The Bhargava Greedoid as a Gaussian Elimination Greedoid

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Abstract

Inspired by Manjul Bhargava's theory of generalized factorials, Grinberg and Petrov have defined the *Bhargava greedoid* – a greedoid (a matroid-like set system on a finite set) assigned to any "ultra triple" (a somewhat extended variant of a finite ultrametric space). Here we show that the Bhargava greedoid of a finite ultra triple is always a *Gaussian elimination greedoid* over any sufficiently large (e.g., infinite) field; this is a greedoid analogue of a representable matroid. We find necessary and sufficient conditions on the size of the field to ensure this.

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Introduction

The notion of a *greedoid* was coined in 1981 by Korte and Lovász, and has since seen significant developments ([11], [2]). It is a type of set system (i.e., a set of subsets of a given ground set) that is required to satisfy some axioms weaker than the matroid axioms – so that, in particular, the independent sets of a matroid form a greedoid.

In [8], Grinberg and Petrov have constructed a greedoid stemming from Bhargava's theory of generalized factorials [1, §2], albeit in a setting significantly more general than Bhargava's. Roughly speaking, the sets that belong to this greedoid are subsets of maximum perimeter (among all subsets of their size) of a finite ultrametric space (which, in Bhargava's work, was a Dedekind ring with a metric coming from a valuation).

More precisely, the setup is more general than that of an ultrametric space: We consider a finite set E, a *distance function* d that assigns a "distance" $d(e,f)$ to any pair (e,f) of distinct elements of E, and a *weight function* w that assigns a "weight" $w(e)$ to each $e \in E$. The distances and weights are required to belong to a totally ordered abelian

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group $\mathbb{V}$ (for example, $\mathbb{R}$). The distances are required to satisfy the symmetry axiom $d(e,f) = d(f,e)$ and the "ultrametric triangle inequality" $d(a,b) \leqslant \max\{d(a,c), d(b,c)\}$. In this setting, any subset S of E has a well-defined *perimeter*, obtained by summing the weights and the pairwise distances of all its elements. The subsets S of E that have maximum perimeter (among all $|S|$-element subsets of E) then form a greedoid, which has been called the *Bhargava greedoid* of (E, w, d) in [8]. This greedoid is furthermore a strong greedoid [8, Theorem 6.1], which implies in particular that for any given $k \leqslant |E|$, the k-element subsets of E that have maximum perimeter are the bases of a matroid.

In the present paper, we prove that the Bhargava greedoid of (E, w, d) is a *Gaussian elimination greedoid* over any sufficiently large (e.g., infinite) field. Roughly speaking, a Gaussian elimination greedoid is a greedoid analogue of a representable matroid[1]. We quantify the "sufficiently large" by providing a sufficient condition for the size of the field. When all weights $w(e)$ are equal, we show that this condition is also necessary.

We note that the Bhargava greedoid can be seen to arise from an optimization problem in phylogenetics: Given a finite set E of organisms and an integer $k \in \mathbb{N}$, we want to choose a k-element subset of E that maximizes some kind of biodiversity. Depending on the definition of biodiversity used, the properties of the maximizing subsets can differ. It appears natural to define biodiversity in terms of distances on the evolutionary tree (which is a finite ultrametric space), and such a definition has been considered by Moulton, Semple and Steel in [15], leading to the result that the maximum-biodiversity sets form a strong greedoid. The Bhargava greedoid is an analogue of their greedoid using a slightly different definition of biodiversity[2]. The present paper potentially breaks this analogy by showing that the Bhargava greedoid is a Gaussian elimination greedoid, whereas this is unknown for the greedoid of Moulton, Semple and Steel. Whether the latter is a Gaussian elimination greedoid as well remains to be understood[3], as does the question of interpolating between the two notions of biodiversity.

This paper is almost self-contained: everything used is proved, except for some elementary facts in linear algebra as well as some straightforward lemmas whose proofs can

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[1] In particular, this entails that all the matroids mentioned in the preceding paragraph are representable.

[2] To be specific: We view the organisms as the leaves of an evolutionary tree $\mathcal{T}$ that obeys a molecular clock assumption (i.e., all its leaves have the same distance from the root). Then, the set E of these organisms is equipped with a distance function (measuring distances along the edges of the tree), which satisfies the "ultrametric triangle inequality". We define the weight function w by setting $w(e) = 0$ for all $e \in E$. Now, the *phylogenetic diversity* of a subset $S \subseteq E$ is defined to be the sum of the edge lengths of the minimal subtree of $\mathcal{T}$ that connects all leaves in S. This phylogenetic diversity is the measure of biodiversity used in [15]. Meanwhile, our notion of perimeter can also be seen as a measure of biodiversity – perhaps even a better one for sustainability questions, as it rewards subsets that are roughly balanced across different clades. To give a trivial example, a zoo optimized for phylogenetic diversity might have dozens of mammals and only one bird, while this would unlikely be considered optimal in terms of perimeter.
The molecular clock assumption can actually be dropped, at the expense of changing the weight function to account for different distances from the root.

[3] This question might have algorithmic significance. At least for polymatroids, representability can make the difference between a problem being NP-hard and in P, as shown by Lovász in [14] for polymatroid matching.

be found in the appendices of the arXiv version of the paper (arXiv:2001.05535). In particular, it can be read independently of [8].

The 12-page extended abstract [9] summarizes the highlights of both [8] and this paper; it is thus a convenient starting point for a reader interested in the subject.

1 Gaussian elimination greedoids

We first recall several notions from the theory of greedoids.

Convention 1. Here and in the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.

Convention 2. If E is any set, then 2^E will denote the powerset of E (that is, the set of all subsets of E).

Convention 3. Let $\mathbb{K}$ be any field, and let $n \in \mathbb{N}$. Then, $\mathbb{K}^n$ shall denote the $\mathbb{K}$-vector space of all column vectors of size n over $\mathbb{K}$.

Now, we recall the definition of greedoids and strong greedoids (see, e.g., [3, §2] or [8, §6.1][4]):

Definition 4. Let E be a finite set. A subset $\mathcal{F}$ of 2^E is said to be a *greedoid* if it satisfies the following three axioms:

- **(i)** We have $\varnothing \in \mathcal{F}$.
- **(ii)** If $B \in \mathcal{F}$ satisfies $|B| > 0$, then there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{F}$.
- **(iii)** If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$.

Furthermore, $\mathcal{F}$ is said to be a *strong greedoid* if it satisfies the following axiom (in addition to axioms **(i)**, **(ii)** and **(iii)**):

- **(iv)** If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$ and $B \setminus \{b\} \in \mathcal{F}$.

Note that axiom **(iii)** in Definition 4 clearly follows from axiom **(iv)**; thus, only axioms **(i)**, **(ii)** and **(iv)** need to be checked in order to convince ourselves that a subset of 2^E is a strong greedoid.

We refer to [11] for many properties and examples of greedoids. We will not have much use for the general theory of greedoids in this paper, as we will only be considering a special class of greedoids – the so-called Gaussian elimination greedoids:

[4]Strong greedoids also appear in [11, §IX.4] under the name of "Gauss greedoids", but they are defined differently. (The equivalence between the two definitions is proved in [3, §2].)

Definition 5. Let E be a finite set.

Let $m \in \mathbb{N}$ be such that $m \geqslant |E|$. Let $\mathbb{K}$ be a field. For each $k \in \{0, 1, \ldots, m\}$, let $\pi_k : \mathbb{K}^m \to \mathbb{K}^k$ be the projection map that removes all but the first k coordinates of a column vector. (That is, $\pi_k \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ for each $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \in \mathbb{K}^m$.)

For each $e \in E$, let $v_e \in \mathbb{K}^m$ be a column vector. The family $(v_e)_{e \in E}$ will be called a *vector family* over $\mathbb{K}$.

Let $\mathcal{G}$ be the subset

$$\left\{ F \subseteq E \ \mid \ \text{the family } \left(\pi_{|F|}(v_e)\right)_{e \in F} \in \left(\mathbb{K}^{|F|}\right)^F \text{ is linearly independent} \right\}$$

of 2^E. Then, $\mathcal{G}$ is called the *Gaussian elimination greedoid* of the vector family $(v_e)_{e \in E}$. It is furthermore called a *Gaussian elimination greedoid on ground set E*.

Example 6. Let $\mathbb{K} = \mathbb{Q}$ and $E = \{1, 2, 3, 4, 5\}$ and $m = 6$. Let $v_1, v_2, v_3, v_4, v_5 \in \mathbb{K}^6$ be the columns of the 6×5-matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

Then, the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E} = (v_1, v_2, v_3, v_4, v_5)$ is the set

$$\{\varnothing, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\}\}.$$

For example, the 3-element set $\{1, 2, 5\}$ belongs to this greedoid because the corresponding family $(\pi_3(v_e))_{e \in \{1,2,5\}} \in (\mathbb{K}^3)^{\{1,2,5\}}$ is linearly independent (indeed, this family consists of the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$).

Our definition of a Gaussian elimination greedoid follows [10, §1.3], except that we are using vector families instead of matrices (but this is equivalent, since any matrix can be identified with the vector family consisting of its columns) and we are talking about linear independence rather than non-singularity of matrices (but this is again equivalent, since a square matrix is non-singular if and only if its columns are linearly independent). The same definition is given in [11, §IV.2.3].

As the name suggests, Gaussian elimination greedoids are greedoids. Even better, they are strong greedoids:

Theorem 7. *The Gaussian elimination greedoid $\mathcal{G}$ in Definition 5 is a strong greedoid.*

Proof. This is implicit in [11, §IX.4] and partly proved in [10, §1.3][5]. We omit the proof here, as this theorem will not be used in the main part of this paper. A detailed proof can be found in an appendix in the arXiv version of this paper. □

Let us briefly connect our classes of greedoids with the (much more mainstream) notion of matroids, even though we will not concern ourselves any further with them in this note. We refer to [17, §1.1–1.2] for the definition of a matroid and of related notions.

Proposition 8. *Let $\mathcal{G}$ be a Gaussian elimination greedoid on a ground set E. Let $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the set of all k-element sets in $\mathcal{G}$. Then, $\mathcal{G}_k$ is either empty or is the collection of bases of a representable matroid on the ground set E.*

Proof. Assume that $\mathcal{G}_k$ is nonempty. Consider the field $\mathbb{K}$ and the vector family $(v_e)_{e\in E}$ over $\mathbb{K}$ such that $\mathcal{G}$ is the Gaussian elimination greedoid of this vector family $(v_e)_{e\in E}$. WLOG assume that $E = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. Let X be the $k \times n$-matrix whose columns are the vectors $\pi_k(v_e) \in \mathbb{K}^k$ for all $e \in E$. Then, it is easy to see that $\mathcal{G}_k$ is the set of bases of the vector matroid corresponding to this matrix X. Details of this argument can be found in an appendix in the arXiv version of this paper. □

Proposition 8 justifies thinking of Gaussian elimination greedoids as a greedoid analogue of representable matroids.

2 $\mathbb{V}$-ultra triples

Definition 9. Let E be a set. Then, $E \underline{\times} E$ shall denote the subset $\{(e, f) \mid e \neq f\}$ of $E \times E$.

Convention 10. Fix a totally ordered abelian group $(\mathbb{V}, +, 0, \leqslant)$ (with ground set $\mathbb{V}$, group operation $+$, zero 0 and smaller-or-equal relation $\leqslant$). The total order on $\mathbb{V}$ is required to be translation-invariant (i.e., if $a, b, c \in \mathbb{V}$ satisfy $a \leqslant b$, then $a + c \leqslant b + c$).

We shall refer to the ordered abelian group $(\mathbb{V}, +, 0, \leqslant)$ simply as $\mathbb{V}$. We will use the standard additive notations for the abelian group $\mathbb{V}$; in particular, we will use the $\sum$ sign for finite sums inside the group $\mathbb{V}$. We will furthermore use the standard order-theoretical notations for the totally ordered set $\mathbb{V}$; in particular, we will use the symbol $\geqslant$ for the reverse relation of $\leqslant$ (that is, $a \geqslant b$ means $b \leqslant a$), and we will use the symbols $<$ and $>$ for the strict versions of the relations $\leqslant$ and $\geqslant$. We will denote the largest element of a nonempty subset S of $\mathbb{V}$ (with respect to the relation $\leqslant$) by $\max S$. Likewise, $\min S$ will stand for the smallest element of S.

We will keep this group $\mathbb{V}$ fixed throughout this paper.

For almost all examples we are aware of, it suffices to set $\mathbb{V}$ to be the abelian group $\mathbb{R}$, or even the smaller abelian group $\mathbb{Z}$. Nevertheless, we shall work in full generality, as it serves to separate objects that would otherwise easily be confused.

[5] To be precise, two paragraphs above [10, Example 1.3.15], it is shown that $\mathcal{G}$ is a greedoid.

Definition 11. A $\mathbb{V}$-*ultra triple* shall mean a triple (E, w, d) consisting of:

- a set E, called the *ground set* of this $\mathbb{V}$-ultra triple;
- a map $w : E \to \mathbb{V}$, called the *weight function* of this $\mathbb{V}$-ultra triple;
- a map $d : E \underline{\times} E \to \mathbb{V}$, called the *distance function* of this $\mathbb{V}$-ultra triple, and required to satisfy the following axioms:
  - **Symmetry:** We have $d(a, b) = d(b, a)$ for any two distinct elements a and b of E.
  - **Ultrametric triangle inequality:** We have $d(a, b) \leqslant \max\{d(a, c), d(b, c)\}$ for any three distinct elements a, b and c of E.

If (E, w, d) is a $\mathbb{V}$-ultra triple and $e \in E$, then the value $w(e) \in \mathbb{V}$ is called the *weight* of e.

If (E, w, d) is a $\mathbb{V}$-ultra triple and e and f are two distinct elements of E, then the value $d(e, f) \in \mathbb{V}$ is called the *distance* between e and f.

Example 12. For this example, let $\mathbb{V} = \mathbb{Z}$, and let E be a subset of $\mathbb{Z}$. Let m be any integer. Define a map $w : E \to \mathbb{V}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by

$$d(a, b) = \begin{cases} 1, & \text{if } a \not\equiv b \bmod m; \\ 0, & \text{if } a \equiv b \bmod m \end{cases} \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple.

Example 13. For this example, let $\mathbb{V} = \mathbb{Z}$ again, and let E be a subset of $\mathbb{Z}$. Fix a prime number p. For each nonzero integer k, let $v_p(k)$ denote the largest $i \in \mathbb{N}$ such that $p^i \mid k$. (For instance, $v_3(45) = 2$.)

Define a map $w : E \to \mathbb{V}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by

$$d(a, b) = -v_p(a - b) \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple.

More generally, we can replace $\mathbb{Z}$ by any integral domain, and v_p by any valuation on this integral domain, and obtain a $\mathbb{V}$-ultra triple, where $\mathbb{V}$ is the target of our valuation.

The notion of a $\mathbb{V}$-ultra triple generalizes the notion of an ultra triple as defined in [8]. More precisely, if $\mathbb{V}$ is the additive group $(\mathbb{R}, +, 0, \leqslant)$ (with the usual addition and the usual total order on $\mathbb{R}$), then a $\mathbb{V}$-ultra triple is the same as what is called an "ultra triple" in [8]. Several properties and examples of ultra triples can be found in [8].

It is straightforward to adapt all the definitions and results stated in [8] for ultra triples to the more general setting of $\mathbb{V}$-ultra triples[6]. Let us specifically extend two definitions from [8] to $\mathbb{V}$-ultra triples: the definition of a perimeter ([8, §3.1]) and the definition of the Bhargava greedoid ([8, §6.2]):

[6] There is one stupid exception: The definition of R in [8, Remark 8.13] requires $\mathbb{V} \neq 0$. But [8, Remark 8.13] is just a tangent without concrete use.

Definition 14. Let (E, w, d) be a $\mathbb{V}$-ultra triple. Let A be a finite subset of E. Then, the *perimeter* of A (with respect to (E, w, d)) is defined to be

$$\sum_{a \in A} w(a) + \sum_{\substack{\{a,b\} \subseteq A; \\ a \neq b}} d(a, b) \in \mathbb{V}.$$

(Here, the second sum ranges over all **unordered** pairs $\{a, b\}$ of distinct elements of A.)
The perimeter of A is denoted by $\operatorname{PER}(A)$.

For example, if $A = \{p, q, r\}$ is a 3-element set, then

$$\operatorname{PER}(A) = w(p) + w(q) + w(r) + d(p, q) + d(p, r) + d(q, r).$$

Definition 15. Let S be any set, and let $k \in \mathbb{N}$. A *k-subset* of S means a k-element subset of S (that is, a subset of S having size k).

Definition 16. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. The *Bhargava greedoid* of (E, w, d) is defined to be the subset

$$\begin{aligned} &\{A \subseteq E \mid A \text{ has maximum perimeter among all } |A| \text{-subsets of } E\} \\ &= \{A \subseteq E \mid \operatorname{PER}(A) \geqslant \operatorname{PER}(B) \text{ for all } B \subseteq E \text{ satisfying } |B| = |A|\} \end{aligned}$$

of 2^E.

Example 17. For this example, let $\mathbb{V} = \mathbb{Z}$ and $E = \{0, 1, 2, 3, 4\}$. Define a map $w : E \to \mathbb{V}$ by setting $w(e) = \max\{e, 1\}$ for each $e \in E$. (Thus, $w(0) = 1$ and $w(e) = e$ for all $e > 0$.) Define a map $d : E \underline{\times} E \to \mathbb{V}$ by setting

$$d(e, f) = \min\{3, \max\{4 - e, 4 - f\}\} \qquad \text{for all } (e, f) \in E \underline{\times} E.$$

Here is a table of values of d:

d	0	1	2	3	4
0		3	3	3	3
1	3		3	3	3
2	3	3		2	2
3	3	3	2		1
4	3	3	2	1	

.

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple. Let $\mathcal{F}$ be its Bhargava greedoid. Thus, $\mathcal{F}$ consists of the subsets A of E that have maximum perimeter among all $|A|$-subsets of E. What are these subsets?

