Colored unavoidable patterns and balanceable graphs

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Abstract

We study a Turán-type problem on edge-colored complete graphs. We show that for any r and t, any sufficiently large r-edge-colored complete graph on n vertices with $\Omega(n^{2-1/tr^r})$ edges in each color contains a member from a certain finite family $\mathcal{F}_t^r$ of r-edge-colored complete graphs.

Next, we study a related problem where the corresponding Turán threshold is linear. We call an edge-coloring of a path with rk edges balanced if each color appears k times in the coloring. We show that any 3-edge-coloring of a large complete graph with $kn + o(n)$ edges in each color contains a balanced P_{3k}. This is tight up to a constant factor of 2. For more colors, the problem becomes surprisingly more delicate. Already for $r = 7$, we show that even $n^{2-o(1)}$ edges from each color do not guarantee the existence of a balanced path on $7k$ edges.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

The basic principle behind Ramsey theory is that no matter how a system is partitioned, there must exist an organized subsystem that is entirely contained inside one of the parts of the partition. Recently, numerous authors have pursued a line of research investigating the emergence of subsystems that are organized, yet meet every single part in the partition. Of course, to do so, one needs to assume that each part in the partition is sufficiently large. In order to state a prototypical result in this direction, we first give the following definition (for results of a similar flavor, see [1, 11, 12, 23, 25]). Let $\mathcal{F}_t$ be the family of two-edge-colored complete graphs on $2t$ vertices where one color forms a clique of size t, or two disjoint cliques of size t. The following was conjectured by Bollobás, and proved by Cutler and Montágh.

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Theorem 1 ([8]). *Let $0 < \varepsilon \leqslant \frac{1}{2}$ be a real number and $t \geqslant 1$ an integer. For large enough n, any two-edge-coloring of K_n with at least $\varepsilon\binom{n}{2}$ edges in each color contains a member of $\mathcal{F}_t$.*

Cutler and Montágh's argument shows that one can take $n \geqslant 4^{t/\varepsilon}$ to find a member of $\mathcal{F}_t$. Fox and Sudakov improved their result by showing that one can take $n \geqslant \varepsilon^{-ct}$ for some absolute constant c, which is tight up to the value of c [15]. In an attempt to generalize these results to a setting where an arbitrary number of color classes are allowed, the following definition was given in [3].

Definition 2 ([3]). Let t and r be positive integers. A complete graph H whose edges are colored with r colors belongs to the family $\mathcal{F}_t^r$ if there is an integer $k \geqslant 1$ and a partition $V(H) = \bigsqcup_{i \in [k]} V_i$ of the vertices of H into k parts V_i with $|V_i| = t$ such that:

1. For all $i, j \in [k]$, $H[V_i]$ and $H[V_i \times V_j]$ are monochromatic.

2. All r-colors are present in H.

3. Not all r colors are present in $H \setminus V_i$, for any $i \in [k]$.

We make a couple of remarks regarding this definition. By (1), the color of any edge in H depends only on the parts the endpoints come from. By (3), it is not hard to see that any graph in $\mathcal{F}_t^r$ can have at most $2r$ parts (see the argument in [3]). Hence, $\mathcal{F}_t^r$ is a finite set. Observe also that $\mathcal{F}_t^2 = \mathcal{F}_t$.

By a *pattern*, we denote a maximal subfamily of $\mathcal{F}_t^r$ consisting of graphs colored all the same up to permutations of their colors. There are two different patterns in $\mathcal{F}_t^2$ and nine different patterns in $\mathcal{F}_t^3$, see an illustration of the latter in Figure 1. In general, the number of different patterns in $\mathcal{F}_t^r$ grows exponentially with r as one can embed the family of all non-isomorphic tournaments on r vertices into $\mathcal{F}_t^r$ by associating to each vertex a K_t of a different color, and giving the bipartite graphs between the K_t's the color of the K_t they point to in the tournament (see [24] for information regarding the number of non-isomorphic tournaments on n vertices).

We can now state the multicolor generalization of the result of Cutler and Montágh, due to Bowen, Lamaison, and Müyesser.

Theorem 3 ([3]). *For any $r \geqslant 2$, there exists a constant $c := c(r)$ such that, for any $\varepsilon > 0$ and $t \geqslant 2$, any r-coloring of a K_n with $n \geqslant \varepsilon^{-ct}$ and $\varepsilon\binom{n}{2}$ edges in each color contains a member of $\mathcal{F}_t^r$.*

1.1 Turán-bounds for colored unavoidable patterns

The bound in Theorem 3, up to the dependence of c on r, is optimal by a random construction, similar to the one given in [15]. However, if one is interested in the minimum density of edges required of each color in order to force a member of $\mathcal{F}_t^r$, assuming that each color has $\Theta(n^2)$ many edges is not necessary. To discuss this extremal aspect of the problem precisely, we first define the following Turán-type parameter.
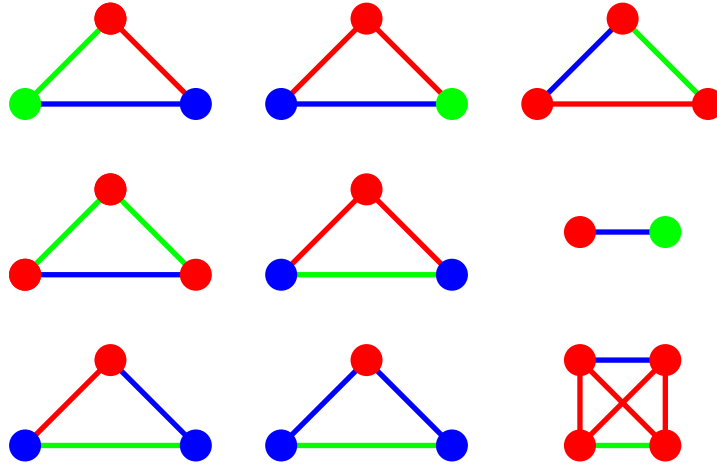
Figure 1: The patterns from $\mathcal{F}_{3,t}$. The colored vertices represent cliques of the corresponding color of size t. Similarly, edges represent complete bipartite graphs of the corresponding color between the associated cliques.

Definition 4. Let r, t, n be positive integers. Let $\mathcal{F}$ be a family of r-edge colored graphs. We denote by $\mathrm{ex}_r(K_n, \mathcal{F})$ the minimum integer m (if it exists) such that, for any r-edge coloring of K_n with more than m edges in each of the r colors, K_n contains a member of $\mathcal{F}$. If there is no such m, we set $\mathrm{ex}_r(K_n, \mathcal{F}) = \infty$.

