

Combinatorics and Representation Theory of Special Cases of Chern Plethysm

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Abstract

Chern plethysm, introduced by Billey, Rhoades, and Tewari, is a geometric way to produce Schur positive symmetric polynomials. We present combinatorial interpretations for the Schur expansions of special cases of Chern plethysm. We also exhibit a symmetric group module whose Frobenius characteristic is a symmetric function analog of one of these cases, generalizing a result of Reiner and Webb.

Mathematics Subject Classifications: 05E05, 05E10

1 Introduction

Schur polynomials, and more generally Schur positive symmetric polynomials, are ubiquitous in algebraic combinatorics and representation theory. The Schur polynomials are the characters of the irreducible polynomial representations of the general linear group, and they are explicitly defined in terms of *semistandard Young tableaux*. They form a basis of the complex vector space of symmetric polynomials in n variables. A symmetric polynomial is *Schur positive* if it can be written as a nonnegative integer linear combination of Schur polynomials.

A Schur polynomial is determined by a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and a positive integer n , the number of variables. When $\lambda = (m)$, the corresponding Schur polynomial s_λ is called the *homogeneous symmetric polynomial*, and when $\lambda = (1^m) := (1, 1, \dots, 1)$, the corresponding Schur polynomial is called the *elementary symmetric polynomial*. In these cases, we say that λ is a *single row* or *single column*, respectively. For example, if $n = 3$ and $\lambda = (1, 1)$, the corresponding Schur polynomial is

$$s_{(1,1)}(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1.$$

Chern plethysm is a geometric way to produce Schur positive symmetric polynomials. It is related to the notion of *classical plethysm* of Schur polynomials; it involves a

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composition operation which appears simple in principle, but produces very complex results. The Schur positivity of Chern plethysm follows from work of Pragacz ([4]) and Fulton-Lazarsfeld ([2]). In [11], Billey–Rhoades–Tewari explored a particular case of Chern plethysm: the *Boolean product polynomials*, defined as follows.

Definition 1. For $1 \leq k \leq n$, let

$$B_{n,k} := \prod_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (x_{i_1} + \dots + x_{i_k}).$$

Of particular interest in that paper is the case $k = n - 1$. In that case, the work by Reiner and Webb in [7] gives a combinatorial interpretation for the Schur coefficients in terms of descents of *standard Young tableaux*. That work also gives a representation-theoretic interpretation of this case in terms of poset homology.

Expanding on [7], in this paper we consider a more general class of Chern plethysms for which $B_{n,n-1}$ is just one example. In general, the result of Chern plethysm depends on two partitions, an *inner* partition μ and an *outer* partition λ , and n , the number of variables involved. This paper concerns the case $\mu = (1^{n-1})$. We can think of the resulting Chern plethysm as plugging in the sums of all but one of the variables at a time into a Schur polynomial. For example, if $\lambda = (1^n)$, we recover the Boolean product polynomial $B_{n,n-1}$.

Our first results (Theorems 8 and 10) give combinatorial expansions for this case of Chern plethysm when λ is a single column or single row. Both results give tableau-theoretic interpretations for the coefficients: in the column case, the coefficients count certain standard tableaux we call *parity tableaux*; in the row case the coefficients count standard tableaux of a given modified shape.

We then turn to finding combinatorial interpretations for the coefficients of specific terms in the Schur expansions of Chern plethysm with $\mu = (1^{n-1})$ and general λ . We are able to find compact combinatorial descriptions for the coefficients indexed by one row, two rows, or a column (17, 18, and 19). The coefficient indexed by one row is simply counted by a set of semistandard tableaux, while the coefficients indexed by two rows are counted by a certain sum of pairs of skew tableaux with conditions. The computation of the coefficient of a column uses the well-known Lindström–Gessel–Viennot lemma for counting lattice paths ([3]), and gives a formula in terms of fillings of Young diagrams which are not quite semistandard. Finding a complete combinatorial description of all of the coefficients for general λ remains open.

We conclude with Theorem 25, which gives a representation-theoretic interpretation of the case when λ is a column. We are no longer able to use homology of posets as in [7], but we are able to use spectral sequences and a result in [9] to relate Chern plethysm to the homology of a certain algebraic complex.

2 Definitions and preliminary results

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a weakly decreasing sequence of nonnegative integers. Each *positive* λ_i is called a *part* of the partition λ , and length $\ell(\lambda)$ of λ is the number of

parts. We use the notation $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$ to mean that $\lambda_1 + \dots + \lambda_l = n$. A *Young diagram* for the partition λ is an upper left justified series of boxes with λ_i boxes in each row. Given two partitions λ and μ with $\mu_i \leq \lambda_i$ for all i , the skew shape λ/μ is the Young diagram for λ with the Young diagram for μ deleted from it.

For integers $a \leq b$, we let $[a, b] := \{a, a+1, \dots, b-1, b\}$, and we write $[1, n]$ as $[n]$. For a set of integers S and a skew partition λ/μ , we define the corresponding set of semistandard Young tableaux, denoted $\text{SSYT}(\lambda/\mu, S)$, to be the set of all fillings of the skew shape λ/μ with the numbers S such that the entries of each filling are weakly increasing along rows and strictly increasing along columns. When $S = [n]$, the brackets are omitted, and when $S = \mathbb{N}$ the argument is omitted entirely. Given $T \in \text{SSYT}(\lambda/\mu, S)$ and integers r, c such that $\mu_r < c \leq \lambda_r$, let $T(r, c)$ denote the label in row r and column c of T .

We define $\text{SYT}(\lambda/\mu, n) \subseteq \text{SSYT}(\lambda/\mu, n)$ to be the elements of $\text{SSYT}(\lambda/\mu, n)$ that use each value at most once and are strictly increasing along rows. Note that we allow entries larger than the number of boxes in this definition. Let $f^{\lambda/\mu, n} = |\text{SYT}(\lambda/\mu, n)|$. When $\mu = (0)$ it is omitted, and when $n = |\lambda| - |\mu|$, n is occasionally omitted from the argument of SYT and $f^{\lambda/\mu, n}$ as well. Finally, when α is not a skew partition, we define f^α to be zero.

Given an element $T \in \text{SSYT}(\lambda, n)$, we define $wt(T) = (wt_1(T), \dots, wt_n(T))$, where $wt_i(T)$ is the number of i 's that appear in T . For a tuple $a = (a_1, \dots, a_n)$, define $x^a := x_1^{a_1} \dots x_n^{a_n}$. The *Schur polynomial* indexed by λ is defined as follows.

Definition 2.

$$s_\lambda(x_1, \dots, x_n) := \sum_{T \in \text{SSYT}(\lambda, n)} x^{wt(T)}.$$

Let $\varepsilon := \sum_{w \in \mathfrak{S}_n} \text{sign}(w) \cdot w$ denote the antisymmetrizing element of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group on n letters. The group \mathfrak{S}_n acts on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ by permuting variable indices and the group algebra acts by linear extension. We have the following useful formula for the Schur polynomial $s_\lambda(x_1, \dots, x_n)$. This is the *bialternant formula*, which is actually the original definition dating back to 1841; see [1].

$$s_\lambda(x_1, \dots, x_n) = \frac{\varepsilon \cdot x^{\lambda+\delta}}{\varepsilon \cdot x^\delta} \tag{1}$$

where $\delta = (n-1, n-2, \dots, 1, 0)$.

