A 2-stable family of triple systems

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Abstract

For many well-known families of triple systems $\mathcal{M}$, there are perhaps many near-extremal $\mathcal{M}$-free configurations that are far from each other in edit-distance. Such a property is called non-stable and is a fundamental barrier to determining the Turán number of $\mathcal{M}$. Liu and Mubayi gave the first finite example that is non-stable. In this paper, we construct another finite family of triple systems $\mathcal{M}$ such that there are two near-extremal $\mathcal{M}$-free configurations that are far from each other in edit-distance. We also prove its Andrásfai-Erdős-Sós type stability theorem: Every $\mathcal{M}$-free triple system whose minimum degree is close to the average degree of the extremal configurations is a subgraph of one of these two near-extremal configurations. As a corollary, our main result shows that the boundary of the feasible region of $\mathcal{M}$ has exactly two global maxima.

Mathematics Subject Classifications: 05C35, 05C65, 05D05

1 Introduction

1.1 Turán number and stability

For $r \geqslant 2$ an r-uniform hypergraph (henceforth r-graph) $\mathcal{H}$ is a collection of r-subsets of some finite set V. Given a family $\mathcal{F}$ of r-graphs we say $\mathcal{H}$ is $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The *Turán density* $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$. A family $\mathcal{F}$ is called *nondegenerate* if $\pi(\mathcal{F}) > 0$.

The study of $\mathrm{ex}(n, \mathcal{F})$ is perhaps the central topic in extremal combinatorics. Much is known about $\mathrm{ex}(n, \mathcal{F})$ when $r = 2$, and one of the most famous results in this regard is Turán's theorem, which states that for every integer $\ell \geqslant 2$ the Turán number $\mathrm{ex}(n, K_{\ell+1})$ is uniquely achieved by the balanced ℓ-partite graph on n vertices, which is called the Turán graph $T(n, \ell)$.

For $r \geqslant 3$ determining $\pi(\mathcal{F})$ for a family $\mathcal{F}$ of r-graphs is known to be notoriously hard in general. Indeed, the problem of determining $\pi(K_\ell^r)$ raised by Turán [32], where

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K_ℓ^r is the complete r-graph on ℓ vertices, is still wide open for all $\ell > r \geqslant 3$. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geqslant 3$ and \$1000 for the determination of all $\pi(K_\ell^r)$ with $\ell > r \geqslant 3$.

Conjecture 1 (Turán [32]). For every integer $\ell \geqslant 3$ we have $\pi(K_{\ell+1}^3) = 1 - 4/\ell^2$.

The case $\ell = 3$ above, which states that $\pi(K_4^3) = 5/9$ has generated a lot of interest and activity over the years [4, 8, 17]. The best bound that $\pi(K_4^3) \leqslant 0.561666$ was given by Razborov [30] using the Flag Algebra machinery.

For a family $\mathcal{F}$ of r-graphs, it is natural to ask for the "continuity" of the discrete $\mathcal{F}$-free r-graphs whose size is close to $\text{ex}(n, \mathcal{F})$. The seminal result in this regard is the Simonovits stability theorem, proved independently by Erdős and Simonovits [31].

Theorem 2 (Erdős, Simonovits [31]). *Fix $\ell \geqslant 2$. For every $\delta > 0$, there exists ϵ and $N_0 = N_0(\epsilon)$ such that the following holds for every $n > N_0$: if G is an n-vertex graph containing no copy of $K_{\ell+1}$ with at least $(1-\epsilon)|T(n,\ell)|$ edges, then G can be transformed to $T(n,\ell)$ by adding and deleting at most δn^2 edges.*

The stability phenomenon plays an important role in determining Turán number exactly. It was first used by Simonovits [31] for graphs, and by several authors [3, 10, 11, 13, 16, 27, 28] for r-graphs. More precisely, one can determine $\text{ex}(n, \mathcal{F})$ exactly by first determining the asymptotic value, then using a stability theorem to prove that any r-graph with the extremal number of edges is the unique extremal r-graph. This is especially valuable in extremal hypergraph theory, where exact results are rare, and any new approach gives insight to the governing phenomenon of the problems.

However, there are many Turán problems for hypergraphs that (perhaps) do not have the stability property. For example, Kostochka [17] showed that there are at least 2^{n-2} nonisomorphic extremal K_4^3-free constructions on $3n$ vertices (assuming Turán's Tetrahedron conjecture is true). The absence of stability seems to be a fundamental barrier in determining Turán numbers of some families. This motivates Mubayi [26] to make the following definition.

Definition 3 (***t***-stable). Let $r \geqslant 2$ and $t \geqslant 1$ be integers. A family $\mathcal{F}$ of r-graphs is t-stable if for every $m \in \mathbb{N}$ there exist r-graphs $\mathcal{G}_1(m), \ldots, \mathcal{G}_t(m)$ on m vertices such that the following holds. For every $\delta > 0$ there exist $\epsilon > 0$ and N_0 such that for all $n \geqslant N_0$ if $\mathcal{H}$ is an $\mathcal{F}$-free r-graph on n vertices with $|\mathcal{H}| > (1-\epsilon)\text{ex}(n, \mathcal{F})$ then $\mathcal{H}$ can be transformed to some $\mathcal{G}_i(n)$ by adding and removing at most δn^r edges. Say $\mathcal{F}$ is stable if it is 1-stable.

Denote by $\xi(\mathcal{F})$ the minimum integer t such that $\mathcal{F}$ is t-stable, and set $\xi(\mathcal{F}) = \infty$ if there is no such t. Call $\xi(\mathcal{F})$ *the stability number* of $\mathcal{F}$.