We begin with a discussion of the case when $r = 2$. Caro, Hansberg and Montejano showed that there exists a $\delta := \delta(t)$ such that $\mathrm{ex}_2(K_n, \mathcal{F}_t) = \Omega(n^{2-\delta})$ when n is large enough [7]. Shortly after, Girão and Narayanan [18] proved that $\delta = 1/t$ is best possible up to the involved constants (we give a shorter proof of their result in Section 2.1), assuming that the well-known conjecture that $\mathrm{ex}(K_{t,t}) = \Omega(n^{2-1/t})$ for all t is true [22]. In [4], an asymmetric version of this result is investigated. Further, in [4], a characterization of unavoidable patterns with respect to the order of magnitude (in terms of n) of the assumed minimum number of edges in each color is given.

For arbitrary $r \geqslant 3$, the structure of the graphs in $\mathcal{F}_t^r$ is a lot more complicated (already for $r = 3$ there are 9 different patterns, see Figure 1). However, it is still natural to suspect that the existence of large bipartite graphs is the only barrier to finding the patterns in $\mathcal{F}_t^r$. In particular, this would imply that $\Omega(n^{2-1/t})$ edges in each color class should guarantee the existence of a member from $\mathcal{F}_t^r$. We conjecture that this is indeed the case. [1]

Conjecture 5. For any $r \geqslant 2$ and $t \geqslant 1$, there exists a constant $C := C(r, t)$ such that, for n large enough, $\mathrm{ex}_r(K_n, \mathcal{F}_t^r) \leqslant Cn^{2-1/t}$.

We show that at this same density (with $\Omega(n^{2-1/t})$ edges) we can find a member from $\mathcal{F}_t^s$, where $s = \left\lfloor \frac{t}{r^r} \right\rfloor$.

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[1] In the process of revision of this paper, this conjecture was confirmed in [17].

Theorem 6. *For any $r \geqslant 2$ and $t \geqslant 1$, there exists a constant $C := C(r,t)$ such that, for n large enough, $\mathrm{ex}_r(K_n, \mathcal{F}_t^r) \leqslant Cn^{2-1/t'}$, where $t' = tr^{r-1}$.*

Observe that the result of Girão and Narayanan [18] does not follow from the above by setting $r = 2$. We give a short proof of the result of Girão and Narayanan in Section 2.1.

1.2 Balanceable graphs

Next, we turn our attention to a more specific problem, concerning balanced colorings of paths. We denote a path on k edges by P_k.. We say that an r-colored graph G is *balanced* if each of the r colors appears in precisely $\lfloor e(G)/r \rfloor$ or $\lceil e(G)/r \rceil$ edges. Further, we say that an r-coloring of $E(K_n)$ contains a *balanced copy* of G if it admits a balanced embedding of G. The following parameter for the case $r = 2$ was introduced by Caro, Hansberg and Montejano [7].

Definition 7. Given a graph G and positive integers r and n, we set $\mathrm{bal}_r(n, G) = \mathrm{ex}_r(K_n, \mathcal{F}_{\mathrm{bal}}(G))$, where $\mathcal{F}_{\mathrm{bal}}(G)$ is the family of all r-colored copies of G in which each of the r colors appears in either $\lfloor e(G)/r \rfloor$ or $\lceil e(G)/r \rceil$ edges. We call $\mathrm{bal}_r(n, G)$ the *r-balancing number* of G and, when $r = 2$, we will put $\mathrm{bal}_2(n, G) = \mathrm{bal}(n, G)$ and call it just the *balancing number* of G. A graph G with $\mathrm{bal}_r(n, G) < \infty$ for every sufficiently large n is called *r-balanceable* or, when $r = 2$, simply *balanceable*.

Caro et al. [7] characterized all balanceable graphs. It follows by this characterization that all balanceable graphs G have $\mathrm{bal}(n, G) = o(n^2)$. See [7, 9, 10] for several examples and non-examples of balanceable graphs. Notably, there are many dense balanceable graphs (like some amoebas [5, 7]) and also many graphs having linear balancing number (in n), like trees [7], and cycles C_k with $k \not\equiv 2 \pmod 4$ [9]. For instance, $\mathrm{bal}(n, P_{2k}) = \left(\lfloor \frac{k-1}{2} \rfloor + o(1) \right) n$, and the value was precisely determined in [7]. Surprisingly, it turns out to be an intricate problem to determine even the order of magnitude of $\mathrm{bal}_r(n, P_{rk})$ for arbitrary r and k, as the next two results show.

Theorem 8. *Let $k \in \mathbb{N}$ be odd. Then there exists infinitely many r such that $\mathrm{bal}_r(n, P_{rk}) = \Omega(n^2)$. In particular, $\mathrm{bal}_7(n, P_{7k}) = \Omega(n^2)$.*

We would expect that, in this scenario, $\mathrm{bal}_r(n, P_{rk})$ is in fact ∞, but this would require giving a construction with exactly the same number of edges from each color class. On the other hand, we have the following positive result:

Theorem 9. *Let $r \geqslant 2$. Then, there exists a k_0 such that for all $k \geqslant k_0$, $\mathrm{bal}_r(n, P_{2rk}) = o(n^2)$.*

We remark that, while the dependence of k_0 on r our proof gives is possibly far from optimal, some threshold is necessary. Indeed, there exist members of $\mathcal{F}_t^r$, for every r, where one color class separates all the remaining colors, thus making it impossible to embed a balanced P_r (a rainbow path) in such graphs. As an example of such a member from $\mathcal{F}_t^r$, one may start with $r - 1$ vertex-disjoint $K_{n,n}$'s, each colored a distinct color,

and color all the edges that remain in the rth color. An illustration can be found for $r = 3$ in the bottom-right part of Figure 1.

Having established that degeneracies arise for large r, we turn our attention to the function $\mathrm{bal}_3(n, P_{3k})$. The following theorem establishes the growth of this function up to a constant factor of 2.

Theorem 10. *For $k \geqslant 1$, $(\frac{1}{2}(k-1) + o(1))n \leqslant \mathrm{bal}_3(n, P_{3k}) \leqslant (k + o(1))n$.*

We believe that the lower bound from Theorem 10 should be tight for arbitrary k but are only able to confirm this when $k \leqslant 2$. When $k = 1$, it is not hard to show $\mathrm{bal}_3(n, P_3) = 0$ (Proposition 16), whereas for $k = 2$ some more work is needed. We state the latter result in the following theorem.

Theorem 11. $\mathrm{bal}_3(n, P_6) = (\frac{1}{2} + o(1))n$.

1.3 Organization of the paper

We begin by proving our general result, Theorem 6. This already implies $\mathrm{bal}_3(P_{3k}) = o(n^2)$, as one can check all members from $\mathcal{F}_t^3$ in Figure 1 admit balanced embeddings of P_{3k}. Afterward, we prove much better bounds on this function, showing that it grows linearly in n (Theorems 10 and 11). We proceed by proving Theorems 8 and 9, which summarize our understanding of the function $\mathrm{bal}_r(n, P_{rk})$ for arbitrary r. We conclude with some open problems in the Discussion section.

2 Proof of Theorem 6

We will use the following version of the dependent random choice lemma. Here, and in subsequent uses, the common neighborhood of a set of vertices is the intersection of the neighborhoods of each vertex in the set.

