Labelled Well-Quasi-Order in Juxtapositions of Permutation Classes

Robert Brignall[a]

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Abstract

The *juxtaposition* of permutation classes $\mathcal{C}$ and $\mathcal{D}$ is the class of all permutations formed by concatenations $\sigma\tau$, such that σ is order isomorphic to a permutation in $\mathcal{C}$, and τ to a permutation in $\mathcal{D}$.

We give simple necessary and sufficient conditions on the classes $\mathcal{C}$ and $\mathcal{D}$ for their juxtaposition to be labelled well-quasi-ordered (lwqo): namely that both $\mathcal{C}$ and $\mathcal{D}$ must themselves be lwqo, and at most one of $\mathcal{C}$ or $\mathcal{D}$ can contain arbitrarily long zigzag permutations. We also show that every class without long zigzag permutations has a growth rate which must be integral.

Mathematics Subject Classifications: 05A05, 06A07

For Sophie

1 Introduction

Let $\mathcal{C}$ and $\mathcal{D}$ be permutation classes. The *juxtaposition* $\mathcal{C}\,\mathcal{D}$ is the permutation class comprising all permutations formed by concatenations $\sigma\tau$, where σ is order isomorphic to a permutation in $\mathcal{C}$ and τ is order isomorphic to a permutation in $\mathcal{D}$.

A *zigzag permutation* (or just *zigzag*) is a permutation $\pi = \pi(1)\cdots\pi(n)$ with the property that there is no index $i \in [n-2]$ such that $\pi(i)\pi(i+1)\pi(i+2)$ forms a monotone increasing or decreasing pattern.[1] The main purpose of this note is to establish the following theorem.

Theorem 1. *The juxtaposition $\mathcal{C}\,\mathcal{D}$ is labelled-well-quasi-ordered if and only if both $\mathcal{C}$ and $\mathcal{D}$ are lwqo, and at least one of $\mathcal{C}$ or $\mathcal{D}$ contains only finitely many zigzag permutations.*

[a] School of Mathematics and Statistics, The Open University, U.K. (`rbrignall@gmail.com`).

[1] Zigzag permutations (sometimes called *alternating permutations*, but we reserve the term 'alternating' for other purposes) have been widely studied in relation to enumerative problems, and are strongly related to the Euler numbers (sequence A000111 of the OEIS [1]) – for a survey, see Stanley [16].

The juxtaposition of permutations was first introduced in Atkinson's foundational work [2], and has since been studied in terms of enumeration (see, for example, [8]) since it represents a natural yet non-trivial way to combine two permutation classes. Indeed, juxtapositions are a special case of *grid classes*, which we define in the next section.

The study of well-quasi-ordering and infinite antichains in permutation classes dates back to the 1970s in the work of Tarjan [17] and Pratt [15], and rose to prominence in the 2000s as a result of works such as Atkinson, Murphy and Ruškuc [3] and Murphy and Vatter [12]. The stronger notion of *labelled* well-quasi-ordering dates back to Pouzet [14], but received little attention in the context of permutation classes until the current author's recent work with Vatter [9].

The rest of this paper is organised as follows. In Section 2 we briefly cover the requisite terminology. In Section 3 we provide a necessary and sufficient characterisation of permutation classes without long zigzags. As a by-product of this characterisation, we show that every permutation class without long zigzags has an integral growth rate. In Section 4 we prove that the juxtaposition of a labelled well-quasi-ordered permutation class with $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$ is again labelled well-quasi-ordered, and this, together with the characterisation from Section 3, enables us to complete our proof of Theorem 1. We finish with some concluding remarks in Section 5.

2 Preliminaries

Permutation classes We provide here only the minimum terminology required for our purposes, and refer the reader to [5] for fuller details.

A permutation of length n, typically denoted $\pi = \pi(1)\cdots\pi(n)$, is an ordering of the symbols in $[n] = \{1, \dots, n\}$. We say that $\sigma = \sigma(1)\cdots\sigma(k)$ is *contained* in π, and write $\sigma \leqslant \pi$, if there exists a subsequence $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that the relative ordering of the points in $\pi(i_1)\cdots\pi(i_k)$ is the same as that of σ. That is, π contains a subsequence that is *order isomorphic* to σ.