- Clearly, $\varnothing$ is the only $|\varnothing|$-subset of E, and thus has maximum perimeter among all $|\varnothing|$-subsets of E. Hence, $\varnothing \in \mathcal{F}$.

- The perimeter of a 1-subset $\{e\}$ of E is just the weight $w(e)$. Thus, the 1-subsets of E having maximum perimeter among all 1-subsets of E are precisely the subsets $\{e\}$ where $e \in E$ has maximum weight. In our example, there is only one $e \in E$ having maximum weight, namely 4. Thus, the only 1-subset of E having maximum perimeter among all 1-subsets of E is $\{4\}$. In other words, the only 1-element set in $\mathcal{F}$ is $\{4\}$.

- What about 2-element sets in $\mathcal{F}$? The perimeter $\operatorname{PER}\{e, f\}$ of a 2-subset $\{e, f\}$ of E is $w(e) + w(f) + d(e, f)$. Thus,

$$\operatorname{PER}\{0, 4\} = w(0) + w(4) + d(0, 4) = 1 + 4 + 3 = 8$$

and similarly $\operatorname{PER}\{1, 4\} = 8$, $\operatorname{PER}\{2, 4\} = 8$, $\operatorname{PER}\{3, 4\} = 8$, $\operatorname{PER}\{0, 3\} = 7$, $\operatorname{PER}\{1, 3\} = 7$, $\operatorname{PER}\{2, 3\} = 7$, $\operatorname{PER}\{0, 2\} = 6$, $\operatorname{PER}\{1, 2\} = 6$, and $\operatorname{PER}\{0, 1\} = 5$. Thus, the 2-subsets of E having maximum perimeter among all 2-subsets of E are $\{0, 4\}$ and $\{1, 4\}$ and $\{2, 4\}$ and $\{3, 4\}$. So these four sets are the 2-element sets in $\mathcal{F}$.

- Similarly, the 3-element sets in $\mathcal{F}$ are $\{0, 1, 4\}$, $\{0, 3, 4\}$, $\{1, 3, 4\}$, $\{0, 2, 4\}$ and $\{1, 2, 4\}$. They have perimeter 15, while all other 3-subsets of E have perimeter 14 or 13.

- Similarly, the 4-element sets in $\mathcal{F}$ are $\{0, 1, 2, 4\}$ and $\{0, 1, 3, 4\}$.

- Clearly, E is the only $|E|$-subset of E, so that $E \in \mathcal{F}$.

Thus, the Bhargava greedoid of (E, w, d) is

$$\begin{aligned}\mathcal{F} = \{&\varnothing, \{4\}, \{0, 4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \\ &\{0, 1, 4\}, \{0, 3, 4\}, \{1, 3, 4\}, \{0, 2, 4\}, \{1, 2, 4\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, E\}.\end{aligned}$$

Example 18. For this example, let $\mathbb{V} = \mathbb{Z}$ and $E = \{1, 2, 3\}$. Define a map $w : E \to \mathbb{V}$ by setting $w(e) = e$ for each $e \in E$. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by setting $d(e, f) = 1$ for each $(e, f) \in E \underline{\times} E$. It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). What is $\mathcal{F}$?

The same kind of reasoning as in Example 17 (but simpler due to the fact that all values of d are the same) shows that

$$\mathcal{F} = \{\varnothing, \{3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

One thing we observed in both of these examples is the following simple fact:

Remark 19. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Then, $E \in \mathcal{F}$.

Proof of Remark 19. The set E obviously has maximum perimeter among all $|E|$-subsets of E (since E is the only $|E|$-subset of E). Hence, $E \in \mathcal{F}$. □

3 The main theorem

In [8, Theorem 6.1], it was proved that the Bhargava greedoid of an ultra triple with finite ground set is a strong greedoid. More generally, this holds for any $\mathbb{V}$-ultra triple with finite ground set (and the same argument can be used to prove this). However, we shall prove a stronger statement:

Theorem 20. *Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field of size $|\mathbb{K}| \geqslant |E|$. Then, $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.*

We will spend the next few sections working towards a proof of this theorem. First, however, let us extend it somewhat by strengthening the $|\mathbb{K}| \geqslant |E|$ bound.

4 Cliques and stronger bounds

For the rest of Section 4, we fix a $\mathbb{V}$-ultra triple (E, w, d).

Let us define a certain kind of subsets of E, which we call *cliques*.

Definition 21. Let $\alpha \in \mathbb{V}$. An *α-clique* of (E, w, d) will mean a subset F of E such that any two distinct elements $a, b \in F$ satisfy $d(a, b) = \alpha$.

Definition 22. A *clique* of (E, w, d) will mean a subset of E that is an α-clique for some $\alpha \in \mathbb{V}$.

Thus, any 1-element subset of E is a clique (and an α-clique for every $\alpha \in \mathbb{V}$). The same holds for the empty subset. Any 2-element subset $\{a, b\}$ of E is a clique and, in fact, a $d(a, b)$-clique.

Note that the notion of a clique (and of an α-clique) depends only on E and d, not on w.

Example 23. For this example, let m, $\mathbb{V}$, E, w and d be as in Example 12. Then:

- **(a)** The 0-cliques of E are the subsets of E whose elements are all mutually congruent modulo m.
- **(b)** The 1-cliques of E are the subsets of E that have no two distinct elements congruent to each other modulo m. Thus, any 1-clique has size $\leqslant m$ if m is positive.
- **(c)** If $\alpha \in \mathbb{V} \setminus \{0, 1\}$, then the α-cliques of E are the subsets of E having size $\leqslant 1$.

Using the notion of cliques, we can assign a number $\operatorname{mcs}(E, w, d)$ to our $\mathbb{V}$-ultra triple (E, w, d):

Definition 24. Let $\operatorname{mcs}(E, w, d)$ denote the maximum size of a clique of (E, w, d). (This is well-defined whenever E is finite, and sometimes even otherwise.)

Clearly, $\operatorname{mcs}(E, w, d) \leqslant |E|$, since any clique of (E, w, d) is a subset of E.

Example 25. Let $\mathbb{V}$, E, w and d be as in Example 17. Then, $\{0, 1, 2\}$ is a 3-clique of (E, w, d) and has size 3; no larger cliques exist in (E, w, d). Thus, $\operatorname{mcs}(E, w, d) = 3$.

Example 26. For this example, let m, $\mathbb{V}$, E, w and d be as in Example 12. Then:

(a) If $m = 2$ and $E = \{1, 2, 3, 4, 5, 6\}$, then $\operatorname{mcs}(E, w, d) = 3$, due to the 0-clique $\{1, 3, 5\}$ having maximum size among all cliques.

(b) If $m = 3$ and $E = \{1, 2, 3, 4, 5, 6\}$, then $\operatorname{mcs}(E, w, d) = 3$, due to the 1-clique $\{1, 2, 3\}$ having maximum size among all cliques.

We can now strengthen Theorem 20 as follows:

Theorem 27. *Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field of size $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. Then, $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.*

Theorem 27 is stronger than Theorem 20 because $|E| \geqslant \operatorname{mcs}(E, w, d)$.
We shall prove Theorem 27 in Section 10.

5 The converse direction

Before that, let us explore the question whether the bound $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$ can be improved. In an important particular case – namely, when the map w is constant[7] –, it cannot, as the following theorem shows:

Theorem 28. *Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Assume that the map w is constant. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field such that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. Then, $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$.*

We shall prove Theorem 28 in Section 11.

When the map w in a $\mathbb{V}$-ultra triple (E, w, d) is constant, Theorems 27 and 28 combined yield an exact characterization of those fields $\mathbb{K}$ for which the Bhargava greedoid of (E, w, d) can be represented as the Gaussian elimination greedoid of a vector family over $\mathbb{K}$: Namely, those fields are precisely the fields $\mathbb{K}$ of size $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. When w is not constant, Theorem 27 gives a sufficient condition; we don't know a necessary condition. Here are two examples:

Example 29. Let $\mathbb{V}$, E, w, d and $\mathcal{F}$ be as in Example 17. Then, $\operatorname{mcs}(E, w, d) = 3$ (as we saw in Example 25). Hence, Theorem 27 shows that $\mathcal{F}$ can be represented as the Gaussian elimination greedoid of a vector family over any field $\mathbb{K}$ of size $|\mathbb{K}| \geqslant 3$. This bound on $|\mathbb{K}|$ is optimal, since the Bhargava greedoid $\mathcal{F}$ is not the Gaussian elimination greedoid of any vector family over the 2-element field $\mathbb{F}_2$. (But this does not follow from Theorem 28, because w is not constant.)

[7] A map $f : X \to Y$ between two sets X and Y is said to be *constant* if all values of f are equal (i.e., if every $x_1, x_2 \in X$ satisfy $f(x_1) = f(x_2)$). In particular, if $|X| \leqslant 1$, then $f : X \to Y$ is automatically constant.

Example 30. Let $\mathbb{V}$, E, w, d and $\mathcal{F}$ be as in Example 18. Then, $\operatorname{mcs}(E, w, d) = 3$, since E itself is a clique. Hence, Theorem 27 shows that $\mathcal{F}$ can be represented as the Gaussian elimination greedoid of a vector family over any field $\mathbb{K}$ of size $|\mathbb{K}| \geqslant 3$. However, this bound on $|\mathbb{K}|$ is not optimal. Indeed, the Bhargava greedoid $\mathcal{F}$ is the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E} = (v_1, v_2, v_3)$ over the field $\mathbb{F}_2$, where $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Question 31. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. How to characterize the fields $\mathbb{K}$ for which the Bhargava greedoid of (E, w, d) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$? Is there a constant $c(E, w, d)$ such that these fields are precisely the fields of size $\geqslant c(E, w, d)$?

Remark 32. Let E, w, d and $\mathcal{F}$ be as in Theorem 20. Let $\mathbb{K}$ be any field. For each $k \in \mathbb{N}$, let $\mathcal{F}_k$ be the set of all k-element sets in $\mathcal{F}$.

If $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$, then each $\mathcal{F}_k$ with $k \in \{0, 1, \ldots, |E|\}$ is the collection of bases of a representable matroid on the ground set E. (Indeed, this follows from Proposition 8, since $\mathcal{F}_k$ is nonempty.) But the converse is not true: It can happen that each $\mathcal{F}_k$ with $k \in \{0, 1, \ldots, |E|\}$ is the collection of bases of a representable matroid on the ground set E, yet $\mathcal{F}$ is not the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. For example, this happens if $E = \{1, 2, 3\}$ and both maps w and d are constant (so that $\mathcal{F} = 2^E$), and $\mathbb{K} = \mathbb{F}_2$.

6 Valadic $\mathbb{V}$-ultra triples

As a first step towards the proof of Theorem 27, we will next introduce a special kind of $\mathbb{V}$-ultra triples which, in a way, are similar to Bhargava's for integers (see [8, Example 2.5 and §9]). We will call them *valadic*[8], and we will see (in Theorem 41) that they satisfy Theorem 20. Afterwards (in Theorem 57), we will prove that any $\mathbb{V}$-ultra triple with finite ground set is isomorphic (in an appropriate sense) to a valadic one over a sufficiently large field. Combining these two facts, we will then readily obtain Theorem 27.

Recall that $\mathbb{V}$ is a totally ordered abelian group (see Convention 10 for details). Let us introduce some further notations that will be used throughout Section 6.

[8]The name is a homage to the notion of a valuation ring, which is latent in the argument that follows (although never used explicitly). Indeed, if we define the notion of a valuation ring as in [4, Exercise 11.1], then the $\mathbb{K}$-algebra $\mathbb{L}_+$ constructed below is an instance of a valuation ring (with $\mathbb{L}$ being its fraction field, and $\operatorname{ord} : \mathbb{L} \setminus \{0\} \to \mathbb{V}$ being its valuation), and many of its properties that will be used below are instances of general properties of valuation rings. If we extended our argument to the more general setting of valuation rings, we would also recover Bhargava's original ultra triples based on integer divisibility (see [8, Example 2.5 and §9]). However, we have no need for this generality (as we only need the construction as a stepping stone towards our proof of Theorem 27), and prefer to remain elementary and self-contained.

Definition 33. We fix a field $\mathbb{K}$. Let $\mathbb{K}[\mathbb{V}]$ denote the group algebra of the group $\mathbb{V}$ over $\mathbb{K}$. This is a free $\mathbb{K}$-module with basis $(t_\alpha)_{\alpha\in\mathbb{V}}$; it becomes a $\mathbb{K}$-algebra with unity t_0 and with multiplication determined by

$$t_\alpha t_\beta = t_{\alpha+\beta} \qquad \text{for all } \alpha, \beta \in \mathbb{V}.$$

This group algebra $\mathbb{K}[\mathbb{V}]$ is commutative, since the group $\mathbb{V}$ is abelian.

Let $\mathbb{V}_{\geqslant 0}$ be the set of all $\alpha \in \mathbb{V}$ satisfying $\alpha \geqslant 0$; this is a submonoid of the group $\mathbb{V}$. Let $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ be the monoid algebra of this monoid $\mathbb{V}_{\geqslant 0}$ over $\mathbb{K}$. This is a $\mathbb{K}$-algebra defined in the same way as $\mathbb{K}[\mathbb{V}]$, but using $\mathbb{V}_{\geqslant 0}$ instead of $\mathbb{V}$. It is clear that $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ is the $\mathbb{K}$-subalgebra of $\mathbb{K}[\mathbb{V}]$ spanned by the basis elements t_α with $\alpha \in \mathbb{V}_{\geqslant 0}$.

Example 34. If $\mathbb{V} = \mathbb{Z}$ (with the usual addition and total order), then $\mathbb{V}_{\geqslant 0} = \mathbb{N}$. In this case, the group algebra $\mathbb{K}[\mathbb{V}]$ is the Laurent polynomial ring $\mathbb{K}[X, X^{-1}]$ in a single indeterminate X over $\mathbb{K}$ (indeed, t_1 plays the role of X, and more generally, each t_α plays the role of X^α), and its subalgebra $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ is the polynomial ring $\mathbb{K}[X]$.

This example should be regarded as a guide; even though $\mathbb{V}$ does not have to be $\mathbb{Z}$, the reader cannot go wrong thinking of $\mathbb{K}[\mathbb{V}]$ as a generalized Laurent polynomial ring and of $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ as a generalized polynomial ring (in a single indeterminate) and of t_α as a generalized monomial X^α. This analogy shall clarify much of what follows.

Definition 35.

- **(a)** Let $\mathbb{L}$ be the commutative $\mathbb{K}$-algebra $\mathbb{K}[\mathbb{V}]$, and let $\mathbb{L}_+$ be its $\mathbb{K}$-subalgebra $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$. Thus, the $\mathbb{K}$-module $\mathbb{L}$ has basis $(t_\alpha)_{\alpha\in\mathbb{V}}$, while its $\mathbb{K}$-submodule $\mathbb{L}_+$ has basis $(t_\alpha)_{\alpha\in\mathbb{V}_{\geqslant 0}}$.
- **(b)** If $a \in \mathbb{L}$ and $\beta \in \mathbb{V}$, then $[t_\beta]\, a$ shall denote the coefficient of t_β in a (when a is expanded in the basis $(t_\alpha)_{\alpha\in\mathbb{V}}$ of $\mathbb{L}$). This is an element of $\mathbb{K}$. For example, $[t_3](t_2 - t_3 + 5t_6) = -1$ (if $\mathbb{V} = \mathbb{Z}$).
- **(c)** If $a \in \mathbb{L}$ is nonzero, then the *order* of a is defined to be the smallest $\beta \in \mathbb{V}$ such that $[t_\beta]\, a \neq 0$. This order is an element of $\mathbb{V}$, and is denoted by $\operatorname{ord} a$. For example, $\operatorname{ord}(t_2 - t_3 + 5t_6) = 2$ (if $\mathbb{V} = \mathbb{Z}$). Note that $\operatorname{ord}(t_\alpha) = \alpha$ for each $\alpha \in \mathbb{V}$.

The notations we just defined generalize standard features of Laurent polynomials: If $\mathbb{V} = \mathbb{Z}$ as in Example 34, then the coefficient $[t_\beta]\, a$ of an element $a \in \mathbb{L} = \mathbb{K}[X, X^{-1}]$ is simply the coefficient of X^β in the Laurent polynomial a, and the order $\operatorname{ord} a$ of a nonzero Laurent polynomial $a \in \mathbb{L}$ is the order of a in the usual sense (i.e., the smallest exponent of a monomial appearing in a). If we substitute X^{-1} for X (thus replacing each monomial X^β by $X^{-\beta}$), then the order of a Laurent polynomial becomes its degree (with a negative sign). In light of this, the following properties of orders should not be surprising:

Lemma 36.

- ***(a)*** *A nonzero element* $a \in \mathbb{L}$ *belongs to* $\mathbb{L}_+$ *if and only if its order* $\operatorname{ord} a$ *is nonnegative (i.e., we have* $\operatorname{ord} a \geqslant 0$*).*
- ***(b)*** *We have* $\operatorname{ord}(-a) = \operatorname{ord} a$ *for any nonzero* $a \in \mathbb{L}$*.*
- ***(c)*** *Let* a *and* b *be two nonzero elements of* $\mathbb{L}$*. Then,* ab *is nonzero and satisfies* $\operatorname{ord}(ab) = \operatorname{ord} a + \operatorname{ord} b$*.*
- ***(d)*** *Let* a *and* b *be two nonzero elements of* $\mathbb{L}$ *such that* $a+b$ *is nonzero. Then,* $\operatorname{ord}(a+b) \geqslant \min\{\operatorname{ord} a, \operatorname{ord} b\}$*.*

The proof of Lemma 36 is easy and entirely analogous to the proof of the corresponding properties of usual polynomials. Thus, we omit it. A detailed proof can be found in an appendix to the arXiv version of this paper.

