitive automorphism group of $\mathcal{D}(\Gamma, G)$.*

Proof. Let $\overline{\Gamma}$ be a union of m isomorphic connected components $\overline{\Gamma}_j$ $(j = 1, 2, \ldots, m, m \geqslant 1)$, and $\overline{G_{j_0}} \leqslant S_d \times S_i$ be an edge-transitive automorphism group of $\overline{\Gamma}_{j_0}$. Let also G_{j_0} be the extension of $\overline{G_{j_0}}$ to the permutation group on $C \times N$ by fixing the added points. For any $\overline{\Gamma}_j$ there exists $g \in G$ such that $\overline{\Gamma}_j = (\overline{\Gamma}_{j_0})^g$ and so $G_j := g^{-1} G_{j_0} g$ induces an edge-transitive automorphism group $G_j^{\overline{\Gamma}_j}$ of $\overline{\Gamma}_j$. Now consider the induced action of G on $\{(\overline{\Gamma}_1)^g : g \in G\}$. Let $\Theta = \{\overline{\Gamma}_1, \overline{\Gamma}_2, \ldots, \overline{\Gamma}_m\}$. By the condition, we can easily get that G_Θ is transitive on Θ. Let M be any transitive subgroup of G_Θ on Θ and let $L = \langle G_1, G_2, \ldots, G_m, M \rangle$. Then L induces an edge-transitive automorphism group on Γ. If $L \leqslant H \leqslant G$, then the incidence structure $D(\Gamma, H)$ admits a flag-transitive automorphism group H. Since $L \leqslant H_\Theta \leqslant G_\Theta$, we get that G is also flag-transitive on $D(\Gamma, G)$. □

Lemma 6. [4, Proposition 1.3] *Let X be a permutation group of the set Ω, having orbits $O_1, \ldots, O_m$ on the set of 2-subsets of Ω, and B a k-subset of Ω. Then (Ω, B^X) is a 2-design if and only if the ratio of the number of members of O_ℓ contained in B to the total number of members of O_ℓ is independent of ℓ.*

In the following Theorem 7, we consider the special case that $H = G = S_c \times S_n$. We give equivalent conditions of the existence of a 2-design admitting a flag-transitive automorphism group G.

Theorem 7. *The incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-design admitting a flag-transitive automorphism group $G = S_c \times S_n$ if and only if there exists a bipartite graph Γ with $\mathcal{D} \cong \mathcal{D}(\Gamma, G)$, and with $\boldsymbol{x}$, $\boldsymbol{y}$ the degree sequences of vertices in N and C respectively:*

$$\boldsymbol{x} = \{d, d, \ldots, d, 0, 0 \ldots, 0\} \text{ and } \boldsymbol{y} = \{i, i, \ldots, i, 0, 0, \ldots, 0\},$$

where $d(\geqslant 2)$ has multiplicity s in $\boldsymbol{x}$ and $i(\geqslant 2)$ has multiplicity $\frac{sd}{i}$ in $\boldsymbol{y}$, such that

(a) *$\overline{\Gamma}$ is a union of isomorphic connected components, and for any two components $\overline{\Gamma}_{j_1}$, $\overline{\Gamma}_{j_2}$ there exists $g \in G$ such that $(\overline{\Gamma}_{j_1})^g = \overline{\Gamma}_{j_2}$;*

(b) *every $G^{\overline{\Gamma}_j}$ is edge-transitive;*

(c) *the 5-tuple (c, n, s, d, i) satisfies* $\begin{cases} c(i-1) = d(s-1), \\ (c-1)(i-1) = (d-1)(n-1). \end{cases}$

Proof. By Lemma 4, for any such flag-transitive 2-design $\mathcal{D}$, there exists Γ such that $\mathcal{D} \cong \mathcal{D}(\Gamma, G)$, where Γ satisfies (a) and (b). Moreover, since $G^{\overline{\Gamma}}$ induces transitive groups on the two bipartite blocks of $\overline{\Gamma}$ respectively, every vertex in a bipartite block of $\overline{\Gamma}$ has the same degree. By the multiple transitivity of symmetric groups, without loss of generality we can assume that Γ has degree sequences

$$\mathbf{x} = \{d, d, \ldots, d, 0, 0 \ldots, 0\} \text{ and } \mathbf{y} = \{i, i, \ldots, i, 0, 0, \ldots, 0\},$$

where d has multiplicity s in $\mathbf{x}$ and i has multiplicity $\frac{sd}{i}$ in $\mathbf{y}$. The action φ has three orbits on 2-subsets of $\mathcal{P}$: $O_1 = \{\{(\alpha, \beta), (\gamma, \beta)\} : \alpha \neq \gamma\}$; $O_2 = \{\{(\alpha, \beta), (\alpha, \gamma)\} : \beta \neq \gamma\}$; $O_3 = \{\{(\alpha, \beta), (\gamma, \delta)\} : \alpha \neq \gamma, \beta \neq \delta\}$. Clearly, $|O_1| = n\binom{c}{2}$, $|O_2| = c\binom{n}{2}$ and $|O_3| = c(c-1)\binom{n}{2}$. Regard the 2-subsets of $\mathcal{P}$ as the unordered pairs of edges in Γ. From Lemma 6 we know that $d, i \geqslant 2$. It is easy to count that the number of members of O_1 contained in B is $s\binom{d}{2}$ and the number of members of O_2 contained in B is $\frac{sd}{i}\binom{i}{2}$. So the number of members of O_3 contained in B is $\binom{sd}{2} - s\binom{d}{2} - \frac{sd}{i}\binom{i}{2}$. By Lemma 6, $\mathcal{D}(\Gamma, G)$ is a 2-design if and only if

$$\frac{s\binom{d}{2}}{n\binom{c}{2}} = \frac{\frac{sd}{i}\binom{i}{2}}{c\binom{n}{2}} = \frac{\binom{sd}{2} - s\binom{d}{2} - \frac{sd}{i}\binom{i}{2}}{c(c-1)\binom{n}{2}}.$$