Next, we recall the definition of the *Frobenius characteristic map* (see e.g. [8]). This definition will be used in Section 6 when we give a representation-theoretic interpretation of the column case. We recall some basic facts about symmetric functions. The *power sum symmetric function* is defined by

$$p_r(x_1, x_2, \dots) := x_1^r + x_2^r + \dots$$

as a formal power series. For a partition $\rho = (\rho_1, \rho_2, \dots, \rho_l) \vdash n$, define $p_\rho := p_{\rho_1} p_{\rho_2} \dots p_{\rho_l}$ to be the power sum symmetric function corresponding to ρ .

Next, we have the *Schur function*

$$s_\rho(x_1, x_2, \dots) := \sum_{T \in \text{SSYT}(\rho)} x^{\text{wt}(T)}.$$

The *cycle type* of a permutation $w \in \mathfrak{S}_n$ is the partition $(\rho_1, \dots, \rho_l) \vdash n$ denoting the lengths of the cycles in a disjoint cycle decomposition of w . Let $z_\rho = \frac{n!}{k_\rho}$, where k_ρ is the number of permutations in \mathfrak{S}_n of cycle type ρ .

Definition 3. Let f be the character of a finite-dimensional complex representation V of \mathfrak{S}_n . Then the Frobenius characteristic of f , written $\text{ch}(f)$ is defined as

$$\text{ch}(f) := \sum_{\rho \vdash n} z_\rho^{-1} f(\rho) p_\rho$$

where $f(\rho)$ denotes the value of f on any element of the conjugacy class corresponding to ρ . We also use the notation $\text{ch}(V)$ to denote the same symmetric function.

It is important to note that if V^λ is the irreducible representation of \mathfrak{S}_n corresponding to the partition λ , then

$$\text{ch}(V^\lambda) = s_\lambda(x_1, x_2, \dots).$$

Let $X = \mathbb{P}^\infty \times \dots \times \mathbb{P}^\infty$ be the n -fold product of infinite-dimensional complex projective space, and let $\mathcal{E} = \ell_1 \oplus \dots \oplus \ell_n$ be the direct sum of the tautological line bundles over the n factors of X . Then we have the presentation $H^\bullet(X) = \mathbb{Z}[x_1, \dots, x_n]$, where the Chern roots of \mathcal{E} are the variables x_1, \dots, x_n .

Let $V = \mathbb{C}^n$ be the standard n -dimensional complex vector space. We refer the reader to Chapter 8 of [5] for the definition of the *Schur functor* $\mathbb{S}^\mu(V)$ (therein referred to as a *Schur module* V^μ). Let $p \in X$ and let \mathcal{E}_p denote the fiber of \mathcal{E} over p . We also define $\mathbb{S}^\mu(\mathcal{E})$ to be the new bundle with fibers $\mathbb{S}^\mu(\mathcal{E})_p := \mathbb{S}^\mu(\mathcal{E}_p)$. We note that the Chern roots of $\mathbb{S}^\mu(\mathcal{E})$ are the sums $\sum_{\square \in T} x_\square$, where T ranges over all elements of $\text{SSYT}(\mu, n)$.

In [11], the authors defined the notion of *Chern plethysm* $F(\mathcal{E})$, where F is a symmetric function. In that paper, and in [4] (using work of Fulton-Lazarsfeld in [2]), the authors showed that if λ and μ are partitions, then $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$ is always Schur positive. We restate the definition of Chern plethysm in this case here:

Definition 4. Given two partitions λ and μ , and a positive integer n , we have

$$s_\lambda(\mathbb{S}^\mu(\mathcal{E})) = s_\lambda \left(\dots, \sum_{\square \in T} x_\square, \dots \right)$$

where the alphabet on which we are evaluating s_λ is indexed by all semistandard Young tableaux T of shape μ and maximum entry n , and $\square \in T$ refers to the multiset of labels in the tableau T .

The problem of determining combinatorial or representation-theoretic interpretations for this Schur positivity remains open in general.

In this paper, we consider the symmetric polynomials $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$ in the special case where $\mu = (1^{n-1})$. Fixing $\mu = (1^{n-1})$, the Chern roots of $\mathbb{S}^\mu(\mathcal{E})$ are $\{(x_1 + \cdots + x_n) - x_i : 1 \leq i \leq n\}$. We will denote $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$ by

$$\bar{s}_\lambda(x_1, \dots, x_n) := s_\lambda((x_1 + \cdots + x_n) - x_1, \dots, (x_1 + \cdots + x_n) - x_n).$$

For example,

$$\bar{s}_{(1)}(x_1, \dots, x_n) = \sum_{i=1}^n (x_1 + \cdots + x_n) - x_i = (n-1)s_{(1)}(x_1, \dots, x_n).$$

We also use the notation

$$[s_\mu] \bar{s}_\lambda(x_1, \dots, x_n)$$

to denote the coefficient of s_μ in the Schur expansion of $\bar{s}_\lambda(x_1, \dots, x_n)$. We record the simple observation that

$$[s_\mu] \bar{s}_\lambda(x_1, \dots, x_n) = 0$$

unless $|\lambda| = |\mu|$.

Note that by Pieri's rule ([8, Equation 5.16]), for any partition μ and positive integer k ,

$$s_\mu(x_1, \dots, x_n) s_{(1)^k}(x_1, \dots, x_n) = \sum_{|\lambda|=|\mu|+k} f^{\lambda/\mu} s_\lambda(x_1, \dots, x_n). \quad (2)$$

For a skew partition λ/μ , let $\lambda/\mu - e_j$ (respectively $\lambda/\mu - e'_j$) represent λ/μ with the rightmost box in the j th row (respectively the lowest box in the j th column) removed if such a box exists, or λ/μ otherwise. Note that this may no longer be a skew partition. Recall our convention that $f^\alpha = 0$ unless α is a valid skew partition. For any skew partition λ/μ ,

$$f^{\lambda/\mu} = \sum_j f^{\lambda/\mu - e_j} = \sum_j f^{\lambda/\mu - e'_j}, \quad (3)$$

because the largest number in a skew tableau can only appear in one of the corner boxes.

3 The Column and Row Cases

3.1 An alternating formula

We begin with a lemma which allows us to express the Schur coefficients of \bar{s}_λ as an alternating sum. Let Par_n denote the set of partitions with at most n parts.

Lemma 5. *For $n \in \mathbb{N}$ and $\lambda \in \text{Par}_n$,*

$$\bar{s}_\lambda(x_1, \dots, x_n) = \sum_{\nu \subseteq \lambda} (-1)^{|\nu|} s_\nu(x_1, \dots, x_n) s_{(1)^{|\lambda/\nu|}}(x_1, \dots, x_n) f^{\lambda/\nu} \frac{\prod_{(r,c) \in \lambda/\nu} (n-r+c)}{|\lambda/\nu|!}$$

where $(r, c) \in \lambda/\nu$ refers to the ordered pair denoting the row and column index of a box in the skew shape λ/ν .

Proof. First consider $s_\lambda(t + x_1, t + x_2, \dots, t + x_n)$, where t is a new indeterminate. Taking the Taylor series of this function with respect to t about $t = 0$, and grouping terms by power of t , we get

$$s_\lambda(t + x_1, \dots, t + x_n) = \sum_{k=0}^n t^k \frac{\nabla^k}{k!} (s_\lambda(x_1, \dots, x_n))$$

where we define $\nabla f(x_1, \dots, x_n) := \sum_{i=1}^n \frac{\partial f}{\partial x_i}$, and ∇^k indicates the operator ∇ applied k times. The following sub-lemma follows from Proposition 1 in [10].

Lemma 6. *For any partition λ and positive integer k ,*

$$\frac{\nabla^k}{k!} (s_\lambda(x_1, \dots, x_n)) = \sum_{\substack{\nu \subseteq \lambda \\ |\nu| = |\lambda| - k}} s_\nu(x_1, \dots, x_n) f^{\lambda/\nu} \frac{\prod_{(r,c) \in \lambda/\nu} (n - r + c)}{|\lambda/\nu|!}.$$

The proof of the lemma follows by substituting $t = -s_{(1)}(x_1, \dots, x_n)$ and multiplying through by $(-1)^{|\lambda|}$. \square

We have the following corollary, which will prove useful in the next two subsections.