A *permutation class* $\mathcal{C}$ is a set of permutations closed downwards under containment. Every such class can be described by its set of minimal forbidden elements, but for our purposes it suffices to record that $\mathsf{Av}(21) = \{1, 12, 123, \dots\}$ is the class of increasing permutations, and $\mathsf{Av}(21) = \{1, 21, 321, \dots\}$ is the class of decreasing permutations. Two other classes we will require are as follows

$$\bigoplus 21 = \{\text{finite subpermutations of } 21436587\cdots\} = \mathsf{Av}(231, 312, 321),$$
$$\bigominus 21 = \{\text{finite subpermutations of } \cdots 78563412\} = \mathsf{Av}(123, 132, 213).$$

One important family of permutation classes in the structural study of permutations are *grid classes*. These are defined by a *gridding matrix* $\mathcal{M}$ of permutation classes, and each permutation in $\mathsf{Grid}(\mathcal{M})$ has the property that its plot can be divided using horizontal and vertical lines into a grid of cells, of the same dimensions as $\mathcal{M}$, and such that the entries in each cell of the plot are order isomorphic to a permutation that belongs to a class in the corresponding cell of $\mathcal{M}$.

Of particular note are *monotone* grid classes, where each cell of $\mathcal{M}$ is $\mathsf{Av}(21)$, $\mathsf{Av}(12)$ or empty, and we say that a permutation class $\mathcal{C}$ is *monotone griddable* if it is the subclass of some monotone grid class. We need the following characterisation.

Theorem 2 (Huczynska and Vatter [11, Theorem 2.5])**.** *A permutation class is monotone griddable if and only if it has finite intersection with* $\bigoplus 21$ *and* $\bigominus 12$*.*

The juxtaposition $\mathcal{C}\,\mathcal{D}$ can alternatively be considered as $\mathsf{Grid}(\mathcal{M})$ where $\mathcal{M} = \begin{bmatrix}\mathcal{C} & \mathcal{D}\end{bmatrix}$. Also of interest to us is the class of *gridded* permutations in a juxtaposition – denoted $\mathcal{C} \mid \mathcal{D}$ – whose members comprise the permutations of $\mathcal{C}\,\mathcal{D}$ together with a vertical line that witnesses the permutation's membership of the juxtaposition. Note that each permutation in $\mathcal{C}\,\mathcal{D}$ can correspond to more than one gridded permutation in $\mathcal{C} \mid \mathcal{D}$. The same notion exists for grid classes defined by larger matrices: if $\mathcal{C} \subseteq \mathsf{Grid}(\mathcal{M})$ then $\mathcal{C}^\sharp$ denotes the set of permutations in $\mathcal{C}$ equipped with horizontal and vertical lines to witness their membership of $\mathsf{Grid}(\mathcal{M})$.

Well-quasi-ordering A quasi-order $(P, \leqslant)$ is *well-quasi-ordered* (wqo) if it contains no infinite descending chain, and no infinite antichain – that is, a set of pairwise incomparable elements. For quasi-ordered classes of combinatorial objects (such as permutation classes or gridded permutation classes), this condition typically reduces to checking for the presence of infinite antichains.

Given a quasi-order $(P, \leqslant)$, let P^* denote the set of finite sequences of P. The set P^* can be ordered using the *generalised subword order*: for $v = v_1 \cdots v_m$ and $w = w_1 \cdots w_n$ in P^*, we say that $v \preceq w$ if there exists a subsequence $1 \leqslant i_1 \leqslant \cdots \leqslant i_m \leqslant n$ such that $v_j \leqslant w_{i_j}$ for all $1 \leqslant j \leqslant m$. One celebrated result that we will need is Higman's lemma:

Lemma 3 (Higman [10])**.** *If* $(P, \leqslant)$ *is a wqo set, then so is* $(P^*, \preceq)$*.*

Another way to combine wqo sets and obtain another wqo set is by taking products:

Proposition 4 (See [9, Proposition 1.2])**.** *Let* $(P, \leqslant_P)$ *and* $(Q, \leqslant_Q)$ *be wqo sets. Then* $P \times Q$ *is wqo under the product order,* $(p_1, q_1) \leqslant (p_2, q_2)$ *if and only if* $p_1 \leqslant_P p_2$ *and* $q_1 \leqslant_Q q_2$*.*

The final piece of core wqo machinery we require is as follows. We say that a mapping $\Phi : P \to Q$ between two quasi-orders is *order preserving* if $p_1 \leqslant_P p_2$ implies $\Phi(p_1) \leqslant_Q \Phi(p_2)$. We have:

Proposition 5 (See [9, Proposition 1.10])**.** *Let* $(P, \leqslant_P)$ *and* $(Q, \leqslant_Q)$ *be quasi-orders, and suppose that* $\Phi : (P, \leqslant_P) \to (Q, \leqslant_Q)$ *is an order-preserving surjection. If* $(P, \leqslant_P)$ *is wqo, then so is* $(Q, \leqslant_Q)$*.*

Labelled well-quasi-ordering Let $(L, \leqslant_L)$ be any quasi-order. An L-labelling of a permutation π of length n (or of a gridded permutation of length n) is a mapping ℓ_π from the indices of π to elements of L. We write the resulting L-labelled permutation as (π, ℓ_π), and the set of all L-labelled permutations from some set (or class) $\mathcal{C}$ is denoted $\mathcal{C} \wr L$.

The set $\mathcal{C} \wr L$ induces a natural ordering: Let $\sigma, \pi \in \mathcal{C}$ be of lengths m and n, respectively. We say that (σ, ℓ_σ) is contained in (π, ℓ_π) if there exists a subsequence $1 \leqslant i_1 < \cdots < i_m \leqslant n$ such that $\pi(i_1) \cdots \pi(i_m)$ is order isomorphic to σ, and $\ell_\sigma(j) \leqslant_L \ell_\pi(i_j)$ for all $j \in [m]$.

Finally, a set or class $\mathcal{C}$ is *labelled well-quasi-ordered* (lwqo) if $\mathcal{C} \wr L$ is a wqo set for every wqo set $(L, \leqslant_L)$. We refer the reader to [9] for a complete treatment of lwqo in permutation classes.

3 Zigzags

A *peak* of a permutation π is a position i such that $\pi(i-1) < \pi(i) > \pi(i+1)$. The *peak set* of π is

$$\mathsf{Peaks}(\pi) = \{i : \pi(i-1) < \pi(i) > \pi(i+1)\}.$$

The peak set has been much studied in enumerative and algebraic combinatorics, see, for example, Nyman [13] and Billey, Burdzy and Sagan [6], although here it simply provides convenient terminology to prove the following result.

Following Bevan [4], a *skinny* grid class is a class $\mathsf{Grid}(\mathcal{M})$ in which the matrix $\mathcal{M}$ comprises a single row, and in which each entry is $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$.[2]

Proposition 6. *Let $\mathcal{C}$ be a permutation class that contains only finitely many zigzags. Then $\mathcal{C}$ is contained in a skinny grid class.*

Proof. Suppose that the longest zigzag in $\mathcal{C}$ has length k. For any $\pi \in \mathcal{C}$ of length n consider the peak set $\mathsf{Peaks}(\pi)$ and let i and j be two consecutive peaks (that is, there is no $h \in \mathsf{Peaks}(\pi)$ such that $i < h < j$). Since there are no peaks between i and j, the sequence $\pi(i) \cdots \pi(j)$ must be a *valley*: that is, it is formed of a decreasing sequence, followed by an increasing sequence. Let v_i be the index such that $i < v_i < j$ for which $\pi(v_i)$ is minimal (the 'bottom of the valley'). Similarly, if ℓ is the leftmost peak in π, then $\pi(1) \cdots \pi(\ell)$ is a valley, and if r is the rightmost peak in π, then $\pi(r) \cdots \pi(n)$ is a valley. In particular, we set v_r to be the index in $[r, n]$ for which $\pi(v_r)$ is minimal.

Since the entries between consecutive peaks (and before the first, and after the last peak) form valleys, we see that the entries of π can be partitioned into a sequence of $2(|\mathsf{Peaks}(\pi)| + 1)$ (possibly empty) intervals of entries, that alternately form decreasing and increasing permutations. By construction, the subpermutation formed on the indices $\mathsf{Peaks}(\pi) \cup \{v_i : i \in \mathsf{Peaks}(\pi)\}$ is a zigzag of length $2|\mathsf{Peaks}(\pi)|$. Thus, $2|\mathsf{Peaks}(\pi)| \leqslant k$ for every $\pi \in \mathcal{C}$, and hence π belongs to the grid class whose matrix is

$$\begin{bmatrix} \mathsf{Av}(12) & \mathsf{Av}(21) & \mathsf{Av}(12) & \mathsf{Av}(21) & \cdots & \mathsf{Av}(12) & \mathsf{Av}(21) \end{bmatrix}$$

comprising $k+2$ cells (if k is even), or $k+1$ cells (if k is odd). □

[2] Skinny grid classes were originally introduced by Atkinson, Murphy and Ruškuc [3] under the term 'W-classes'.