Lemma 12 ([14]). *For all $K, t \in \mathbb{N}$, there exists a constant C such that any graph with at least $Cn^{2-1/t}$ edges contains a set S of K vertices in which each subset $X \subseteq S$ with t vertices has a common neighborhood of size at least K.*

We will also use the following, which is essentially a consequence of iterating a bipartite version of the dependent random choice lemma from [3] and Ramsey's theorem.

Lemma 13 (Corollary 2.6 in [3]). *Let r and t be positive integers, $r \geqslant 2$. There exists $N = N(r, t)$ such that the following holds for all $n \geqslant N$. Let $A_1, A_2, \ldots, A_t$ partition the vertex set of an r-colored complete graph, where $|A_i| = n$ for all $i \in [t]$. Then, there exist subsets $X_i \subset A_i$, of size $|X_i| = \frac{1}{2^{t+1}r} \log_r n$, such that every set X_i induces a monochromatic clique and every complete bipartite graph between X_i and X_j is monochromatic.*

We emphasize that the proof structure of the result in this section will be very similar to that of the main result from [3].

Proof of Theorem 6. Start with given integers $r \geqslant 2$, $t \geqslant 1$ and set $t' = tr^{r-1}$. We want to show that there exists a constant $C := C(r, t)$ such that, for n large enough, $\mathrm{ex}_r(K_n, \mathcal{F}_t^r) \leqslant Cn^{2-1/t'}$. We choose a C with the benefit of hindsight, large enough to make sure the following calculations go through. Similarly, we choose a sufficiently large n and consider an r-coloring of a K_n with $Cn^{2-1/t'}$ edges in each color class.

First, we choose a K' (with hindsight) such that $K' \gg t'$, and apply, for each color class, Lemma 12 with parameters t' and rK' to find a collection of rK'-sized sets $\{S_i\}_{i=1}^r$ such that each t'-subset of S_i has rK' common neighbors in color i (this can be done if C is large enough). The different sets $S_1, S_2, \cdots, S_r$ can intersect, but we can choose, up to renaming, suitable subsets of each one in order to have r disjoint sets each with $|S_i| = K'$.

We now apply Lemma 13 to the collection $\{S_i\}_{i=1}^r$, obtaining sets $\{X_i\}_{i=1}^r$ such that, for each $i \in [r]$, $X_i \subset S_i$, $|X_i| = \frac{1}{2^{t'+1}r} \log_r(K')$, X_i induces a monochromatic clique (of some color, not necessarily i), and the complete bipartite graph connecting vertices between X_i and X_j, with $i \neq j$, is monochromatic. Here, we make sure to select K' large enough and potentially throw out vertices to ensure that $|X_i| = t'$.

Now, since $X_i \subset S_i$, we use the property of the sets S_i to find rK' common neighbors of X_i in color i. Call these sets Y_i and, again, choose suitable subsets (without renaming) to ensure that we get a collection $\{Y_i\}_{i=1}^r$ of disjoint sets, where $|Y_i| = K'$ for each $i \in [r]$.

Our aim is now to find monochromatic complete bipartite graphs between Y_i and X_j for all $i \neq j$. Associate to each vertex y of Y_i a $t'r$-tuple with entries in $[r]$ listing the color of the edges (y, x) where $x \in \bigcup X_i$ (recall $|\bigcup X_i| = t'r$). As there are at most $r^{t'r}$ such tuples, there must be $|Y_i|/r^{t'r}$ vertices in each Y_i (call them Y_i') such that the associated tuple is identical. This means that, for each $x \in X_j$, all the edges from x to vertices in Y_i' are of the same color.

We may now fix subsets $X_i' \subseteq X_i$ of size $|X_i|/r^{r-1}$ so that the edges between Y_j' and X_i', for $i \neq j$, are monochromatic (recall that these bipartite subgraphs are already monochromatic in color i when $i = j$). In particular, to do this we associate to each vertex x of X_i an $(r-1)$-tuple with entries in $[r]$ listing the color of the edges from x to the sets Y_j' for $j \neq i$ (recall that all edges from x to Y_j' are monochromatic). So, for at least $|X_i|/r^{r-1}$ of the vertices, its tuples are all identical. Call these sets X_i' and note that $|X_i'| \geqslant t'/r^{r-1} = t$ for every $i \in [r]$.

To finish, we apply Lemma 13 to the collection $\{Y_i'\}_{i=1}^r$, obtaining sets $\{Y_i''\}_{i=1}^r$ such that, for each $i \in [r]$, $Y_i'' \subset Y_i'$, $|Y_i''| = \frac{1}{2^{t'+1}r} \log_r(K'/r^{t'r})$, each Y_i'' induces a monochromatic clique, and the complete bipartite graph connecting vertices between Y_i'' and Y_j'', for $i \neq j$, is monochromatic. We select K' in advance to make sure this quantity exceeds t. Looking at the graph induced by the sets $\{X_i'\}_{i=1}^r$ and $\{Y_i''\}_{i=1}^r$, we thus have a blow-up of a complete graph of order $2r$ with each blow-up of size at least t, where all r colors are used, so this structure must contain as a subgraph a member from $\mathcal{F}_t^r$. Hence, $\mathrm{ex}_r(K_n, \mathcal{F}_t^r) \leqslant Cn^{2-1/t'}$. $\qquad \square$

2.1 The case of $r = 2$

In this subsection, in an attempt to demonstrate the relative simplicity of Conjecture 5 when $r = 2$, we give a very short proof of this particular case, originally proved in [18]

using a more involved approach. Our version of the proof will essentially be an adaptation of an argument from [7], replacing the usage of the Kővari-Sós-Turán theorem with the dependent random choice lemma (Lemma 12).

Before we begin, we recall two definitions. Denote by $R(k)$ the classical Ramsey number, that is, the smallest integer for which every 2-edge-coloring of K_n with $n \geqslant R(k)$ contains a monochromatic K_k. Denote by $BR(k)$ the bipartite Ramsey number, that is, the smallest integer for which every 2-edge-coloring of $K_{n,n}$, with $n \geqslant BR(k)$, contains a monochromatic $K_{k,k}$. All we need in the following proof is that $R(k)$ and $BR(k)$ are constants depending only on k, which is a well-known fact.