It is obvious that the first equation is equivalent to $(c-1)(i-1) = (d-1)(n-1)$, and the second equation is equivalent to $c(i-1) = d(s-1)$. The "if" part follows from Lemmas 5 and 6. $\square$

3 On the subgroups of $D_{2c} \times S_n$ or $S_c \times D_{2n}$

Note that the symmetric group is the largest permutation group on a given set, with extremely high transitivity. We may also need to consider some much "smaller" transitive subgroups. The "smallest" transitive groups would be cyclic groups or regular groups and the slightly "bigger" groups would be dihedral groups. It is a natural thought to ask what may happen if H is a relatively small subgroup of $G = S_c \times S_n$. For example, we can start from $Z_c \times S_n$ or $D_{2c} \times S_n$. In this section, we prove that if H is a flag-transitive automorphism group of a non-trivial 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, then H cannot be any subgroup of $D_{2c} \times S_n$ or $S_c \times D_{2n}$, by showing that H cannot project into a dihedral group. Here we define ρ_1 as the left projection map from H to S_c, i.e., $\rho_1(H) = \{h_1 \in S_c : (h_1, h_2) \in H\}$ and ρ_2 is defined as the right projection map similarly. We denote by $V_1(\overline{\Gamma})(\subseteq C)$ and $V_2(\overline{\Gamma})(\subseteq N)$ the two bipartite blocks of $\overline{\Gamma}$, respectively.

Lemma 9, Theorem 10 and their proofs provide an example of analysing specific subgroups of G. We believe that the technique applied in the proof could be generalised to analyse other types of subgroups in some way. The following lemma is a well known technique.

Lemma 8. [4, Proposition 1.1] *Let X be a permutation group on Ω, and $B \subseteq \Omega$. If $\mathcal{D} = (\Omega, B^X)$ is a 2-(v, k, λ) design, then for any M with $X \leqslant M \leqslant Sym(\Omega)$, $\mathcal{D} = (\Omega, B^M)$ is a 2-(v, k, λ^*) design for some λ^*.*

Lemma 9. *Let $H \leqslant D_{2c} \times S_n$ where $c > 3$ is odd and $\mathcal{D}(\Gamma, H)$ be a non-trivial 2-design for some Γ. If each $\alpha \in V_2(\overline{\Gamma})$ has the same valency d and $(H_\Gamma)_\alpha$ acts transitively on the neighborhood $\Gamma(\alpha)$ of α, then $1 < d < \frac{c+1}{2}$ and $d \nmid c$.*

Proof. The group D_{2c} acting on C has a point-stabilizer of order 2, which is a complement of Z_c. So D_{2c} acts on C as its natural action on a regular n-gon. Without loss of generality, assume that $D_{2c} = Z_c : Z_2$ with $Z_c = \langle (1\,2\,\ldots\,c) \rangle$. Let $\alpha \in N$ and $(\delta, \alpha) \in E(\Gamma)$. Then $\delta^{\rho_1((H_{\overline{\Gamma}})_\alpha)}$ are those points in C which are adjacent to α. So $|\delta^{\rho_1((H_{\overline{\Gamma}})_\alpha)}|$ is the valency d of α in Γ. Note that ρ_1 and ρ_2 are homomorphisms and so $\rho_1((H_{\overline{\Gamma}})_\alpha) \leqslant \rho_1(H) \leqslant D_{2c}$. Each $\rho_1((H_{\overline{\Gamma}})_\alpha)$ is isomorphic to either a subgroup Z_{c_0} of Z_c or a subgroup $D_{2c_0} = Z_{c_0} : Z_2$ of D_{2c}, or a complement of Z_c of order 2.

If for each $\alpha \in V_2(\overline{\Gamma})$, $|\Gamma(\alpha)| = d \geqslant 3$, then $\rho_1((H_{\overline{\Gamma}})_\alpha)$ has order greater than or equal to 3 and so is isomorphic to either a subgroup Z_{c_0} of Z_c or a subgroup $D_{2c_0} = Z_{c_0} : Z_2$ of D_{2c}. If $\rho_1((H_{\overline{\Gamma}})_\alpha) \cong Z_{c_0} \leqslant Z_c$, $\Gamma(\alpha)$ is an orbit of the unique subgroup Z_{c_0} of Z_c. Hence, for $m \in V_1(\overline{\Gamma})$ with $(m, \alpha) \in E(\Gamma)$, $m^{\rho_1((H_{\overline{\Gamma}})_\alpha)} = m^{Z_{c_0}} = \Gamma(\alpha)$ equals

$$\Theta_{m,c_0} := \{m, m + \frac{c}{c_0}, m + 2\frac{c}{c_0}, \ldots, m + (c_0 - 1)\frac{c}{c_0}\},$$

where the addition is performed modulo c. If $\rho_1((H_{\overline{\Gamma}})_\alpha) \cong D_{2c_0} < D_{2c}$, then by the action of $D_{2c_0} < D_{2c}$ on a regular n-gon (as a Frobenius group), D_{2c_0} has an orbit Θ_{u,c_0} on C for some $u \in C$ whereas other orbits of D_{2c_0} on $C \setminus \Theta_{u,c_0}$ are all regular orbits of length $2c_0$. Hence, $\Gamma(\alpha)$ is either the Θ_{u,c_0} or a regular orbit of D_{2c_0} on $C \setminus \Theta_{u,c_0}$.