The classical Erdős–Stone–Simonovits theorem [7, 6] and Erdős–Simonovits stability theorem [31] imply that every nondegenerate family of graphs is stable. Families that are non-stable and whose Turán densities can be determined were constructed only very recently. In [21], Liu and Mubayi constructed the first finite 2-stable family of 3-graphs. Later in [22], Liu, Mubayi and Reiher constructed the first finite t-stable family of triple systems (3-graphs) for every integer $t \geqslant 3$. Recently, together with Liu and Mubayi, the authors [12] gave the first exact and stability results for a hypergraph Turán problem with infinitely many extremal constructions that are far from each other in edit-distance. Liu and Pikhurko [24] used another method to construct finite families with an infinite stability number.

In this paper, we give another finite family $\mathcal{M}$ of 3-graphs with $\xi(\mathcal{M}) = 2$ and give its Andrásfai–Erdős–Sós type [1] stability theorem. Note that an r-graph $\mathcal{H}$ is a *blowup* of an r-graph $\mathcal{G}$ if there exists a map $\psi\colon V(\mathcal{H}) \to V(\mathcal{G})$ so that $\psi(E) \in \mathcal{G}$ iff $E \in \mathcal{H}$. We say $\mathcal{H}$ is $\mathcal{G}$-*colorable* if there exists a map $\phi\colon V(\mathcal{H}) \to V(\mathcal{G})$ so that $\phi(E) \in \mathcal{G}$ for all $E \in \mathcal{H}$, and we call such a map ϕ a *homomorphism* from $\mathcal{H}$ to $\mathcal{G}$. In other words, $\mathcal{H}$ is $\mathcal{G}$-colorable if and only if $\mathcal{H}$ occurs as a subgraph in some blowup of $\mathcal{G}$. An r-graph $\mathcal{H}$ is called bipartite if $V(\mathcal{H})$ has a partition $A \cup B$ such that $\mathcal{H}[A] = \mathcal{H}[B] = \emptyset$. Let $\mathcal{G}^2_{12}$ be a 3-graph on 12 vertices whose complement is a perfect matching. The following is our main result.

Theorem 4. *There exists a finite family $\mathcal{M}$ of 3-graphs such that the following statements hold.*

(a) $\mathrm{ex}(n, \mathcal{M}) \leqslant n^3/8$ *for all positive integers n, and equality holds if and only if $12 \mid n$.*

(b) There exist constants $\epsilon > 0$ and N_0 such that the following holds for every integer $n \geqslant N_0$. Every n-vertex $\mathcal{M}$-free 3-graph with minimum degree at least $(3/8 - \epsilon)\, n^2$ is either bipartite or $\mathcal{G}^2_{12}$-colorable. In other words, $\xi(\mathcal{M}) = 2$.

1.2 Feasible region

Our result has an application on feasible region function which was introduced by Liu and Mubayi [20] to understand the extremal properties of $\mathcal{F}$-free hypergraphs beyond just the determination of $\pi(\mathcal{F})$. Given an r-graph $\mathcal{H}$ on n vertices, the *shadow* of $\mathcal{H}$ is defined as

$$\partial\mathcal{H} = \left\{ A \in \binom{V(\mathcal{H})}{r-1} \colon \text{there is } B \in \mathcal{H} \text{ such that } A \subset B \right\}.$$

The *edge density* of $\mathcal{H}$ is defined as $\rho(\mathcal{H}) = |\mathcal{H}|/\binom{v(\mathcal{H})}{r}$, and the *shadow density* of $\mathcal{H}$ is defined as $\rho(\partial\mathcal{H}) = |\partial\mathcal{H}|/\binom{v(\mathcal{H})}{r-1}$. For a family $\mathcal{F}$ the *feasible region* $\Omega(\mathcal{F})$ of $\mathcal{F}$ is the set of points $(x, y) \in [0,1]^2$ such that there exists a sequence of $\mathcal{F}$-free r-graphs $(\mathcal{H}_k)_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} v(\mathcal{H}_k) = \infty, \quad \lim_{k\to\infty} \rho(\partial\mathcal{H}_k) = x, \quad \text{and} \quad \lim_{k\to\infty} \rho(\mathcal{H}_k) = y.$$

The feasible region unifies and generalizes many classical problems such as the Kruskal–Katona theorem [14, 18] and the Turán problem. For some constant $c(\mathcal{F}) \in [0,1]$ the projection to the first coordinate,

$$\mathrm{proj}\Omega(\mathcal{F}) = \{x \colon \text{there is } y \in [0,1] \text{ such that } (x,y) \in \Omega(\mathcal{F})\},$$

is the interval $[0, c(\mathcal{F})]$. Moreover, there is a left-continuous almost everywhere differentiable function $g(\mathcal{F})\colon \mathrm{proj}\Omega(\mathcal{F}) \to [0,1]$ such that

$$\Omega(\mathcal{F}) = \left\{(x,y) \in [0, c(\mathcal{F})] \times [0,1] \colon 0 \leqslant y \leqslant g(\mathcal{F})(x)\right\}.$$

Let us call $g(\mathcal{F})$ the *feasible region function* of $\mathcal{F}$. It was shown in [20] that $g(\mathcal{F})$ is not necessarily continuous, and in [19], it was shown that $g(\mathcal{F})$ can have infinitely many local maxima even for some simple and natural families $\mathcal{F}$.

Using Theorem 4, we have the following result.