Corollary 37. *The ring* $\mathbb{L}$ *is an integral domain.*

Proof. This follows from Lemma 36 **(c)**. □

Applying Lemma 36 **(c)** many times, we also obtain the following:

Corollary 38. *The map* $\operatorname{ord} : \mathbb{L} \setminus \{0\} \to \mathbb{V}$ *transforms (finite) products into sums. In more detail: If* $(a_i)_{i \in I}$ *is any finite family of nonzero elements of* $\mathbb{L}$*, then the product* $\prod_{i \in I} a_i$ *is nonzero and satisfies*

$$\operatorname{ord}\left(\prod_{i \in I} a_i\right) = \sum_{i \in I} \operatorname{ord}(a_i).$$

Proof. Induction on $|I|$. The induction step uses Lemma 36 **(c)**. □

We can now assign a $\mathbb{V}$-ultra triple to each subset of $\mathbb{L}$:

Definition 39. Let E be a subset of $\mathbb{L}$. Define a distance function $d : E \underline{\times} E \to \mathbb{V}$ by setting

$$d(a,b) = -\operatorname{ord}(a-b) \qquad \text{for all } (a,b) \in E \underline{\times} E.$$

(Recall that $E \underline{\times} E$ means the set $\{(a,b) \in E \times E \mid a \neq b\}$.)

Then, (E, w, d) is a $\mathbb{V}$-ultra triple whenever $w : E \to \mathbb{V}$ is a function (by Lemma 40 below). Such a $\mathbb{V}$-ultra triple (E, w, d) will be called *valadic*.

Lemma 40. *In Definition 39, the triple* (E, w, d) *is indeed a* $\mathbb{V}$*-ultra triple.*

Lemma 40 follows easily from Lemma 36; the details are left to the reader. A detailed proof can be found in an appendix to the arXiv version of this paper.

Now, we claim that the Bhargava greedoid of a valadic $\mathbb{V}$-ultra triple (E, w, d) with finite E is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$:

Theorem 41. *Let E be a finite subset of $\mathbb{L}$. Define d as in Definition 39. Let $w : E \to \mathbb{V}$ be a function. Then, the Bhargava greedoid of the $\mathbb{V}$-ultra triple (E, w, d) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.*

In order to prove this theorem, we will need a determinantal identity:

Lemma 42. *Let R be a commutative ring. Consider the polynomial ring $R[X]$. Let $m \in \mathbb{N}$. Let $f_1, f_2, \ldots, f_m$ be m polynomials in $R[X]$. Assume that f_j is a monic polynomial of degree $j-1$ for each $j \in \{1, 2, \ldots, m\}$. Let $u_1, u_2, \ldots, u_m$ be m elements of R. Then,*

$$\det\left(\left(f_j\left(u_i\right)\right)_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m}\right) = \prod_{\substack{(i,j) \in \{1,2,\ldots,m\}^2; \\ i > j}} \left(u_i - u_j\right).$$

Here, we are using the notation $(b_{i,j})_{1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q}$ for the $p \times q$-matrix whose (i, j)-th entry is $b_{i,j}$ for all $i \in \{1, 2, \ldots, p\}$ and all $j \in \{1, 2, \ldots, q\}$.

Lemma 42 is a classical generalization of the famous Vandermonde determinant. In this form, it is a particular case of [6, Theorem 2] (applied to $P_j = f_j$ and $a_i = u_i$), because the coefficient of X^{j-1} in a monic polynomial of degree $j-1$ is 1. It also appears in [5, Exercise 267] (where it is stated for the transpose of the matrix we are considering here), in [16, Chapter XI, Exercise 2 in Set XVIII] (where it, too, is stated for the transpose of the matrix), in [12, Proposition 1], and in [7, Exercise 6.62].

We need two more simple lemmas for our proof of Theorem 41:

Lemma 43. *The map*

$$\pi : \mathbb{L}_+ \to \mathbb{K}, \qquad x \mapsto [t_0]\, x$$

is a $\mathbb{K}$-algebra homomorphism.

Lemma 44. *Consider the map $\pi : \mathbb{L}_+ \to \mathbb{K}$ from Lemma 43. Let $a \in \mathbb{L}_+$ be nonzero. Then, $\pi(a) \neq 0$ holds if and only if $\operatorname{ord} a = 0$.*

The proofs of these lemmas are easy (and, again, analogous to those of analogous properties of polynomials), and are thus omitted. Detailed proofs can be found in an appendix to the arXiv version of this paper.

Proof of Theorem 41. Let $m = |E|$. Consider the $\mathbb{V}$-ultra triple (E, w, d); all perimeters discussed in this proof are defined with respect to this $\mathbb{V}$-ultra triple.

We construct a list $(c_1, c_2, \ldots, c_m)$ of elements of E by the following recursive procedure:

- For each $i \in \{1, 2, \ldots, m\}$, we choose c_i (assuming that the entries $c_1, c_2, \ldots, c_{i-1}$ are already constructed) to be an element of $E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$ that maximizes the perimeter $\operatorname{PER}\{c_1, c_2, \ldots, c_i\}$.

This procedure can indeed be carried out, since at each step we can find an element $c_i \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$ that maximizes the perimeter $\operatorname{PER}\{c_1, c_2, \ldots, c_i\}$ (indeed, $m = |E|$ ensures that we never run out of elements, whereas the finiteness of E guarantees the existence of the maximum). Clearly, this procedure constructs an m-tuple $(c_1, c_2, \ldots, c_m)$ of elements of E. The m entries $c_1, c_2, \ldots, c_m$ of this m-tuple are distinct[9], and thus are m distinct elements of E; but E has only m elements altogether (since $m = |E|$). Hence, the m entries $c_1, c_2, \ldots, c_m$ must cover the whole set E. In other words, $E = \{c_1, c_2, \ldots, c_m\}$.

Furthermore, for each $i \in \{1, 2, \ldots, m\}$ and each $x \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, we have

$$\operatorname{PER}\{c_1, c_2, \ldots, c_i\} \geqslant \operatorname{PER}\{c_1, c_2, \ldots, c_{i-1}, x\} \tag{1}$$

(due to how c_i is chosen). Thus, in the parlance of [8, §3.2], the m-tuple $(c_1, c_2, \ldots, c_m)$ is a greedy m-permutation of E.

For each $j \in \{1, 2, \ldots, m\}$, define a $\rho_j \in \mathbb{V}$ by

$$\rho_j = w(c_j) + \sum_{i=1}^{j-1} d(c_i, c_j). \tag{2}$$

(This is precisely what is called $\nu_j^\circ(C)$ in [8], where $C = E$.)

Consider the polynomial ring $\mathbb{L}[X]$. For each $j \in \{1, 2, \ldots, m\}$, define a polynomial $f_j \in \mathbb{L}[X]$ by

$$f_j = (X - c_1)(X - c_2) \cdots (X - c_{j-1}) = \prod_{i=1}^{j-1} (X - c_i).$$

This is a monic polynomial of degree $j - 1$.

Next we claim the following:

Claim 1: Let $e \in E$ and $j \in \{1, 2, \ldots, m\}$. Then, $t_{\rho_j - w(e)} f_j(e) \in \mathbb{L}_+$.

[*Proof of Claim 1:* We have $f_j = \prod_{i=1}^{j-1} (X - c_i)$ and thus $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$. Hence, if $e \in \{c_1, c_2, \ldots, c_{j-1}\}$, then $f_j(e) = 0$ and thus our claim $t_{\rho_j - w(e)} f_j(e) \in \mathbb{L}_+$ is obvious. Thus, we WLOG assume that $e \notin \{c_1, c_2, \ldots, c_{j-1}\}$. Hence, $\prod_{i=1}^{j-1} (e - c_i)$ is a product of nonzero elements of $\mathbb{L}$, and thus is itself nonzero (since Corollary 37 says that $\mathbb{L}$ is an integral domain). In other words, $f_j(e)$ is nonzero (since $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$). Hence, $t_{\rho_j - w(e)} f_j(e)$ is nonzero as well (since $t_{\rho_j - w(e)}$ is nonzero, and since $\mathbb{L}$ is an integral domain).

Moreover, from $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$, we obtain

$$\operatorname{ord}(f_j(e)) = \operatorname{ord}\left(\prod_{i=1}^{j-1} (e - c_i)\right) = \sum_{i=1}^{j-1} \operatorname{ord}(e - c_i) \tag{3}$$

[9] since each c_i is chosen to be an element of $E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$

(by Corollary 38).

From $e \in E$ and $e \notin \{c_1, c_2, \ldots, c_{j-1}\}$, we obtain $e \in E \setminus \{c_1, c_2, \ldots, c_{j-1}\}$. Hence, (1) (applied to $i = j$ and $x = e$) yields

$$\operatorname{PER}\{c_1, c_2, \ldots, c_j\} \geqslant \operatorname{PER}\{c_1, c_2, \ldots, c_{j-1}, e\}. \tag{4}$$

But $c_1, c_2, \ldots, c_j$ are distinct[10]. Hence, the definition of the perimeter yields

$$\begin{aligned}
\operatorname{PER}\{c_1, c_2, \ldots, c_j\} &= \underbrace{\sum_{i=1}^{j} w(c_i)}_{=\sum_{i=1}^{j-1} w(c_i) + w(c_j)} + \underbrace{\sum_{1 \leqslant i < p \leqslant j} d(c_i, c_p)}_{=\sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, c_j)} \\
&= \sum_{i=1}^{j-1} w(c_i) + w(c_j) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, c_j) \\
&= \underbrace{w(c_j) + \sum_{i=1}^{j-1} d(c_i, c_j)}_{\substack{=\rho_j \\ \text{(by (2))}}} + \sum_{i=1}^{j-1} w(c_i) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) \\
&= \rho_j + \sum_{i=1}^{j-1} w(c_i) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p)
\end{aligned}$$

and

$$\operatorname{PER}\{c_1, c_2, \ldots, c_{j-1}, e\} = \sum_{i=1}^{j-1} w(c_i) + w(e) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, e)$$

(since $c_1, c_2, \ldots, c_{j-1}, e$ are distinct[11]). Hence, (4) rewrites as

$$\begin{aligned}
&\rho_j + \sum_{i=1}^{j-1} w(c_i) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) \\
&\geqslant \sum_{i=1}^{j-1} w(c_i) + w(e) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, e).
\end{aligned}$$

After cancelling equal terms, this inequality transforms into

$$\rho_j \geqslant w(e) + \sum_{i=1}^{j-1} d(c_i, e).$$

[10] This is because $c_1, c_2, \ldots, c_m$ are distinct.

[11] This is because $c_1, c_2, \ldots, c_m$ are distinct and $e \notin \{c_1, c_2, \ldots, c_{j-1}\}$.

In view of

$$\sum_{i=1}^{j-1} \underbrace{d(c_i, e)}_{\substack{=d(e,c_i) \\ \text{(by the "Symmetry"} \\ \text{axiom in the definition} \\ \text{of a } \mathbb{V}\text{-ultra triple)}}} = \sum_{i=1}^{j-1} \underbrace{d(e, c_i)}_{\substack{=-\operatorname{ord}(e-c_i) \\ \text{(by the definition of } d)}} = -\underbrace{\sum_{i=1}^{j-1} \operatorname{ord}(e - c_i)}_{\substack{=\operatorname{ord}(f_j(e)) \\ \text{(by (3))}}} = -\operatorname{ord}(f_j(e)),$$

this rewrites as

$$\rho_j \geqslant w(e) - \operatorname{ord}(f_j(e)).$$

In other words, $\operatorname{ord}(f_j(e)) \geqslant w(e) - \rho_j$. Now, Lemma 36 **(c)** yields

$$\operatorname{ord}\left(t_{\rho_j - w(e)} f_j(e)\right) = \underbrace{\operatorname{ord}\left(t_{\rho_j - w(e)}\right)}_{=\rho_j - w(e)} + \underbrace{\operatorname{ord}(f_j(e))}_{\geqslant w(e) - \rho_j} \geqslant \rho_j - w(e) + w(e) - \rho_j = 0.$$

Hence, Lemma 36 **(a)** (applied to $a = t_{\rho_j - w(e)} f_j(e)$) shows that $t_{\rho_j - w(e)} f_j(e)$ belongs to $\mathbb{L}_+$. This proves Claim 1.]

For each $e \in E$ and $j \in \{1, 2, \ldots, m\}$, we define an $a(e, j) \in \mathbb{L}_+$ by

$$a(e, j) = t_{\rho_j - w(e)} f_j(e). \tag{5}$$

(This is well-defined, due to Claim 1.)

We now claim the following:

Claim 2: Let $k \in \mathbb{N}$. Let $u_1, u_2, \ldots, u_k$ be any k distinct elements of E. Let $U = \{u_1, u_2, \ldots, u_k\}$. Then,

$$\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right) \text{ is a nonzero element of } \mathbb{L}_+ \tag{6}$$

and

$$\operatorname{ord}\left(\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)\right) = \sum_{j=1}^{k} \rho_j - \operatorname{PER}(U). \tag{7}$$

[*Proof of Claim 2:* We have $|E| \geqslant k$ (since $u_1, u_2, \ldots, u_k$ are k distinct elements of E). Hence, $k \leqslant |E| = m$. Therefore, $\{1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, m\}$. In other words, for each $j \in \{1, 2, \ldots, k\}$, we have $j \in \{1, 2, \ldots, m\}$.

Hence, $a(u_i, j) \in \mathbb{L}_+$ for any $i, j \in \{1, 2, \ldots, k\}$ (since we defined $a(e, j)$ to satisfy $a(e, j) \in \mathbb{L}_+$ for any $e \in E$ and $j \in \{1, 2, \ldots, m\}$). In other words, all entries of the matrix $(a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}$ belong to $\mathbb{L}_+$. Hence, its determinant $\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)$ belongs to $\mathbb{L}_+$ as well (since $\mathbb{L}_+$ is a ring).

Lemma 42 (applied to $\mathbb{L}$ and k instead of R and m) yields

$$\det\left((f_j(u_i))_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant k}\right) = \prod_{\substack{(i,j) \in \{1,2,\ldots,k\}^2; \\ i > j}} (u_i - u_j)$$
$$= \prod_{1 \leqslant j < i \leqslant k} (u_i - u_j). \tag{8}$$

It is known that the determinant of a matrix equals the determinant of its transpose. Thus,

$$\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)=\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}\right)$$

(since the matrix $\left(f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}$ is the transpose of the matrix $\left(f_j\left(u_i\right)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}$).

But when we scale a column of a matrix by a scalar λ, then its determinant also gets multiplied by λ. Hence,

$$\begin{aligned}\det\left(\left(t_{-w(u_i)}f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)&=\left(\prod_{i=1}^{k}t_{-w(u_i)}\right)\cdot\underbrace{\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)}_{\substack{=\det\left(\left(f_j(u_i)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}\right)\\ =\prod_{1\leqslant j<i\leqslant k}(u_i-u_j)\\ \text{(by (8))}}}\\&=\left(\prod_{i=1}^{k}t_{-w(u_i)}\right)\cdot\prod_{1\leqslant j<i\leqslant k}\left(u_i-u_j\right).\end{aligned}$$

Furthermore, when we scale a row of a matrix by a scalar λ, then its determinant also gets multiplied by λ. Hence,

$$\begin{aligned}\det\left(\left(t_{\rho_j}t_{-w(u_i)}f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)&=\left(\prod_{j=1}^{k}t_{\rho_j}\right)\cdot\underbrace{\det\left(\left(t_{-w(u_i)}f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)}_{=\left(\prod_{i=1}^{k}t_{-w(u_i)}\right)\cdot\prod_{1\leqslant j<i\leqslant k}(u_i-u_j)}\\&=\left(\prod_{j=1}^{k}t_{\rho_j}\right)\left(\prod_{i=1}^{k}t_{-w(u_i)}\right)\cdot\prod_{1\leqslant j<i\leqslant k}\left(u_i-u_j\right).\end{aligned}$$

However, for every $i\in\{1,2,\ldots,k\}$ and $j\in\{1,2,\ldots,k\}$, we have

$$\begin{aligned}a\left(u_i,j\right)&=\underbrace{t_{\rho_j-w(u_i)}}_{=t_{\rho_j}t_{-w(u_i)}}f_j\left(u_i\right)\qquad\text{(by the definition of }a\left(u_i,j\right))\\&=t_{\rho_j}t_{-w(u_i)}f_j\left(u_i\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\det\left(\left(\underbrace{a\left(u_i,j\right)}_{=t_{\rho_j}t_{-w(u_i)}f_j(u_i)}\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\\&=\det\left(\left(t_{\rho_j}t_{-w(u_i)}f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\\&=\left(\prod_{j=1}^{k}t_{\rho_j}\right)\left(\prod_{i=1}^{k}t_{-w(u_i)}\right)\cdot\prod_{1\leqslant j<i\leqslant k}\left(u_i-u_j\right).\end{aligned}\tag{9}$$

The right hand side of this equality is a product of nonzero elements of $\mathbb{L}$ (since $u_1, u_2, \ldots, u_k$ are distinct), and thus is nonzero (by Corollary 37). Hence, the left hand side is nonzero. In other words, $\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ is nonzero. This proves (6) (since we already know that $\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ belongs to $\mathbb{L}_+$).