Theorem 14 ([18])**.** *For any $t \geqslant 1$, there exists a constant $C := C(t)$ such that, for n large enough, $\mathrm{ex}_2(K_n, \mathcal{F}_t) \leqslant Cn^{2-1/t}$.*

Proof. Given t, consider a 2-edge-coloring of K_n (for n large) with $Cn^{2-1/t}$ edges in each color class, where C is the constant given in Lemma 12 for t and $K = R(BR(t))$. By Lemma 12, we can find a red $K_{t,BR(t)}$ where both parts of the bipartite graph are monochromatic cliques. If either of these cliques is blue, we find a member of $\mathcal{F}_t$. So, we may assume that both are red and thus there is a red clique of size $BR(t)$. Likewise, it can be inferred that there exists another complete graph of the same order, colored blue, which is disjoint from the previous one. Consider now the 2-edge colored complete bipartite graph, $K_{BR(t),BR(t)}$, induced by the vertices of those two cliques. By definition, there is a monochromatic $K_{t,t}$ in such a complete bipartite graph, yielding the desired graph contained in $\mathcal{F}_t$. $\square$

We remark that the proof above is quite specific to the $r = 2$ case. Indeed, if we employed a similar strategy for the $r = 3$ case, for each monochromatic $K_{t,BR(t)}$ with both parts being monochromatic cliques, all we could conclude is that the entire structure does not use all three colors. In particular, instead of finding three large cliques of three different colors, we could end up with three large cliques all of the same color.

3 Proofs of Theorems 10 and 11

In this section, we focus on upper and lower bounds for the function $\mathrm{bal}_3(n, P_{3k})$. For convenience, we recall the corresponding result from the 2-color case, this time in its most precise formulation.

Theorem 15 ([7])**.** *Let $k \geqslant 1$ be even and let $n \geqslant \frac{9}{8}k^2 + \frac{1}{2}k + 1$. Then*

$$\mathrm{bal}_2(n, P_k) = \begin{cases} \frac{(k-2)n}{4} - \frac{k^2}{32} + \frac{1}{8} & k \equiv 2 \pmod 4 \\ \frac{(k-4)n}{4} - \frac{k^2}{32} + \frac{k}{8} + 1 & k \equiv 0 \pmod 4 \end{cases}$$

We remark that in [7] the extremal family of 2-edge-colored P_{2k}-avoiding complete graphs were explicitly characterized.

3.1 Preliminaries

In preparation for the proofs of Theorems 10 and 11, we first collect some helpful lemmas. In the proofs below, we always assume that the host graph has sufficiently many vertices. We start giving the 3-balancing number for P_3. Observe that a 3-balanced P_3 is simply a rainbow P_3.

Proposition 16. $\mathrm{bal}_3(n, P_3) = 0$.

Proof. Assume that we have a large 3-edge-colored complete graph with at least 1 edge from each color. There must exist a triangle with exactly two of the colors represented, say red and blue. Let u be a vertex on this triangle adjacent to both a red and blue edge. No green edge can be adjacent to this triangle (otherwise we can easily construct a rainbow P_3), but there must be a green edge with end vertices v, w. We can form a balanced P_3 by walking from v to w to u and then ending on either the red or blue edge adjacent to u, depending on the color of the edge wu. □

We will use the notation $P = x_0 x_1 \ldots x_k$ to represent a k-path with edges $x_i x_{i+1}$ for $i \in \{0, \ldots, k-1\}$. We will call a 3-balanced K_3 a *rainbow triangle* because this is the term that is used in connection to Gallai-colorings, which we will use later.

Lemma 17. *Let $k \geqslant 2$. If a 3-edge-colored complete graph contains a balanced P_{3k-3} and a vertex-disjoint rainbow triangle, then the graph contains a balanced P_{3k}.*

Proof. Let $P = x_0 x_1 \ldots x_{3k-3}$ be a balanced path which is vertex-disjoint to a rainbow triangle $\{x_r, x_g, x_b\}$ where $x_g x_b$ is red, $x_r x_g$ is blue and $x_b x_r$ is green. If $x_0 x_r$ or $x_{3k-3} x_r$ is blue or green, we we can easily construct a balanced P_{3k} (take $P x_r x$ or $x x_r P$ where $x = x_b$ or $x = x_g$ according to the color we choose). Thus, we may assume that both $x_0 x_r$ and $x_{3k-3} x_r$ are red. Similarly, avoiding balanced P_{3k}'s, the colors of $x_0 x_b$, $x_{3k-3} x_b$, $x_0 x_g$ and $x_{3k-3} x_g$ are determined to be blue and green respectively. Suppose now, without loss of generality, that the first edge of the path P, $x_0 x_1$, is red. Then the path $x_1 \ldots x_{3k-3} x_b x_g x_0 x_r$ is a balanced P_{3k}. □

Now we show the lower bound from Theorem 10. Let $k \geqslant 2$ and consider a 3-edge-coloring of K_n where we split $V(K_n)$ into three parts A, B, and C, with $|A| = k - 1$, $|B| = \lfloor \frac{n-k+1}{2} \rfloor$, and $|C| = \lceil \frac{n-k+1}{2} \rceil$, and we color the edges as follows: edges between A and B are red, edges between A and C are blue, and all remaining edges are green. Since any balanced P_{3k} requires k red edges and k blue edges, and any such edge is incident on A, this graph contains no balanced P_{3k}. Further, the graph has at least $(k-1)(n-k)/2$ edges in each color class, implying

$$\mathrm{bal}_3(n, P_{3k}) \geqslant \left(\frac{k-1}{2} + o(1) \right) n \tag{1}$$

as desired.

Before proving the upper bound of Theorem 10, we will first determine the 3-balancing number for P_6. This will show that we are able to match the lower bound (1) when $k = 2$.

3.2 Proof of Theorem 11

In order to prove Theorem 11 we need the following lemmas.

Lemma 18. *If a 3-edge-colored complete graph contains two vertex-disjoint balanced P_3's whose middle edges have different colors, then the graph contains a balanced P_6.*

Proof. Without loss of generality, assume that the first path $P = x_0 x_1 x_2 x_3$ is red-blue-green, and the second path $P' = x_4 x_5 x_6 x_7$ is red-green-blue. If $x_3 x_4$ is blue or red, we have a balanced P_6 (take PP' and remove either x_0 or x_7 according to the color of $x_3 x_4$). Similarly, if $x_0 x_7$ is green or red, we have a balanced P_6. Assuming these edges are green and blue respectively, we can see that $x_7 x_0 x_1 x_2 x_3 x_4 x_5$ is a balanced P_6. $\square$

The above lemma will allow us to conclude that all balanced P_3's in a 3-edge-colored complete graph without balanced P_6's must be color isomorphic. This fact combined with the next lemma will allow us to conclude that in a 3-edge-colored complete graph with several balanced P_3's and no balanced P_6, the color of the middle edge of any of the P_3's must be dense in the complete graph.