Theorem 5. *The set* $\mathrm{proj}\Omega(\mathcal{M}) = [0,1]$*, and* $g(\mathcal{M})(x) \leqslant 3/4$ *for all* $x \in [0,1]$*. Moreover,* $g(\mathcal{M})(x) = 3/4$ *if and only if* $x = 11/12$ *or* 1*.*

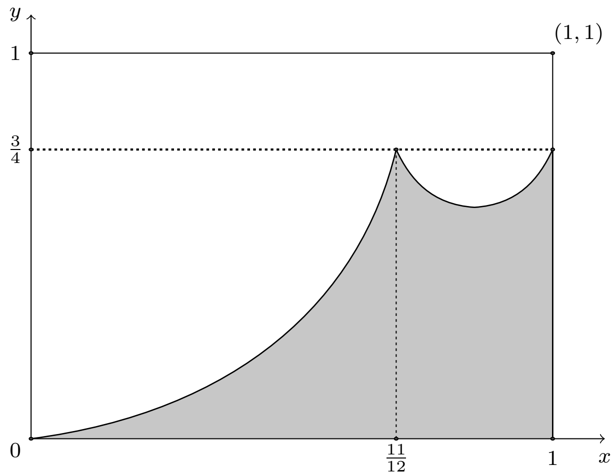

Figure 1: The function $g(\mathcal{M})$ has exactly two global maxima.

This paper is organized as follows. The following section gives the construction of $\mathcal{M}$. In Section 3 we determine the Turán number of $\mathcal{M}$ and prove Theorem 4 (a). In Section 4, we study the stability of $\mathcal{M}$ and prove Theorem 4 (b). The proof of Theorem 5 is presented in Section 5. Section 6 contains some concluding remarks.

2 Construction of $\mathcal{M}$

In this section, we will construct a finite family $\mathcal{M}$ of 3-graphs with $\mathrm{ex}(n, \mathcal{M}) \leqslant n^3/8$ and $\xi(\mathcal{M}) = 2$. At first, we give some definitions and notations. For positive integers n_1 and n_2 with $n_1 \leqslant n_2$, let $[n_1]\colon = \{1, \dots, n_1\}$ and $[n_1, n_2]\colon = [n_2] \setminus [n_1 - 1]$. We identify an r-graph $\mathcal{H}$ with its edge set, use $V(\mathcal{H})$ to denote its vertex set, and denote by $v(\mathcal{H})$ the size of $V(\mathcal{H})$. For a vertex $v \in V(\mathcal{H})$, the *link* $L_{\mathcal{H}}(v)$ of v in $\mathcal{H}$ is

$$L_{\mathcal{H}}(v) = \{A \in \partial\mathcal{H} \colon A \cup \{v\} \in \mathcal{H}\}.$$

The *degree* of v in $\mathcal{H}$ is $d_{\mathcal{H}}(v) = |L_{\mathcal{H}}(v)|$. Denote by $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ the minimum degree and maximum degree of $\mathcal{H}$, respectively. The *neighborhood* $N_{\mathcal{H}}(v)$ of v is defined as

$$N_{\mathcal{H}}(v) = \{u \in V(\mathcal{H}) \setminus \{v\} : \text{ there exists } e \in \mathcal{H} \text{ such that } \{u, v\} \subseteq e\}.$$

For a set $A \subseteq V(\mathcal{H})$ denote by $\mathcal{H}[A]$ the induced subgraph of $\mathcal{H}$ with A. We will omit the subscript $\mathcal{H}$ from our notations if it is clear from the context. For a graph G and two disjoint sets $X, Y \subseteq V(G)$ denote by $G[X, Y]$ the induced bipartite subgraph of G with two parts X and Y.

Let $\ell \geqslant r \geqslant 2$ and $\mathcal{K}^r_{\ell+1}$ be the collection of all r-graphs F with at most $\binom{\ell+1}{2}$ edges such that for some $(\ell+1)$-set S, which will be called the *core* of F, every pair $\{u, v\} \subset S$ is covered by an edge in F. We notice that $\mathcal{K}^r_{\ell+1}$ is a finite family. The *Fano plane*, hereafter denoted by Fano, is the unique hypergraph with 7 triples on 7 vertices in which every pair of vertices is contained in a unique triple, i.e., the 3-graph on the vertex set $[7]$ with the edge set

$$\{123, 345, 561, 174, 275, 376, 246\}.$$

The Turán density of a Fano was determined by De Caen and Füredi in [5]. Later Keevash and Sudakov [16] and, independently, Füredi and Simonovits [10] proved the stability theorem of a Fano by showing

Theorem 6 (Füredi and Simonovits [10]). *There exist $\gamma > 0$ and N_0 such that the following holds. If $\mathcal{H}$ is a Fano-free 3-graph on $n > N_0$ vertices with $\delta(\mathcal{H}) \geqslant (3/8 - \gamma)n^2$, then $\mathcal{H}$ is bipartite.*

Using Theorem 6, the Turán number of a Fano was determined for large n [10]. The complete determination of its Turán number was obtained by Bellmann and Reiher [2].

Theorem 7 (Bellmann and Reiher [2]). *For every integer $n \geqslant 7$, we have*

$$\mathrm{ex}(n, \mathrm{Fano}) = \frac{n-2}{2} \cdot \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Recall that $\mathcal{G}^2_{12}$ is the 3-graph on 12 vertices whose complement is a perfect matching. The following two 3-graphs play a key role in our proof.

Definition 8. Let $\mathcal{G}^1_n$ be a 3-graph on n vertices whose vertex set is partitioned into two parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, and whose edges consist of all triples intersecting both parts. For $n \geqslant 12$, let $\mathcal{G}^2_n$ be a 3-graph on n vertices which is a blowup of $\mathcal{G}^2_{12}$ with the maximum number of edges.

Remarks.