Corollary 7. *For $n \in \mathbb{N}$ and $\lambda, \mu \in \text{Par}_n$,*

$$\begin{aligned} [s_\mu] \bar{s}_\lambda(x_1, \dots, x_n) &= \sum_{\nu \subseteq \lambda \cap \mu} (-1)^{|\nu|} \frac{f^{\lambda/\nu} f^{\mu/\nu} \prod_{(r,c) \in \lambda/\nu} (n - r + c)}{|\lambda/\nu|!} \\ &= \sum_{\nu \subseteq \lambda \cap \mu} (-1)^{|\nu|} \binom{|\lambda|}{|\nu|} f^{\mu/\nu} \frac{f^\nu f^{\lambda/\nu}}{f^\lambda} \frac{|\text{SSYT}(\lambda, n)|}{|\text{SSYT}(\nu, n)|}. \end{aligned}$$

Proof. The first equality follows from Equation 2 applied to each term in Lemma 5. The second equality follows from applying both the hook length formula for standard Young tableaux and the hook content formula for semistandard Young tableaux (see [6, Theorem 7.21.2, Corollary 7.21.6]). \square

3.2 The column case

Denote by $P(\lambda, n)$ the set of elements of $\text{SYT}(\lambda, n)$ such that the smallest label not appearing in the first column is odd. We call the elements of $P(\lambda, n)$ *parity tableaux*. The following theorem gives a combinatorial interpretation for the Schur coefficients of $\bar{s}_{(1^k)}(x_1, \dots, x_n)$, generalizing [7, Proposition 2.3] to the cases $k \leq n$.

Theorem 8. *For any positive integer k ,*

$$\bar{s}_{(1^k)}(x_1, \dots, x_n) = \sum_{|\mu|=k} |P(\mu, n)| s_\mu(x_1, \dots, x_n).$$

Proof. We first apply Corollary 7. Note that if $\nu \subseteq \lambda \cap \mu$, then ν is a column, say of length i . Then the terms $f^{\lambda/\nu}$, f^ν , and f^λ are all equal to 1.

Next, we have $|\text{SSYT}(\lambda, n)| = \binom{n}{k}$, $|\text{SSYT}(\nu, n)| = \binom{n}{i}$, and $\binom{|\lambda|}{|\nu|} = \binom{k}{i}$. So we have $\binom{k}{i} \cdot \binom{n}{k} / \binom{n}{i} = \binom{n-i}{n-k}$. We conclude that

$$[s_\mu] \bar{s}_{(1)^k}(x_1, \dots, x_n) = \sum_{i=0}^k (-1)^i \binom{n-i}{k-i} f^{\mu/(1^i)}.$$

Note that $\binom{n-i}{k-i} f^{\mu/(1^i)}$ counts the elements of $\text{SYT}(\mu/(1^i), n-i)$ - or equivalently, elements of $\text{SYT}(\mu, n)$ such that the numbers from 1 through i are all in the leftmost column.

This implies that $\binom{n-2j}{k-2j} f^{\mu/(1^{2j})} - \binom{n-2j}{k-2j} f^{\mu/(1^{2j+1})}$ counts the number of elements of $\text{SYT}(\mu, n)$ such that the numbers from 1 through $2j$ are all in the first column, but $2j+1$ is not, so $2j+1$ is the smallest number not appearing in the first column.

As a result, summing over all i , we get that $\sum_{i=0}^k (-1)^i \binom{n-i}{k-i} f^{\mu/(1^i)}$ is the number of elements of $\text{SYT}(\mu, n)$ such that the smallest number not appearing in the first column is odd. This recovers the definition of $P(\mu, n)$, so we are done. \square

3.3 The row case

We now turn to examining the case $\bar{s}_{(k)}(x_1, \dots, x_n)$. This case will prove to be somewhat more complex. Given a partition μ and an integer p such that $p + |\mu| \geq 0$, we define a new partition $\mu^{(p)}$ as follows: Let

$$a = \min\{i \in \mathbb{N} : p \geq \mu'_i - i\}.$$

Then the i th column of $\mu^{(p)}$ is given by:

$$(\mu^{(p)})'_i = \begin{cases} \mu'_i - 1 & i < a, \\ p + a - 1 & i = a, \\ \mu'_{i-1} & i > a. \end{cases}$$

We proceed with the following lemma:

Lemma 9. *Fix a partition μ and $p \in \mathbb{Z}$ such that $p + |\mu| \geq 0$. Then*

$$(-1)^{a+1} f^{\mu^{(p)}} = \sum_{i=0}^{|\mu|} (-1)^i \binom{p+|\mu|}{p+i} f^{\mu/(i)}.$$

In particular, if $p = \mu'_a - a$, then

$$0 = \sum_{i=0}^{|\mu|} (-1)^i \binom{p+|\mu|}{p+i} f^{\mu/(i)}.$$

Proof. By the definition of a , we have that the sequence $(\mu'_i - i)_{i \in \mathbb{N}}$ is strictly decreasing, and $\mu^{(p)}$ is a partition if and only if $p > \mu'_a - a$. We proceed by induction on $|\mu| + p$. In the base case, $|\mu| + p = 0$, which implies that

$$\sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{\mu/(i)} = (-1)^{|\mu|} f^{\mu/(|\mu|)}.$$

If $\mu \neq (|\mu|)$, then $\mu'_{|\mu|} = 0$ so we have that $p = -|\mu| = -|\mu| + 0 = -|\mu| + \mu'_{|\mu|}$ which implies that $a = |\mu|$, and so $(\mu^{(-|\mu|)})'_{|\mu|} = p + a - 1 = -1$, so $\mu^{(-|\mu|)}$ is not a partition, so $f^{\mu^{(-|\mu|)}} = 0$. Similarly, if $\mu \neq (|\mu|)$, then $f^{\mu/(|\mu|)} = 0$, so $\sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{\mu/(i)} = 0 = f^{\mu^{(-|\mu|)}}$, completing the base case in this case.

On the other hand, if $\mu = (|\mu|)$, then $\mu'_{|\mu|} - |\mu| = 1 - |\mu| > p > -|\mu| - 1 = \mu'_{|\mu|+1} - (|\mu| + 1)$, so $a = |\mu| + 1$. Therefore, $(|\mu|)^{(-|\mu|)} = (0)$, and indeed

$$\begin{aligned} \sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{(|\mu|)/(i)} &= (-1)^{|\mu|} f^{(|\mu|)/(|\mu|)} \\ &= (-1)^{|\mu|} \\ &= (-1)^{|\mu|} f^{(0)} \\ &= (-1)^{-|\mu|-1+1} f^{(|\mu|)^{(-|\mu|)}} \end{aligned}$$

which completes the proof of the base case.