Moreover, (9) yields

$$\begin{aligned}
&\operatorname{ord}\left(\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)\\
&= \operatorname{ord}\left(\left(\prod_{j=1}^{k} t_{\rho_j}\right)\left(\prod_{i=1}^{k} t_{-w(u_i)}\right)\cdot \prod_{1\leqslant j<i\leqslant k}\left(u_i - u_j\right)\right)\\
&= \sum_{j=1}^{k} \underbrace{\operatorname{ord}\left(t_{\rho_j}\right)}_{=\rho_j} + \sum_{i=1}^{k} \underbrace{\operatorname{ord}\left(t_{-w(u_i)}\right)}_{=-w(u_i)} + \sum_{1\leqslant j<i\leqslant k} \operatorname{ord}\left(u_i - u_j\right)\\
&\qquad \text{(by Lemma 36 \textbf{(c)} and Corollary 38)}\\
&= \sum_{j=1}^{k} \rho_j - \sum_{i=1}^{k} w\left(u_i\right) + \sum_{1\leqslant j<i\leqslant k} \operatorname{ord}\left(u_i - u_j\right)\\
&= \sum_{j=1}^{k} \rho_j - \left(\sum_{i=1}^{k} w\left(u_i\right) - \sum_{1\leqslant j<i\leqslant k} \operatorname{ord}\left(u_i - u_j\right)\right). \qquad (10)
\end{aligned}$$

But recall that $U = \{u_1, u_2, \ldots, u_k\}$ with $u_1, u_2, \ldots, u_k$ distinct. The definition of perimeter thus yields

$$\begin{aligned}
\operatorname{PER}(U) &= \sum_{i=1}^{k} w\left(u_i\right) + \sum_{1\leqslant i<j\leqslant k} d\left(u_i, u_j\right) = \sum_{i=1}^{k} w\left(u_i\right) + \sum_{1\leqslant j<i\leqslant k} \underbrace{d\left(u_j, u_i\right)}_{\substack{=d(u_i,u_j)\\ \text{(by the ``Symmetry''}\\ \text{axiom in the definition}\\ \text{of a } \mathbb{V}\text{-ultra triple)}}}\\
&\qquad \text{(here, we have renamed the index } (i,j) \text{ as } (j,i) \text{ in the second sum)}\\
&= \sum_{i=1}^{k} w\left(u_i\right) + \sum_{1\leqslant j<i\leqslant k} \underbrace{d\left(u_i, u_j\right)}_{\substack{=-\operatorname{ord}(u_i-u_j)\\ \text{(by the definition of } d)}}\\
&= \sum_{i=1}^{k} w\left(u_i\right) - \sum_{1\leqslant j<i\leqslant k} \operatorname{ord}\left(u_i - u_j\right). \qquad (11)
\end{aligned}$$

Hence, we can rewrite (10) as

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right) = \sum_{j=1}^{k} \rho_j - \operatorname{PER}(U).$$

So (7) is proved. This proves Claim 2.]

As a consequence of Claim 2, we obtain the following:

> *Claim 3:* Let $k \in \{0, 1, \ldots, m\}$. Then, $\sum_{j=1}^{k} \rho_j$ is the maximum perimeter of a k-subset of E.

[*Proof of Claim 3:* The elements $c_1, c_2, \ldots, c_m$. Hence, $\{c_1, c_2, \ldots, c_k\}$ is a k-subset of E.

Adding up the equalities (2) for all $j \in \{1, 2, \ldots, k\}$, we obtain

$$\sum_{j=1}^{k} \rho_j = \sum_{j=1}^{k} \left(w(c_j) + \sum_{i=1}^{j-1} d(c_i, c_j) \right)$$
$$= \sum_{j=1}^{k} w(c_j) + \sum_{1 \leqslant i < j \leqslant k} d(c_i, c_j) = \operatorname{PER}\{c_1, c_2, \ldots, c_k\}$$

(since $c_1, c_2, \ldots, c_k$ are distinct). Since $\{c_1, c_2, \ldots, c_k\}$ is a k-subset of E, we thus conclude that $\sum_{j=1}^{k} \rho_j$ is the perimeter of some k-subset of E. Thus, in order to prove Claim 3, we need only to show that $\sum_{j=1}^{k} \rho_j \geqslant \operatorname{PER}(U)$ for every k-subset U of E.

So let U be a k-subset of E. Write the k-subset U in the form $U = \{u_1, u_2, \ldots, u_k\}$ for k distinct elements $u_1, u_2, \ldots, u_k$ of E. Claim 2 thus yields that

$$\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \right) \text{ is a nonzero element of } \mathbb{L}_+$$

and

$$\operatorname{ord}\left(\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \right) \right) = \sum_{j=1}^{k} \rho_j - \operatorname{PER}(U). \tag{12}$$

Thus, in particular, $\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \right)$ belongs to $\mathbb{L}_+$. Hence,

$$\operatorname{ord}\left(\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \right) \right) \geqslant 0$$

(by Lemma 36 **(a)**, applied to $\det\left((a(u_i, j))_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \right)$ instead of a). In view of (12), this rewrites as $\sum_{j=1}^{k} \rho_j - \operatorname{PER}(U) \geqslant 0$. In other words, $\sum_{j=1}^{k} \rho_j \geqslant \operatorname{PER}(U)$. This completes the proof of Claim 3.]

Now, recall the $\mathbb{K}$-algebra homomorphism $\pi : \mathbb{L}_+ \to \mathbb{K}$ from Lemma 43. For each $e \in E$, define a column vector $v_e \in \mathbb{K}^m$ by

$$v_e = \begin{pmatrix} \pi\left(a\left(e,1\right)\right) \\ \pi\left(a\left(e,2\right)\right) \\ \vdots \\ \pi\left(a\left(e,m\right)\right) \end{pmatrix} = \left(\pi\left(a\left(e,j\right)\right)\right)_{1 \leqslant j \leqslant m}.$$

We thus have a vector family $(v_e)_{e \in E}$ over $\mathbb{K}$. Let $\mathcal{G}$ be the Gaussian elimination greedoid of this family. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Our goal is to prove that $\mathcal{F} = \mathcal{G}$ (since this will yield Theorem 41).

In order to do so, it suffices to show that if U is any subset of E, then we have the logical equivalence

$$(U \in \mathcal{F}) \iff (U \in \mathcal{G}). \tag{13}$$

So let us do this. Let U be a subset of E. We must prove the equivalence (13).

Write the subset U in the form $U = \{u_1, u_2, \ldots, u_k\}$ with $u_1, u_2, \ldots, u_k$ distinct. Thus, $|U| = k$. In other words, U is a k-subset of E. Claim 2 thus yields that

$$\det\left(\left(a\left(u_i, j\right)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right) \text{ is a nonzero element of } \mathbb{L}_+ \tag{14}$$

and

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i, j\right)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)\right) = \sum_{j=1}^{k} \rho_j - \operatorname{PER}(U). \tag{15}$$

Also, recall that π is a $\mathbb{K}$-algebra homomorphism (by Lemma 43), thus a ring homomorphism. Hence,

$$\det\left(\left(\pi\left(a\left(u_i, j\right)\right)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right) = \pi\left(\det\left(\left(a\left(u_i, j\right)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)\right) \tag{16}$$

(since ring homomorphisms respect determinants).

Also, U is a subset of E; hence, $|U| \leqslant |E| = m$. Thus, $k = |U| \leqslant m$. Hence, $k \in \{0, 1, \ldots, m\}$. Therefore, Claim 3 shows that $\sum_{j=1}^{k} \rho_j$ is the maximum perimeter of a k-subset of E. In other words, $\sum_{j=1}^{k} \rho_j$ is the maximum perimeter of a $|U|$-subset of E (since $|U| = k$).

The definition of the Gaussian elimination greedoid $\mathcal{G}$ shows that we have the following equivalence: [12]

$$\begin{aligned} &(U \in \mathcal{G}) \\ \iff &\left(\text{the family } \left(\pi_{|U|}\left(v_e\right)\right)_{e \in U} \in \left(\mathbb{K}^{|U|}\right)^U \text{ is linearly independent}\right) \end{aligned}$$

[12] The words "linearly independent" should always be understood to mean "$\mathbb{K}$-linearly independent" here.

$$\begin{aligned}
&\iff \left(\text{the family } \left(\pi_k\left(v_e\right)\right)_{e\in U} \in \left(\mathbb{K}^k\right)^U \text{ is linearly independent}\right) \\
&\qquad (\text{since } |U| = k) \\
&\iff (\text{the vectors } \pi_k\left(v_{u_1}\right), \pi_k\left(v_{u_2}\right), \ldots, \pi_k\left(v_{u_k}\right) \text{ are linearly independent}) \\
&\qquad (\text{since } U = \{u_1, u_2, \ldots, u_k\} \text{ with } u_1, u_2, \ldots, u_k \text{ distinct}) \\
&\iff \left(\text{the columns of the matrix } \left(\pi\left(a\left(u_i, j\right)\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k} \text{ are linearly independent}\right) \\
&\qquad \left(\begin{array}{c}\text{since the vectors } \pi_k\left(v_{u_1}\right), \pi_k\left(v_{u_2}\right), \ldots, \pi_k\left(v_{u_k}\right) \\ \text{are the columns of the matrix } \left(\pi\left(a\left(u_i, j\right)\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\end{array}\right) \\
&\iff \left(\text{the matrix } \left(\pi\left(a\left(u_i, j\right)\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k} \text{ is invertible}\right) \\
&\iff \left(\det\left(\left(\pi\left(a\left(u_i, j\right)\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \neq 0 \text{ in } \mathbb{K}\right) \\
&\iff \left(\pi\left(\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right) \neq 0 \text{ in } \mathbb{K}\right) \qquad (\text{by (16)}) \\
&\iff \left(\operatorname{ord}\left(\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right) = 0\right) \\
&\qquad \left(\text{by Lemma 44, applied to } \det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \text{ instead of } a\right) \\
&\iff \left(\sum_{j=1}^{k} \rho_j - \operatorname{PER}(U) = 0\right) \qquad (\text{by (15)}) \\
&\iff \left(\operatorname{PER}(U) = \sum_{j=1}^{k} \rho_j\right) \\
&\iff (\operatorname{PER}(U) \text{ is the maximum perimeter of a } |U|\text{-subset of } E) \\
&\qquad \left(\text{since } \sum_{j=1}^{k} \rho_j \text{ is the maximum perimeter of a } |U|\text{-subset of } E\right) \\
&\iff (U \text{ has maximum perimeter among all } |U|\text{-subsets of } E) \\
&\iff (U \in \mathcal{F})
\end{aligned}$$

(by the definition of the Bhargava greedoid $\mathcal{F}$). Thus, the equivalence (13) is proven. This concludes the proof of Theorem 41. ☐

7 Isomorphism

Next, we introduce the notion of a set system. This elementary notion will play a purely technical role in what follows.

Definition 45. Let E be a set. A *set system* on ground set E shall mean a subset of 2^E. (Recall that 2^E means the powerset of E.)

Thus:

- The Gaussian elimination greedoid of a vector family $(v_e)_{e \in E}$ (over any field $\mathbb{K}$) is a set system on ground set E.

- The Bhargava greedoid of any $\mathbb{V}$-ultra triple (E, w, d) is a set system on ground set E.

- Any greedoid $\mathcal{F} \subseteq 2^E$ is a set system on ground set E.

We shall use the following two simple concepts of isomorphism:

Definition 46. Let (E, w, d) and (F, v, c) be two $\mathbb{V}$-ultra triples.

(a) A bijective map $f : E \to F$ is said to be an *isomorphism of* $\mathbb{V}$*-ultra triples from* (E, w, d) *to* (F, v, c) if it satisfies $v \circ f = w$ and

$$c(f(a), f(b)) = d(a, b) \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

(b) The $\mathbb{V}$-ultra triples (E, w, d) and (F, v, c) are said to be *isomorphic* if there exists an isomorphism $f : E \to F$ of $\mathbb{V}$-ultra triples from (E, w, d) to (F, v, c). (Note that being isomorphic is clearly a symmetric relation.)

Definition 47. Let $\mathcal{E}$ and $\mathcal{F}$ be two set systems on ground sets E and F, respectively.

(a) A bijective map $f : E \to F$ is said to be an *isomorphism of set systems from* $\mathcal{E}$ *to* $\mathcal{F}$ if the bijection $2^f : 2^E \to 2^F$ induced by it (i.e., the bijection that sends each $S \in 2^E$ to $f(S) \in 2^F$) satisfies $2^f(\mathcal{E}) = \mathcal{F}$.

(b) The set systems $\mathcal{E}$ and $\mathcal{F}$ are said to be *isomorphic* if there exists an isomorphism $f : E \to F$ of set systems from $\mathcal{E}$ to $\mathcal{F}$. (Note that being isomorphic is clearly a symmetric relation.)

The intuitive meaning of both of these two definitions is simple: Two $\mathbb{V}$-ultra triples are isomorphic if and only if one can be obtained from the other by relabeling the elements of the ground set. The same holds for two set systems.

The following two propositions are obvious:

Proposition 48. *Let* (E, w, d) *and* (F, v, c) *be two isomorphic* $\mathbb{V}$*-ultra triples such that* E *and* F *are finite. Then, the Bhargava greedoids of* (E, w, d) *and* (F, v, c) *are isomorphic as set systems.*

Proposition 49. *Let* $\mathbb{K}$ *be a field. Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be two isomorphic set systems. If* $\mathcal{F}$ *is the Gaussian elimination greedoid of a vector family over* $\mathbb{K}$*, then so is* $\mathcal{E}$*.*

8 Decomposing a $\mathbb{V}$-ultra triple

For the rest of Section 8, we fix a $\mathbb{V}$-ultra triple (E, w, d).

We are going to study the structure of this $\mathbb{V}$-ultra triple. We recall the notions of α-cliques and cliques (defined in Definition 21 and Definition 22, respectively). We shall next define another kind of subsets of E: the *open balls*.

Definition 50. Let $\alpha \in \mathbb{V}$ and $e \in E$. The *open ball* $B^\circ_\alpha(e)$ is defined to be the subset

$$\{f \in E \mid f = e \text{ or else } d(f, e) < \alpha\}$$

of E.

Clearly, for each $\alpha \in \mathbb{V}$ and each $e \in E$, we have $e \in B^\circ_\alpha(e)$, so that the open ball $B^\circ_\alpha(e)$ contains at least the element e.

Proposition 51. *Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e, f) < \alpha$. Then, $B^\circ_\alpha(e) = B^\circ_\alpha(f)$.*

Proof. The "symmetry" axiom in Definition 11 yields that $d(f, e) = d(e, f) < \alpha$.

By the definition of $B^\circ_\alpha(f)$, we have $e \in B^\circ_\alpha(f)$ (since $e \neq f$ and $d(e, f) < \alpha$) and $f \in B^\circ_\alpha(f)$.

Let $x \in B^\circ_\alpha(e)$. We shall show that $x \in B^\circ_\alpha(f)$.

If $x = e$, then this follows immediately from $e \in B^\circ_\alpha(f)$. If $x = f$, then this follows immediately from $f \in B^\circ_\alpha(f)$. It remains to consider the remaining case, i.e., the case when x is neither e nor f. In this case, the points e, f, x are distinct (since $e \neq f$), and we have $d(x, e) < \alpha$ (since $x \in B^\circ_\alpha(e)$). Now, the ultrametric triangle inequality yields $d(x, f) \leqslant \max\{d(x, e), d(f, e)\} < \alpha$ (since $d(x, e) < \alpha$ and $d(f, e) < \alpha$), and therefore $x \in B^\circ_\alpha(f)$ again (by the definition of $B^\circ_\alpha(f)$).

Now, forget that we fixed x. We thus have shown that $x \in B^\circ_\alpha(f)$ for each $x \in B^\circ_\alpha(e)$. In other words, $B^\circ_\alpha(e) \subseteq B^\circ_\alpha(f)$.

But our situation is symmetric in e and f (since $f \neq e$ and $d(f, e) < \alpha$). Hence, the same argument that let us prove $B^\circ_\alpha(e) \subseteq B^\circ_\alpha(f)$ can be applied with the roles of e and f interchanged; thus we obtain $B^\circ_\alpha(f) \subseteq B^\circ_\alpha(e)$. Combining this with $B^\circ_\alpha(e) \subseteq B^\circ_\alpha(f)$, we obtain $B^\circ_\alpha(e) = B^\circ_\alpha(f)$. This proves Proposition 51. ☐

Corollary 52. *Let $\alpha \in \mathbb{V}$ and $e \in E$. Let $f \in B^\circ_\alpha(e)$. Then, $B^\circ_\alpha(e) = B^\circ_\alpha(f)$.*

Proof. WLOG assume that $e \neq f$. Hence, $d(f, e) < \alpha$ (since $f \in B^\circ_\alpha(e)$). Thus, the "symmetry" axiom in Definition 11 yields that $d(e, f) = d(f, e) < \alpha$. Hence, Proposition 51 yields $B^\circ_\alpha(e) = B^\circ_\alpha(f)$. Qed. ☐

Proposition 53. *Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e, f) \geqslant \alpha$. Then:*

(a) *If $a \in B^\circ_\alpha(e)$ and $b \in B^\circ_\alpha(f)$, then $a \neq b$ and $d(a, b) \geqslant \alpha$.*

(b) *The open balls $B^\circ_\alpha(e)$ and $B^\circ_\alpha(f)$ are disjoint.*

Proof. **(a)** Let $a \in B_\alpha^\circ(e)$ and $b \in B_\alpha^\circ(f)$. We must prove that $a \neq b$ and $d(a,b) \geqslant \alpha$.

Assume the contrary. Thus, either $a = b$ or else $d(a,b) < \alpha$. Hence, $a \in B_\alpha^\circ(b)$ (by the definition of $B_\alpha^\circ(b)$). Thus, $B_\alpha^\circ(b) = B_\alpha^\circ(a)$ (by Corollary 52, applied to b and a instead of e and f). Also, from $a \in B_\alpha^\circ(e)$, we obtain $B_\alpha^\circ(e) = B_\alpha^\circ(a)$ (by Corollary 52, applied to a instead of f). Furthermore, from $b \in B_\alpha^\circ(f)$, we obtain $B_\alpha^\circ(f) = B_\alpha^\circ(b)$ (by Corollary 52, applied to f and b instead of e and f). Finally, the definition of $B_\alpha^\circ(e)$ yields $e \in B_\alpha^\circ(e) = B_\alpha^\circ(a) = B_\alpha^\circ(b) = B_\alpha^\circ(f)$. Since $e \neq f$, this entails $d(e,f) < \alpha$ (by the definition of $B_\alpha^\circ(f)$). This contradicts $d(e,f) \geqslant \alpha$. This contradiction shows that our assumption was false. Hence, Proposition 53 **(a)** follows.

(b) This is simply the "$a \neq b$" part of Proposition 53 **(a)**. □

The next proposition shows how a finite $\mathbb{V}$-ultra triple (of size > 1) can be decomposed into several smaller $\mathbb{V}$-ultra triples; this will later be used for recursive reasoning:[13]

Proposition 54. *Assume that E is finite and $|E| > 1$. Let $\alpha = \max(d(E \underline{\times} E))$.*

Pick any maximum-size α-clique, and write it in the form $\{e_1, e_2, \ldots, e_m\}$ for some distinct elements $e_1, e_2, \ldots, e_m$ of E.