Lemma 19. *If a 3-edge-colored complete graph without a balanced P_6 contains two vertex-disjoint balanced P_3's whose middle edges are of the same color, say blue, then any other edge between the vertices of these two paths (including edges contained within a single path) is blue.*

Proof. Let $P = x_0 x_1 x_2 x_3$ and $P' = x_4 x_5 x_6 x_7$ be two vertex-disjoint red-blue-green paths. If $x_3 x_4$ is red or green, we can easily construct a balanced P_6 (take PP' and remove either x_0 or x_7 according to the color of $x_3 x_4$). Hence, $x_3 x_4$ is blue. The same argument allows us to conclude that $x_0 x_7$ is blue. Observe that $C = x_0 x_1 \ldots x_7 x_0$ is a red-blue-green-blue-red-blue-green-blue cycle. If $x_0 x_2$ is red then $x_0 x_2 x_3 x_4 x_5 x_6 x_7$ is a balanced P_6. Also, by Lemma 17, $x_0 x_2$ can't be green (see triangle $\{x_0, x_1, x_2\}$ and path $x_4 x_5 x_6 x_7$). Thus, $x_0 x_2$ must be blue. A similar argument allows us to prove that all edges $x_i x_{i+2}$, with $i \in \{0, \ldots, 7\}$ (where addition is taken modulo 8) are blue. Note now that $x_3 x_7$ can't be red (respectively, green) by $P x_7 x_6 x_5$ (respectively, $P x_7 x_5 x_4$). Thus, $x_3 x_7$ must be blue. Again, by taking advantage of the symmetry of C, we can conclude that all edges of the form $x_i x_{i+4}$, with $i \in \{0, \ldots, 7\}$ and addition modulo 8, are blue. Finally, to conclude that the remaining edges are blue, we consider a new cycle $C' = x_0 x_1 x_3 x_2 x_4 x_5 x_7 x_6 x_0$ which is color isomorphic to C, and repeat the arguments. $\square$

We are now in a position to prove Theorem 11, which states that $\mathrm{bal}_3(n, P_6) = (\frac{1}{2} + o(1))n$.

Proof of Theorem 11. Let $\varepsilon > 0$ be arbitrary, and consider a 3-edge-coloring of K_n (for n sufficiently large) with each color class having at least $(\frac{1}{2} + \varepsilon)n$ edges. By Proposition 16, we can find a balanced P_3. Let I be the vertex set of a maximum sized family of vertex-disjoint balanced P_3's and let $D = V(K_n) \setminus I$. Define $i = |I|/4$ as the number of vertex-disjoint balanced P_3's. By Proposition 16, one of the color classes must not appear

in D. We will handle the case where $i = 1$ and $i \geqslant 2$ separately.

Case 1: Assume $i = 1$.

Let P be the unique balanced 3-path in K_n. Assume without loss of generality that red is missing in D. We first show that it cannot be the case that D is almost missing another color as well.

Subcase 1.1: Suppose D is monochromatic except for at most one edge.

Assume, without loss of generality, that all edges in D but possibly one edge, say uv, are blue (so uv is either blue or green). Observe that between $D \setminus \{u, v\}$ and I we have almost all (but a constant number) of the red and the green edges. Since all color classes have at least $(\frac{1}{2} + \varepsilon)n$ edges, it cannot be the case that all green and red edges are incident to the same vertex in P. Thus, there are two distinct vertices $x, y \in I$ and five distinct vertices (since n is sufficiently large) $x_1, x_2, y_1, y_2, z \in D \setminus \{u, v\}$ such that xx_1 and xx_2 are green, yy_1 and yy_2 are red, and so $x_1 x x_2 z y_1 y y_2$ is a balanced P_6.

Subcase 1.2: Suppose D has at least two blue edges and at least two green edges.

By Theorem 15, the 2-color balancing number of P_4 is 2, so we can find a blue-green-balanced P_4 in D, say Q. Observe that between $V(P)$ and $D \setminus V(Q)$ we have almost all (but a constant) of the red edges. Since all color classes have at least $(\frac{1}{2} + \varepsilon)n$ edges, there must be two distinct vertices $x, y \in D \setminus V(Q)$ and one vertex in $z \in V(P)$ such that zx and zy are red. Let a and b be the end-vertices of the path Q. Consider now the cycle $C = zxQyz$. We know that both edges xa and yb are either blue or green. Suppose first that xa and yb are both of the same color, say blue. Then the path xQy has 4 blue edges and 2 green edges. Thus, it must contain two consecutive blue edges, say rs and st. Then $C - s$ is a balanced 3-colored 6-path. If, on the other side, xa and yb are one blue and one green, then we can take two consecutive edges rs and st from Q such that they have different colors and we can see that $C - s$ is a balanced 3-colored 6-path.

Case 2: Assume $i \geqslant 2$.

By Lemma 18, we may assume, without loss of generality, that all vertex disjoint balanced paths in I are of the form red-blue-green. By Lemma 19, all remaining edges induced by vertices in I are blue. In this case, we will conclude that either there are too few red or green edges, or we can find a balanced P_6.

Subcase 2.1: D has no red or no green edges.

Assume, without loss of generality, that red is missing in D. Since we have at least $(\frac{1}{2} + \varepsilon)n$ red edges in K_n and amongst them only i in I and none in D, the remaining red edges are all in $E(I, D)$. Let uv be a red edge with $u \in I$ and $v \in D$. Then u belongs to a red or a green edge ux in I. Assume first that ux is green. Consider another green edge $x_1 x_2$ and a red edge $y_1 y_2$ in I. Since all these edges form a matching and all other edges between the vertices u, x_1, x_2, y_1, y_2 are blue, we can see easily that $vuxx_1 x_2 y_1 y_2$ is a balanced P_6.

The case that ux is red is completely analogous by taking two green edges x_1x_2 and y_1y_2.

Subcase 2.2: *D has at least a red edge and a green edge.*
The blue has to be missing in D. Let uvw be a green-red path. Consider one red edge x_1x_2 and one green edge y_1y_2 in I. Let $z \in I \setminus \{x_1, x_2, y_1, y_2\}$. Then all other edges between the vertices x_1, x_2, y_1, y_2 and z are blue. Casing upon the color of the edge wx_1, it is easily seen that there is a balanced P_6 within these vertices. For example, if wx_1 is red, then $uvwx_1y_1y_2z$ is a balanced P_6. The other two cases are done similarly. $\qquad\square$

3.3   Proof of the upper bound from Theorem 10

In general, the best upper bound for $\mathrm{bal}_3(n, P_{3k})$ we have is:

$$\mathrm{bal}_3(n, P_{3k}) \leqslant (k + o(1))n. \tag{2}$$

In order to prove (2), we will use the following theorem. Recall that the extremal number $\mathrm{ex}(n, H)$ is the maximum number of edges in an H-free graph on n vertices. Erdős and Gallai [13] determined the extremal number for paths. From their result, it follows that

$$\mathrm{ex}(n, P_k) \leqslant \frac{k-1}{2}n. \tag{3}$$

Given a coloring on the edges of K_n with red (r), blue (b), and green (g), we will use the following notation: For $c \in \{r, b, g\}$ and a set $S \subseteq V(K_n)$, we will denote by $e_c(S)$ the number of c-colored edges with both vertices in the set S. If $S, T \subseteq V(K_n)$ are two disjoint sets, we set $E(S, T)$ for the set of edges with one vertex in S and one in T, and $E_c(S, T)$ for the set of c-colored edges from $E(S, T)$. Moreover $|E(S, T)| = e(S, T)$, $|E_c(S, T)| = e_c(S, T)$, and, for a vertex v, $E_c(v, S) = E_c(\{v\}, S)$.