Let $H^* := D_{2c} \times S_n$ and then $\mathcal{D}(\Gamma, H^*)$ is also a 2-design by Lemma 8. Consider the action of H^* on the 2-subsets of edges of $K_{c,n}$. Let O_1 be the orbit of H^* containing $\{(1,1),(2,1)\}$, i.e., $O_1 := \{(1,1),(2,1)\}^{H^*}$. By the natural action of a dihedral group on a n-gon, the distance of any two points in C is invariant. So O_1 contains all pairs of edges of $K_{c,n}$ such that the two edges in each such pair have the two coordinates in C with difference 1 (modulo c) and the same coordinate in N. This gives

$$O_1 = \{\{(\ell,\beta),(\ell+1,\beta)\} : \ell \in C, \beta \in N\}.$$

Similarly, we define another orbit $O_{c/c_0} := \{(1,1),(1+\frac{c}{c_0},1)\}^{H^*}$:

$$O_{c/c_0} = \{\{(\ell,\beta),(\ell+\frac{c}{c_0},\beta)\} : \ell \in C, \beta \in N\}.$$

It is easy to calculate that $|O_1| = |O_{c/c_0}| = cn$.

If $d \mid c$ and $d > 1$, then $d \geqslant 3$, and for each $\alpha \in V_2(\overline{\Gamma})$, $\Gamma(\alpha) = \Theta_{u,c_0}$ for some $u \in C$, and $\rho_1((H_{\overline{\Gamma}})_\alpha)$ is isomorphic to $Z_0 \leqslant Z_c$ or $D_{2c_0} \leqslant D_{2c}$. By Lemma 6, every orbit of H^* on 2-subsets of $\mathcal{P}$ should have at least one member contained in a block. If $c_0 < c$, then every pair of edges in $\overline{\Gamma}$ which have a common coordinate in N have the two coordinates in C with difference at least $\frac{c}{c_0}$ (modulo c). It follows that O_1 has no member contained in $E(\Gamma)$, a contradiction. Thus $c_0 = c$ and then $\Gamma(\alpha) = C$. So $\overline{\Gamma}$ is then a complete bipartite graph and we get that $\mathcal{D}(\Gamma, G)$ is a flag-transitive designs by Lemmas 5 and 8. Apply $i = s$ and $c = d$ to Theorem 7(c) and we obtain $i = s = n$, which implies that $\overline{\Gamma} = \Gamma = K_{c,n}$. So $\mathcal{D}(\Gamma, H)$ is trivial, a contradiction. If each element $\alpha \in V_2(\overline{\Gamma})$ has valency $d = 1$, then each connected component of $\overline{\Gamma}$ is $K_{1,i}$ for some i. Again, it is easy to see that no member of O_1 is contained in $E(\Gamma)$, a desired contradiction.

If $d \geqslant \frac{c+1}{2}$, then clearly $d \nmid c$ and $d \geqslant 3$ since $c > 3$ is odd and $d \neq c$. So, for each $\alpha \in V_2(\overline{\Gamma})$, $\Gamma(\alpha)$ must be a regular orbit of $\rho_1((H_{\overline{\Gamma}})_\alpha) \cong D_{2c_0} < D_{2c}$ with $d = 2c_0 < c$. By our assumption, $d = 2c_0 \geqslant \frac{c+1}{2}$. Since c is odd, having a proper factor c_0, there exists $t \geqslant 3$ such that $c = c_0 t$. So $4c_0 \geqslant c+1 = c_0 t + 1$, which implies that $t = 3$ and $c = 3c_0$. Now there are exactly two orbits of $\rho_1((H_{\overline{\Gamma}})_\alpha) \cong D_{2c_0}$ on C, namely, Θ_{u,c_0} and $C \setminus \Theta_{u,c_0}$ for some u. Thus $\Gamma(\alpha) = C \setminus \Theta_{u,c_0}$. Recall that the edge set $E(\Gamma)$ of Γ is a block B of $\mathcal{D}(\Gamma, H^*)$. Let $s := |V_2(\overline{\Gamma})|$. Denote by $B^{\{2\}}$ the set of 2-subsets of B and we then have $|B^{\{2\}} \cap O_1| = (\frac{c}{c_0} - 2)c_0 s$ and $|B^{\{2\}} \cap O_{c/c_0}| = (\frac{c}{c_0} - 1)c_0 s$. So

$$\frac{|B^{\{2\}} \cap O_1|}{|O_1|} \neq \frac{|B^{\{2\}} \cap O_{c/c_0}|}{|O_{c/c_0}|},$$

which contradicts Lemma 6. □

Theorem 10. *No non-trivial 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ admits a flag-transitive automorphism group H with $H \leqslant D_{2c} \times S_n$ or $H \leqslant S_c \times D_{2n}$.*