For each $i \in \{1, 2, \ldots, m\}$, let E_i be the open ball $B_\alpha^\circ(e_i)$, and let (E_i, w_i, d_i) be the $\mathbb{V}$-ultra triple $\left(E_i, w\mid_{E_i}, d\mid_{E_i \underline{\times} E_i}\right)$.

Then:

- ***(a)*** *We have $m > 1$.*
- ***(b)*** *The sets $E_1, E_2, \ldots, E_m$ form a set partition of E. (This means that these sets $E_1, E_2, \ldots, E_m$ are disjoint and nonempty and their union is E.)*
- ***(c)*** *We have $|E_i| < |E|$ for each $i \in \{1, 2, \ldots, m\}$.*
- ***(d)*** *If $i \in \{1, 2, \ldots, m\}$, and if $a, b \in E_i$ are distinct, then $d(a,b) < \alpha$.*
- ***(e)*** *If i and j are two distinct elements of $\{1, 2, \ldots, m\}$, and if $a \in E_i$ and $b \in E_j$, then $a \neq b$ and $d(a,b) = \alpha$.*
- ***(f)*** *Let $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ for each $i \in \{1, 2, \ldots, m\}$. Then,*

$$\operatorname{mcs}(E, w, d) = \max\{m, n_1, n_2, \ldots, n_m\}.$$

Proof. Let us first check that α is well-defined. Indeed, the set $E \underline{\times} E$ is nonempty (since $|E| > 1$) and finite (since E is finite). Hence, the set $d(E \underline{\times} E)$ is nonempty and finite as well. Thus, its largest element $\max(d(E \underline{\times} E))$ is well-defined. In other words, α is well-defined.

We have

$$d(a,b) \leqslant \alpha \qquad \text{for each } (a,b) \in E \underline{\times} E \tag{17}$$

(since $\alpha = \max(d(E \underline{\times} E))$).

[13] See Definition 24 for the meaning of $\operatorname{mcs}(E, w, d)$.

The set $\{e_1, e_2, \ldots, e_m\}$ is a maximum-size α-clique (by its definition), but its size is m (since $e_1, e_2, \ldots, e_m$ are distinct). Hence, the maximum size of an α-clique is m. Thus, every α-clique has size $\leqslant m$.

(a) From $\alpha = \max(d(E \underline{\times} E)) \in d(E \underline{\times} E)$, we conclude that there exist two distinct elements u and v of E satisfying $d(u, v) = \alpha$. Consider these u and v. Then, $\{u, v\}$ is an α-clique. This α-clique must have size $\leqslant m$ (since every α-clique has size $\leqslant m$). Hence, $|\{u, v\}| \leqslant m$. Thus, $m \geqslant |\{u, v\}| = 2$ (since u and v are distinct), so that $m \geqslant 2 > 1$. This proves Proposition 54 **(a)**.

(b) If i and j are two distinct elements of $\{1, 2, \ldots, m\}$ then $e_i \neq e_j$ (since $e_1, e_2, \ldots, e_m$ are distinct) and thus $d(e_i, e_j) = \alpha$ (since $\{e_1, e_2, \ldots, e_m\}$ is an α-clique), and therefore the open balls $B^{\circ}_{\alpha}(e_i)$ and $B^{\circ}_{\alpha}(e_j)$ are disjoint (by Proposition 53 **(b)**, applied to $e = e_i$ and $f = e_j$). In other words, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then the open balls E_i and E_j are disjoint (since $E_i = B^{\circ}_{\alpha}(e_i)$ and $E_j = B^{\circ}_{\alpha}(e_j)$).

Hence, the open balls $E_1, E_2, \ldots, E_m$ are disjoint. Furthermore, they are nonempty (since each open ball $E_i = B^{\circ}_{\alpha}(e_i)$ contains at least the element e_i).

Furthermore, the union of these balls $E_1, E_2, \ldots, E_m$ is the whole set E.

[*Proof:* Assume the contrary. Thus, there exists some $a \in E$ that belongs to none of these balls $E_1, E_2, \ldots, E_m$. Consider this a. Then, for each $i \in \{1, 2, \ldots, m\}$, we have $a \notin E_i = B^{\circ}_{\alpha}(e_i)$. By the definition of $B^{\circ}_{\alpha}(e_i)$, this entails that $a \neq e_i$ and $d(a, e_i) \geqslant \alpha$, hence $d(a, e_i) = \alpha$ (since (17) yields $d(a, e_i) \leqslant \alpha$). Thus, we have shown that $d(a, e_i) = \alpha$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, $\{a, e_1, e_2, \ldots, e_m\}$ is an α-clique. This α-clique has size $m + 1$ (since $e_1, e_2, \ldots, e_m$ are distinct, and since $a \neq e_i$ for each $i \in \{1, 2, \ldots, m\}$). But this contradicts the fact that every α-clique has size $\leqslant m$. This contradiction shows that our assumption was wrong. Hence, the union of the balls $E_1, E_2, \ldots, E_m$ is the whole set E.]

We have now proved that the balls $E_1, E_2, \ldots, E_m$ are disjoint and nonempty and their union is the whole set E. In other words, they form a set partition of E. This proves Proposition 54 **(b)**.

(c) Part **(c)** follows from part **(b)**, since $m > 1$.

(d) Let $i \in \{1, 2, \ldots, m\}$. Let $a, b \in E_i$ be distinct. We must prove that $d(a, b) < \alpha$.

We have $b \in E_i = B^{\circ}_{\alpha}(e_i)$ (by the definition of E_i) and thus $B^{\circ}_{\alpha}(e_i) = B^{\circ}_{\alpha}(b)$ (by Corollary 52, applied to e_i and b instead of e and f). But $a \in E_i = B^{\circ}_{\alpha}(e_i) = B^{\circ}_{\alpha}(b)$. In other words, $a = b$ or else $d(a, b) < \alpha$ (by the definition of $B^{\circ}_{\alpha}(b)$). Hence, $d(a, b) < \alpha$ (since $a \neq b$). This proves Proposition 54 **(d)**.

(e) Let i and j be two distinct elements of $\{1, 2, \ldots, m\}$. Let $a \in E_i$ and $b \in E_j$. We must prove that $a \neq b$ and $d(a, b) = \alpha$.

We have $a \in E_i = B^{\circ}_{\alpha}(e_i)$ (by the definition of E_i) and similarly $b \in B^{\circ}_{\alpha}(e_j)$. Furthermore, $e_i \neq e_j$ (since $i \neq j$ and since $e_1, e_2, \ldots, e_m$ are distinct) and thus $d(e_i, e_j) = \alpha$ (since $\{e_1, e_2, \ldots, e_m\}$ is an α-clique). Hence, Proposition 53 **(a)** (applied to e_i and e_j instead of e and f) yields $a \neq b$ and $d(a, b) \geqslant \alpha$. Combining $d(a, b) \geqslant \alpha$ with (17), we obtain $d(a, b) = \alpha$. This proves Proposition 54 **(e)**.

(f) Let us first notice that the map d_i (for each $i \in \{1, 2, \ldots, m\}$) is defined to be a restriction of the map d. Thus, for any $i \in \{1, 2, \ldots, m\}$, we have

$$d_i(e, f) = d(e, f) \qquad \text{for any two distinct } e, f \in E_i. \tag{18}$$

The $\mathbb{V}$-ultra triple (E, w, d) has a clique of size m (namely, $\{e_1, e_2, \ldots, e_m\}$), and a clique of size n_i for each $i \in \{1, 2, \ldots, m\}$ (indeed, $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ shows that the $\mathbb{V}$-ultra triple (E_i, w_i, d_i) has such a clique; but this clique must of course be a clique of (E, w, d) as well[14]). Thus, the $\mathbb{V}$-ultra triple (E, w, d) has a clique of size $\max\{m, n_1, n_2, \ldots, n_m\}$. Thus,

$$\operatorname{mcs}(E, w, d) \geqslant \max\{m, n_1, n_2, \ldots, n_m\}.$$

It remains to prove the reverse inequality: $\operatorname{mcs}(E, w, d) \leqslant \max\{m, n_1, n_2, \ldots, n_m\}$.

Assume the contrary. Thus, $\operatorname{mcs}(E, w, d) > \max\{m, n_1, n_2, \ldots, n_m\}$. The definition of $\operatorname{mcs}(E, w, d)$ shows that the $\mathbb{V}$-ultra triple (E, w, d) has a clique C of size $\operatorname{mcs}(E, w, d)$. Consider this C. Thus, $|C| = \operatorname{mcs}(E, w, d) > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant 0$. Hence, the set C is nonempty. In other words, there exists some $a \in C$. Consider this a.

But recall that $E_1, E_2, \ldots, E_m$ form a set partition of E. Thus, $E_1 \cup E_2 \cup \cdots \cup E_m = E$. Now, $a \in C \subseteq E = E_1 \cup E_2 \cup \cdots \cup E_m$. In other words, $a \in E_i$ for some $i \in \{1, 2, \ldots, m\}$. Consider this i.

We are in one of the following two cases:

Case 1: We have $C \subseteq E_i$.

Case 2: We have $C \not\subseteq E_i$.

Let us first consider Case 1. In this case, we have $C \subseteq E_i$. Recall that C is a clique of (E, w, d). Thus, from $C \subseteq E_i$, we conclude that C is a clique of the $\mathbb{V}$-ultra triple (E_i, w_i, d_i) (because of (18)). But the maximum size of such a clique is n_i (since we have $\operatorname{mcs}(E_i, w_i, d_i) = n_i$). Hence, $|C| \leqslant n_i$. This contradicts $|C| > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant n_i$. Thus, we have obtained a contradiction in Case 1.

Let us next consider Case 2. In this case, we have $C \not\subseteq E_i$. In other words, there exists some $b \in C$ such that $b \notin E_i$. Consider this b. From $b \notin E_i$, we conclude that $b \in E_j$ for some $j \neq i$ (because $b \in C \subseteq E = E_1 \cup E_2 \cup \cdots \cup E_m$). Consider this j. Thus, Proposition 54 **(e)** yields that $a \neq b$ and $d(a, b) = \alpha$. Now, $a \neq b$ shows that a and b are two distinct elements of C. But C is a clique, and thus is an α-clique (since its two elements a and b satisfy $d(a, b) = \alpha$). Thus, $|C| \leqslant m$ (since every α-clique has size $\leqslant m$). This contradicts $|C| > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant m$. Thus, we have obtained a contradiction in Case 2.

We have thus found a contradiction in each of the two Cases 1 and 2. Thus, we always have a contradiction. This completes the proof of Proposition 54 **(f)**. □

Remark 55. Here are some additional observations on Proposition 54, which we will not need (and thus will not prove):

[14] because of (18)

(a) The set partition $\{E_1, E_2, \ldots, E_m\}$ constructed in Proposition 54 **(b)** does not depend on the choice of $e_1, e_2, \ldots, e_m$. Indeed, $E_1, E_2, \ldots, E_m$ are precisely the maximal (with respect to inclusion) subsets F of E satisfying
$(d(a,b) < \alpha$ for any distinct $a, b \in F)$.

(b) Applying Proposition 54 iteratively, we can see that a $\mathbb{V}$-ultra triple (E, w, d) with finite E has a recursive structure governed by a tree. This idea is not new; see [13] and [18, §2–§3] for related results.

9 Valadic representation of $\mathbb{V}$-ultra triples

For the whole Section 9, we fix a field $\mathbb{K}$, and we let $\mathbb{V}_{\geqslant 0}$, $\mathbb{L}$, $\mathbb{L}_+$ and t_α be as in Section 6. We also recall Definition 39.

Definition 56. Let $\gamma \in \mathbb{V}$ and $u \in \mathbb{L}$. We say that a valadic $\mathbb{V}$-ultra triple (E, w, d) is (γ, u)*-positioned* if

$$E \subseteq u + t_{-\gamma}\mathbb{L}_+.$$

In other words, a valadic $\mathbb{V}$-ultra triple (E, w, d) is (γ, u)-positioned if and only if each element of E has the form $u + \sum\limits_{\substack{\beta\in\mathbb{V};\\ \beta\geqslant-\gamma}} p_\beta t_\beta$ for some $p_\beta \in \mathbb{K}$.

Theorem 57. *Let (E, w, d) be a $\mathbb{V}$-ultra triple such that the set E is finite. Let $\gamma \in \mathbb{V}$ be such that*

$$d(a,b) \leqslant \gamma \qquad \text{for each } (a,b) \in E \underline{\times} E. \tag{19}$$

Let $u \in \mathbb{L}$.

Assume that $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. Then, there exists a (γ, u)-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d).

Proof of Theorem 57. We proceed by strong induction on $|E|$.

If $|E| = 1$, then this is clear (just take the obvious valadic $\mathbb{V}$-ultra triple on the set $\{u\} \subseteq \mathbb{L}$, which is clearly (γ, u)-positioned). The case $|E| = 0$ is even more obvious. Thus, WLOG assume that $|E| > 1$. Thus, the set $E \underline{\times} E$ is nonempty. Hence, $d(E \underline{\times} E)$ is a nonempty finite subset of $\mathbb{V}$, and therefore has a largest element. In other words, $\max(d(E \underline{\times} E))$ is well-defined.

Let $\alpha = \max(d(E \underline{\times} E))$. Thus, $\alpha \leqslant \gamma$ (by (19)).

Pick any maximum-size α-clique, and write it in the form $\{e_1, e_2, \ldots, e_m\}$ for some distinct elements $e_1, e_2, \ldots, e_m$ of E. For each $i \in \{1, 2, \ldots, m\}$, let E_i be the open ball $B^\circ_\alpha(e_i)$, and let (E_i, w_i, d_i) be the $\mathbb{V}$-ultra triple $\left(E_i, w\mid_{E_i}, d\mid_{E_i \underline{\times} E_i}\right)$.

Then, Proposition 54 **(a)** shows that $m > 1$. Moreover, Proposition 54 **(b)** shows that the m sets $E_1, E_2, \ldots, E_m$ form a set partition of E. Hence, $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$ (an internal disjoint union). Each set E_i is the open ball $B^\circ_\alpha(e_i)$ and thus contains e_i.

Let $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ for each $i \in \{1, 2, \ldots, m\}$. Then,

$$|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d) = \max\{m, n_1, n_2, \ldots, n_m\} \qquad (\text{by Proposition 54 } \textbf{(f)}).$$

For any subset Z of $\mathbb{L}$, we define a distance function $\overline{d}_Z : Z \underline{\times} Z \to \mathbb{V}$ by setting

$$\overline{d}_Z (a, b) = -\operatorname{ord}(a-b) \qquad \text{for all } (a,b) \in Z \underline{\times} Z. \tag{20}$$

Note that the distance function of any valadic $\mathbb{V}$-ultra triple is precisely $\overline{d}_Z$, where Z is the ground set of this $\mathbb{V}$-ultra triple.

We have $|\mathbb{K}| \geqslant \max\{m, n_1, n_2, \ldots, n_m\} \geqslant m$. Hence, there exist m distinct elements $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $\mathbb{K}$. Fix m such elements. Define m elements $u_1, u_2, \ldots, u_m$ of $\mathbb{L}$ by

$$u_i = u + \lambda_i t_{-\alpha} \qquad \text{for each } i \in \{1, 2, \ldots, m\}. \tag{21}$$

Let $\mathbb{L}_{++}$ denote the $\mathbb{K}$-submodule of $\mathbb{L}_+$ generated by t_δ for all positive $\delta \in \mathbb{V}$. (Of course, $\delta \in \mathbb{V}$ is said to be positive if and only if $\delta > 0$.) It is easy to see that $\mathbb{L}_{++}$ is an ideal of $\mathbb{L}_+$. (Actually, $\mathbb{L}_{++} = \operatorname{Ker}\pi$, where π is as defined in Lemma 43.) Hence, $\mathbb{L}_{++}\mathbb{L}_+ \subseteq \mathbb{L}_{++}$.

Let $i \in \{1, 2, \ldots, m\}$. We shall work under the assumption that $|E_i| > 1$; we will later explain how to proceed without it.

The set $E_i \underline{\times} E_i$ is nonempty (since $|E_i| > 1$) and finite (since E_i is finite). Hence, the set $d(E_i \underline{\times} E_i)$ is nonempty and finite as well. Thus, it has a well-defined largest element $\max(d(E_i \underline{\times} E_i))$. Let us denote this largest element by β.

Proposition 54 **(d)**, shows that any element of $d(E_i \underline{\times} E_i)$ is $< \alpha$. Hence, in particular, $\beta < \alpha$ (since β is the largest element of $d(E_i \underline{\times} E_i)$). Thus, $\alpha - \beta > 0$ and therefore $t_{\alpha-\beta} \in \mathbb{L}_{++}$. Hence,

$$t_{-\beta} = t_{-\alpha} \underbrace{t_{\alpha-\beta}}_{\in \mathbb{L}_{++}} \in t_{-\alpha}\mathbb{L}_{++}. \tag{22}$$

We have

$$d_i(a,b) \leqslant \beta \qquad \text{for each } (a,b) \in E_i \underline{\times} E_i$$

(because for each $(a,b) \in E_i \underline{\times} E_i$, we have $d_i(a,b) = d(a,b)$ (since $d_i = d\mid_{E_i \underline{\times} E_i}$) and thus $d_i(a,b) \in d(E_i \underline{\times} E_i)$, so that $d_i(a,b) \leqslant \max(d(E_i \underline{\times} E_i)) = \beta$). Also, $|E_i| < |E|$ (by Proposition 54 **(c)**) and

$$|\mathbb{K}| \geqslant \max\{m, n_1, n_2, \ldots, n_m\} \geqslant n_i = \operatorname{mcs}(E_i, w_i, d_i).$$

Hence, the induction hypothesis shows that we can apply Theorem 57 to β, u_i and (E_i, w_i, d_i) instead of γ, u and (E, w, d). We thus conclude that there exists a (β, u_i)-positioned valadic $\mathbb{V}$-ultra triple $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$ isomorphic to (E_i, w_i, d_i). Consider this $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$. The $\mathbb{V}$-ultra triple (E_i, w_i, d_i) is isomorphic to $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$ (since $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$ is isomorphic to (E_i, w_i, d_i), but being isomorphic is a symmetric relation). In other words, there exists an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$. Consider this f_i. Since the $\mathbb{V}$-ultra triple $(\overline{E}_i, \overline{w}_i, \overline{d}_i)$ is (β, u_i)-positioned, we have

$$\overline{E}_i \subseteq u_i + \underbrace{t_{-\beta}}_{\substack{\in t_{-\alpha}\mathbb{L}_{++} \\ \text{(by (22))}}} \mathbb{L}_+ \subseteq u_i + t_{-\alpha} \underbrace{\mathbb{L}_{++}\mathbb{L}_+}_{\subseteq \mathbb{L}_{++}} \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}.$$

Thus, we have found a $\mathbb{V}$-ultra triple (E_i, w_i, d_i) and a valadic $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ satisfying

$$\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++},$$

and an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. We have done so assuming that $|E_i| > 1$; but this is even easier when $|E_i|$ is $\leqslant 1$ instead[15].