- Simple calculation shows that each part in $\mathcal{G}^2_n$ has size either $\lfloor n/12 \rfloor$ or $\lceil n/12 \rceil$.
- For $i = 1, 2$, let $g_i(n) = |\mathcal{G}^i_n|$. Then $\lim_{n\to\infty} g_i(n)/n^3 = 1/8$.
- Transforming $\mathcal{G}^1_n$ to $\mathcal{G}^2_n$ requires us to delete and add $\Omega(n^3)$ edges. Indeed, $\partial\mathcal{G}^1_n$ is a complete graph, whereas the clique number of $\partial\mathcal{G}^2_n$ is 12. By Turán's theorem, one must delete at least $(1 - \pi(K_{13}))\binom{n}{2} = \Omega(n^2)$ edges from $\partial\mathcal{G}^1_n$ to obtain a copy of $\partial\mathcal{G}^2_n$. Since every edge in $\partial\mathcal{G}^1_n$ is covered by $\Omega(n)$ edges in $\mathcal{G}^1_n$, we need to remove at least $\Omega(n^3)$ edges from $\mathcal{G}^1_n$ before getting $\mathcal{G}^2_n$.

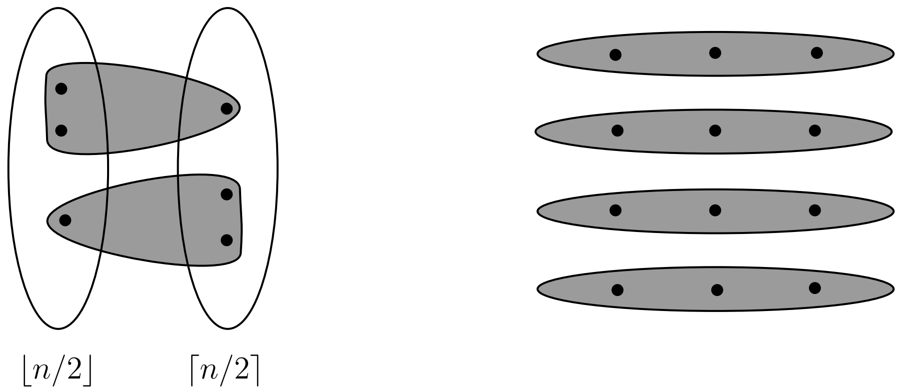

Figure 2: The 3-graph $\mathcal{G}_n^1$ and the complement of $\mathcal{G}_{12}^2$.

Both $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ has at most $n^3/8$ edges. Now we find a finite family $\mathcal{M}$ of 3-graphs such that both $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ are $\mathcal{M}$-free and the extremal constructions of $\mathcal{M}$ is close to either $\mathcal{G}_n^1$ or $\mathcal{G}_n^2$. For $i = 1, 2$, let $\mathcal{F}_\infty(\mathcal{G}_n^i)$ be the (infinite) family of all r-graphs F such that $\mathcal{G}_n^i$ is F-free for all positive integers n, i.e.,

$$\mathcal{F}_\infty(\mathcal{G}_n^i) = \left\{r\text{-graph } F\colon \mathcal{G}_n^i \text{ is } F\text{-free for all positive integers } n\right\}.$$

For every positive integer m, let $\mathcal{F}_m(\mathcal{G}_n^i)$ consist of all members of $\mathcal{F}_\infty(\mathcal{G}_n^i)$ with at most m vertices, i.e.,

$$\mathcal{F}_m(\mathcal{G}_n^i) = \left\{F \in \mathcal{F}_\infty(\mathcal{G}_n^i)\colon v(F) \leqslant m\right\}.$$

We set

$$\mathcal{M} = \mathcal{F}_{91}(\mathcal{G}_n^1) \cap \mathcal{F}_{91}(\mathcal{G}_n^2). \tag{1}$$

Remarks.

- Each member $\mathcal{J}$ of $\mathcal{K}_{13}^3$ is not contained in any blow up of $\mathcal{G}_{12}^2$. As $\nu(\mathcal{J}) \leqslant 91$, that is why we choice $m = 91$ in (1).
- $\mathcal{G}_n^1$ is Fano-free. So, if we can find $\mathcal{J} \in \mathcal{K}_{13}^3$ such that $\mathcal{J}$ contains a copy of Fano, then $\mathcal{J} \in \mathcal{M}$.

3 Turán number of $\mathcal{M}$

In this section we prove Theorem 4 (a) using a idea given by Liu, Mubayi and Reiher [23]. First we present some definitions and lemmas which will be used later. Let $\mathcal{H}$ be an r-graph with $V(\mathcal{H}) = [n]$. For $x = (x_1, \ldots, x_n)$ define the *multilinear polynomial* of $\mathcal{H}$ as

$$p_{\mathcal{H}}(x)\colon = \sum_{E \in \mathcal{H}} \prod_{i \in E} x_i.$$

Denote by Δ_{n-1} the standard $(n-1)$-dimensional simplex, i.e.,

$$\Delta_{n-1} = \left\{ x \in [0,1]^n \colon \sum_{i=1}^{n} x_i = 1 \right\}.$$

The *Lagrangian* of $\mathcal{H}$ is

$$\lambda(\mathcal{H})\colon = \max\{p_{\mathcal{H}}(x)\colon x \in \Delta_{n-1}\}.$$

Note that Δ_{n-1} is compact in $[0,1]^n$ and $p_{\mathcal{H}}(x)$ is continuous, so $\lambda(\mathcal{H})$ is well-defined.

The following lemma is about the relationship between $\lambda(\mathcal{H})$ and the maximum number of edges in a blowup of $\mathcal{H}$ (e.g. see Frankl and Füredi [9] or Keevash's survey [15, Section 3]).