Proof. The group D_{2c} acting on $C = \{1, 2, \ldots, c\}$ has a point-stabilizer of order 2, which is clearly not a center of D_{2c} as D_{2c} is faithful on C. So D_{2c} acts on C as its natural action on a n-gon. By Lemma 8, we only need to deal with $H = D_{2c} \times S_n$ and $S_c \times D_{2n}$ since the non-existence of flag-transitive designs admitting $D_{2c} \times S_n$ or $S_c \times D_{2n}$ will imply the non-existence for any subgroup of them. It suffices to assume that $H = D_{2c} \times S_n$ with $Z_c = \langle (1\,2\,\ldots\,c) \rangle$ and $D_{2c} = Z_c : Z_2$. Now, by Lemma 4, suppose for the contrary that $\mathcal{D} = \mathcal{D}(\Gamma, H)$ is a non-trivial 2-design admitting a flag-transitive automorphism group H with Γ a bipartite graph with bipartite blocks $C = \{1, 2, \ldots, c\}$ and $N = \{1, 2, \ldots, n\}$, where $E(\Gamma)$ is exactly a block of $\mathcal{D}(\Gamma, H)$. Without loss of generality, assume that $(1, 1) \in E(\Gamma)$. All $\rho_1((H_{\overline{\Gamma}})_\alpha)$ with $\alpha \in V_2(\overline{\Gamma})$ are isomorphic to each other by the transitivity of $H_{\overline{\Gamma}}$ on $V_2(\overline{\Gamma})$. The edge-transitivity of $H^{\overline{\Gamma}}$ implies that $(H_\Gamma)_\alpha$ acts transitively on $\Gamma(\alpha)$.

If c is even, let $x := (1, 1) \in \mathcal{P}$, then $H_x = H_{(1,1)}$ stabilizes the point $y := (\frac{c}{2}+1, 1) \in \mathcal{P}$. Now H_x will then acts transitively on the set of blocks through x and y, which implies that $r = \lambda$, a contradiction. Hence, in the following we assume that c is odd. If $c > 3$, then the condition of Lemma 9 is clearly satisfied.

It is folklore that every orbit of H_x in $\mathcal{P} \setminus \{x\}$ intersects non-trivially with a given block containing x, by the two facts that H_x acts transitively on the set of blocks containing x and that for any point $y \in \mathcal{P} \setminus \{x\}$ there exists a block containing both x and y. Note that $H_x = H_{(1,1)}$ has orbits $\{(m, 1), (c - m + 2, 1)\}$, where $m \in \{2, 3, \ldots, \frac{c+1}{2}\}$. These $\frac{c-1}{2}$ orbits are exactly all those containing elements with coordinate $1 \in N$, excluding the trivial orbit. It follows that there are at least $\frac{c-1}{2} + 1 = \frac{c+1}{2}$ points in C adjacent to $1 \in N$. However, by Lemma 9, we have $|\Gamma(\alpha)| = d < \frac{c+1}{2}$ when $c > 3$, which yields a contradiction. Hence, $c = 3$.

Now, all element in $V_2(\overline{\Gamma})$ have valency $d \in \{2, 3\}$ as $|\Gamma(\alpha)| \geqslant \frac{c+1}{2} = 2$. If each connected component of $\overline{\Gamma}$ is a complete bipartite graph, then $\overline{\Gamma}$ is clearly connected and a complete bipartite graph. By Lemma 8, since $\mathcal{D}(\Gamma, H)$ is a 2-design, $\mathcal{D}(\Gamma, G)$ is also a 2-design. Since $H^{\overline{\Gamma}}$ is edge-transitive, $G^{\overline{\Gamma}}$ is also edge-transitive and thus G is flag-transitive on $\mathcal{D}(\Gamma, G)$. It follows that Theorem 7 holds. Apply $i = s$ to Theorem 7(c) and we get that $c = d$ and $i = s = n$, which implies that $\overline{\Gamma} = \Gamma = K_{c,n}$. So $\mathcal{D}(\Gamma, H)$ is trivial, a contradiction. If each component of $\overline{\Gamma}$ is not complete, then it immediately follows that $V_1(\overline{\Gamma}) = C$ and $d = 2$. Apply $d = 2$ to the first equation of Theorem 7(c) and we get that $2 \mid i - 1$ since c is odd. So i is odd. Then $k = ci$ is also odd, which contradicts the fact that $k = sd = 2s$ is even. □

The following problem arises very naturally.

Problem 11. Characterize flag-transitive 2-designs $\mathcal{D}(\Gamma, H)$ for some special $H < S_c \times S_n$.

4 The constructions related to complete bipartite subgraphs and cycles

4.1 The construction related to complete bipartite subgraphs

In this section, we construct flag-transitive 2-designs with the natural imprimitive product action of $G = S_c \times S_n$, by defining Γ as a graph whose non-trivial components are complete bipartite graphs.