For the induction step, assume $p + |\mu| > 0$, and assume for our inductive hypothesis that the statement is true for smaller values of $p + |\mu|$. Using Equation 3 and a well-known identity of binomial coefficients, we have that

$$\begin{aligned} \sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{\mu/(i)} &= \sum_{i=0}^{|\mu|} (-1)^i \left(\binom{p + |\mu| - 1}{p + i - 1} + \binom{p + |\mu| - 1}{p + i} \right) f^{\mu/(i)} \\ &= \sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu| - 1}{p + i - 1} f^{\mu/(i)} + \sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu| - 1}{p + i} f^{\mu/(i)} \\ &= \sum_{i=0}^{|\mu|} (-1)^i \binom{(p - 1) + |\mu|}{(p - 1) + i} f^{\mu/(i)} \\ &\quad + \sum_{i=0}^{|\mu|-1} (-1)^i \binom{p + (|\mu| - 1)}{p + i} f^{\mu/(i)} \\ &= \sum_{i=0}^{|\mu|} (-1)^i \binom{(p - 1) + |\mu|}{(p - 1) + i} f^{\mu/(i)} \\ &\quad + \sum_{i=0}^{|\mu|-1} (-1)^i \binom{p + (|\mu| - 1)}{p + i} \left(\sum_{j=1}^{\infty} f^{(\mu/(i)) - e'_j} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{|\mu|} (-1)^i \binom{(p-1) + |\mu|}{(p-1) + i} f^{\mu/(i)} \\
&\quad + \sum_{j=1}^{\infty} \sum_{i=0}^{|\mu|-1} (-1)^i \binom{p + (|\mu| - 1)}{p + i} f^{(\mu/(i)) - e'_j} \\
&= (-1)^{a(\mu, p-1)+1} f^{\mu^{(p-1)}} + \sum_{j=1}^{\infty} (-1)^{a(\mu - e'_j, p)+1} f^{(\mu - e'_j)^{(p)}}
\end{aligned}$$

where we define $a(\mu, p) = \min\{i \in \mathbb{N} : p \geq \mu'_i - i\}$, and in the last step we have used the inductive hypothesis. We break this up into two cases, depending on whether or not $p = \mu'_{a(\mu, p)} - a(\mu, p)$.

If $p = \mu'_{a(\mu, p)} - a(\mu, p)$, then by construction $a(\mu, p-1) = a(\mu, p) + 1$, and for all j such that $\mu - e'_j$ is a partition, $a(\mu - e'_j, p) = a(\mu, p)$. Additionally, observe that for all $j \neq a(\mu, p)$, $f^{\mu^{(p)}} = f^{(\mu - e'_j)^{(p)}} = 0$, and $\mu^{(p-1)} = (\mu - e'_{a(\mu, p)})^{(p)}$. As a result, we have that

$$\begin{aligned}
\sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{\mu/(i)} &= (-1)^{a(\mu, p-1)+1} f^{\mu^{(p-1)}} + \sum_{j=1}^{\infty} (-1)^{a(\mu - e'_j, p)+1} f^{(\mu - e'_j)^{(p)}} \\
&= (-1)^{a(\mu, p)} (f^{\mu^{(p-1)}} - f^{(\mu - e'_{a(\mu, p)})^{(p)}}) \\
&= (-1)^{a(\mu, p)} (f^{\mu^{(p-1)}} - f^{\mu^{(p-1)}}) \\
&= 0 \\
&= (-1)^{a(\mu, p)+1} f^{\mu^{(p)}},
\end{aligned}$$

completing the induction in this case.

Finally, we consider the case where $p \neq \mu'_{a(\mu, p)} - a(\mu, p)$. This means that $p > \mu'_{a(\mu, p)} - a(\mu, p)$, and in particular $a(\mu, p-1) = a(\mu, p)$, with $\mu^{(p-1)} = \mu^{(p)} - e'_{a(\mu, p)}$. For all j , either $a(\mu - e'_j, p) = a(\mu, p)$, or $a(\mu - e'_j, p) = a(\mu, p) - 1$. In the former case, $(\mu - e'_j)^{(p)} = \mu^{(p)} - e'_j$ for $j < a(\mu, p)$ and $(\mu - e'_j)^{(p)} = \mu^{(p)} - e'_{j+1}$ for $j \geq a(\mu, p)$. On the other hand, if $a(\mu - e'_j, p) = a(\mu, p) - 1$, then $j = a(\mu, p) - 1$, with $p = \mu'_j - j - 1$, so in this case $(\mu - e'_j)^{(p)}$ is not a partition, so $f^{(\mu - e'_j)^{(p)}} = 0 = f^{\mu^{(p)} - e'_j}$. Combining these two cases, we see that $\sum_j f^{(\mu - e'_j)^{(p)}} = \sum_{j \neq a(\mu, p)} f^{\mu^{(p)} - e'_j}$. As a result,

$$\begin{aligned}
\sum_{i=0}^{|\mu|} (-1)^i \binom{p + |\mu|}{p + i} f^{\mu/(i)} &= (-1)^{a(\mu, p-1)+1} f^{\mu^{(p-1)}} + \sum_{j=1}^{\infty} (-1)^{a(\mu - e'_j, p)+1} f^{(\mu - e'_j)^{(p)}} \\
&= (-1)^{a(\mu, p)+1} f^{\mu^{(p-1)}} + \sum_{j=1}^{\infty} (-1)^{a(\mu, p)+1} f^{(\mu - e'_j)^{(p)}} \\
&= (-1)^{a(\mu, p)+1} f^{\mu^{(p)} - e'_{a(\mu, p)}} + \sum_{j \neq a(\mu, p)} (-1)^{a(\mu, p)+1} f^{\mu^{(p)} - e'_j}
\end{aligned}$$

$$\begin{aligned}
&= \sum_j (-1)^{a(\mu,p)+1} f^{\mu^{(p)} - e'_j} \\
&= (-1)^{a(\mu,p)+1} f^{\mu^{(p)}}
\end{aligned}$$

completing the proof by induction in this case. \square

We now have the tools we need to provide a combinatorial interpretation for the Schur expansion in the row case.

Theorem 10. *For any positive integer k ,*

$$\bar{s}_{(k)}(x_1, \dots, x_n) = \sum_{\substack{|\mu|=k \\ \ell(\mu) < n}} f^{\mu+(1^{n-1})} s_{\mu}(x_1, \dots, x_n).$$

Proof. We again use Corollary 7. Note that if $\nu \subseteq \lambda \cap \mu$, then ν is a row, say of length i .

Then we have that $f^{\lambda/\nu}$, f^{ν} , and f^{λ} are all equal to 1, and $\binom{|\lambda|}{|\nu|} = \binom{k}{i}$, $|\text{SSYT}(\lambda, n)| = \binom{n+k-1}{k}$, and $|\text{SSYT}(\nu, n)| = \binom{n+i-1}{i}$. We conclude that

$$[s_{\mu}] \bar{s}_{(k)}(x_1, \dots, x_n) = \sum_{i=0}^k (-1)^i \binom{n+k-1}{n+i-1} f^{\mu/(i)}. \tag{4}$$

By Theorem 9, letting $p = n - 1$, when $\ell(\mu) = n$ the right-hand side of 4 is zero, and when $\ell(\mu) < n$ the right-hand side of 4 is equal to $f^{\mu+(1^{n-1})}$, and so we are done. \square

The following definition will be used in Corollary 18. There, we will consider general λ . For now, we note that this definition gives us another combinatorial interpretation of the row case.

Definition 11. For λ, μ, n , with $|\lambda| = |\mu|$, define $G(\lambda, \mu, n)$ to be the set of pairs of tableaux $(S, T) \in \text{SSYT}(\lambda, n) \times \text{SYT}(\mu)$ such that if i appears in row r of T , then the i^{th} smallest value in S is greater than r . Here, ties for i^{th} smallest value are broken by first declaring identical entries in a lower row to be larger, and then declaring identical entries in a further-right column to be larger. This definition extends to the case where λ and/or μ are replaced with skew partitions.