Proof of (2). Let $0 < \epsilon < \frac{1}{2}$ be arbitrary and let n be large enough such that all inequalities hold. Consider a 3-edge-coloring of K_n with at least $(k + \epsilon)n$ edges of each color class. We proceed by induction on k to prove that there is a balanced P_{3k}. The case $k = 2$ was already done in Theorem 11. We assume that $\mathrm{bal}_3(n, P_{3k'}) \leqslant (k' + o(1))n$ for any $2 \leqslant k' < k$. Suppose now we have a complete graph on n vertices whose edges are colored with red, blue, and green, such that there are at least $(k + \varepsilon)n$ edges from each color. By the induction hypothesis, there is a balanced $P_{3(k-1)}$, say P. Let $S = V(P)$ and $D = V(K_n) \setminus S$. We will distinguish two cases.

Case 1: *D has no edges from at least one color.*
Assume, without loss of generality, that red is missing in D. Then $e_r(S, D) = e_r(K_n) - e_r(S) \geqslant kn$, if n is large enough.

Subcase 1.1: *Suppose that $e_b(D) \geqslant \left(\frac{k-1}{2} + \epsilon\right)n$ and $e_g(D) \geqslant \left(\frac{k-1}{2} + \epsilon\right)n$.*
Then, by Theorem 15, there is a balanced blue-green $2k$-path, say Q, contained in D. Moreover, since n is large, we still have $e_r(S, D \setminus V(Q)) \geqslant kn$. Hence, because of (3), there is a red k-path, say R, contained in the red graph induced by the edge set $E_r(S, D \setminus V(Q))$.

Let $a, b \in Q$ be the end-vertices of the path Q, and let $u, v \in V(R)$ be the end-vertices of the path R. We will show that there is a way of connecting both paths to have a balanced 3-colored P_{3k}. Consider the cycle C formed by the two paths Q and R and the edges au and bv. Let $v'v$ be the last edge in path R. If the edge ua is red, we can easily see that $C - v$ is a balanced 3-colored $3k$-path. Hence, au is either green or blue. The same occurs with the edge bv. Suppose now au and bv are both of the same color, say blue. Then the path consisting of the blue-green path Q together the edges au and bv has $k + 2$ blue edges and k green edges. Thus, it has to contain two consecutive blue edges, say xy and yz. Then $C - y$ is a balanced 3-colored $3k$-path. If, on the other side, au and bv are one blue and one green, then we can take two consecutive edges xy and yz from Q such that they have different colors and we can see that $C - y$ is a balanced 3-colored $3k$-path.

Subcase 1.2: Suppose $e_b(D) < \left(\frac{k-1}{2} + \epsilon\right) n$ or $e_g(D) < \left(\frac{k-1}{2} + \epsilon\right) n$.
Without loss of generality, we assume $e_b(D) < \left(\frac{k-1}{2} + \epsilon\right) n$. Then, since for n large enough we have $e_b(K_n) - e_b(S) \geqslant kn$, and it follows that

$$e_b(S, D) = e_b(K_n) - e_b(S) - e_b(D) \geqslant kn - \left(\frac{k-1}{2} + \epsilon\right) n = \left(\frac{k+1}{2} - \epsilon\right) n.$$

We will show first that there is a blue, a red, and a green k-path, pairwise disjoint. By the above inequality, $e_b(S, D) \geqslant \left(\frac{k+1}{2} - \epsilon\right) n$. and so by (3), the graph induced by the edges contained in $E_b(S, D)$ contains a path of length $k + 1$, say B'. If k is odd, let x be one of the two end-vertices of B'. If k is even, B' has one end-vertex in S and one in D. In this case, let x be the end-vertex of B' contained in S. Now define $B = B' - x$. Observe that, for k even or odd, we have

$$|S \cap V(B)| = \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{4}$$

We will now show that $e_r(S \setminus V(B), D) \geqslant \frac{k-1}{2}n$ by using the fact that

$$e_r(S \cap V(B), D) \leqslant |S \cap V(B)||D| \leqslant \left\lfloor \frac{k+1}{2} \right\rfloor n,$$

which holds because of (4). Indeed,

$$e_r(S \setminus V(B), D) = e_r(S, D) - e_r(S \cap V(B), D) \geqslant kn - \left\lfloor \frac{k+1}{2} \right\rfloor n \geqslant \frac{k-1}{2}n.$$

Hence, by Erdős-Gallai (3), there is a red path on k edges, say R, contained in $E_r(S \setminus V(B), D)$. Finally, consider the set $D' = D \setminus (V(B) \cup V(R))$. Since there are only green and blue edges in D and $e_b(D) < \left(\frac{k-1}{2} - \epsilon\right) n$ by hypothesis of this case, we have

$$e_g(D') = \binom{|D'|}{2} - e_b(D') \geqslant \binom{|D'|}{2} - \frac{k-1}{2}n = \Theta(n^2).$$

Hence, the set D' has $\Theta(n^2)$ green edges and thus clearly it has to contain a green k-path G, again by (3).

We will now show that, with these three paths B, R and G, we can construct a balanced 3-colored P_{3k}. Let b, b' be the end-vertices of B, r, r' the end-vertices of R and g, g' the end-vertices of G and consider the cycle Q formed by the three paths together with the edges $b'r$, $r'g$ and $g'b$. Suppose two of the edges $b'r$, $r'g$, $g'b$ are of the same color, say green. We call c the color of the third edge. Then, no matter where this c-colored edge is situated, we can easily see that there is a 3-path $wxyz$ contained in the cycle such that wx and xy are green and yz has color c. Deleting the vertices x and y from Q gives a balanced P_{3k} and we are done. Hence we may assume that the edges $b'r$, $r'g$, and $g'b$ have all a different color. If $b'r$ is green, we can consider the $3k$-path that is formed by deleting vertices b' and r from Q. In a similar manner, a comparable situation arises if $r'g$ is blue or $g'b$ is red. Hence, we may assume that the cycle Q consists of a blue, a green, and a red P_{k+1} glued together. Without loss of generality, we assume that $b'r$ is red, $r'g$ is green, and $g'b$ is blue. Let r'' be the neighbor of r' on the red path R and consider the edge gr'' and the cycle $Q' = (Q - r') + gr''$. If gr'' is blue, then $Q' - b$ is a balanced P_{3k}. If gr'' is red, then $Q' - b'$ is a balanced P_{3k}. Finally, if gr'' is green, then $Q' - g'$ is a balanced $3k$-path.

Case 2: D has edges from all three colors.

If there is a rainbow triangle in D, we are done by Lemma 17. If there is no rainbow triangle in D, then the 3-coloring in D is a Gallai coloring (see [16, 20, 21]) and, by [2] (see also [19]), we know that the graph induced by the edges of one of the colors is spanning in D. Without loss of generality, assume that the spanning color in D is green. Let x and y be the end-vertices of the path P.