Construction 4.1. Let $\mathcal{A} = (c, n, s, d, i)$ be a 5-tuple. Define the incidence structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ where $\Gamma(\mathcal{A})$ is a bipartite graph with bipartite blocks of size c and n, satisfying the following:

(a) $\overline{\Gamma(\mathcal{A})}$ is a union of $\frac{s}{i}$ many disjoint complete bipartite subgraphs $K_{d,i}$;

(b) $n \geqslant s \geqslant 2$, $c \geqslant d \geqslant 2$, $i \geqslant 2$, $i \mid s$ and $c \geqslant \frac{sd}{i}$. $(*)$

Remark 12. If a 5-tuple $\mathcal{A} = (c, n, s, d, i)$ satisfies condition $(*)$, then clearly $\Gamma(\mathcal{A})$ is well-defined. Recall that $\mathbf{x}$ and $\mathbf{y}$ denote the degree sequences of vertices in N and C, respectively. So, $\Gamma(\mathcal{A})$ has degree sequences

$$\mathbf{x} = \{d, d, \ldots, d, 0, 0 \ldots, 0\} \text{ and } \mathbf{y} = \{i, i, \ldots, i, 0, 0, \ldots, 0\},$$

where d has multiplicity s in $\mathbf{x}$ and i has multiplicity $\frac{sd}{i}$ in $\mathbf{y}$. It is easy to see that $v = nc$ and $k = sd$.

Example 13. Here we give an example to clarify the notation better: the bipartite graph $\Gamma(\mathcal{A})$ with $\mathcal{A} = (9, 5, 4, 3, 2)$ is described in Figure 1.

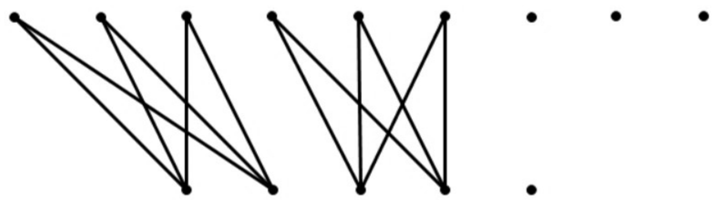

Figure 1: The bipartite graph $\Gamma(\mathcal{A})$ with $\mathcal{A} = (9, 5, 4, 3, 2)$

For any component $\overline{\Gamma(\mathcal{A})}_j$ of $\Gamma(\mathcal{A})$, we have that $G^{\overline{\Gamma(\mathcal{A})}_j}$ is edge-transitive. By Lemma 5, G acts flag-transitively on the incidence structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ and then Theorem 7(a) and (b) hold. Moreover, if $\mathcal{A} = (c, n, s, d, i)$ satisfies the equations in Theorem 7(c), then $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a flag-transitive 2-design.

Proposition 14. *The number of blocks of* $\mathcal{D}(\Gamma(\mathcal{A}), G)$ *in construction 4.1 is equal to*

$$b = |\mathcal{B}| = \frac{n!c!}{(n-s)!(\frac{s}{i})!(d!)^{\frac{s}{i}}(i!)^{\frac{s}{i}}(c - \frac{sd}{i})!}.$$

Proof. The result is obtained by simple counting argument. First take s many vertices on N which will occur in a block. There are $\binom{n}{s}$ possibilities. Then separate these s vertices into $\frac{s}{i}$ classes each of size i. There are $\frac{\binom{s}{i}\binom{s-i}{i}\binom{s-2i}{i}\cdots\binom{i}{i}}{(\frac{s}{i})!}$ choices. Finally, for each of these $\frac{s}{i}$ classes, we arrange d vertices from the set C such that each of these vertices is adjacent to every vertex of the chosen class. There are $\binom{c}{d}\binom{c-d}{d}\cdots\binom{c-(\frac{s}{i}-1)d}{d}$ many possibilities. Therefore,

$$b = \binom{n}{s} \cdot \frac{\binom{s}{i}\binom{s-i}{i}\binom{s-2i}{i}\cdots\binom{i}{i}}{(\frac{s}{i})!} \cdot \binom{c}{d}\binom{c-d}{d}\cdots\binom{c-(\frac{s}{i}-1)d}{d}.$$

Simplify it and we have the result. □

Proposition 15. *If $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a non-trivial 2-design, then $\overline{\Gamma(\mathcal{A})}$ is not connected.*

Proof. If $\overline{\Gamma(\mathcal{A})}$ is connected, then it has only one component and so $i = s$. By Theorem 7(c), we have $c = d$ and $i = n = s$. This implies that

$$\mathcal{A} = (c, n, s, d, i) = (c, n, n, c, n).$$

Then $\Gamma(\mathcal{A}) = K_{c,n}$ and so $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is trivial. □

In the following we present constructions in three special cases: (1) $i = 2$, (2) $d = 2$ and (3) $i = d$, since it turns out that these cases generate some infinite families of 2-designs with very nice arithmetical properties. To be specific, for any given one or two integers, we can construct corresponding designs, with their parameters (v, k, λ) known explicitly.

Theorem 16. *Suppose $\mathcal{A} = \{c, n, s, d, 2\}$. The incidence structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if $\mathcal{A} = (td^2 - td + d, td + 2, td - t + 2, d, 2)$, where $d \geqslant 2$, $t \geqslant 0$ and $t(d-1)$ is even.*

Proof. By Theorem 7, the structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if

$$\begin{cases} d(s-1) = c, \\ (n-1)(d-1) = c - 1, \\ (*):\ n \geqslant s \geqslant 2,\ c \geqslant d \geqslant 2,\ 2 \mid s \ \text{and}\ c \geqslant \dfrac{sd}{2}. \end{cases}$$

are satisfied. By the first two equations, we have

$$n = \frac{d(s-1)-1}{d-1} + 1 = \frac{d(s-1)-d+d-1}{d-1} + 1 = \frac{d(s-2)}{d-1} + 2.$$