Example 12. Let $\lambda = (4)$, $\mu = (2, 2)$, and $n = 3$. The $G(\lambda, \mu, n)$ consists of the following five pairs of tableaux:

$$\begin{aligned}
&\left(\begin{array}{|c|c|c|c|} \hline 2 & 2 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) & \left(\begin{array}{|c|c|c|c|} \hline 2 & 3 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) \\
&\left(\begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) & \left(\begin{array}{|c|c|c|c|} \hline 2 & 3 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right)
\end{aligned}$$

$$\left(\begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right).$$

An alternative interpretation of the row case is given by the following corollary.

Corollary 13. *For any positive integers k and n , and any partition μ with $|\mu| = k$,*

$$[s_\mu] \bar{s}_{(k)}(x_1, \dots, x_n) = |G((k), \mu, n)|.$$

Proof. We construct a bijection from $\text{SYT}(\mu + (1^{n-1}))$ to $G((k), \mu, n)$ to prove this statement. Fix an element $S \in \text{SYT}(\mu + (1^{n-1}))$, and let $1 = a_1 < \dots < a_{n-1}$ be the values in the leftmost column of S , and let $a_n = n + |\mu| = n + k$. Let S' be the unique element of $\text{SSYT}((k), n)$ such that, for each $2 \leq i \leq n$, S' has exactly $a_i - a_{i-1} - 1$ i 's. Define T to be the unique element of $\text{SYT}(\mu)$ such that the labels of T are in the same relative order as S with the first column deleted. This transformation from S to (S', T) is clearly an injection, and the inverse transformation is also an injection, so this is a bijection.

For example, the pairs in Example 12 correspond respectively to the standard Young tableaux

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

of shape $(2, 2) + (1, 1) = (3, 3)$. □

The following corollary will be used in the proof of Theorem 19.

Corollary 14. *For any $k \leq n$ and $1 \leq p \leq n - k + 1$,*

$$[s_{(1^k)}] \bar{s}_{(k)}(x_1, \dots, x_n) = |\{S \in \text{SSYT}((k), n) : S(1, 1) > 1, (S(1, i), S(1, i + 1)) \neq (p + i, p + i)\}|.$$

Proof. We construct a bijection from the desired set to $G((k), (1^k), n)$. Let $S \in \text{SSYT}((k), n)$ be such that $S(1, 1) > 1$ and $(S(1, i), S(1, i + 1)) \neq (p + i, p + i)$ for any $1 \leq i < k$. Break the bijection into two cases depending on whether there exists some i for which $S(1, i) \leq p + i - 1$.

If $S(1, i) > p + i - 1$ for all i , then the bijection maps S to (S, T) , where T is the unique element of $\text{SYT}((1^k))$. To show that it is well-defined, observe that since $S(1, i) > p + i - 1$ for all i , in particular $S(1, i) > i$, so $(S, T) \in G((k), (1^k), n)$.

On the other hand, if there exists some i for which $S(1, i) \leq p + i - 1$, let $a = \max\{i : S(1, i) \leq p + i - 1\}$. Let $S' \in \text{SSYT}((k), n)$ be defined as $S'(1, i) = S(1, i)$ for $i > a$, and $S'(1, i) = p + a + 1 - S(1, a + 1 - i)$ for $i \leq a$. The bijection will map S to (S', T) , where T is the unique element of $\text{SYT}((1^k))$.

For example, suppose $k = 3$ and $n = 4$. First let $p = 1$, so the elements of the set in question are

2	3	4	2	4	4	3	3	4
3			4			4		

For each of these tableau S , we have $S(1, i) > p + i - 1 = i$, so the bijection will map S to (S, T) , where T is the unique element of $\text{SYT}((1^k))$. Now suppose instead that $p = 2$. Then the mapping of S to S' looks as follows:

S	S'
2	4
2	4
2	4
2	3
2	4
2	4
2	3
3	3
3	4
2	2
3	3
4	4

We can see that the elements (S', T) are again the elements of $G((3), (1^3), 4)$, and so the bijection will map the tableau S to the pair (S', T) . We proceed by proving that this map is indeed a bijection. We start with a lemma.

Lemma 15. *For all $i \leq a$, $S(1, i) \leq p + i - 1$.*

Proof. Assume for the sake of contradiction that there exists $i \leq a$ such that $S(1, i) > p + i - 1$, and let j be the maximal such i . By the definition of a , $j < a$, and so

$$p + j \geq S(1, j + 1) \geq S(1, j) > p + j - 1$$

so $S(1, j) = S(1, j + 1) = p + j$, which contradicts the construction of S . □

Next, we show that the map is well-defined in this case. Since S is weakly increasing along the row, the only cells where it is not obvious that S' is weakly increasing is from $S'(1, a)$ to $S'(1, a + 1)$. For here, observe that since $S(1, 1) \geq 2$ and $S'(1, a + 1) > p + a$, that $S'(1, a) \leq p + a + 1 - 2 = p + a - 1 < p + a + 1 < S'(1, a + 1)$. By Lemma 15, for all $i \leq a$

we have that $S'(1, i) = p + a + 1 - S(1, a + 1 - i) \geq p + a + 1 - (p + (a + 1 - i) - 1) = i + 1 > i$. Also, for $i > a$, $S'(1, i) \geq p + i > i$. As a result, this map is well-defined.

To show that this map is a bijection, it suffices to show that it is invertible. The inverse map is defined as follows: given $(S', T) \in G((k), (1^k), n)$, return S if $S'(1, i) > p + i - 1$ for all i . If there exists some i such that $S'(1, i) \leq p + i - 1$, then let $a = \max\{i : S'(1, i) \leq p + i - 1\}$. In this case, the map returns $S \in \text{SSYT}((k))$ such that $S(1, i) = S'(1, i)$ for all $i > a$, and $S(1, i) = p + a + 1 - S'(1, a + 1 - i)$. In the first case, since $S'(1, i) > p + i - 1$, in particular $S'(1, i + 1) \neq p + i$ for any i , so $(S(1, i), S(1, i + 1)) \neq (p + i, p + i)$, so this map is well defined.

For the second case, since S' is weakly increasing, the only place where it is not obvious that S is weakly increasing is from $S(1, a)$ to $S(1, a + 1)$. However, here

$$S(1, a + 1) > p + (a + 1) - 1 = p + a = p + a + 2 - 2 > p + a + 1 - S'(1, a) = S(1, a)$$

so it is weakly increasing everywhere. Also, notice that for all $i > a$, $S'(1, i) > p + i - 1$, so in particular $S'(1, i) \neq p + i - 1$ for any i , so $(S(1, i - 1), S(1, i)) \neq (p + i - 1, p + i - 1)$. Similarly, for all $i \leq a$, $S'(1, i) > i$, so

$$S(1, i) = p + a + 1 - S'(1, a + 1 - i) < p + a + 1 - (a + 1 - i) = p + i$$

so in particular $S'(1, i) \neq p + i$. so $(S(1, i), S(1, i + 1)) \neq (p + i, p + i)$. This implies that the reverse map is well-defined.

The fact that these two maps are inverses of each other follows from the observation that the two a -values in the maps will be the same, so the map is a bijection. So the two sets in question are the same size, completing the proof. \square

4 Multiplicity of one or two rows

This section and the next provide some partial results on the Schur expansion of \bar{s}_λ for general λ . In particular, we compute the coefficients of $s_{(a,b)}$ and $s_{1|\lambda|}$.

For a $T \in \text{SSYT}(\lambda)$, let $wt_{>i}(T) = \sum_{j>i} wt_j(T)$. We start with the following lemma.