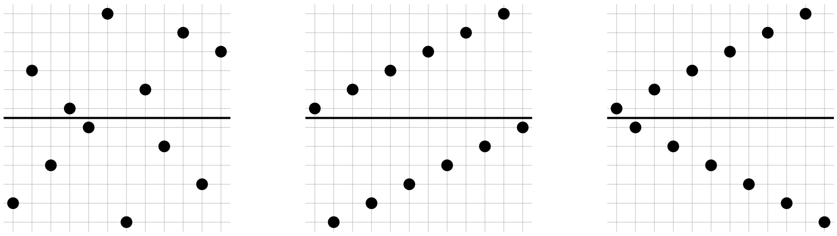

Figure 1: From left to right: a vertical alternation, a parallel alternation, and a wedge alternation.

Our next result establishes a more precise characterisation of classes without long zigzags. A *vertical alternation* is a permutation in which every odd-indexed entry lies above every even-indexed entry, or vice-versa. Some simple applications of the Erdős-Szekeres Theorem shows that every sufficiently long vertical alternation contains a long *parallel* or *wedge* alternation – see Figure 1.

Lemma 7. *The permutation class $\mathcal{C}$ contains only finitely many zigzags if and only if $\mathcal{C}$ is monotone griddable and does not contain arbitrarily long vertical alternations.*

Proof. If $\mathcal{C}$ is not monotone griddable then it contains $\bigoplus 21$ or $\bigominus 12$ by Theorem 2. In particular, for every $n \geqslant 1$, $\mathcal{C}$ contains either $2143 \cdots (2n)(2n-1)$ or $(2n-1)(2n) \cdots 12$, both of which are zigzags. Similarly, if $\mathcal{C}$ contains arbitrarily long vertical alternations then for every $n \geqslant 1$ it contains a permutation of the form $a_1 b_1 a_2 b_2 \cdots a_n b_n$ where $\{a_1, \ldots, a_n\} = \{n+1, \ldots, 2n\}$ and $\{b_1, \ldots, b_n\} = \{1, \ldots, n\}$, all of which are zigzags.

Conversely, Proposition 6 shows that a class $\mathcal{C}$ with bounded length zigzags is contained in a skinny grid class, and this demonstrates both that $\mathcal{C}$ is monotone griddable and that it cannot contain arbitrarily long vertical alternations. $\square$

We finish this section by recording an interesting consequence of the above theorem. The *growth rate* of a permutation class $\mathcal{C}$ (or gridded permutation class $\mathcal{C}^\sharp$), if it exists, is $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$, where $\mathcal{C}_n$ denotes the set of permutations in $\mathcal{C}$ of length n. The existence of the growth rate of a class in general depends upon whether the *upper* and *lower* growth rates coincide, that is, whether $\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} = \liminf_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$.

Corollary 8. *Let $\mathcal{C}$ be a class that contains only finitely many zigzags. Then* $\mathrm{gr}(\mathcal{C})$ *exists and is integral.*

We need two auxiliary results. The first tells us that when a class is $\mathcal{M}$-griddable, then it suffices to consider the upper and lower growth rates of the gridded permutations.

Proposition 9 (Vatter [18, Proposition 2.1])**.** *For a matrix of permutation classes $\mathcal{M}$ and a class $\mathcal{C} \subseteq \mathsf{Grid}(\mathcal{M})$, the upper or lower growth rate of $\mathcal{C}$ is equal, respectively, to the upper or lower growth rate of $\mathcal{C}^\sharp$.*

The second result is attributed to Albert in one of Vatter's seminal works regarding the growth rates of permutation classes.

Proposition 10 (Attributed to Albert – see Vatter [19, Proposition 7.4])**.** *The growth rate of every subword-closed language exists and is integral.*

Proof of Corollary 8. By Proposition 6, we may suppose that $\mathcal{C}$ is contained in a skinny grid class $\mathsf{Grid}(\mathcal{M})$ whose defining matrix comprises (say) m cells.

The set $\mathcal{C}^\sharp$ of all $\mathcal{M}$-gridded permutations in $\mathcal{C}$ is in bijection with a subword-closed language over an alphabet of size m (see, for example, the description in Section 7 of Vatter [19]), and in this bijection, the set of words corresponding to $\mathcal{C}$ is also subword-closed. By Proposition 10, the growth rate of $\mathcal{C}^\sharp$ exists and is integral, and thus by Proposition 9 the same is true of the growth rate of $\mathcal{C}$. □

4 Juxtapositions and lwqo

Since a class that contains only finitely many zigzags is contained in a skinny grid class, we now want to understand what happens when we juxtapose an arbitrary lwqo class $\mathcal{C}$ with such a grid class. The bulk of the remaining work lies in the next theorem, which establishes that lwqo is preserved whenever we juxtapose an lwqo class with $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$.