Forget that we fixed i. Thus, for each $i \in \{1, 2, \ldots, m\}$, we have constructed a $\mathbb{V}$-ultra triple (E_i, w_i, d_i) and a valadic $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ satisfying

$$\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$$

and an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$.

For later use, let us observe the following:

> *Claim 1:* Let i and j be two distinct elements of $\{1, 2, \ldots, m\}$. Let $a \in \overline{E}_i$ and $b \in \overline{E}_j$. Then, $a - b \neq 0$ and $\operatorname{ord}(a - b) = -\alpha$.

[*Proof of Claim 1:* From $i \neq j$, we conclude that λ_i and λ_j are two distinct elements of $\mathbb{K}$ (since $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct elements of $\mathbb{K}$). Hence, $\lambda_i - \lambda_j$ is a nonzero element of $\mathbb{K}$.

We have $a \in \overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$, so that $a - u_i \in t_{-\alpha}\mathbb{L}_{++}$. Similarly, $b - u_j \in t_{-\alpha}\mathbb{L}_{++}$. Also, (21) yields $u_i = u + \lambda_i t_{-\alpha}$. Likewise, $u_j = u + \lambda_j t_{-\alpha}$. Subtracting the latter two equalities from one another, we obtain

$$u_i - u_j = (u + \lambda_i t_{-\alpha}) - (u + \lambda_j t_{-\alpha}) = (\lambda_i - \lambda_j)\, t_{-\alpha}.$$

Now,

$$\begin{aligned} a - b &= \underbrace{(u_i - u_j)}_{=(\lambda_i - \lambda_j)t_{-\alpha}} + \underbrace{(a - u_i)}_{\in t_{-\alpha}\mathbb{L}_{++}} - \underbrace{(b - u_j)}_{\in t_{-\alpha}\mathbb{L}_{++}} \\ &\in (\lambda_i - \lambda_j)\, t_{-\alpha} + \underbrace{t_{-\alpha}\mathbb{L}_{++} - t_{-\alpha}\mathbb{L}_{++}}_{\subseteq t_{-\alpha}\mathbb{L}_{++}} \subseteq (\lambda_i - \lambda_j)\, t_{-\alpha} + t_{-\alpha}\mathbb{L}_{++}. \end{aligned}$$

In other words, $a - b$ can be written in the form

$$a - b = (\lambda_i - \lambda_j)\, t_{-\alpha} + (\text{a } \mathbb{K}\text{-linear combination of } t_\eta \text{ with } \eta > -\alpha)$$

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[15] *Proof.* Assume that $|E_i| \leqslant 1$. Since e_i is an element of E_i (because $E_i = B_\alpha^\circ(e_i)$), we thus have $E_i = \{e_i\}$. Now, set $\overline{E}_i = \{u_i\}$; let $\overline{w}_i : \overline{E}_i \to \mathbb{V}$ be the map that sends u_i to $w_i(e_i)$; let $\overline{d}_i : \overline{E}_i \underline{\times} \overline{E}_i \to \mathbb{V}$ be the distance function $\overline{d}_{\{u_i\}}$; and let $f_i : E_i \to \overline{E}_i$ be the map that sends e_i to u_i. Then, it is easy to see that $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is a valadic $\mathbb{V}$-ultra triple satisfying $\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$ (since $\overline{E}_i = \{u_i\} = u_i + \underbrace{0}_{\subseteq t_{-\alpha}\mathbb{L}_{++}} \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$), and that the map $f_i : E_i \to \overline{E}_i$ is an isomorphism of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ (since $E_i = \{e_i\}$ and $\overline{E}_i = \{u_i\}$, so that the maps d_i and $\overline{d}_i$ both have no values, whereas the map $\overline{w}_i$ is defined in such a way that $\overline{w}_i(u_i) = w_i(e_i)$ and thus $\overline{w}_i \circ f_i = w_i$). Thus, everything we constructed above still exists when $|E_i| \leqslant 1$.

(since the elements of $t_{-\alpha}\mathbb{L}_{++}$ are precisely the $\mathbb{K}$-linear combinations of t_η with $\eta > -\alpha$).

From this, we obtain two things: First, we obtain that $[t_{-\alpha}](a-b) = \lambda_i - \lambda_j$ (since $\lambda_i - \lambda_j \in \mathbb{K}$), hence $[t_{-\alpha}](a-b) = \lambda_i - \lambda_j \neq 0$ (since $\lambda_i - \lambda_j$ is nonzero), and thus $a - b \neq 0$. Furthermore, we obtain that $\operatorname{ord}(a-b) = -\alpha$ (again since $\lambda_i - \lambda_j \neq 0$). Thus, Claim 1 is proved.]

Let $\overline{E}$ denote the subset

$$\overline{E}_1 \cup \overline{E}_2 \cup \cdots \cup \overline{E}_m$$

of $\mathbb{L}$. Note that the sets $\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m$ are disjoint[16]. Hence, $\overline{E} = \overline{E}_1 \sqcup \overline{E}_2 \sqcup \cdots \sqcup \overline{E}_m$ (an internal disjoint union). Moreover, each $i \in \{1, 2, \ldots, m\}$ satisfies

$$\begin{aligned}\overline{E}_i &\subseteq \underbrace{u_i}_{\substack{=u+\lambda_i t_{-\alpha} \\ \text{(by (21))}}} + t_{-\alpha}\mathbb{L}_{++} = u + \underbrace{\lambda_i t_{-\alpha} + t_{-\alpha}\mathbb{L}_{++}}_{=t_{-\alpha}(\lambda_i + \mathbb{L}_{++})} = u + \underbrace{t_{-\alpha}}_{\substack{\in t_{-\gamma}\mathbb{L}_+ \\ \text{(since } -\alpha \geqslant -\gamma \\ \text{(because } \alpha \leqslant \gamma))}} \underbrace{(\lambda_i + \mathbb{L}_{++})}_{\substack{\subseteq \mathbb{L}_+ \\ \text{(since } \lambda_i \in \mathbb{K} \subseteq \mathbb{L}_+)}} \\ &\subseteq u + t_{-\gamma} \underbrace{\mathbb{L}_+ \mathbb{L}_+}_{\subseteq \mathbb{L}_+} \subseteq u + t_{-\gamma}\mathbb{L}_+. \end{aligned} \tag{23}$$

Now,

$$\overline{E} = \overline{E}_1 \cup \overline{E}_2 \cup \cdots \cup \overline{E}_m = \bigcup_{i=1}^{m} \underbrace{\overline{E}_i}_{\substack{\subseteq u + t_{-\gamma}\mathbb{L}_+ \\ \text{(by (23))}}} \subseteq \bigcup_{i=1}^{m} (u + t_{-\gamma}\mathbb{L}_+) \subseteq u + t_{-\gamma}\mathbb{L}_+.$$

Now, recall that $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$ and $\overline{E} = \overline{E}_1 \sqcup \overline{E}_2 \sqcup \cdots \sqcup \overline{E}_m$. Thus, we can glue the bijections $f_i : E_i \to \overline{E}_i$ together to a single bijection

$$\begin{aligned} f : E &\to \overline{E}, \\ a &\mapsto f_i(a), \qquad \text{where } i \in \{1, 2, \ldots, m\} \text{ is such that } a \in E_i. \end{aligned}$$

Consider this f. Define a weight function $\overline{w} : \overline{E} \to \mathbb{V}$ by setting $\overline{w} = w \circ f^{-1}$. Then, $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$ is a (γ, u)-positioned valadic $\mathbb{V}$-ultra triple. (It is (γ, u)-positioned, since $\overline{E} \subseteq u + t_{-\gamma}\mathbb{L}_+$.) Moreover, it is easy to see that every $i, j \in \{1, 2, \ldots, m\}$ and any two distinct elements $a \in \overline{E}_i$ and $b \in \overline{E}_j$ satisfy

$$\overline{d}_{\overline{E}}(a, b) = \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i}(a, b), & \text{if } i = j \end{cases} \tag{24}$$

[16] *Proof.* Assume the contrary. Thus, there exist some distinct $i, j \in \{1, 2, \ldots, m\}$ such that $\overline{E}_i \cap \overline{E}_j \neq \varnothing$. Consider these i and j. There exists some $a \in \overline{E}_i \cap \overline{E}_j$ (since $\overline{E}_i \cap \overline{E}_j \neq \varnothing$). Consider this a. We have $a \in \overline{E}_i \cap \overline{E}_j \subseteq \overline{E}_i$ and similarly $a \in \overline{E}_j$. Hence, Claim 1 (applied to $b = a$) yields $a - a \neq 0$ and $\operatorname{ord}(a - a) = -\alpha$. But $a - a \neq 0$ clearly contradicts $a - a = 0$. This contradiction proves that our assumption was false, qed.

[17]. From this, it is easy to see that

$$\overline{d}_{\overline{E}}(f(a), f(b)) = d(a,b) \qquad \text{for all } (a,b) \in E \underline{\times} E \tag{25}$$

[18]. Moreover, $\overline{w} \circ f = w$ (since $\overline{w} = w \circ f^{-1}$). Hence, the bijection $f : E \to \overline{E}$ is an isomorphism of $\mathbb{V}$-ultra triples from (E, w, d) to $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$. Hence, there exists a (γ, u)-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d) (namely, $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$). This proves Theorem 57 for our (E, w, d). Thus, the induction step is complete, and Theorem 57 is proven. ☐

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[17] *Proof of (24):* Let $i, j \in \{1, 2, \ldots, m\}$, and let $a \in \overline{E}_i$ and $b \in \overline{E}_j$ be two distinct elements. We must prove (24). It clearly suffices to verify the equality

$$\operatorname{ord}(a-b) = \begin{cases} -\alpha, & \text{if } i \neq j; \\ \operatorname{ord}(a-b), & \text{if } i = j \end{cases}$$

(since both $\overline{d}_{\overline{E}}$ and $\overline{d}_{\overline{E}_i}$ are given by the formula (20)). This equality is obviously true in the case when $i = j$; on the other hand, it follows from Claim 1 in the case when $i \neq j$. Hence, (24) is proved.

[18] *Proof of (25):* Let $(a,b) \in E \underline{\times} E$. Thus, a and b are two distinct elements of E. Let $i, j \in \{1, 2, \ldots, m\}$ be such that $a \in E_i$ and $b \in E_j$. (These i and j clearly exist, because a and b both belong to $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$.) Then, the definition of f yields $f(a) = f_i(a)$ and $f(b) = f_j(b)$. Hence,

$$\begin{aligned} \overline{d}_{\overline{E}}(f(a), f(b)) &= \overline{d}_{\overline{E}}(f_i(a), f_j(b)) \\ &= \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i}(f_i(a), f_j(b)), & \text{if } i = j \end{cases} \end{aligned} \tag{26}$$

(by (24), applied to $f_i(a)$ and $f_j(b)$ instead of a and b). If $i \neq j$, then this becomes

$$\overline{d}_{\overline{E}}(f(a), f(b)) = \alpha = d(a,b) \qquad (\text{since Proposition 54 } \textbf{(e)} \text{ yields } d(a,b) = \alpha),$$

and thus (25) is proven in this case. Hence, for the rest of this proof of (25), we WLOG assume that $i = j$. Hence, $b \in E_j = E_i$ (since $j = i$). Recall that $f_i : E_i \to \overline{E}_i$ is an isomorphism of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. Hence,

$$\overline{d}_i(f_i(a), f_i(b)) = d_i(a,b) = d(a,b) \qquad (\text{by the definition of } d_i).$$

Also, $\overline{d}_i = \overline{d}_{\overline{E}_i}$ (since the $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is valadic). Now, (26) becomes

$$\begin{aligned} \overline{d}_{\overline{E}}(f(a), f(b)) &= \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i}(f_i(a), f_j(b)), & \text{if } i = j \end{cases} \\ &= \overline{d}_{\overline{E}_i}(f_i(a), f_j(b)) \qquad (\text{since } i = j) \\ &= \underbrace{\overline{d}_{\overline{E}_i}}_{=\overline{d}_i}(f_i(a), f_i(b)) \qquad (\text{since } j = i) \\ &= \overline{d}_i(f_i(a), f_i(b)) = d(a,b). \end{aligned}$$

Thus, (25) is proven.

10 Proof of the main theorem

We can now prove Theorem 27 (from which we will immediately obtain Theorem 20).

Proof of Theorem 27. Pick some $\gamma \in \mathbb{V}$ such that

$$d(a,b) \leqslant \gamma \qquad \text{for each } (a,b) \in E \underline{\times} E.$$

(Such a γ clearly exists, since the set $E \underline{\times} E$ is finite and the set $\mathbb{V}$ is totally ordered.)

Theorem 57 (applied to $u = 0$) thus yields that there exists a $(\gamma, 0)$-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d). Consider this valadic $\mathbb{V}$-ultra triple, and denote it by (F, v, c). Let $\mathcal{E}$ denote the Bhargava greedoid of this $\mathbb{V}$-ultra triple (F, v, c). The set F is a subset of $\mathbb{L}$ (since (F, v, c) is a valadic $\mathbb{V}$-ultra triple) and is finite (since (F, v, c) is isomorphic to (E, w, d), whence $|F| = |E| < \infty$). Moreover, the distance function c of the $\mathbb{V}$-ultra triple (F, v, c) is the function d from Definition 39 (since (F, v, c) is a valadic $\mathbb{V}$-ultra triple). Hence, Theorem 41 (applied to (F, v, c) instead of (E, w, d)) yields that the Bhargava greedoid of (F, v, c) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. In other words, $\mathcal{E}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$ (since the Bhargava greedoid of (F, v, c) is $\mathcal{E}$).

But Proposition 48 yields that the Bhargava greedoids of (E, w, d) and (F, v, c) are isomorphic. In other words, the set systems $\mathcal{F}$ and $\mathcal{E}$ are isomorphic (since $\mathcal{F}$ and $\mathcal{E}$ are the Bhargava greedoids of (E, w, d) and (F, v, c), respectively).

Hence, Proposition 49 shows that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. This proves Theorem 27. ☐

Proof of Theorem 20. This follows from Theorem 27, since $|\mathbb{K}| \geqslant |E| \geqslant \operatorname{mcs}(E, w, d)$. ☐

11 Proof of Theorem 28

For the rest of Section 11, we fix a $\mathbb{V}$-ultra triple (E, w, d).

Our next goal is to prove Theorem 28. We have to build several tools to this purpose.

11.1 Closed balls

We will use a counterpart to the concept of open balls: the notion of *closed balls*. To wit, it is defined as follows:

Definition 58. Let $\alpha \in \mathbb{V}$ and $e \in E$. The *closed ball* $B_\alpha(e)$ is defined to be the subset

$$\{f \in E \mid f = e \text{ or else } d(f,e) \leqslant \alpha\}$$

of E.

Clearly, for each $\alpha \in \mathbb{V}$ and each $e \in E$, we have $e \in B_\alpha(e)$, so that the closed ball $B_\alpha(e)$ contains at least the element e.

Most properties of open balls have analogues for closed balls. In particular, here is an analogue of Proposition 51:

Proposition 59. *Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e,f) \leqslant \alpha$. Then, $B_\alpha(e) = B_\alpha(f)$.*

Proof. Replace all "$<$" signs by "$\leqslant$" signs in the above proof of Proposition 51. □

Next comes an analogue of Corollary 52:

Corollary 60. *Let $\alpha \in \mathbb{V}$ and $e \in E$. Let $f \in B_\alpha(e)$. Then, $B_\alpha(e) = B_\alpha(f)$.*

Proof. Replace all "$<$" signs by "$\leqslant$" signs in the above proof of Corollary 52. □

Knowing these properties, we can easily obtain the following lemma:

Lemma 61. *Let $\beta \in \mathbb{V}$. Let C be a β-clique.*

(a) *The closed balls $B_\beta(c)$ for all $c \in C$ are identical.*

Now, let $B = B_\beta(c)$ for some $c \in C$. Then:

(b) *We have $C \subseteq B$.*

(c) *For any distinct elements $p, q \in B$, we have $d(p,q) \leqslant \beta$.*

(d) *For any $n \in E \setminus B$ and any $p, q \in B$, we have $d(n,p) = d(n,q)$.*

(Intuitively, it helps to think of a clique C as an ultrametric analogue of a sphere, and of the set B constructed in Lemma 61 as being the whole closed ball whose boundary is this sphere. Of course, this must not be taken literally; in particular, every point in this ball serves as the "center" of this ball, so to speak.)

Proof of Lemma 61. We know that C is a β-clique. In other words, C is a subset of E such that

$$\text{any two distinct elements } a, b \in C \text{ satisfy } d(a,b) = \beta \tag{27}$$

(by the definition of a "β-clique").

(a) We must prove that $B_\beta(e) = B_\beta(f)$ for any $e, f \in C$.

So let $e, f \in C$. We must prove that $B_\beta(e) = B_\beta(f)$. We WLOG assume that $e \neq f$. Thus, (27) (applied to $a = e$ and $b = f$) yields $d(e,f) = \beta$. Thus, the "symmetry" axiom in Definition 11 yields that $d(f,e) = d(e,f) = \beta \leqslant \beta$. Hence, $f = e$ or else $d(f,e) \leqslant \beta$. In other words, $f \in B_\beta(e)$ (by the definition of $B_\beta(e)$). Hence, Corollary 60 (applied to $\alpha = \beta$) yields $B_\beta(e) = B_\beta(f)$.