By $d - 1 \mid d(s-2)$, we have $d - 1 \mid s - 2$. There exists an integer $t \geqslant 0$ such that $s - 2 = t(d-1)$. Then the parameters n, c and s are determined by d and t in the first two equations. Thus any solution of the equations has form $(c, n, s, d) = (td(d-1)+d, td+2, t(d-1)+2, d)$. On the other hand, it is easy to check that for any $d > 1$ and $t \geqslant 0$, the above tuple (c, n, s, d) satisfies the first two equations. Moreover, if $s = t(d-1)+2$ is even, then $i = 2$ divides s and so (c, n, s, d) satisfies conditions $(*)$. □

Theorem 17. *Suppose $\mathcal{A} = \{c, n, s, 2, i\}$. The incidence structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if $\mathcal{A} = (it + 2, i^2t - it + i, \frac{i^2t - it + 2i}{2}, 2, i)$, where $i \geqslant 2$, $t \geqslant 0$ and $t(i-1)$ is even.*

Proof. By Theorem 7, the structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if

$$\begin{cases} c(i-1) = 2(s-1), \\ (c-1)(i-1) = n - 1, \\ (*):\ n \geqslant s \geqslant 2,\ c \geqslant 2,\ i \geqslant 2,\ i \mid s \ \text{and}\ c \geqslant \dfrac{2s}{i}. \end{cases}$$

are satisfied. By the first equation and $i \mid s$, we have $i \mid c-2$. So there exists an integer $t \geqslant 0$ such that $it = c-2$. Then the parameters n, c and s are determined by i and t in the first two equations. Thus any solution of the equations has form $(c,n,s,i) = (it+2, (it+1)(i-1)+1, \frac{(it+2)(i-1)}{2}+1, i)$. On the other hand, it is easy to check that for any $i \geqslant 2$ and $t \geqslant 0$, the tuple (c,n,s,i) satisfies the first two equations. Moreover, if $t(i-1)$ is even, then $c-t = t(i-1)+2$ is even. By the first equation, we have $2s = i(c-t)$. Then $2 \mid c-t$ implies that $i \mid s$. The condition $(*)$ is satisfied. $\square$

The following theorem shows an interesting fact that if $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design with $\mathcal{A} = (c,n,s,2,2)$, then it is flag-transitive with $v = (k-2)^2$. So these 2-designs attain the Cameron-Praeger upper bound.

Theorem 18. *Suppose $\mathcal{A} = (c,n,s,2,2)$. The structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if $\mathcal{A} = (c, c, \frac{c+2}{2}, 2, 2)$ with $c \equiv 2 \pmod 4$. The parameters of $\mathcal{D}(\Gamma(\mathcal{A}), G)$ are*

$$(v,k,\lambda) = (c^2, c+2, \frac{(c+2)(c-1)!(c-2)!}{2^{\frac{c+2}{2}}(\frac{c+2}{4})!((\frac{c-2}{2})!)^2}).$$

Proof. This is an application of Theorem 16. Let $i = d = 2$. Then $(c,n,s,d) = (2t+2, 2t+2, t+2, 2) = (c, c, \frac{c+2}{2}, 2)$. Since t is even, we have $c \equiv 2 \pmod 4$. The parameters v and k follow from $v = nc$ and $k = sd$. By Proposition 14, the parameter λ is calculated in the following:

$$\lambda = \frac{bk(k-1)}{v(v-1)} = \frac{n!c!}{(n-s)!(\frac{s}{i})!(d!)^{\frac{s}{i}}(i!)^{\frac{s}{i}}(c-\frac{sd}{i})!} \cdot \frac{sd(sd-1)}{cn(cn-1)}.$$

Reduce the equation and the result is obtained. $\square$

The next theorem deals with the case $i = d$. This case also gives a family of 2-designs admitting flag-transitive automorphism groups $S_c \wr S_2$, which is a primitive group of product action type on $\mathcal{P}$. Note that all finite primitive permutation groups are classified into several types by the O'Nan-Scott theorem and the explicit description of these types of groups can be found in [9] and [11], for example.

Theorem 19. *Suppose $\mathcal{A} = (c,n,s,d,d)$. The structure $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if $\mathcal{A} = (td^2+d, td^2+d, td^2-td+d, d, d)$ with $t \geqslant 0$.*

Proof. Let $i = d$. By Theorem 7, $\mathcal{D}(\Gamma(\mathcal{A}), G)$ is a 2-design if and only if

$$\begin{cases} c(d-1) = d(s-1), \\ c = n, \\ (*):\ n \geqslant s \geqslant 2,\ c \geqslant d \geqslant 2,\ d \mid s \ \text{and}\ c \geqslant s. \end{cases}$$

are satisfied. Then $d-1 \mid s-1$. There exists $q \geqslant 0$ such that $(d-1)q = s-1$. By $(*)$, we also have $d \mid s$, and so $d \mid q-1$. There exists $t \geqslant 0$ such that $dt = q-1$. This implies $s = (d-1)q+1 = (d-1)(dt+1)+1 = td^2-td+d$ and $c = n = dq = td^2+d$. So $(c,s) = (td^2+d, td^2-td+d)$. On the other hand, for given integers $t \geqslant 0$ and $d \geqslant 2$, it is easy to verify that such (c,s) with $c = n$ is a solution of the equations. $\square$

Let $G^* = S_c \wr S_2 = (S_c \times S_c) \rtimes \langle \pi \rangle$, acting on $\mathcal{P} = C \times C$ by its primitive product action, i.e., $(\alpha, \beta)^{(g_1,g_2)\pi} = (\beta^{g_2}, \alpha^{g_1})$ and $(\alpha, \beta)^{(g_1,g_2)} = (\alpha^{g_1}, \beta^{g_2})$. For the designs constructed in Theorem 19, it is easy to see that

$$\mathcal{D}(\Gamma(\mathcal{A}), G) = \mathcal{D}(\Gamma(\mathcal{A}), G^*)$$

since all components of $\overline{\Gamma(\mathcal{A})}$ are complete bipartite graphs whose bipartite blocks have the same size. Hence, this is also a family of flag-transitive 2-designs admitting primitive automorphism groups of product action type.