Lemma 9 (Frankl and Füredi [9])**.** *Let $r \geqslant 2$ and $\mathcal{G}$ and $\mathcal{H}$ be two r-graphs. Suppose that $\mathcal{G}$ is a blowup of $\mathcal{H}$ with $v(\mathcal{G}) = n$. Then $|\mathcal{G}| \leqslant \lambda(\mathcal{H})n^r$.*

The following lemmas give Lagrangians of two hypergraphs by an easy calculation.

Lemma 10. *Let $\mathcal{H}$ be a complete bipartite 3-graph on n vertices. Then $\lambda(\mathcal{H}) < 1/8$.*

Proof. Suppose that $\mathcal{H}$ is partitioned into two parts A_1 and A_2 with $V(A_1) = [s]$ and $V(A_2) = [s+1, n]$ such that $\mathcal{H}[A_i] = \emptyset$ for $i = 1, 2$. For $x \in \Delta_{n-1}$, let $\sum_{i=1}^{s} x_i = a$ and $\sum_{i=s+1}^{n} x_i = 1 - a$. Then, by the Maclaurin's inequality, we have

$$\begin{aligned}
p_{\mathcal{H}}(x) &= \sum_{i=1}^{s} x_i \left(\sum_{\{j,k\} \subseteq [s,n]} x_j x_k \right) + \sum_{i=s+1}^{n} x_i \left(\sum_{\{j,k\} \subseteq [s]} x_j x_k \right) \\
&\leqslant a \binom{n-s}{2} \left(\frac{1-a}{n-s} \right)^2 + (1-a) \binom{s}{2} \left(\frac{a}{s} \right)^2 \\
&< \frac{1}{2} a (1-a)^2 + \frac{1}{2} (1-a) a^2 \\
&= \frac{1}{2} a (1-a) \\
&\leqslant \frac{1}{8},
\end{aligned}$$

which implies that $\lambda(\mathcal{H}) < 1/8$. □

Lemma 11. $\lambda(\mathcal{G}_{12}^2) = 1/8.$

Proof. For $x \in \Delta_{11}$, let $x_{3i-2} + x_{3i-1} + x_{3i} = a_i$ for $i \in [4]$. Then, we have

$$\begin{aligned}
p_{\mathcal{G}_{12}^2}(x) &= \sum_{\{i,j,k\} \subset [12]} x_i x_j x_k - (x_1x_2x_3 + x_4x_5x_6 + x_7x_8x_9 + x_{10}x_{11}x_{12}) \\
&= \sum_{i=1}^{4} \left(\prod_{j \in [4]\setminus\{i\}} a_j + (x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i}) \sum_{k \in [4]\setminus\{i\}} a_k \right) \\
&\leqslant \sum_{i=1}^{4} \left(\prod_{j \in [4]\setminus\{i\}} a_j + 3\left(\frac{a_i}{3}\right)^2 \sum_{k \in [4]\setminus\{i\}} a_k \right) \\
&= \frac{1}{3} \sum_{\{j,k\} \subset [4]} a_j a_k \\
&\leqslant 2 \left(\frac{\sum_{i=1}^{4} a_i}{4} \right)^2 \\
&= \frac{1}{8},
\end{aligned}$$

where the first and second inequalities both follow from the Maclaurin's inequality. Moreover, equality holds only if $a_1 = a_2 = a_3 = a_4 = 1/4$ and $x_i = 1/12$ for $i \in [12]$. □

For two r-graphs F and $\mathcal{H}$, recall that we say $f\colon V(F) \to V(\mathcal{H})$ is a homomorphism if $f(E) \in \mathcal{H}$ for all $E \in F$. We say $\mathcal{H}$ is *F-hom-free* if there is no homomorphism from F to $\mathcal{H}$. This is equivalent to say that every blowup of $\mathcal{H}$ is F-free. For a family $\mathcal{F}$ of r-graphs we say $\mathcal{H}$ is *$\mathcal{F}$-hom-free* if it is F-hom-free for all $F \in \mathcal{F}$. An r-graph F is 2-covered if every $\{u, v\} \subset V(F)$ is contained in some $E \in F$, and a family $\mathcal{F}$ is 2-*covered* if all $F \in \mathcal{F}$ are 2-covered. An easy observation is that if an r-graph F is 2-covered, then $\mathcal{H}$ is F-free if and only if it is F-hom-free.

Definition 12 (Blowup-invariant)**.** A family $\mathcal{F}$ of r-graphs is blowup-invariant if every $\mathcal{F}$-free r-graph is also $\mathcal{F}$-hom-free.

The following simple lemma is a special case of Lemma 15 [24], which is also an extension of Lemma 8 [29].

Lemma 13 (see Liu and Pikhurko [24])**.** *The family $\mathcal{M} = \mathcal{F}_{91}(\mathcal{G}_n^1) \cap \mathcal{F}_{91}(\mathcal{G}_n^2)$ is blowup-invariant.*

Let $\mathcal{H}$ be an r-graph and $\{u, v\} \subset V(\mathcal{H})$ be two non-adjacent vertices (i.e., no edge contains both u and v). We say u and v are *equivalent* if $L_{\mathcal{H}}(u) = L_{\mathcal{H}}(v)$. Otherwise we say they are *non-equivalent.* An equivalence class of $\mathcal{H}$ is a maximal vertex set in which every pair of vertices are equivalent. We say $\mathcal{H}$ is *symmetrized* if for any two non-equivalent vertices u, $v \in V(\mathcal{H})$ there is an edge of $\mathcal{H}$ containing both of them. In [23], Liu, Mubayi and Reiher summarized the well known method of Zykov [33] symmetrization for solving Turán problems into the following statement.