Hence, we have proved $B_\beta(e) = B_\beta(f)$; thus, our proof of Lemma 61 **(a)** is complete.

In preparation for the proofs of parts **(b)**, **(c)** and **(d)**, let us observe the following:

We have defined B to be $B_\beta(c)$ for some $c \in C$. Consider this c. Thus, $B = B_\beta(c)$. Because of Lemma 61 **(a)**, we thus have

$$B = B_\beta(a) \qquad \text{for } \textbf{every } a \in C. \tag{28}$$

(b) Let $a \in C$. Then, $a \in B_\beta(a)$ (by the definition of $B_\beta(a)$, since $a = a$). Hence, $a \in B_\beta(a) = B$ (by (28)).

Forget that we fixed a. We thus have proved that $a \in B$ for each $a \in C$. In other words, $C \subseteq B$. This proves Lemma 61 **(b)**.

(c) Let $p, q \in B$ be distinct. We must prove that $d(p,q) \leqslant \beta$.

We have $p \in B = B_\beta(c)$. Hence, Corollary 60 (applied to $\alpha = \beta$, $e = c$ and $f = p$) yields $B_\beta(c) = B_\beta(p)$. Now, $q \in B = B_\beta(c) = B_\beta(p)$. In other words, we have $q = p$ or else $d(q,p) \leqslant \beta$ (by the definition of $B_\beta(p)$). Since $q = p$ is impossible (because p and q are distinct), we thus obtain $d(q,p) \leqslant \beta$. Thus, the "symmetry" axiom in Definition 11 yields that $d(p,q) = d(q,p) \leqslant \beta$. This proves Lemma 61 **(c)**.

(d) Let $n \in E \setminus B$ and $p, q \in B$. We must prove that $d(n,p) = d(n,q)$. If $p = q$, then this is obvious. Thus, we WLOG assume that $p \neq q$. Hence, Lemma 61 **(c)** yields $d(p,q) \leqslant \beta$.

We have $n \in E \setminus B$. In other words, $n \in E$ and $n \notin B$.

But we have $q \in B = B_\beta(c)$. Hence, Corollary 60 (applied to $\alpha = \beta$, $e = c$ and $f = q$) yields $B_\beta(c) = B_\beta(q)$. Now, $n \notin B = B_\beta(c) = B_\beta(q)$. In other words, we don't have ($n = q$ or else $d(n,q) \leqslant \beta$) (by the definition of $B_\beta(q)$). In other words, we have $n \neq q$ and $d(n,q) > \beta$. Thus, $d(n,q) > \beta \geqslant d(p,q)$ (since $d(p,q) \leqslant \beta$).

We have $p \neq n$ (since $p \in B$ but $n \notin B$) and similarly $q \neq n$. Hence, the elements n, p and q of E are distinct (since $p \neq n$ and $q \neq n$ and $p \neq q$). The ultrametric triangle inequality (applied to n, p and q instead of a, b and c) thus yields

$$d(n,p) \leqslant \max\{d(n,q), d(p,q)\} = d(n,q) \qquad (\text{since } d(n,q) > d(p,q)).$$

The same argument (with the roles of p and q interchanged) yields $d(n,q) \leqslant d(n,p)$. Combining these two inequalities, we obtain $d(n,p) = d(n,q)$. This proves Lemma 61 **(d)**. ☐

11.2 Exchange results for sets intersecting a ball

From now on, for the rest of Section 11, we assume that E is finite.

Corollary 62. *Let $\beta \in \mathbb{V}$. Let C be a β-clique. Let $B = B_\beta(c)$ for some $c \in C$.*

Let N be a subset of $E \setminus B$.

Let P and Q be two subsets of B such that $|P| = |Q|$. Then:

- ***(a)*** *We have* $\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) = \operatorname{PER}(Q) - \operatorname{PER}(P)$.
- ***(b)*** *Assume that the map $w : E \to \mathbb{V}$ is constant. Assume further that Q is a subset of C. Then,* $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$.

Proof of Corollary 62. Let $m = |P| = |Q|$. Let $p_1, p_2, \ldots, p_m$ be all the m elements of P (listed without repetition). Let $q_1, q_2, \ldots, q_m$ be all the m elements of Q (listed without repetition).

(a) We have

$$d(n, p_i) = d(n, q_i) \qquad \text{for each } n \in N \text{ and } i \in \{1, 2, \ldots, m\}. \tag{29}$$

[*Proof of (29):* Let $n \in N$ and $i \in \{1, 2, \ldots, m\}$. Then, $n \in N \subseteq E \setminus B$ and $p_i \in P \subseteq B$ and $q_i \in Q \subseteq B$. Thus, Lemma 61 **(d)** (applied to $p = p_i$ and $q = q_i$) yields $d(n, p_i) = d(n, q_i)$. This proves (29).]

The set N is disjoint from B (since N is a subset of $E \setminus B$), and thus disjoint from P as well (since $P \subseteq B$). Hence, the definition of perimeter yields

$$\begin{aligned} \operatorname{PER}(N \cup P) &= \operatorname{PER}(N) + \operatorname{PER}(P) + \sum_{n \in N} \underbrace{\sum_{p \in P} d(n, p)}_{\substack{= \sum_{i=1}^{m} d(n, p_i) \\ \text{(since } p_1, p_2, \ldots, p_m \text{ are all} \\ \text{the } m \text{ elements of } P \\ \text{(listed without repetition))}}} \\ &= \operatorname{PER}(N) + \operatorname{PER}(P) + \sum_{n \in N} \sum_{i=1}^{m} d(n, p_i). \end{aligned}$$

Likewise,

$$\operatorname{PER}(N \cup Q) = \operatorname{PER}(N) + \operatorname{PER}(Q) + \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i).$$

Subtracting the first of these two equalities from the second (and cancelling the $\operatorname{PER}(N)$ addends), we obtain

$$\begin{aligned} &\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) \\ &= \operatorname{PER}(Q) - \operatorname{PER}(P) + \underbrace{\sum_{n \in N} \sum_{i=1}^{m} d(n, q_i) - \sum_{n \in N} \sum_{i=1}^{m} d(n, p_i)}_{\substack{=0 \\ \text{(by (29))}}} \\ &= \operatorname{PER}(Q) - \operatorname{PER}(P). \end{aligned}$$

This proves Corollary 62 **(a)**.

(b) We know that C is a β-clique. But $p_1, p_2, \ldots, p_m$ are m distinct elements of P (by their definition). Hence, $p_1, p_2, \ldots, p_m$ are m distinct elements of B (since $P \subseteq B$). Thus, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then p_i and p_j are two distinct elements of B, and therefore satisfy

$$d(p_i, p_j) \leqslant \beta \tag{30}$$

(by Lemma 61 **(c)**, applied to $p = p_i$ and $q = p_j$).

On the other hand, $q_1, q_2, \ldots, q_m$ are m distinct elements of Q (by their definition). Hence, $q_1, q_2, \ldots, q_m$ are m distinct elements of C (since $Q \subseteq C$). Thus, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then q_i and q_j are two distinct elements of C, and therefore satisfy

$$d(q_i, q_j) = \beta \tag{31}$$

(since C is a β-clique).

Recall that $p_1, p_2, \ldots, p_m$ are all the m elements of P (listed without repetition). Hence, the definition of perimeter yields

$$\begin{aligned}\operatorname{PER}(P) &= \sum_{i=1}^{m} \underbrace{w(p_i)}_{\substack{=w(q_i) \\ \text{(since } w \text{ is constant)}}} + \sum_{1 \leqslant i<j \leqslant m} \underbrace{d(p_i, p_j)}_{\substack{\leqslant \beta \\ \text{(by (30))}}} \leqslant \sum_{i=1}^{m} w(q_i) + \sum_{1 \leqslant i<j \leqslant m} \underbrace{\beta}_{\substack{=d(q_i,q_j) \\ \text{(by (31))}}} \\ &= \sum_{i=1}^{m} w(q_i) + \sum_{1 \leqslant i<j \leqslant m} d(q_i, q_j) = \operatorname{PER}(Q)\end{aligned}$$

(by the definition of Q, since $q_1, q_2, \ldots, q_m$ are all the m elements of Q (listed without repetition)).

But Corollary 62 **(a)** yields

$$\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) = \operatorname{PER}(Q) - \operatorname{PER}(P) \geqslant 0$$

(since $\operatorname{PER}(P) \leqslant \operatorname{PER}(Q)$). In other words, $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$. This proves Corollary 62 **(b)**. □

Corollary 63. *Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Assume that the map $w : E \to \mathbb{V}$ is constant.*

Let $\beta \in \mathbb{V}$. Let C be a β-clique. Let $B = B_\beta(c)$ for some $c \in C$.

Let N be a subset of $E \setminus B$.

Let P and Q be two subsets of B such that $|P| = |Q|$ and $Q \subseteq C$ and $N \cup P \in \mathcal{F}$. Then, $N \cup Q \in \mathcal{F}$.

Proof of Corollary 63. Corollary 62 **(b)** yields $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$.

Also, the set N is disjoint from B (since $N \subseteq E \setminus B$), and thus is disjoint from P as well (since P is a subset of B). Hence, $|N \cup P| = |N| + |P|$. The same argument (applied to Q instead of P) shows that $|N \cup Q| = |N| + |Q|$. Hence, $|N \cup P| = |N| + \underbrace{|P|}_{=|Q|} = |N| + |Q| = |N \cup Q|$.

We know that $\mathcal{F}$ is the Bhargava greedoid of (E, w, d). In other words,

$$\mathcal{F} = \{A \subseteq E \mid A \text{ has maximum perimeter among all } |A|\text{-subsets of } E\} \tag{32}$$

(by Definition 16). Hence, from $N \cup P \in \mathcal{F}$, we conclude that the set $N \cup P$ has maximum perimeter among all $|N \cup P|$-subsets of E. In other words, the set $N \cup P$ has maximum perimeter among all $|N \cup Q|$-subsets of E (since $|N \cup P| = |N \cup Q|$). Since $N \cup Q$ is a further $|N \cup Q|$-subset of E, we thus conclude that $\operatorname{PER}(N \cup P) \geqslant \operatorname{PER}(N \cup Q)$. Combining this with $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$, we obtain $\operatorname{PER}(N \cup Q) = \operatorname{PER}(N \cup P)$.

In other words, the subsets $N \cup Q$ and $N \cup P$ of E have the same perimeter. Therefore, the set $N \cup Q$ has maximum perimeter among all $|N \cup Q|$-subsets of E (because the set $N \cup P$ has maximum perimeter among all $|N \cup Q|$-subsets of E). In view of (32), this entails that $N \cup Q \in \mathcal{F}$. This proves Corollary 63. $\square$

11.3 Gaussian elimination greedoids in terms of determinants

Next, we introduce some notations for matrices.

Definition 64. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A = (a_{i,j})_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m}$ be an $n \times m$-matrix. Let $i_1, i_2, \ldots, i_u$ be some elements of $\{1, 2, \ldots, n\}$; let $j_1, j_2, \ldots, j_v$ be some elements of $\{1, 2, \ldots, m\}$. Then, we define $\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_v} A$ to be the $u \times v$-matrix $\left(a_{i_x,j_y}\right)_{1 \leqslant x \leqslant u,\ 1 \leqslant y \leqslant v}$.

When $i_1 < i_2 < \cdots < i_u$ and $j_1 < j_2 < \cdots < j_v$, the matrix $\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_v} A$ can be obtained from A by crossing out all rows other than the i_1-th, the i_2-th, etc., the i_u-th row and crossing out all columns other than the j_1-th, the j_2-th, etc., the j_v-th column. Thus, in this case, $\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_v} A$ is called a *submatrix* of A.

Example 65. If $n = 3$ and $m = 4$ and $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \end{pmatrix}$, then $\operatorname{sub}_{1,3}^{1,3,4} A = \begin{pmatrix} a & c & d \\ i & k & \ell \end{pmatrix}$ (this is a submatrix of A) and $\operatorname{sub}_{2,3}^{3,2,1} A = \begin{pmatrix} g & f & e \\ k & j & i \end{pmatrix}$ (this is not, in general, a submatrix of A).

We can now describe Gaussian elimination greedoids in terms of determinants:

Lemma 66. *Let $n \in \mathbb{N}$. Let E be the set $\{1, 2, \ldots, n\}$.*

Let $m \in \mathbb{N}$ be such that $m \geqslant |E|$. Let $\mathbb{K}$ be a field. For each $e \in E$, let $v_e \in \mathbb{K}^m$ be a column vector. Let A be the $m \times n$-matrix whose columns (from left to right) are $v_1, v_2, \ldots, v_n$.

Let $\mathcal{G}$ be the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E}$.

Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in E$ be p distinct numbers. Let $I = \{i_1, i_2, \ldots, i_p\}$. Then,

$$I \in \mathcal{G} \text{ holds if and only if } \det\left(\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) \neq 0.$$

Proof of Lemma 66. We have $I = \{i_1, i_2, \ldots, i_p\}$. Thus, $|I| = p$ (since $i_1, i_2, \ldots, i_p$ are distinct) and $I \subseteq E$ (since $i_1, i_2, \ldots, i_p \in E$).

Define the maps π_k for all $k \in \{0, 1, \ldots, m\}$ as in Definition 5. Since $I \subseteq E$, we have $|I| \leqslant |E|$. Hence, $p = |I| \leqslant |E| \leqslant m$ (since $m \geqslant |E|$), so that $p \in \{0, 1, \ldots, m\}$. Thus, the map $\pi_p : \mathbb{K}^m \to \mathbb{K}^p$ is well-defined.

Write the $m \times n$-matrix A in the form $A = (a_{i,j})_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$. Then, the definition of $\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A$ yields

$$\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A = \left(a_{x,i_y}\right)_{1 \leqslant x \leqslant p,\ 1 \leqslant y \leqslant p}.$$

Thus, for each $k \in \{1, 2, \ldots, p\}$, we have

$$\left(\text{the } k\text{-th column of } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) = \begin{pmatrix} a_{1,i_k} \\ a_{2,i_k} \\ \vdots \\ a_{p,i_k} \end{pmatrix}. \tag{33}$$

The columns of the matrix A (from left to right) are $v_1, v_2, \ldots, v_n$. Thus, for each $\ell \in \{1, 2, \ldots, n\}$, we have

$$v_\ell = (\text{the } \ell\text{-th column of } A) = \begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{m,\ell} \end{pmatrix}$$

(since $A = (a_{i,j})_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$) and therefore

$$\pi_p(v_\ell) = \pi_p\left(\begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{m,\ell} \end{pmatrix}\right) = \begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{p,\ell} \end{pmatrix} \tag{34}$$

(by the definition of π_p). Hence, for each $k \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned} \pi_p(v_{i_k}) &= \begin{pmatrix} a_{1,i_k} \\ a_{2,i_k} \\ \vdots \\ a_{p,i_k} \end{pmatrix} && \text{(by (34), applied to } \ell = i_k\text{)} \\ &= \left(\text{the } k\text{-th column of } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) && \text{(by (33))}. \end{aligned}$$

In other words, $\pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p\left(v_{i_p}\right)$ are the columns of the matrix $\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A$.

We know that $\mathcal{G}$ is the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E}$. Thus, Definition 5 shows that

$$\mathcal{G} = \left\{F \subseteq E \ \mid \ \text{the family } \left(\pi_{|F|}(v_e)\right)_{e \in F} \in \left(\mathbb{K}^{|F|}\right)^F \text{ is linearly independent}\right\}.$$

Hence, we have the following chain of logical equivalences:

$$\begin{aligned} (I \in \mathcal{G}) &\iff \left(\text{the family } \left(\pi_{|I|}(v_e)\right)_{e \in I} \in \left(\mathbb{K}^{|I|}\right)^I \text{ is linearly independent}\right) \\ &\iff \left(\text{the family } \left(\pi_p(v_e)\right)_{e \in I} \in \left(\mathbb{K}^p\right)^I \text{ is linearly independent}\right) \\ &\qquad \text{(since } |I| = p\text{)} \end{aligned}$$

$$\iff \left(\text{the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p}) \text{ are linearly independent}\right)$$
$$\left(\begin{array}{c}\text{since the vectors } \pi_p(v_e) \text{ for } e \in I \text{ are precisely} \\ \text{the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p})\end{array}\right)$$
$$\iff \left(\text{the columns of the matrix } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A \text{ are linearly independent}\right)$$
$$\left(\begin{array}{c}\text{since the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p}) \text{ are} \\ \text{the columns of the matrix } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\end{array}\right)$$
$$\iff \left(\det\left(\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) \neq 0\right)$$

(since the columns of a square matrix are linearly independent if and only if this matrix is invertible). This proves Lemma 66. □

We can leverage Lemma 66 to obtain a criterion that, roughly speaking, says that if a Gaussian elimination greedoid over a field $\mathbb{K}$ contains a certain "constellation" (in an appropriate sense), then $|\mathbb{K}|$ must be $\geqslant$ to a certain value. Namely:

Lemma 67. *Let $\mathbb{K}$ be a field. Let E be a finite set. Let $\mathcal{F}$ be the Gaussian elimination greedoid of a vector family $(v_e)_{e\in E}$ over $\mathbb{K}$. Let N and C be two disjoint subsets of E. Assume that the following three conditions hold:*

(i) *For any $i \in C$, we have $N \cup \{i\} \in \mathcal{F}$.*

(ii) *For any distinct $i, j \in C$, we have $N \cup \{i, j\} \in \mathcal{F}$.*

(iii) *For any $p \in N$ and any distinct $i, j \in C$, we have $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$.*

Then, $|\mathbb{K}| \geqslant |C|$.

Proof of Lemma 67. Let $m \in \mathbb{N}$ be such that the vectors v_e belong to $\mathbb{K}^m$. Then, $m \geqslant |E|$ (since otherwise, the Gaussian elimination greedoid of $(v_e)_{e\in E}$ would not be well-defined).