Theorem 11. *Let $\mathcal{C}$ be an arbitrary lwqo class, and let $\mathcal{D}$ be a monotone class. Then $\mathcal{C}\,\mathcal{D}$ is lwqo.*

Proof. By symmetry, we can assume that $\mathcal{D} = \mathsf{Av}(21)$. Furthermore, it suffices to show that the gridded permutations, $\mathcal{C}\,|\,\mathcal{D}$, are lwqo, since for any quasi-order L, the mapping $\Phi : \mathcal{C}\,|\,\mathcal{D} \wr L \to \mathcal{C}\,\mathcal{D} \wr L$ that removes the gridline is an order-preserving surjection, and thus by Proposition 5, if $\mathcal{C}\,|\,\mathcal{D} \wr L$ is wqo then so is $\mathcal{C}\,\mathcal{D} \wr L$.

Let $(L, \leqslant_L)$ be an arbitrary wqo set of labels. By Higman's lemma, $(L^*, \preceq)$ is wqo. Furthermore, by Proposition 4 the product $L \times L^*$ is also wqo, and thus $\mathcal{C} \wr (L \times L^*)$ is wqo since $\mathcal{C}$ is lwqo. Finally, another application of Proposition 4 shows that $\mathcal{C} \wr (L \times L^*) \times L^*$ is wqo.

A typical element of $\mathcal{C} \wr (L \times L^*) \times L^*$ has the form $\mathfrak{P} = ((\pi, k_\pi), z_1 \cdots z_q)$ where $\pi \in \mathcal{C}$ (of length n, say), $z_1, \ldots, z_q \in L$, where $k_\pi : [n] \to L \times L^*$ is given by

$$k_\pi(i) = (\ell(i), \lambda_{i1} \cdots \lambda_{in_i})$$

for all $i \in [n]$, in which $\ell : [n] \to L$, $\lambda_{ij} \in L$, and $n_i \geqslant 0$.

We now construct an order-preserving surjection Ψ from $\mathcal{C} \wr (L \times L^*) \times L^*$ to $\mathcal{C}\,|\,\mathcal{D} \wr L$. This mapping takes an object $\mathfrak{P} = ((\pi, k_\pi), z_1 \cdots z_q)$ and outputs an L-labelled permutation in $\mathcal{C}\,|\,\mathcal{D} \wr L$ of length $n + \sum_{i=1}^n n_i + q$. Specifically, in $\Psi(\mathfrak{P})$:

- There are n points to the left of the gridline, order isomorphic to π.
- For $i \in [n]$, the ith point from the left is labelled by $\ell(i)$.
- There are $\sum_{i=1}^n n_i + q$ points to the right of the gridline, forming an increasing sequence.

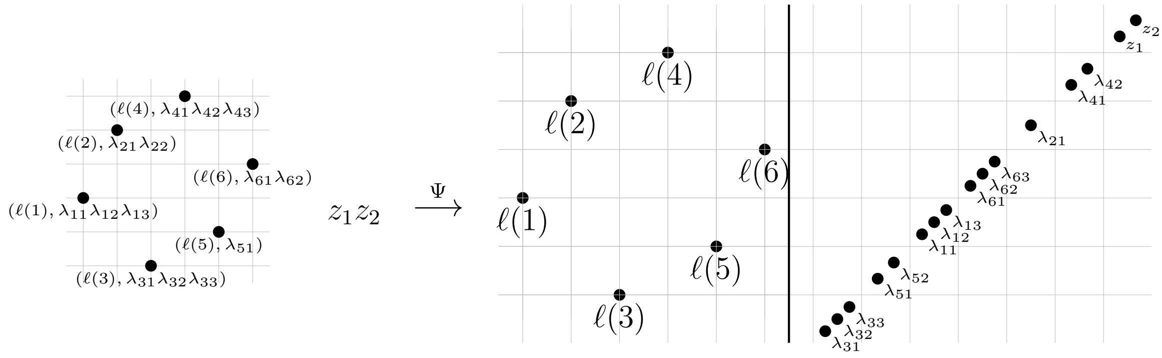

Figure 2: The mapping $\Psi : \mathcal{C} \wr (L \times L^*) \times L^* \to \mathcal{C} \,|\, \mathcal{D} \wr L$.