From here, we build the proof by contradiction, assuming that there is no 3-colored balanced P_{3k}.

Claim 1: There is no red-blue P_2 in D.

If abc is a red-blue 2-path in D we can construct a balanced P_{3k} as follows. Since there are no rainbow triangles, assume without loss of generality that ac is red. Since the color green is spanning in D, we know there are vertices $d, d' \in D$ such that da and $d'b$ are green (where $d = d'$ is possible). Observe now that, depending on the color of cx, one of $abcP$, $d'bcP$ or $dacP$ will be a balanced P_{3k}. $\diamond$

Claim 2: All edges from $\{x, y\}$ to D are green.

Suppose that there is a vertex $c \in D$ such that xc is not green, say, without loss of generality, it is blue. If there is a vertex $b \in D$ such that bc is red, then we can easily build a balanced $3k$-path, namely $abcP$, where a is such that ab is green (recall that color green is spanning in D). On the other hand, if there is no red edge in D incident to c, consider a red edge ab in $D \setminus \{c\}$. By Claim 1, ac is not blue, so it has to be green and, again, we can easily build a balanced P_{3k}. Hence, all edges from x to D are green. By symmetry, all edges from y to D are green, too. $\diamond$

For the next claims, we consider a balanced 3-path $abcd$ in D, which exists because of Lemma 16. By Claim 1, the middle edge is green and, moreover, all remaining edges

induced by $\{a, b, c, d\}$ are green.

Claim 3: The end-edges of P are not green.
Suppose one end-edge of P is green, say the edge incident to y. Then the path $abcdP - y$ is a balanced $3k$-path since, according to Claim 2, dx is green. ◇

Claim 4: There are no consecutive green edges in P.
If uvw is a green-green path in P, consider the cycle $Q = abcPa$. Then, $Q - v$ is a balanced P_{3k}. ◇

Claim 5: The edge xy is green.
Suppose that xy is not green, say, without loss of generality, it is red. Consider the cycle $Q = Px$ that has $k - 1$ blue edges, $k - 1$ green edges, and k red edges. Note that, for any red edge uv in P, the path obtained from Q by removing the edge uv plays the same role as P. Thus, by Claim 3, the end-edges of this new path, $Q - uv$, are not green. This means that no red edge can be next to a green edge in Q. Then, since green edges form a matching, each green edge has to be preceded and succeeded by a blue edge, implying that there have to be more blue edges than green edges in Q, which is a contradiction. ◇

Now that we know that the edge xy is green, we are going to work with the cycle $Q = Px$, which has $k - 1$ blue edges, $k - 1$ red edges, and k green edges, where the green edges form a matching. Recall that $V(P) = S = V(Q)$.

Claim 6: There are at most $k - 2$ vertices in S incident to a red or a blue edge from $E(S, D)$.
Let uv be a green edge in Q and consider the path $Q - uv$. Note that this new path, $Q - uv$, plays the same role as P. Then, by Claim 2, all edges from $\{u, v\}$ to D are green. Since there are no adjacent green edges in Q, this leaves us $(3k - 2) - 2k = k - 2$ vertices allowed to send red or blue edges to D. ◇

By a simple counting argument, it must be that Q contains a subpath $uvwz$ which is green-notgreen-green. Suppose, without loss of generality, that vw is red.

Claim 7: The graph induced by the red edges in D is a matching.
If this is not the case, take a red-red path, abc, in D. Consider a blue edge de in D. By Claim 1, we know that $\{d, e\} \cap \{a, b, c\} = \emptyset$ and the edge cd is green. Let $P' = Q - \{v, w\}$. Note that P' is a $(3k - 5)$-path with $k - 1$ blue edges, $k - 2$ red edges and $k - 2$ green edges. Hence, $abcdeP'$ is a balanced P_{3k}, as the edges from u (or z) to D are green, by considering $Q - wz$ (or $Q - uv$). ◇

Now by using Claims 6 and 7, we can count the maximum number of red edges in

order to get a contradiction.

$$e_r(K_n) = e_r(S, D) + e_r(D) + e_r(S)$$
$$\leqslant (k-2)|D| + \frac{|D|}{2} + o(n)$$
$$\leqslant \left(k - \frac{3}{2}\right)n + o(n)$$
$$= \left(\left(k - \frac{3}{2}\right) + o(1)\right)n,$$

which is not possible by hypothesis. Hence, we have shown that, for any $0 < \epsilon < \frac{1}{2}$, $\mathrm{bal}_3(n, P_{3k}) < (k + \epsilon)n$, which gives the desired bound (2). $\square$

4 Balanced paths with many colors

In this section, our goal is to prove Theorems 8 and 9. The former shows that it is hopeless to find a straightforward generalization of our bounds on $\mathrm{bal}_r(n, P_{rk})$ when $r = 3$ to arbitrary r, since, for any odd k, there are infinitely many r with $\mathrm{bal}_r(n, P_{rk}) = \Omega(n^2)$. On the other hand, the latter will give some hope of generalizing our bounds under the assumption that k is even and sufficiently large with respect to r.

Proof of Theorem 8. Given an odd k, we consider an r of the form $r = \binom{l}{2} + 1$, with $l \geqslant 4$, and l even. As there are infinitely many such r, it will be enough to construct an r-colored complete graph on n vertices with $\Theta(n^2)$ edges in each color that avoids a balanced embedding of a P_{rk}.

We divide the vertices of K_n into l sets of size as close to equal as possible and call them V_i, $i \in [l]$. We color each complete bipartite graph $V_i \times V_j$, $i \neq j$, a different color. With the remaining color, say c_l, we color everything else. That is, we color all edges contained in any of the V_i's the same color.

Towards a contradiction, assume that we have a balanced embedding P of P_{rk} into this coloring. We will now define an auxiliary multi-graph H obtained from P by identifying the vertices of P contained in each V_i with one vertex v_i and contracting all edges of color c_l. Then H contains exactly k edges in each of the colors except for c_l, and each vertex has degree $k(l-1)$. Following the edges along the linear order given by P, we see that these edges form an Eulerian trail in H (possibly closed). Hence, all vertices but possibly the start and end vertices of the Eulerian trail must have even degrees in H. But this is not possible, as $k(l-1)$ is odd. $\square$

We now prove Theorem 9, which says that $\mathrm{bal}_r(P_{2rk}) = o(n^2)$ for k sufficiently large with respect to r. We mention that we make no attempt to optimize the constants throughout the proof.

Proof of Theorem 9. Given $r \geqslant 2$, our goal is to find a sufficiently large k_0 such that in any r-edge-coloring of K_n with at least $n^{2-\varepsilon}$ edges in each color class, there is a balanced

copy of P_{2rk} for every $k \geqslant k_0$ and n sufficiently large. To this aim, we will use Theorem 6. Before that, we need to prove the following.