Theorem 14 (Liu, Mubayi and Reiher [23]). *Suppose that $\mathcal{F}$ is a blowup-invariant family of r-graphs. If $\mathfrak{H}$ denotes the class of all symmetrized $\mathcal{F}$-free r-graphs, then $ex(n,\mathcal{F}) = \mathfrak{h}(n)$ holds for every $n \in \mathbb{N}^+$, where $\mathfrak{h}(n) = \max\{|\mathcal{H}|: \mathcal{H} \in \mathfrak{H}$ and $v(\mathcal{H}) = n\}$.*

Now prove Theorem 4 (a) by showing

Theorem 15. *Let $\mathcal{M} = \mathcal{F}_{91}(\mathcal{G}_n^1) \cap \mathcal{F}_{91}(\mathcal{G}_n^2)$. Then $\mathrm{ex}(n,\mathcal{M}) \leqslant n^3/8$ for all positive integers n, and equality holds if and only if $12 \mid n$.*

Proof. Let $\mathfrak{H}$ be the collection of all symmetrized $\mathcal{M}$-free r-graphs. Define

$$\mathfrak{h}(n) = \max\{|\mathcal{H}|: \mathcal{H} \in \mathfrak{H} \text{ and } v(\mathcal{H}) = n\}.$$

Combining Lemma 13 and Theorem 14, it suffices to prove that $\mathfrak{h}(n) \leqslant n^3/8$, and equality holds if and only if $12 \mid n$.

For each $\mathcal{H} \in \mathfrak{H}$, let $T \subseteq V(\mathcal{H})$ be the set that contains exactly one vertex from each equivalence class of $\mathcal{H}$ and let $\mathcal{T} = \mathcal{H}[T]$. Then $\mathcal{H}$ is a blowup of $\mathcal{T}$. To prove $|\mathcal{H}| \leqslant n^3/8$, we divide our remaining argument into the following two cases according to the cardinality of T.

Case 1: $|T| \leqslant 12$.

By Lemma 9, it suffices to show that $\lambda(\mathcal{T}) \leqslant 1/8$. Since $\mathcal{H}$ does not contain any member of $\mathcal{M}$ as a subgraph, either $\mathcal{T} \subseteq \mathcal{G}_m^1$ or $\mathcal{T} \subseteq \mathcal{G}_m^2$ for some positive integer m. If it is the former case, then $\lambda(\mathcal{T}) \leqslant \lambda(\mathcal{G}_m^1) < 1/8$ by Lemma 10. Otherwise, due to the fact that $\mathcal{T}$ is 2-covered, $\mathcal{T} \subseteq \mathcal{G}_{12}^2$. It follows from Lemma 11 that $\lambda(\mathcal{T}) \leqslant \lambda(\mathcal{G}_{12}^2) = 1/8$, and the equality holds if and only if $\mathcal{T}$ is a copy of $\mathcal{G}_{12}^2$.

Case 2: $|T| \geqslant 13$.

If $|\mathcal{H}| \geqslant n^3/8$, then $\mathcal{H}$ contains a copy of a Fano by Theorem 7. Since the Fano is 2-covered, $\mathcal{T}$ also contains a copy of a Fano. Thus, we can find a subgraph $F \subseteq \mathcal{T}$ such that $F \in \mathcal{K}_{13}^3$ and F contains a copy of a Fano, a contradiction to the fact that $F \in \mathcal{M}$. ☐

4 Stability of $\mathcal{M}$

In this section we always suppose that $\mathcal{M} = \mathcal{F}_{91}(\mathcal{G}_n^1) \cap \mathcal{F}_{91}(\mathcal{G}_n^2)$, and prove Theorem 4 (b) using a ingenious machinery provided in [23]. The following two definitions play a key role in our proof.

Definition 16 (Symmetrized-stability). Let $\mathcal{F}$ be a family of r-graphs and let $\mathfrak{H}$ be a class of $\mathcal{F}$-free r-graphs. We say that $\mathcal{F}$ is symmetrized-stable with respect to $\mathfrak{H}$ if there exist $\epsilon > 0$ and N_0 such that every symmetrized $\mathcal{F}$-free r-graphs $\mathcal{H}$ on $n \geqslant N_0$ vertices with $\delta(\mathcal{H}) \geqslant \big(\pi(\mathcal{F})/(r-1)! - \epsilon\big)n^{r-1}$ is a subgraph of a member of $\mathfrak{H}$.

Definition 17 (Vertex-extendibility). Let $\mathcal{F}$ be a family of r-graphs and let $\mathfrak{H}$ be a class of $\mathcal{F}$-free r-graphs. We say that $\mathcal{F}$ is *vertex-extendable* with respect to $\mathfrak{H}$ if there exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that for every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geqslant N_0$ vertices satisfying $\delta(\mathcal{H}) \geqslant \big(\pi(\mathcal{F})/(r-1)! - \zeta\big)n^{r-1}$ the following holds: if $\mathcal{H} - v$ is a subgraph of a member of $\mathfrak{H}$ for some vertex $v \in V(\mathcal{H})$, then $\mathcal{H}$ is a subgraph of a member of $\mathfrak{H}$ as well.

In [23], Liu, Mubayi and Reiher developed a machinery that reduces the proof of stability of certain families $\mathcal{F}$ to the simpler question of checking that an $\mathcal{F}$-free hypergraph $\mathcal{H}$ with large minimum degree is vertex-extendable.