Clearly, the claim we are proving will not change if we rename the elements of E. Thus, we WLOG assume that $E = \{1, 2, \ldots, n\}$ and $N = \{1, 2, \ldots, r\}$ for some nonnegative integers $r \leqslant n$ (since $N \subseteq E$). Thus, $E \setminus N = \{r+1, r+2, \ldots, n\}$ and $n = |E|$ and $r = |N|$.

Since the subsets N and C of E are disjoint, we have $C \subseteq E \setminus N = \{r+1, r+2, \ldots, n\}$.

We must prove that $|\mathbb{K}| \geqslant |C|$. If $|C| \leqslant 1$, then this is obvious (since $|\mathbb{K}| \geqslant 1$). Hence, we WLOG assume that $|C| > 1$ from now on. As a consequence, we easily see that $r + 2 \leqslant n$, and therefore $r + 2 \leqslant n = |E| \leqslant m$.

Let A be the $m \times n$-matrix whose columns (from left to right) are $v_1, v_2, \ldots, v_n$. Write this matrix A in the form $A = (a_{i,j})_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n}$.

We now make a few observations:

Claim 1: We have $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \neq 0$ for each $i \in C$.

[*Proof of Claim 1:* Let $i \in C$. Then, $i \in C \subseteq \{r+1, r+2, \ldots, n\}$. Thus, the $r+1$ numbers $1, 2, \ldots, r, i$ are distinct. Also, $N \cup \{i\} \in \mathcal{F}$ (by condition **(i)** in Lemma 67). But $N \cup \{i\} = \{1, 2, \ldots, r, i\}$ (since $N = \{1, 2, \ldots, r\}$). Thus, Lemma 66 (applied to $\mathcal{F}$, $N \cup \{i\}$, $r+1$ and $(1, 2, \ldots, r, i)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $N \cup \{i\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \neq 0$. Thus, we have $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \neq 0$ (since $N \cup \{i\} \in \mathcal{F}$). This proves Claim 1.]

Now, for each $i \in C$, we define a scalar $r_i \in \mathbb{K}$ by

$$r_i = \frac{a_{r+2,i}}{\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right)}. \tag{35}$$

This is well-defined, since Claim 1 yields $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \neq 0$ (and since $r+2 \leqslant m$).

Claim 2: The scalars r_i for all $i \in C$ are distinct.

[*Proof of Claim 2:* Let us fix two distinct $i, j \in C$. We must prove that $r_i \neq r_j$.

We have $i, j \in C \subseteq \{r+1, r+2, \ldots, n\}$. Thus, the $r+2$ numbers $1, 2, \ldots, r, i, j$ are distinct (since i and j are distinct). Also, $N \cup \{i, j\} \in \mathcal{F}$ (by condition **(ii)** in Lemma 67). But $N \cup \{i, j\} = \{1, 2, \ldots, r, i, j\}$ (since $N = \{1, 2, \ldots, r\}$). Thus, Lemma 66 (applied to $\mathcal{F}$, $N \cup \{i, j\}$, $r+2$ and $(1, 2, \ldots, r, i, j)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $N \cup \{i, j\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0$. Thus, we have

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0 \tag{36}$$

(since $N \cup \{i, j\} \in \mathcal{F}$).

Let us agree to use the following notation: If $p \in \{1, 2, \ldots, r\}$ is arbitrary, then the notation "$1, 2, \ldots, \widehat{p}, \ldots, r, i, j$" will denote the list $1, 2, \ldots, r, i, j$ with the entry p omitted (i.e., the list $1, 2, \ldots, p-1, p+1, p+2, \ldots, r, i, j$).

Now, let us use the Laplace expansion to expand the determinant of the $(r+2) \times (r+2)$-matrix $\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A$ along its last row. We thus obtain

$$\begin{aligned}
&\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \\
&= \sum_{p=1}^{r} (-1)^{(r+2)+p} a_{r+2,p} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \\
&\qquad + (-1)^{(r+2)+(r+1)} a_{r+2,i} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + (-1)^{(r+2)+(r+2)} a_{r+2,j} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right).
\end{aligned} \tag{37}$$

Now, let us simplify the entries in the $\sum_{p=1}^{r}$ sum on the right hand side of (37).

Let $p \in \{1, 2, \ldots, r\}$. Then, $p \in \{1, 2, \ldots, r\} = N$. Hence, $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$ (by condition **(iii)** in Lemma 67). But from $N \cup \{i, j\} = \{1, 2, \ldots, r, i, j\}$, we obtain $(N \cup \{i, j\}) \setminus \{p\} = \{1, 2, \ldots, r, i, j\} \setminus \{p\} = \{1, 2, \ldots, \widehat{p}, \ldots, r, i, j\}$ (since the $r+2$ numbers $1, 2, \ldots, r, i, j$ are distinct). Of course, the $r+1$ numbers $1, 2, \ldots, \widehat{p}, \ldots, r, i, j$ are distinct (since $1, 2, \ldots, r, i, j$ are distinct). Thus, Lemma 66 (applied to $\mathcal{F}$, $(N \cup \{i, j\}) \setminus \{p\}$, $r+1$ and $(1, 2, \ldots, \widehat{p}, \ldots, r, i, j)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $(N \cup \{i, j\}) \setminus \{p\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \neq 0$. Since we have $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$, we thus conclude that

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) = 0. \tag{38}$$

Forget that we fixed p. We thus have proved (38) for each $p \in \{1, 2, \ldots, r\}$. Now, (37) becomes

$$\begin{aligned}
&\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \\
&= \sum_{p=1}^{r} (-1)^{(r+2)+p} a_{r+2,p} \underbrace{\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right)}_{\substack{=0 \\ \text{(by (38))}}} \\
&\qquad + \underbrace{(-1)^{(r+2)+(r+1)}}_{=-1} a_{r+2,i} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + \underbrace{(-1)^{(r+2)+(r+2)}}_{=1} a_{r+2,j} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= - \underbrace{a_{r+2,i}}_{\substack{=\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \cdot r_i \\ \text{(by (35))}}} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) + \underbrace{a_{r+2,j}}_{\substack{=\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot r_j \\ \text{(by (35), applied} \\ \text{to } j \text{ instead of } i)}} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= -\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \cdot r_i \cdot \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot r_j \cdot \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot (r_j - r_i).
\end{aligned}$$

Hence,

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot (r_j - r_i) = \det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0$$

(by (36)). Thus, $r_j - r_i \neq 0$, so that $r_i \neq r_j$. This proves Claim 2.]

Claim 2 shows that the scalars r_i for all $i \in C$ are distinct. Thus, we have found $|C|$ distinct elements of $\mathbb{K}$ (namely, these r_i). Therefore, $|\mathbb{K}| \geqslant |C|$. Thus, Lemma 67 is proved. ☐

11.4 Proving the theorem

We can now prove Theorem 28:

Proof of Theorem 28. The definition of $\operatorname{mcs}(E, w, d)$ shows that $\operatorname{mcs}(E, w, d)$ is the maximum size of a clique of (E, w, d). Thus, there exists a clique C of size $\operatorname{mcs}(E, w, d)$. Consider this C.

We must prove that $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. In other words, we must prove that $|\mathbb{K}| \geqslant |C|$ (since $\operatorname{mcs}(E, w, d) = |C|$). If $|C| \leqslant 1$, then this is obvious (since $|\mathbb{K}| \geqslant 1$). Thus, we WLOG assume that $|C| > 1$. Hence, $|C| \geqslant 2$.

The set C is a clique. In other words, there exists a $\beta \in \mathbb{V}$ such that C is a β-clique. Consider this β.

Choose some $c \in C$. (We can do this, since $|C| \geqslant 2 > 0$.) Set $B = B_\beta(c)$. Lemma 61 **(b)** yields $C \subseteq B$, so that $B \supseteq C$.

A *secant set* shall mean a subset S of E satisfying $S \in \mathcal{F}$ and $|S \cap B| \geqslant 2$.

The set E itself satisfies $E \in \mathcal{F}$ (by Remark 19) and $|E \cap B| \geqslant 2$ (since $E \cap B = B \supseteq C$ and therefore $|E \cap B| \geqslant |C| \geqslant 2$). In other words, E is a secant set. Hence, there exists at least one secant set. Thus, there exists a secant set of smallest size (since there are only finitely many secant sets). Consider such a secant set, and call it S.

Thus, S is a secant set of smallest size. Hence, S is a secant set; in other words, S is a subset of E satisfying $S \in \mathcal{F}$ and $|S \cap B| \geqslant 2$. Since $S \cap B$ is a subset of S, we have $|S \cap B| \leqslant |S|$, so that $|S| \geqslant |S \cap B| \geqslant 2 > 0$. Hence, axiom **(ii)** in Definition 4 (applied to $B = S$) yields that there exists $b \in S$ such that $S \setminus \{b\} \in \mathcal{F}$ (since $\mathcal{F}$ is a greedoid). Consider this b.

We now shall show several claims:

Claim 1: We have $|(S \cap B) \setminus \{b\}| < 2$.

[*Proof of Claim 1:* The set $S \setminus \{b\}$ has smaller size than S (since $b \in S$), and thus cannot be a secant set (since S is a secant set of smallest size). In other words, $S \setminus \{b\}$ cannot be a subset of E satisfying $S \setminus \{b\} \in \mathcal{F}$ and $|(S \setminus \{b\}) \cap B| \geqslant 2$ (by the definition of "secant set"). Hence, we cannot have $|(S \setminus \{b\}) \cap B| \geqslant 2$ (because $S \setminus \{b\}$ is a subset of E satisfying $S \setminus \{b\} \in \mathcal{F}$). In other words, we have $|(S \setminus \{b\}) \cap B| < 2$.

But we have $(X \cap Y) \setminus Z = (X \setminus Z) \cap Y$ for any three sets X, Y and Z. Thus, $(S \cap B) \setminus \{b\} = (S \setminus \{b\}) \cap B$. Hence, $|(S \cap B) \setminus \{b\}| = |(S \setminus \{b\}) \cap B| < 2$. This proves Claim 1.]

Claim 2: We have $b \in S \cap B$.

[*Proof of Claim 2:* Claim 1 yields $|(S \cap B) \setminus \{b\}| < 2 \leqslant |S \cap B|$ (since $|S \cap B| \geqslant 2$). Hence, $|(S \cap B) \setminus \{b\}| \neq |S \cap B|$, so that $(S \cap B) \setminus \{b\} \neq S \cap B$. Therefore, $b \in S \cap B$. This proves Claim 2.]

Claim 3: We have $|S \cap B| = 2$.

[*Proof of Claim 3:* Claim 1 says that $|(S \cap B) \setminus \{b\}| < 2$. But Claim 2 yields $b \in S \cap B$. Thus, $|(S \cap B) \setminus \{b\}| = |S \cap B| - 1$. Hence, $|S \cap B| = \underbrace{|(S \cap B) \setminus \{b\}|}_{<2} + 1 < 2 + 1 = 3$. In other words, $|S \cap B| \leqslant 2$ (since $|S \cap B|$ is an integer). Combining this with $|S \cap B| \geqslant 2$, we find $|S \cap B| = 2$. This proves Claim 3.]

Claim 3 shows that the set $S \cap B$ has exactly two elements. One of these two elements is b (since Claim 2 says that $b \in S \cap B$); let a be the other element. Thus, $a \neq b$ and $S \cap B = \{a, b\}$. Hence, $\{a, b\} = S \cap B \subseteq B$. Also, $|\{a, b\}| = 2$ (since $a \neq b$).

Let $N = S \setminus B$. By general properties of sets, we have

$$S = (S \setminus B) \cup (S \cap B) \qquad \text{and} \qquad (S \setminus B) \cap (S \cap B) = \varnothing.$$

In view of $S \setminus B = N$ and $S \cap B = \{a, b\}$, we can rewrite these two equalities as

$$S = N \cup \{a, b\} \qquad \text{and} \qquad N \cap \{a, b\} = \varnothing.$$

Hence,

$$|S| = |N| + \underbrace{|\{a, b\}|}_{=2} = |N| + 2. \tag{39}$$

Moreover,

$$N = \underbrace{S}_{\subseteq E} \setminus B \subseteq E \setminus \underbrace{B}_{\supseteq C} \subseteq E \setminus C.$$

Hence, the two subsets N and C of E are disjoint.

Now, we have the following:

Claim 4: Let $i \in C$. Then, $N \cap \{i\} = \varnothing$ and $N \cup \{i\} \in \mathcal{F}$.

[*Proof of Claim 4:* If we had $i \in N$, then we would have $i \notin B$ (since $i \in N = S \setminus B$), which would contradict $i \in C \subseteq B$. Hence, we cannot have $i \in N$. Thus, $N \cap \{i\} = \varnothing$. It remains to prove that $N \cup \{i\} \in \mathcal{F}$.

If we had $b \in N$, then we would have $b \notin B$ (since $b \in N = S \setminus B$), which would contradict $b \in \{a, b\} \subseteq B$. Hence, we cannot have $b \in N$. Thus, $b \notin N$, so that $b \notin N \cup \{a\}$ as well (since $b \neq a$).

We have $S = N \cup \{a, b\} = (N \cup \{a\}) \cup \{b\}$, thus $S \setminus \{b\} = N \cup \{a\}$ (since $b \notin N \cup \{a\}$). Hence, $N \cup \{a\} = S \setminus \{b\} \in \mathcal{F}$. But $\{a\}$ and $\{i\}$ are subsets of B (since $a \in \{a, b\} \subseteq B$ and $i \in C \subseteq B$), and satisfy $|\{a\}| = |\{i\}|$ (since both $|\{a\}|$ and $|\{i\}|$ equal 1) and $\{i\} \subseteq C$ (since $i \in C$) and $N \cup \{a\} \in \mathcal{F}$. Hence, Corollary 63 (applied to $P = \{a\}$ and $Q = \{i\}$) yields $N \cup \{i\} \in \mathcal{F}$. This finishes the proof of Claim 4.]

Claim 5: Let $i, j \in C$ be distinct. Then, $N \cap \{i, j\} = \varnothing$ and $N \cup \{i, j\} \in \mathcal{F}$.

[*Proof of Claim 5:* If we had $i \in N$, then we would have $i \notin B$ (since $i \in N = S \setminus B$), which would contradict $i \in C \subseteq B$. Hence, we cannot have $i \in N$. In other words, $i \notin N$. Likewise, $j \notin N$. Combining $i \notin N$ with $j \notin N$, we obtain $N \cap \{i, j\} = \varnothing$. It remains to prove that $N \cup \{i, j\} \in \mathcal{F}$.

Note that $|\{a, b\}| = 2$ and $|\{i, j\}| = 2$ (since i and j are distinct). Hence, $|\{a, b\}| = 2 = |\{i, j\}|$. Also, $i, j \in C$, so that $\{i, j\} \subseteq C \subseteq B$.

We have $S = N \cup \{a, b\}$, thus $N \cup \{a, b\} = S \in \mathcal{F}$. But $\{a, b\}$ and $\{i, j\}$ are subsets of B (since $\{a, b\} \subseteq B$ and $\{i, j\} \subseteq B$), and satisfy $|\{a, b\}| = |\{i, j\}|$ and $\{i, j\} \subseteq C$ and $N \cup \{a, b\} \in \mathcal{F}$. Hence, Corollary 63 (applied to $P = \{a, b\}$ and $Q = \{i, j\}$) yields $N \cup \{i, j\} \in \mathcal{F}$. This finishes the proof of Claim 5.]

Claim 6: Let $p \in N$. Let $i, j \in C$ be distinct. Then, $(N \cup \{i,j\}) \setminus \{p\} \notin \mathcal{F}$.

[*Proof of Claim 6:* Assume the contrary. Hence, $(N \cup \{i,j\}) \setminus \{p\} \in \mathcal{F}$.

From $p \in N = S \setminus B$, we obtain $p \notin B$. Contrasting this with $i \in C \subseteq B$, we see that $i \neq p$.

Now, $i \in (N \cup \{i,j\}) \setminus \{p\}$ (since $i \in \{i,j\} \subseteq N \cup \{i,j\}$ and $i \neq p$). Combining this with $i \in C \subseteq B$, we see that i belongs to the intersection $((N \cup \{i,j\}) \setminus \{p\}) \cap B$. Likewise, j belongs to the same intersection. Since i and j are distinct, this shows that this intersection $((N \cup \{i,j\}) \setminus \{p\}) \cap B$ has at least two elements (viz., i and j). Thus, $|((N \cup \{i,j\}) \setminus \{p\}) \cap B| \geqslant 2$. Since we also know that $(N \cup \{i,j\}) \setminus \{p\} \in \mathcal{F}$, we thus conclude that $(N \cup \{i,j\}) \setminus \{p\}$ is a secant set (by the definition of "secant set"). Hence,

$$|(N \cup \{i,j\}) \setminus \{p\}| \geqslant |S| \tag{40}$$

(since S was defined to be a secant set of smallest size).

We have $|\{i,j\}| = 2$ (since i and j are distinct). But Claim 5 yields $N \cap \{i,j\} = \varnothing$, so that $|N \cup \{i,j\}| = |N| + \underbrace{|\{i,j\}|}_{=2} = |N| + 2 = |S|$ (by (39)). But $p \in N \subseteq N \cup \{i,j\}$ and thus $|(N \cup \{i,j\}) \setminus \{p\}| = |N \cup \{i,j\}| - 1 < |N \cup \{i,j\}| = |S|$. This contradicts (40). This contradiction shows that our assumption was false. Hence, Claim 6 is proved.]

But recall that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$ (by assumption). Let $(v_e)_{e \in E}$ be this vector family. Recall that N and C are two disjoint subsets of E. Moreover, the following facts hold:

- **(i)** For any $i \in C$, we have $N \cup \{i\} \in \mathcal{F}$ (by Claim 4).
- **(ii)** For any distinct $i, j \in C$, we have $N \cup \{i,j\} \in \mathcal{F}$ (by Claim 5).
- **(iii)** For any $p \in N$ and any distinct $i, j \in C$, we have $(N \cup \{i,j\}) \setminus \{p\} \notin \mathcal{F}$ (by Claim 6).

Hence, Lemma 67 shows that $|\mathbb{K}| \geqslant |C|$. This proves Theorem 28. □

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