Claim 20. *Let $r \geqslant 2$ and let t be sufficiently large. For every $K \in \mathcal{F}_t^r$ there exists $k_0' = k_0'(K)$ such that K contains a balanced copy of P_{2rk} for every $k \geqslant k_0'$.*

To see that the claim is true, fix some $K \in \mathcal{F}_t^r$. Recall that $V(K) = \bigcup_{i \in [l]} V_i$ where the number of parts $l \leqslant 2r - 2$ (by the remark in the Introduction), $|V_i| = t$ and, for every $i \neq j$, $i, j \in [l]$, both $K[V_i]$ and $K[V_i \times V_j]$ are monochromatic. Let $c_{i,j}$ be the color of the edges between V_i and V_j and define $\mathcal{C}$ to be the set of colors that appear on such edges. That is

$$\mathcal{C} := \{c_{i,j} \in [r] \colon i, j \in [l], i \neq j \text{ and } c_{i,j} \text{ is the color of edges in } K[V_i \times V_j]\}.$$

For each $c \in \mathcal{C}$, let $t(c)$ be the number of pairs, $i, j \in [l]$, $i \neq j$, such that the edges of $K[V_i \times V_j]$ have color c. Finally, let

$$k_0' = k_0'(K) = \operatorname{lcm}\{t(c) \mid c \in \mathcal{C}\}$$

where lcm denotes the least common multiple of a set. Let $k \geqslant k_0'$ and set $q = 2|\mathcal{C}|k$. We proceed to find an embedding of P_q in K that uses only edges from the complete bipartite graphs $K[V_i \times V_j]$, where $i, j \in [l]$ and $i \neq j$, and meets every part V_i. We achieve this by associating to K an auxiliary multigraph which is Eulerian. Define a multigraph where the vertices correspond to the cliques V_i ($i \in [l]$), and put, for every pair $i, j \in [l]$, $i \neq j$, $2k_0'/t(c_{i,j})$ edges between the vertices corresponding to V_i and V_j. As $2k_0'/t(c)$ is an even integer for all $c \in \mathcal{C}$, we can find an Eulerian circuit in our multigraph, which naturally corresponds to an embedding of a balanced $P_{2|\mathcal{C}|k_0'}$ in K with colors from $\mathcal{C}$. Note that, we can choose t sufficiently large so that each V_i is large enough to guarantee that the Eulerian circuit in the auxiliary multigraph can be translated into a path (and not a walk) in K. Moreover, if t is large enough, we can extend such balanced path of length $2|\mathcal{C}|k_0'$ to a balance path of length $2|\mathcal{C}|k$ (to see this, for each color $c_{i,j} \in \mathcal{C}$, choose an edge $e \in V_i \times V_j$ and replace it with a path containing $2(k - k_0') + 1$ edges between V_i and V_j using vertices that are disjoint from the vertices already used in the path). So far, we have find a balanced path P_q with edges in $E(K) \setminus \bigcup_{i \in [l]} E(V_i)$ (balanced with respect to the colors in $\mathcal{C}$) which meets every V_i in at least one vertex. If $|\mathcal{C}| = r$ we have finished the proof of the claim. If not, we can extend P_q to a balanced path P_{2rk} as follows. Since $|\mathcal{C}| < r$, there are colors that appear only inside one of the parts V_i. For any such color and part V_i, consider a vertex $v \in V_i$ such that v belongs to $P_q = v_1 v_2 \cdots v_j v v_{j+1} \cdots v_q v_{q+1}$. Take a path P of length $2k$ in V_i that is vertex disjoint with P_q, noting that there is space to do this if t is sufficiently large. Now $v_1 v_2 \cdots v_j P v_{j+1} \cdots v_{q+1}$ is a balanced path including this missing color. Repeating this operation for each missing color shows that the claim is true.

Note that we are not interested in estimating how large t should be for the arguments in the previous proof to be valid; we only want to ensure that if t is sufficiently large, then

the arguments work. Since the family of graphs $\mathcal{F}_t^r$ is finite (regardless of the value of t), we can set

$$k_0 := \max\{k_0'(K)\colon K \in \mathcal{F}_t^r\}.$$

Let $k \geqslant k_0$ and let C_1 be a sufficiently large constant such that, by Claim 20, every element of $\mathcal{F}_{C_1 rk}^r$ contains a balanced copy of P_{2rk}. Finally, by Theorem 6, there existe a constant $C_2 = C_2(r, t)$, where $t = C_1 rk$, such that, for n large enough, any r-edge-coloring of K_n with at least $C_2 n^{2-1/C_1 r^r k} = o(n^2)$ edges in each color class contains a member K from $\mathcal{F}_{C_1 rk}^r$, which contains a balanced copy of P_{2rk}, by Claim 20. $\qquad\square$

5 Discussion

We remark that, during the process of revision of this paper, Conjecture 5 was proved in [17].

Closing the gap from Theorem 10 is an open problem. We conjecture that the lower bound should be the truth, but new ideas are needed to improve on our upper bound.

A perhaps more elementary question that remains is determining for which graphs G we have $\mathrm{bal}_r(n, G) < \infty$. We call an embedding of G into an r-edge-colored K_n a *balanced embedding* if all r color classes are almost equally represented as in Definition 7. We have the following abstract characterization.

Proposition 21. *Let G be any graph of order q, and $r \geqslant 2$ an integer. Then $\mathrm{bal}_r(n, G) = o(n^2)$ if and only if there exists a balanced embedding of G into all patterns from $\mathcal{F}_q^r$.*

Proof. If G admits a balanceable embedding into all patterns from $\mathcal{F}_q^r$, by Theorem 6 we will be able to find balanced copies of G in sufficiently large complete graphs with $\Theta(n^{2-1/v(G)})$ edges in each color. On the other hand, if G does not admit a balanced embedding into a particular pattern from $\mathcal{F}_q^r$, we can use this pattern to construct a K_n with $\Theta(n^2)$ edges in each color class, hence it cannot be that $\mathrm{bal}_r(n, G) = o(n^2)$. $\qquad\square$

This characterization does not rule out the possibility that there might be balanceable graphs with $\mathrm{bal}_r(n, G) = \Theta(n^2)$. We find this to be unlikely, but we are unable to prove it for $r \geqslant 3$, in contrast to the case $r = 2$, where it was shown that balanceable graphs always satisfy $\mathrm{bal}(n, G) = o(n^2)$ [7]. For example, by investigating Figure 1, one can verify that $\mathrm{bal}_3(n, C_{6k}) = o(n^2)$, and $\mathrm{bal}_3(n, C_{6k+3}) = \Omega(n^2)$. We leave it as an open problem to determine whether $\mathrm{bal}_3(n, C_{6k+3}) = \infty$.

It would in general be interesting to obtain results for cycles for $r \geqslant 3$ (see [9] for the case $r = 2$).

Acknowledgements

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