Theorem 18 (Liu, Mubayi and Reiher [23]). *Suppose that $\mathcal{F}$ is a blowup-invariant nondegenerate family of r-graphs and that $\mathfrak{H}$ is a hereditary class of $\mathcal{F}$-free r-graphs. If $\mathcal{F}$ is symmetrized-stable and vertex-extendable with respect to $\mathfrak{H}$, then the following statement holds. There exist $\varepsilon > 0$ and N_0 such that every $\mathcal{F}$-free r-graph on $n \geqslant N_0$ vertices with minimum degree at least $(\pi(\mathcal{F})/(r-1)! - \varepsilon)\, n^{r-1}$ is contained in $\mathfrak{H}$.*

Let

$$\mathfrak{H} = \{\text{3-graph } \mathcal{H} : \mathcal{H} \text{ is either bipartite or } \mathcal{G}^2_{12}\text{-colorable}\}. \tag{2}$$

We prove that $\mathcal{M}$ is symmetrized-stable with respect to $\mathfrak{H}$.

Lemma 19. *There exist $\epsilon > 0$ and N_0 such that every symmetrized $\mathcal{M}$-free 3-graphs $\mathcal{H}$ on $n \geqslant N_0$ vertices with minimum degree $\delta(\mathcal{H}) \geqslant \big(3/8 - \epsilon\big)n^2$ belongs to $\mathfrak{H}$.*

Proof. We choose ϵ, N_0 satisfying the condition of Theorem 6 and let $\mathcal{H}$ be a symmetrized $\mathcal{M}$-free 3-graphs $\mathcal{H}$ on $n \geqslant N_0$ vertices with $\delta(\mathcal{H}) \geqslant \big(3/8 - \epsilon\big)n^2$. Let $T \subseteq V(\mathcal{H})$ be a set that contains exactly one vertex from each equivalence class of $\mathcal{H}$ and $\mathcal{T} = \mathcal{H}[T]$. Then $\mathcal{H}$ is a blowup of $\mathcal{T}$. If $|T| \leqslant 12$, then either $\mathcal{T} \subseteq \mathcal{G}^1_m$ or $\mathcal{T} \subseteq \mathcal{G}^2_m$ for some positive integer m as $\mathcal{H}$ does not contain any member of $\mathcal{M}$ as a subgraph. This means $\mathcal{H} \in \mathfrak{H}$. Now let $|T| \geqslant 13$. If $\mathcal{H}$ is Fano-free. Then $\mathcal{H}$ is bipartite by Theorem 6 and we are done. Otherwise, $\mathcal{T}$ also contains a copy of a Fano as the Fano is 2-covered. This means we can find a subgraph $F \subseteq \mathcal{T}$ such that $F \in \mathcal{K}^3_{13}$ and F contains a copy of a Fano. However, $F \in \mathcal{M}$ by the construction of $\mathcal{M}$, a contradiction. ☐

We prove Theorem 4 (b) briefly. The details can be found in Subsections 4.1 and 4.2.

Proof of Theorem 4 (b). Let $\epsilon > 0$ be a sufficiently small constant and N_0 be a sufficiently large integer. Suppose that $\mathcal{H}$ is an $\mathcal{M}$-free 3-graph on $n \geqslant N_0$ vertices with $\delta(\mathcal{H}) \geqslant (3/8 - \epsilon)\, n^2$. It follows from Lemma 13 that $\mathcal{M}$ is blowup-invariant. By Theorem 18 and Lemma 19, it suffices to show that $\mathcal{M}$ is vertex-extendable with respect to $\mathfrak{H}$. If there is a vertex v such that $\mathcal{H} - v$ is bipartite, then $\mathcal{H}$ is also bipartite by Lemma 20. Otherwise, if $\mathcal{H} - v$ is $\mathcal{G}^2_{12}$-colorable, then $\mathcal{H}$ is also $\mathcal{G}^2_{12}$-colorable by Lemma 26. ☐

4.1 Bipartite

In this subsection, we prove the following lemma.

Lemma 20. *There exist $\zeta > 0$ and N_0 such that every $\mathcal{M}$-free 3-graph $\mathcal{H}$ on $n > N_0$ vertices which has minimum degree $\delta(\mathcal{H}) > (3/8 - \zeta)\, n^2$ and possesses a vertex v such that $\mathcal{H}' := \mathcal{H} - v$ is bipartite is bipartite itself.*

Proof. Let $\epsilon > 0$ be a sufficiently small constant and N_0 be a sufficiently large integer. Let $\zeta = \min\{\gamma, \epsilon/2\}$, where γ is the constant in Theorem 6. Suppose that $\mathcal{H}$ is an $\mathcal{M}$-free 3-graph on $n > N_0$ vertices with $\delta(\mathcal{H}) > (3/8 - \zeta)n^2$, and there exists exists a vertex $v \in V(\mathcal{H})$ such that the 3-graph $\mathcal{H}' = \mathcal{H} - v$ is bipartite. Then $\mathcal{H}'$ can be partitioned into two parts A_1 and A_2 such that $\mathcal{H}'[A_i]$ is empty for $i = 1, 2$, and

$$\delta(\mathcal{H}') \geqslant \delta(\mathcal{H}) - n \geqslant (3/8 - \epsilon)\, n^2.$$

Claim 21. *We have* $||A_i| - n/2| < 2\epsilon n$ *for* $i = 1, 2$.