- For $i \in [n]$, there are n_i points to the right of the gridline that lie below the ith entry on the left, and above the next highest entry on the left (if this exists). These n_i points are labelled $\lambda_{i1}, \ldots, \lambda_{in_i}$ from bottom to top.

- Above the highest entry on the left of the gridline, there are q points to the right of the gridline, labeled $z_1, \ldots, z_q$ from bottom to top.

See Figure 2. The proof will be completed by showing that Ψ is an order-preserving surjection.

First, any labelled gridded permutation in $\mathcal{C} \,|\, \mathcal{D} \wr L$ comprises a set of points to the left of the gridline (that form a permutation from $\mathcal{C}$ with labels from L), interleaved by sequences of points to the right of the gridline (that form an increasing permutation, also with labels from L). With this in mind, for any specified element of $\mathcal{C} \,|\, \mathcal{D} \wr L$ it is straightforward to identify a suitable preimage in $\mathcal{C} \wr (L \times L^*) \times L^*$, which shows that Ψ is surjective.

Now consider $\mathfrak{S} = ((\sigma, k_\sigma), w_1 \cdots w_p)$ and $\mathfrak{P} = ((\pi, k_\pi), z_1 \cdots z_q)$ in $\mathcal{C} \wr (L \times L^*) \times L^*$, such that $\mathfrak{S} \leqslant \mathfrak{P}$. This means that there is an embedding of the underlying permutation σ of $\mathfrak{S}$ into the underlying permutation π of $\mathfrak{P}$, such that the labels of each point of $\mathfrak{S}$ precede the labels of the corresponding point of $\mathfrak{P}$. Before we proceed, let us fix some notation for the various components of $\mathfrak{S}$ and $\mathfrak{P}$.

Let σ have length m and π length n. Since $\sigma \leqslant \pi$ as labelled permutations, there exists a subsequence $1 \leqslant i_1 < \cdots < i_m \leqslant n$ such that $\pi(i_1) \cdots \pi(i_m)$ is order isomorphic to σ, and $k_\sigma(j) \leqslant k_\pi(i_j)$ for all $j \in [m]$. If we write $k_\sigma(j) = (\ell_\sigma(j), \lambda_{j1} \cdots \lambda_{jm_j})$ and $k_\pi(i) = (\ell_\pi(i), \kappa_{i1} \cdots \kappa_{in_i})$, then $k_\sigma(j) \leqslant k_\pi(i_j)$ means that $\ell_\sigma(j) \leqslant_L \ell_\pi(i_j)$ and $\lambda_{j1} \cdots \lambda_{jm_j} \preceq \kappa_{i_j 1} \cdots \kappa_{i_j n_{i_j}}$ in generalised subword order. Finally, we also require $w_1 \cdots w_p \preceq z_1 \cdots z_q$.

To complete the proof, we show that $\Psi(\mathfrak{S}) \leqslant \Psi(\mathfrak{P})$ as L-labelled gridded permutations. Intuitively, this is simply a matter of demonstrating that the embedding of $\mathfrak{S}$ into $\mathfrak{P}$ witnesses a copy of $\Psi(\mathfrak{S})$ in $\Psi(\mathfrak{P})$, according to the following commutative diagram (we use the symbol $\hookrightarrow$ to denote an embedding).

$$\begin{array}{ccc} \mathfrak{S} = ((\sigma, k_\sigma), w_1 \cdots w_p) & \hookrightarrow & ((\pi, k_\pi), z_1 \cdots z_q) = \mathfrak{P} \\ \downarrow \Psi & & \downarrow \Psi \\ \Psi(\mathfrak{S}) & \hookrightarrow & \Psi(\mathfrak{P}) \end{array}$$