We have constructed three infinite families of flag-transitive, point-imprimitive designs: $\mathcal{D}(\Gamma(\mathcal{A}), G)$ with $\mathcal{A} = (c, n, s, d, 2)$, $(c, n, s, 2, i)$ and (c, n, s, d, d), respectively. The family constructed in Theorem 18 is actually a subfamily of the family constructed in Theorem 19. Corollaries 20-23 are direct corollaries of Theorems 16-19, respectively.

Corollary 20. *For any given integers $d \geqslant 2$ and $t \geqslant 0$, if $t(d-1)$ is even, then there exists a 2-$((td^2 - td + d)(td + 2), td^2 - td + 2d, \lambda)$ design admitting a flag-transitive, point-imprimitive automorphism group $S_{td(d-1)+d} \times S_{td+2}$, where $\lambda = \frac{bsd(sd-1)}{nc(nc-1)}$ and*

$$b = \frac{n!c!}{2^{\frac{s}{2}}(n-s)!(\frac{s}{2})!(d!)^{\frac{s}{2}}(c - \frac{sd}{2})!} \text{ with } (c, n, s) = (td^2 - td + d,\ td + 2,\ td - t + 2).$$

Corollary 21. *For any given integers $i \geqslant 2$ and $t \geqslant 0$, if $t(i-1)$ is even, then there exists a 2-$((it+2)(i^2t - it + i), i^2t - it + 2i, \lambda)$ design admitting a flag-transitive, point-imprimitive automorphism group $S_{it+2} \times S_{(it+1)(i-1)+1}$, where $\lambda = \frac{bsd(sd-1)}{nc(nc-1)}$ and*

$$b = \frac{n!c!}{2^{\frac{s}{i}}(n-s)!(\frac{s}{i})!(i!)^{\frac{s}{i}}(c - \frac{2s}{i})!} \text{ with } (c, n, s) = (it + 2,\ i^2t - it + i,\ \frac{i^2t - it + 2i}{2}).$$

Corollary 22. *For any given integer $c > 2$ with $c \equiv 2 \pmod 4$, there exists a 2-$(c^2, c + 2, \frac{(c+2)(c-1)!(c-2)!}{2^{\frac{c+2}{2}}(\frac{c+2}{4})!((\frac{c-2}{2})!)^2})$ design admitting a flag-transitive, point-imprimitive automorphism group $S_c \times S_c$, attaining the Cameron-Praeger upper bound.*

Corollary 23. *For any given integers $d \geqslant 2$ and $t \geqslant 0$, there exists a 2-$((td^2 + d)^2, td^3 - td^2 + d^2, \lambda)$ design admitting a flag-transitive, point-imprimitive automorphism group $S_{td^2+d} \times S_{td^2+d}$, and a flag-transitive, point-primitive automorphism group $S_{td^2+d} \wr S_2$, where $\lambda = \frac{bsd(sd-1)}{c^2(c^2-1)}$ and*

$$b = \frac{(c!)^2}{(\frac{s}{d})!(d!)^{\frac{2s}{d}}((c-s)!)^2} \text{ with } (c, s) = (td^2 + d, td^2 - td + d).$$

4.2 The construction related to cycles

In the following Construction 4.2, the non-trivial connected components of Γ are cycles. This actually generalizes the construction in Theorem 18. We will see that if such a construction forms a design, then it must be a flag-transitive design attaining the Cameron-Praeger upper bound $v = (k-2)^2$.

Construction 4.2. Let $\mathcal{E} = (c, n, s, e)$ be a 4-tuple. Define $\mathcal{D}(C(\mathcal{E}), G)$ as the incidence structure where $C(\mathcal{E})$ is a bipartite graph with bipartite block of size c and n, satisfying the following:

(a) $\overline{C(\mathcal{E})}$ is a union of e many disjoint cycles, and each has $\frac{2s}{e}$ vertices;

(b) $n \geqslant s \geqslant 2$, $c \geqslant s \geqslant 2$, $s > e \geqslant 1$ and $e \mid s$. $(**)$

Remark 24. If a 4-tuple $\mathcal{E} = (c, n, s, e)$ satisfies condition $(**)$, then clearly $C(\mathcal{E})$ is well-defined. Moreover, $C(\mathcal{E})$ has degree sequences

$$\mathbf{x} = \{2, 2, \ldots, 2, 0, 0 \ldots, 0\} \textit{ and } \mathbf{y} = \{2, 2, \ldots, 2, 0, 0, \ldots, 0\},$$

where the number 2 has multiplicity s in both $\mathbf{x}$ and $\mathbf{y}$.

Example 25. Here we give an example to clarify the notation better: the bipartite graph $C(\mathcal{E})$ with $\mathcal{E} = (9, 7, 6, 2)$ is described in Figure 2.