Lemma 16. *For $a \geq b \geq 0$ such that $a + b = |\lambda|$,*

$$[x_1^a x_2^b] \bar{s}_\lambda(x_1, \dots, x_n) = \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{a+b-|\nu|}{a-d}.$$

Proof. By the definition of \bar{s}_λ and the binomial theorem, we have

$$\begin{aligned} [x_1^a x_2^b] \bar{s}_\lambda(x_1, \dots, x_n) &= [x_1^a x_2^b] \sum_{T \in \text{SSYT}(\lambda)} \prod_{i=1}^n (x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_n)^{wt_i(T)} \\ &= \sum_{T \in \text{SSYT}(\lambda)} [x_1^a x_2^b] x_2^{wt_1(T)} x_1^{wt_2(T)} (x_1 + x_2)^{wt_{>2}(T)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{T \in \text{SSYT}(\lambda)} [x_1^{a-wt_2(T)} x_2^{b-wt_1(T)}] (x_1 + x_2)^{wt_{>2}(T)} \\
&= \sum_{T \in \text{SSYT}(\lambda)} \binom{wt_{>2}(T)}{a - wt_2(T)} \\
&= \sum_{c+d=0}^{a+b} \sum_{\substack{T \in \text{SSYT}(\lambda) \\ wt_1(T)=c, wt_2(T)=d}} \binom{wt_{>2}(T)}{a - wt_2(T)} \\
&= \sum_{c+d=0}^{a+b} \binom{a+b-c-d}{a-d} |\{T \in \text{SSYT}(\lambda) : wt_1(T) = c, wt_2(T) = d\}|.
\end{aligned}$$

For c, d , an element $T \in \text{SSYT}(\lambda)$ with $wt_1(T) = c, wt_2(T) = d$ must have all 1's in the first row, and can have at most $\min(c, d)$ 2's in the second row, with the rest in the first row to the right of the 1's. Beyond that, the numbers larger than 2 can be arranged in any way, so

$$|\{T \in \text{SSYT}(\lambda) : wt_1(T) = c, wt_2(T) = d\}| = \sum_{i=\max(c,d)}^{c+d} |\text{SSYT}(\lambda/(i, c+d-i), [3, n])|.$$

As a result,

$$\begin{aligned}
[x_1^a x_2^b] \bar{s}_\lambda(x_1, \dots, x_n) &= \sum_{c+d=0}^{a+b} \binom{a+b-c-d}{a-d} |\{T \in \text{SSYT}(\lambda) : wt_1(T) = c, wt_2(T) = d\}| \\
&= \sum_{c+d=0}^{a+b} \binom{a+b-c-d}{a-d} \sum_{i=\max(c,d)}^{c+d} |\text{SSYT}(\lambda/(i, c+d-i), [3, n])| \\
&= \sum_{k=0}^{a+b} \sum_{d=0}^k \binom{a+b-k}{a-d} \sum_{i=\max(k-d,d)}^k |\text{SSYT}(\lambda/(i, k-i), [3, n])| \\
&= \sum_{k=0}^{a+b} \sum_{d=0}^k \sum_{i=\max(k-d,d)}^k \binom{a+b-k}{a-d} |\text{SSYT}(\lambda/(i, k-i), [3, n])| \\
&= \sum_{k=0}^{a+b} \sum_{i=\lceil k/2 \rceil}^k \sum_{d=k-i}^i \binom{a+b-k}{a-d} |\text{SSYT}(\lambda/(i, k-i), [3, n])| \\
&= \sum_{k=0}^{a+b} \sum_{i=\lceil k/2 \rceil}^k |\text{SSYT}(\lambda/(i, k-i), [3, n])| \sum_{d=k-i}^i \binom{a+b-k}{a-d} \\
&= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{a+b-|\nu|}{a-d},
\end{aligned}$$

completing the proof. □

As a corollary, we can express the coefficient indexed by a row in the expansion of \bar{s}_λ as a simple count of semistandard Young tableaux.

Corollary 17. *For any partition λ and any positive integer $n \geq 2$,*

$$[s_{(|\lambda|)}]\bar{s}_\lambda(x_1, \dots, x_n) = |\text{SSYT}(\lambda, [2, n])|.$$

Proof. By Lemma 16,

$$\begin{aligned} [s_{(|\lambda|)}(x_1, \dots, x_n)]\bar{s}_\lambda(x_1, \dots, x_n) &= [x_1^{|\lambda|} x_2^0]\bar{s}_\lambda(x_1, \dots, x_n) \\ &= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq |\lambda|}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{|\lambda| - |\nu|}{|\lambda| - d} \\ &= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq |\lambda|}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{|\lambda| - |\nu|}{d - |\nu|} \\ &= \sum_{\substack{\nu \in \text{Par}_1 \\ |\nu| \leq |\lambda|}} |\text{SSYT}(\lambda/\nu, [3, n])| \\ &= |\text{SSYT}(\lambda, [2, n])|, \end{aligned}$$

completing the proof. □

We are finally able to give a combinatorial expression for the coefficients indexed by two rows in the expansion of \bar{s}_λ . Recall the definition of $G(\lambda, \mu, n)$ (Definition 11).

Corollary 18. *For $a \geq b \geq 0$,*

$$[s_{(a,b)}]\bar{s}_\lambda(x_1, \dots, x_n) = \sum_p G(\lambda/(p, p), (a, b)/(p, p), n).$$

Proof. By Lemma 16,

$$\begin{aligned} [s_{(a,b)}]\bar{s}_\lambda(x_1, \dots, x_n) &= [x_1^a x_2^b]\bar{s}_\lambda(x_1, \dots, x_n) - [x_1^{a+1} x_2^{b-1}]\bar{s}_\lambda(x_1, \dots, x_n) \\ &= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{a+b-|\nu|}{a-d} \\ &\quad - \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \binom{a+b-|\nu|}{a+1-d} \\ &= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \sum_{d=\nu_2}^{\nu_1} \left(\binom{a+b-|\nu|}{a-d} - \binom{a+b-|\nu|}{a+1-d} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| \left(\binom{a+b-|\nu|}{a-\nu_1} - \binom{a+b-|\nu|}{a+1-\nu_2} \right) \\
&= \sum_{\substack{\nu \in \text{Par}_2 \\ |\nu| \leq a+b}} |\text{SSYT}(\lambda/\nu, [3, n])| f^{(a,b)/\nu} \\
&= \sum_{\nu \subseteq (a,b)} |\text{SSYT}(\lambda/\nu, [3, n])| f^{(a,b)/\nu}.
\end{aligned}$$

One can think of $|\text{SSYT}(\lambda/\nu, [3, n])|$ as the number of elements of $\text{SSYT}(\lambda/(\nu_2, \nu_2), [2, n])$ that have $\nu_1 - \nu_2$ 2's, all in the first row. Similarly, one can think of $f^{(a,b)/\nu}$ as the number of elements of $\text{SYT}((a,b)/(\nu_2, \nu_2))$ such that $1, \dots, \nu_1 - \nu_2$ all appear in the first row. As a result, for a fixed $p \leq b$, $\sum_{\nu: \nu_2=p} |\text{SSYT}(\lambda/\nu, [3, n])| f^{(a,b)/\nu}$ equals the number of pairs $(S, T) \in \text{SSYT}(\lambda/(p, p), [2, n]) \times \text{SYT}((a,b)/(p, p))$ such that all of the 2's in S are in the first row, and if i is in the second row of T , then the i th smallest element of S is at least 3. Summing over all p gives us the desired answer. \square

We note that this computation relied on the relatively simple relationship between the Schur coefficients indexed by two rows and the monomial coefficients indexed by corresponding powers of x_1 and x_2 (see the first line of the proof of Corollary 18). We were not able to use the same method to compute the coefficients corresponding to μ for μ having more than two rows.

5 Multiplicity of a Column

In this section we compute the coefficient of $s_{(1|\lambda)}$ in the expansion of \bar{s}_λ . This proof uses the Lindström-Gessel-Viennot lemma for counting lattice paths, (see e.g. [3]).