The points to the left of the gridline in $\Psi(\mathfrak{S})$ and $\Psi(\mathfrak{P})$ form the L-labelled permutations (σ, ℓ_σ) and (π, ℓ_π), respectively. The subsequence $1 \leqslant i_1 < \cdots < i_m \leqslant n$ witnesses both that $\sigma \leqslant \pi$, and that $\ell_\sigma(j) \leqslant_L \ell_\pi(i_j)$, and hence $(\sigma, \ell_\sigma) \leqslant (\pi, \ell_\pi)$. We now consider the points to the right of the gridline. In $\Psi(\mathfrak{S})$, for each $j \in [m]$ the points immediately below the entry on the left corresponding to $\sigma(j)$ form an increasing sequence of length m_j labelled by $\lambda_{j1}, \ldots, \lambda_{jm_j}$. Similarly, in $\Psi(\mathfrak{P})$, the points immediately below the entry corresponding to $\pi(i_j)$ form an increasing sequence of length n_{i_j} labelled by $\kappa_{i_j1}, \ldots, \kappa_{i_jn_{i_j}}$. Since $\lambda_{j1} \cdots \lambda_{jm_j} \preceq \kappa_{i_j1} \cdots \kappa_{i_jn_{i_j}}$, we can embed these m_j labelled points of $\Psi(\mathfrak{S})$ in the n_{i_j} labelled points of $\Psi(\mathfrak{P})$.

Finally, in $\Psi(\mathfrak{S})$, there are p labelled entries to the right of the gridline that lie above all entries to the left of the grid line. Since $w_1 \cdots w_p \preceq z_1 \cdots z_q$, these p entries can be embedded in the q entries of $\Psi(\mathfrak{P})$ in the top-right. We have now embedded every labelled entry of $\Psi(\mathfrak{S})$ in $\Psi(\mathfrak{P})$, and the proof is complete. $\square$

Our approach to resolve one direction of Theorem 1 will be to apply the preceding theorem iteratively. For the other direction, we appeal to pre-existing antichain constructions, which are succinctly summarised by the following theorem.

The *cell graph* of a matrix $\mathcal{M}$ is the graph whose vertices are $\{(i,j) : M_{ij} \neq \varnothing\}$ (corresponding to the non-empty cells of $\mathcal{M}$), and $(i,j) \sim (k,\ell)$ if and only if $i = k$ or $j = \ell$, and there are no non-empty cells between these M_{ij} and $M_{k\ell}$ in their common row or column.

Theorem 12 (See Brignall [7, Theorem 1.1])**.** *Let $\mathcal{M}$ be a gridding matrix where every non-empty cell is an infinite permutation class. Then $\mathsf{Grid}(\mathcal{M})$ is not well-quasi-ordered whenever the cell graph of $\mathcal{M}$ has a cycle, or a component containing two or more cells that are not monotone griddable.*

Note that the 'cyclic' case of the above theorem is originally due to Murphy and Vatter [12].

Proof of Theorem 1. If one of $\mathcal{C}$ or $\mathcal{D}$ is not lwqo, then clearly neither is $\mathcal{C}\,\mathcal{D}$ since it contains both $\mathcal{C}$ and $\mathcal{D}$ as subclasses. So now suppose both $\mathcal{C}$ and $\mathcal{D}$ are lwqo, but contain arbitrarily long zigzags. By Lemma 7 each of $\mathcal{C}$ and $\mathcal{D}$ either is not monotone griddable, or contains arbitrarily long vertical alternations (or both).

If neither $\mathcal{C}$ nor $\mathcal{D}$ is monotone griddable, then $\mathcal{C}\,\mathcal{D}$ is not wqo (and thus also not lwqo) by Theorem 12. (See Figure 3 (left) for a typical antichain element in this case.)

Now suppose, without loss of generality, that $\mathcal{C}$ is monotone griddable but contains long vertical alternations, and $\mathcal{D}$ is not monotone griddable. By Theorem 2, the class

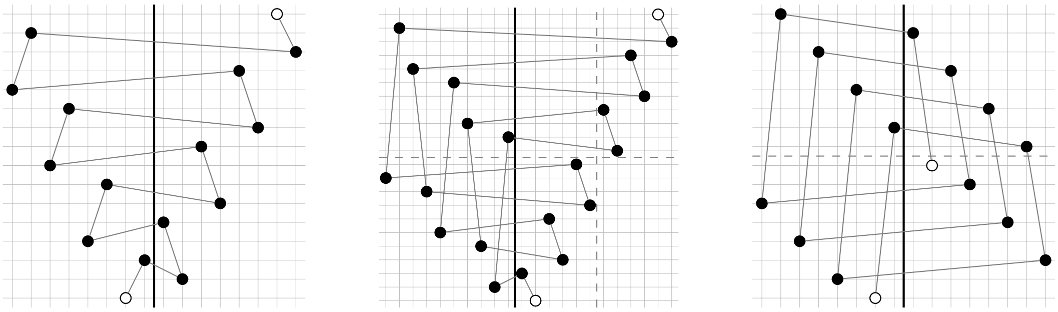

Figure 3: Typical labelled antichain elements arising in juxtaposition classes. Here, we may take $L = \{\bullet, \circ\}$ to be an antichain of size 2.