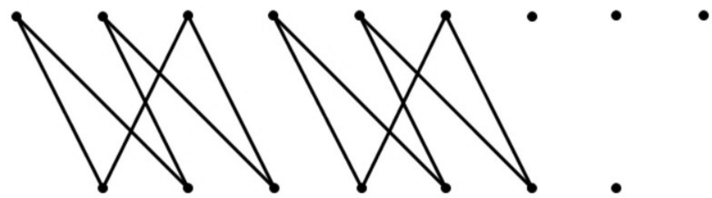

Figure 2: The bipartite graph $C(\mathcal{E})$ with $\mathcal{E} = (9, 7, 6, 2)$

Proposition 26. *The number of blocks of* $\mathcal{D}(C(\mathcal{E}), G)$ *in Construction 4.2 is equal to*

$$b = |\mathcal{B}| = \frac{n!c!}{e!(n-s)!(c-s)!(\frac{2s}{e})^e}.$$

Proof. The counting argument is similar to Proposition 26. First take s many vertices on N which will occur in a block. There are $\binom{n}{s}$ possibilities. Then separate these s vertices into e classes each of size $\frac{s}{e}$. Let $u = \frac{s}{e}$. There are $\frac{\binom{s}{u}\binom{s-u}{u}\binom{s-2u}{u}\cdots\binom{u}{u}}{e!}$ choices. Next, for each of these e classes, we arrange u vertices from the set C such that these u vertices and the vertices of the chosen class form a cycle. There are $\binom{c}{u}\binom{c-u}{u}\cdots\binom{c-(e-1)u}{u}$ choices. At last, make cycles in each of these chosen pairs of u-sets. There are $(\frac{1}{2}u(u-1)^2 \cdots 2^2 1^2)^e$ many possibilities. Note that $(\frac{1}{2}u(u-1)^2 \cdots 2^2 1^2)^e = (\frac{(u!)^2}{2u})^e$. Therefore,

$$b = \binom{n}{s} \cdot \frac{\binom{s}{u}\binom{s-u}{u}\binom{s-2u}{u}\cdots\binom{u}{u}}{e!} \cdot \binom{c}{u}\binom{c-u}{u}\cdots\binom{c-(e-1)u}{u}\left(\frac{(u!)^2}{2u}\right)^e.$$

Simplify the expression and we have the result. □

The theorem below states that if such an incidence structure $\mathcal{D}(C(\mathcal{E}), G)$ forms a 2-design, then it must be a flag-transitive design attaining the Cameron-Praeger upper bound $v = (k-2)^2$.

Theorem 27. *The structure $\mathcal{D}(C(\mathcal{E}), G)$ is a 2-design if and only if $\mathcal{E} = (c, c, \frac{c+2}{2}, e)$ with $1 \leqslant e < \frac{c+2}{2}$ and $2e \mid c+2$.*

Proof. Theorem 7(a) and (b) clearly hold since all components $\overline{C(\mathcal{E})}_j$ are isomorphic cycles and each $G^{\overline{C(\mathcal{E})}_j}$ is edge-transitive. Apply $i = d = 2$ to Theorem 7(c). Then $\mathcal{E} = (c, n, s, e) = (c, c, \frac{c+2}{2}, e)$. The condition $(**)$ gives $c \geqslant 2$ and $e \mid s$, we have $2e \mid c+2$ and $2e < c+2$. $\square$

Clearly, we have $v = cn = c^2$ and $k = sd = 2s = c+2$ and so these 2-designs attain the Cameron-Praeger upper bound.

Corollary 28. *For any given integers $e \geqslant 1$ and $t \geqslant 2$, there exists a 2-$((2et-2)^2, 2et, \lambda)$ design admitting a flag-transitive, point-imprimitive automorphism group $S_{2et-2} \times S_{2et-2}$, attaining the Cameron-Praeger upper bound. This design also admits a flag-transitive, point-primitive automorphism group $S_{2et-2} \wr S_2$. Here*

$$\lambda = \frac{(2et-3)!(2et-4)!}{(2t)^{e-1}(e-1)!((et-2)!)^2}.$$

Proof. Set $t \geqslant 2$, $e \geqslant 1$ and let $c := 2et-2$, $s := et = \frac{c+2}{2}$ and $n := c$. So $\mathcal{E} = (c, n, s, e) = (c, c, \frac{c+2}{2}, e)$ satisfies the sufficient condition of Theorem 27. Hence, $\mathcal{D}(C(\mathcal{E}), G)$ forms a 2-design admitting a flag-transitive automorphism group G. The parameters v and k derive from $v = cn = c^2 = (2et-2)^2$ and $k = sd = 2et$. By Proposition 26, the parameter λ is

$$\lambda = \frac{bk(k-1)}{v(v-1)} = \frac{n!c!}{e!(n-s)!(c-s)!(\frac{2s}{e})^e} \cdot \frac{sd(sd-1)}{cn(cn-1)}.$$

Simplify the expression and we have the result.

Let $G^* := S_c \wr S_2$ act on $\mathcal{P} = C \times C$ by its primitive product action. It is easy to see that

$$\mathcal{D}(C(\mathcal{E}), G) = \mathcal{D}(C(\mathcal{E}), G^*)$$

since all components of $\overline{C(\mathcal{E})}$ are isomorphic cycles. So G^* also acts as a flag-transitive automorphism group on the structure. $\square$

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