Theorem 19. *For any $\ell(\lambda) \leq p \leq n - \lambda_1$, $[s_{(1|\lambda)}] \bar{s}_\lambda(x_1, \dots, x_n)$ equals the number of fillings T of λ with numbers in $[2, n]$ such that for all $(r, c) \in \lambda$:*

1. $T(r, c) \leq T(r+1, c)$, and they are not both equal to $p - r + c$
2. Either $T(r-1, c) < T(r, c)$ or $T(r-1, c) = T(r, c) = p - r + c$.

Proof. By the Jacobi-Trudi formula,

$$s_\lambda(x_1, \dots, x_n) = \det(s_{(\lambda_i - i + j)}(x_1, \dots, x_n))_{1 \leq i, j \leq \ell(\lambda)},$$

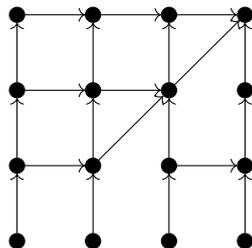
so

$$\bar{s}_\lambda(x_1, \dots, x_n) = \det(\bar{s}_{(\lambda_i - i + j)}(x_1, \dots, x_n))_{1 \leq i, j \leq \ell(\lambda)},$$

and so in particular

$$[s_{(1|\lambda)}] \bar{s}_\lambda(x_1, \dots, x_n) = \det([s_{(1^{\lambda_i - i + j})}] \bar{s}_{(\lambda_i - i + j)}(x_1, \dots, x_n))_{1 \leq i, j \leq \ell(\lambda)}. \quad (5)$$

Consider the directed graph with vertex set $V = [n] \times [n]$, with directed edges from (x, y) to $(x, y + 1)$ for all x, y , from (x, x) to $(x + 1, x + 1)$ for each $x > 1$, and from (x, y) to $(x + 1, y)$ for $y \neq 1$ and $x \neq y$. Here is a picture of this planar network when $n = 4$:



Let $P_{a,b}$ be the number of directed paths from $(a, 1)$ to (b, n) .

Lemma 20. For any $1 \leq p \leq n - k$,

$$[s_{(1^k)}] \bar{s}_{(k)}(x_1, \dots, x_n) = P_{p,p+k}.$$

Proof. For each path counted by $P_{p,p+k}$, consider the sequence of y -coordinates at which the non-vertical steps in the path begin. Because there are no down-steps, this sequence will be unique among all paths counted by $P_{a,b}$, and will be weakly increasing. Additionally, after performing a non-vertical step starting at point (x, y) , the next non-vertical step can start at any y -coordinate strictly larger than y , or can start at y -coordinate y as long as the last step performed was not a diagonal step *i.e.*, $x \neq y$. As a result, each path counted by $P_{p,p+k}$ corresponds to a weakly increasing sequence $1 < c_1 \leq \dots \leq c_k \leq n$ such that for all i , c_i and c_{i+1} do not both equal $p + i - 1$. If we fill the partition (k) with the sequence c_1, \dots, c_k , we get exactly the elements of $\{T \in \text{SSYT}((k), n) : T(1, 1) > 1, (T(1, i), T(1, i + 1)) \neq (p + i, p + i)\}$, which by Corollary 14 has $[s_{(1^k)}] \bar{s}_{(k)}(x_1, \dots, x_n)$ elements, proving the lemma. \square

Using Lemma 20 and Equation 5, we have that for any $\ell(\lambda) \leq p \leq n - \lambda_1$

$$[s_{(1^{|\lambda|})}] \bar{s}_\lambda(x_1, \dots, x_n) = \det(P_{p+1-j, p+1+\lambda_i-i})_{1 \leq i, j \leq \ell(\lambda)}.$$

By Lindström-Gessel-Viennot ([3]), and the fact that the graph is planar, this determinant is equal to the number of tuples of non-intersecting paths that connect each $(p + 1 - j, 1)$ to $(p + 1 + \lambda_i - i, n)$ for $1 \leq i \leq \ell(\lambda)$. Consider one such tuple, and construct a filling T of shape λ such that $T(r, c)$ equals the y -coordinate of the c th non-vertical step of the path that goes from $(p + 1 - r, 1)$ to $(p + 1 + \lambda_r - r, n)$. From Lemma 20, this means that the content of each row of T will be a weakly increasing sequence, except that $T(r, c)$ and $T(r, c + 1)$ cannot both be $p + c - r$. Similarly, two adjacent paths will intersect if and only if there is some c such that the the endpoint of c th non-vertical step of the left path is weakly below the start point of the c th non-vertical step of the right path. As a result, the paths are non-intersecting if and only if for all $(r, c) \in \lambda$, either $T(r - 1, c) < T(r, c)$ or $T(r - 1, c) = T(r, c) = p + c - r$. As a result, the number of tuples of non-intersecting lattice paths will be equal to the number of such fillings, completing the proof. \square

6 A representation-theoretic interpretation of the column case

We first introduce the concepts of chain complexes and homology. Suppose we have a collection of \mathfrak{S}_k -modules C_i , for $0 \leq i \leq k$. A sequence of \mathfrak{S}_k -equivariant maps $\partial_i : C_i \rightarrow C_{i-1}$ is called a *chain complex of \mathfrak{S}_k -modules* if $\partial_i \circ \partial_{i+1} = 0$ for $1 \leq i \leq k-1$. The data of a chain complex will typically be denoted C_* . The i^{th} *homology group* H_i of a chain complex is defined to be the quotient

$$H_i := \ker(\partial_i) / \text{im}(\partial_{i+1}).$$

The main tool in this discussion will be the Hopf trace formula. Suppose we have a chain complex

$$C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

of \mathfrak{S}_k -modules, and suppose we know the Frobenius images $\text{ch}(C_i)$ for each i . Note that each respective homology group H_i will also be an \mathfrak{S}_k -module, since the chain maps are \mathfrak{S}_k -equivariant. Then from the Hopf trace formula, we have the equation

$$\sum_{i=0}^k (-1)^{k-i} \text{ch}(C_i) = \sum_{i=0}^k (-1)^{k-i} \text{ch}(H_i).$$

This formula becomes particularly useful when paired with a vanishing result on the homology groups of the right side.

We proceed by passing to infinitely many variables in order to view $\bar{s}_{(1^k)}$ as a symmetric *function*, instead of a symmetric polynomial. In doing so, we allow the integers n and k to remain as parameters in our definition. We use the expression in Lemma 5 above, replacing i with $k-i$ in order to more clearly interpret the binomial coefficient appearing in the definition.

Definition 21. Let $n \geq k$ be positive integers. Let $c_{n,k}(x) = c_{n,k}(x_1, x_2, \dots)$ be the symmetric function

$$c_{n,k}(x) = \sum_{i=0}^k (-1)^{k-i} \binom{n-k+i}{i} s_{(1^{k-i})} s_{(1)}^i.$$

Note that $c_{n,k}(x_1, \dots, x_n, 0, 0, \dots) = \bar{s}_{(1^k)}(x_1, \dots, x_n)$. We proceed by exhibiting an \mathfrak{S}_k -module whose Frobenius characteristic is exactly $c_{n,k}$ (recall Definition 3).

The following discussion generalizes Theorem 2.4 in [7]. The appearance of the binomial coefficient in Definition 21 precludes the use of poset techniques in what follows.

Let A and M be finite nonempty sets of integers. Let $\Delta_i(A, M)$ denote the set of pairs (w, S) , where w is an *injective word* of length i on the alphabet A , and S is a nondecreasing sequence of length i of elements of the set M . Here by injective word of length i on the alphabet A , we mean a sequence of elements of A where no element is repeated. We write such a sequence by simply concatenating its elements. Let $C_i(A, M)$ denote the \mathbb{C} -vector space with basis $\Delta_i(A, M)$.