$\mathcal{D}$ contains $\bigoplus 21$ or $\bigominus 12$. Consequently, $\mathcal{C}\,\mathcal{D}$ contains $\mathsf{Grid}(\mathcal{M})$ for a matrix $\mathcal{M}$ of the following form:

$$\mathcal{M} = \begin{bmatrix} \mathcal{E}_1 & & \bigoplus 21 \\ \mathcal{E}_2 & \bigoplus 21 & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathcal{E}_1 & \bigominus 12 & \\ \mathcal{E}_2 & & \bigominus 12 \end{bmatrix}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are each either $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$. In any case, the cell graph of $\mathcal{M}$ comprises a component containing two cells that are not monotone griddable (again by Theorem 2), and hence $\mathsf{Grid}(\mathcal{M})$ is not wqo by Theorem 12. (See Figure 3 (middle) for a typical antichain element in this case.)

Finally for this direction, suppose that both $\mathcal{C}$ and $\mathcal{D}$ are monotone griddable, but both contain arbitrarily long vertical alternations. In this case, $\mathcal{C}\,\mathcal{D}$ contains $\mathsf{Grid}(\mathcal{M})$ for a matrix $\mathcal{M}$ of the following form

$$\mathcal{M} = \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \\ \mathcal{E}_3 & \mathcal{E}_4 \end{bmatrix}$$

where $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$ are each either $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$. In any case, the cell graph of $\mathcal{M}$ comprises a component that is a cycle, so $\mathsf{Grid}(\mathcal{M})$ is once again not wqo by Theorem 12, and hence neither is $\mathcal{C}\,\mathcal{D}$. (See Figure 3 (right) for a typical antichain element in this case.)

For the other direction, suppose (without loss of generality) that $\mathcal{C}$ is lwqo, and $\mathcal{D}$ contains only bounded length zigzags. By Proposition 6, there exists a skinny grid class $\mathcal{E}$ such that $\mathcal{D} \subseteq \mathcal{E}$. We claim that $\mathcal{C}\,\mathcal{E}$ is lwqo.

Write $\mathcal{E} = \mathsf{Grid}(\mathcal{M})$ where $\mathcal{M} = \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 & \cdots & \mathcal{E}_k \end{bmatrix}$ for classes $\mathcal{E}_i$ each equal to $\mathsf{Av}(21)$ or $\mathsf{Av}(12)$ $(1 \leqslant i \leqslant k)$. Let $\mathcal{C}_0 = \mathcal{C}$, and for $1 \leqslant i \leqslant k$ set

$$\mathcal{C}_i = \mathsf{Grid}(\begin{bmatrix} \mathcal{C} & \mathcal{E}_1 & \cdots & \mathcal{E}_i \end{bmatrix}).$$

Now $C_0 = \mathcal{C}$ is lwqo, and it follows by induction and Theorem 11 that $\mathcal{C}_i = \mathcal{C}_{i-1}\,\mathcal{E}_i$ is lwqo for each $i = 1, \ldots, k$. In particular $\mathcal{C}_k = \mathcal{C}\,\mathcal{E}$ is lwqo. The result now follows since $\mathcal{C}\,\mathcal{D} \subseteq \mathcal{C}\,\mathcal{E}$. $\square$

5 Concluding remarks

The methods and ideas in this note can almost certainly be adapted to a characterisation of lwqo in grid classes, although it would likely be technically and notationally awkward to do so.

A more interesting future direction is to consider lwqo in *subclasses* of these grid classes. For example, while the juxtaposition of $\bigominus 12$ with $\bigoplus 21$ contains the infinite antichain comprising elements of the form shown on the left of Figure 3, there exist subclasses of this juxtaposition that are lwqo. Individual cases such as this are relatively easy to characterise, but a general answer seems further out of reach.

Can a similar characterisation can be achieved for (unlabelled) wqo? Although the antichain elements depicted in Figure 3 use two labels, the proof of Theorem 1 in fact uses only unlabelled antichains, so aspects of this question already have an answer. However, if $\mathcal{C}$ is a wqo-but-not-lwqo class, then it is sometimes possible to break wqo by juxtaposing $\mathcal{C}$ with the class containing just the singleton permutation, while in other cases, $\mathcal{C}$ must be juxtaposed with two entries. In general, we cannot hope to make progress on this question without a significantly deeper understanding of wqo in permutation classes.

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