Example 22. Let $A = [2]$ and $M = [2]$. Then the elements of $\Delta_2(A, M)$ are

$$\{(12, (1, 1)), (12, (1, 2)), (12, (2, 2)), (21, (1, 1)), (21, (1, 2)), (21, (2, 2))\}.$$

We define an \mathfrak{S}_k action on $C_i([k], [n - k + 1])$ by letting

$$\sigma \cdot (w_1 \dots w_i, S) = \text{sign}(\sigma)(\sigma(w_1) \dots \sigma(w_i), S).$$

Note that \mathfrak{S}_k acts trivially on S . This action has a familiar-looking Frobenius characteristic:

Proposition 23. *The Frobenius characteristic of the \mathfrak{S}_k -module $C_i([k], [n - k + 1])$ is given by*

$$\text{ch}(C_i([k], [n - k + 1])) = \binom{n - k + i}{i} s_{(1^{k-i})} s_{(1)}^i.$$

Proof. Let Γ denote the collection of nondecreasing sequences of elements of the set $[n - k + 1]$, and note that $|\Gamma| = \binom{n - k + i}{i}$. As \mathfrak{S}_k -modules, we have that

$$C_i([k], [n - k + 1]) \cong \bigoplus_{S \in \Gamma} (\text{sign}_{k-i} \otimes \mathbb{C}[\mathfrak{S}_i]) \uparrow_{\mathfrak{S}_{k-i} \times \mathfrak{S}_i}^{\mathfrak{S}_k}$$

where sign_{k-i} is the sign representation of \mathfrak{S}_{k-i} , and $\mathbb{C}[\mathfrak{S}_i]$ is the regular representation of \mathfrak{S}_i . The Frobenius image of each summand of the right-hand side is the product

$$\text{ch}(\text{sign}_{k-i}) \text{ch}(\mathbb{C}[\mathfrak{S}_i]) = s_{(1^{k-i})} s_{(1)}^i.$$

Taking the direct sum indexed by Γ gives the factor $\binom{n - k + i}{i}$. □

We proceed to define an algebraic chain complex and study its homology. For $1 \leq i \leq k$, let $\partial_i : C_i(A, M) \rightarrow C_{i-1}(A, M)$ be defined by

$$\partial_i(w_1 \dots w_i, (j_1, \dots, j_i)) = \sum_{l=1}^i (-1)^{l-1} (w_1 \dots \widehat{w}_l \dots w_i, (j_1, \dots, \widehat{j}_l, \dots, j_i)).$$

Looking at the case $A = [k]$, $M = [n - k + 1]$, the maps ∂_i are \mathfrak{S}_k -equivariant and for $1 \leq i \leq k - 1$, $\partial_i \circ \partial_{i+1} = 0$, so the modules $C_i([k], [n - k + 1])$ form a chain complex of \mathfrak{S}_k -modules. For arbitrary nonempty sets A, M , we let $C_*(A, M)$ denote the corresponding chain complex of vector spaces.

The following lemma is an adaptation of Theorem 3 in [9].

Lemma 24. *Let $H_i = H_i(C_*(A, M))$ denote the homology of the chain complex $C_*(A, M)$ in degree i . Then $H_i = 0$ for $0 \leq i < k$.*

Proof. The proof is essentially the same as that of Theorem 3 in [9]: We use induction on $|A|$, and modify the induction hypothesis so that the set M is not fixed. We define the same filtration and now have that

$$F_p C_*(A, M) / F_{p-1} C_*(A, M) \cong \bigoplus_{(w_1 \dots w_{p-1}, (j_1, \dots, j_i)) \in \Delta_p(A, M)} C_{*-p}(A \setminus \{w_1, \dots, w_{p-1}, \max(A)\}, M \setminus [j_p - 1])$$

and we can apply the induction hypothesis to the right-hand side.

Continuing as in [9], we have that the map

$$(w_1 \dots w_r, (j_1, \dots, j_r)) \mapsto (\max(A)w_1 \dots w_r, (1, j_1, \dots, j_r))$$

gives a null-homotopy for the inclusion map $C_*(A \setminus \max(A), M) \rightarrow C_*(A, M)$. So the edge map $E_{0,k-1}^1 \rightarrow E_{0,k-1}^\infty \subset H_{k-1}(C_*(A, M))$ is zero, and hence $E_{0,k-1}^\infty = 0$. Here we have used the theory of spectral sequences in the same way as [9]. \square

We are now able to state the main result of this section.

Theorem 25. *For any positive integers n and k ,*

$$c_{n,k}(x) = \sum_{i=0}^k (-1)^{k-i} ch(C_i) = ch(H_k).$$

Proof. The first equality is given by Definition 21, and the second equality follows from Lemma 24 and the Hopf trace formula. \square

Also, one can see that the Schur function expansion of $c_{n,k}$ is the same as the Schur polynomial expansion of $\bar{s}_{(1^k)}(x_1, \dots, x_n)$, so Theorem 25 gives a representation-theoretic proof of the Schur positivity of $\bar{s}_{(1^k)}(x_1, \dots, x_n)$.

7 Conclusion and open problems

We conclude with some explicit open problems related to this work. We have found combinatorial expressions for $[s_\mu] \bar{s}_\lambda(x_1, \dots, x_n)$ for μ being a single column, or one or two rows, and for λ being a single row or single column. This naturally leads to

Problem 26. Find a combinatorial expression for $[s_\mu] \bar{s}_\lambda(x_1, \dots, x_n)$ for general μ and λ .

There is also a natural approach to the representation theory of the case when λ is a single row. Again let A and M denote finite nonempty sets of integers. Let $\Delta'_i(A, M)$ be the set of pairs (w, S) of injective words of length i on the alphabet A and *increasing* sequences S of length i of elements of the set M . Let $C'_i(A, M)$ denote the \mathbb{C} -vector space with basis $\Delta'_i(A, M)$.

Now define an \mathfrak{S}_k action on $C'_i([k], [n+k-1])$ by letting

$$\sigma \cdot (w_1 \dots w_k, S) = (\sigma(w_1) \dots \sigma(w_k), S).$$

Note that we no longer have a “sign” term in this action.

By a similar argument as in Proposition 23, we have that the Frobenius characteristic of $C'_i([k], [n+k-1])$ is given by

$$ch(C'_i([k], [n+k-1])) = \binom{n+k-1}{i} s_{(k-i)} s_{(1)}^i.$$

Finally, we define the symmetric function

$$r_{n,k}(x_1, x_2, \dots) = \sum_{i=0}^k (-1)^{k-i} \binom{n+k-1}{i} s_{(k-i)} s_{(1)}^i.$$

Then using Lemma 5, we see that $r_{n,k}(x_1, \dots, x_n, 0, 0, \dots) = \bar{s}_{(k)}(x_1, \dots, x_n)$. And so the Schur function expansion of $r_{n,k}$ will be the same as the Schur polynomial expansion of $\bar{s}_{(k)}$ as long as $n \geq k$ (since otherwise the Schur functions appearing in the former expansion may contain terms indexed by partitions of length greater than n). We can define a chain complex in the same way as above, with the exact same chain maps. Attempting to follow the same argument as in Section 6, we are led to the following

Problem 27. Let H_i denote the i^{th} homology group in this chain complex. Prove that $H_i = 0$ for $i < k$, as long as $|M| \geq 2|A| - 1$.

We have some computational evidence for this conjecture, but were not able to use the same filtration and spectral sequence method to prove it. A proof of this conjecture would give a homological model for the row case.

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