Proof of Claim 21. Let $\alpha = |A_1|$. Since $\mathcal{H}'$ is bipartite, for every vertex $y \in A_2$, we have

$$d(y) \leqslant \alpha(n - \alpha) + \binom{\alpha}{2}.$$

On the other hand, $d(y) \geqslant \delta(\mathcal{H}') \geqslant (3/8 - \epsilon)\, n^2$. Therefore,

$$\left(\frac{3}{8} - \epsilon\right) n^2 \leqslant \alpha(n - \alpha) + \binom{\alpha}{2},$$

where implies that $\alpha > n/2 - 2\epsilon n$. Similarly, we have $|A_2| > n/2 - 2\epsilon n$. Since $|A_1| + |A_2| = n - 1$, we have $||A_i| - n/2| < 2\epsilon n$ for $i = 1, 2$. □

Denote $\widehat{\mathcal{H}}$ by the complete bipartite 3-graph on $V(\mathcal{H}')$ with two parts A_1 and A_2. For $w \in V(\mathcal{H}')$ let $M_w = L_{\widehat{\mathcal{H}}}(w) \setminus L_{\mathcal{H}'}(w)$. Members in M_w are called *missing edges* of $L_{\mathcal{H}'}(w)$.

Claim 22. *For each* $w \in V(\mathcal{H}')$, $|M_w| \leqslant 10\epsilon n^2$.

Proof of Claim 22. For $w \in A_i$, $L_{\widehat{\mathcal{H}}}(w)$ is a complete graph on A_{3-i} and added a complete bipartite graph with two parts A_1 and A_2 except w. By Claim 21, we have

$$|M_w| \leqslant \binom{n/2 + 2\epsilon n}{2} + \left(\frac{1}{2}n + 2\epsilon n\right)^2 - \left(\frac{3}{8} - \epsilon\right) n^2 \leqslant 10\epsilon n^2.$$

□

Claim 23. *For each* $w \in V(\mathcal{H}')$, $|N_{\mathcal{H}}(w) \cap A_2| \geqslant |A_2| - n/100$.

Proof of Claim 23. It suffices to prove $|N_{\mathcal{H}'}(w) \cap A_2| \geqslant |A_2| - n/100$. Let $S \subseteq A_2$ denote the set of isolated vertices of $L_{\mathcal{H}'}(w)$. Then $N_{\mathcal{H}}(w) \cap A_2 = A_2 - S$. Note that each vertex $u \in S$ is incident with $|A_1| + |A_2|$ (or $|A_2|$) missing edges of $L_{\mathcal{H}'}(w)$ if $w \in A_1$ (or $w \in A_2$). On the other hand, there are at most $10\epsilon n^2$ missing edges by Claim 22. Thus, if $w \in A_1$, then $|S| \leqslant 2 \times 10\epsilon n^2/(|A_1| + |A_2|) < n/100$; if $w \in A_2$, then $|S| \leqslant 10\epsilon n^2/|A_1| < n/100$ by Claim 21. □

Claim 24. *For* $i \in [2]$, $|N_{\mathcal{H}}(v) \cap A_i| \geqslant ((\sqrt{3} - 1)/2 - 7\epsilon/2)n$.

Proof of Claim 24. Note that

$$\binom{|N_{\mathcal{H}}(v)|}{2} \geqslant d_{\mathcal{H}}(v) \geqslant \delta(\mathcal{H}) \geqslant \left(\frac{3}{8} - \frac{\epsilon}{2}\right) n^2.$$

This implies that $|N_{\mathcal{H}}(v)| \geqslant (\sqrt{3}/2 - 3\epsilon/2)n$, which together with Claim 21 yields

$$|N_{\mathcal{H}}(v) \cap A_i| \geqslant \left(\frac{\sqrt{3}}{2} - \frac{3}{2}\epsilon\right) n - \left(\frac{1}{2} + 2\epsilon\right) n = \left(\frac{\sqrt{3}-1}{2} - \frac{7}{2}\epsilon\right) n.$$

□

If $\mathcal{H}$ is Fano-free, then we are done by Theorem 6. Suppose that $S \subseteq V(\mathcal{H})$ is a set of size 7 such that $\mathcal{H}[S]$ is a Fano. Then $v \in S$ as $\mathcal{H}'$ is Fano-free. Let $S \setminus \{v\} = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and define

$$A_2' = A_2 \cap N_{\mathcal{H}}(v) \cap \left(\cap_{i\in[6]} N_{\mathcal{H}}(w_i)\right).$$

Then Claims 23 and 24 imply that $|A_2'| \geqslant ((\sqrt{3}-1)/2 - 7\epsilon/2)n - 6 \times n/100 > n/5$. Fix a vertex $u_1 \in A_1$ (it is possible that $u_1 \in \{w_1, w_2, w_3, w_4, w_5, w_6\}$). There exists an edge $w_7w_8 \in L_{\mathcal{H}'}(u_1)[A_2']$ by Claim 22. Let $E_1 \subseteq \mathcal{H}$ be a set of edges that cover all pairs in $S \times \{w_7, w_8\}$, and let $F_1 = \mathcal{H}[S] \cup \{u_1w_7w_8\} \cup E_1$. Then $F_1 \subseteq \mathcal{H}$, $F_1 \in \mathcal{K}_9^3$ and F_1 contains a copy of the Fano. Repeating this process twice, we can get a subgraph $F_3 \subseteq \mathcal{H}$ such that $F_3 \in \mathcal{K}_{13}^3$ contains a copy of a Fano. However, $F_3 \in \mathcal{M}$ by the construction of $\mathcal{M}$, a contradiction. □

4.2 $\mathcal{G}_{12}^2$-colorable

In this subsection, we consider the case that $\mathcal{H}'$ is $\mathcal{G}_{12}^2$-colorable. We need the following lemma given by Liu, Mubayi and Reiher [22] (also see [25] Lemma 3.2).

Lemma 25 ([22, 25]). *Fix a real $\eta \in (0,1)$ and integers $m, n \geqslant 1$. Let $\mathcal{G}$ be a 3-graph with vertex set $[m]$ and let $\mathcal{H}$ be a further 3-graph with $v(\mathcal{H}) = n$. Consider a vertex partition $V(\mathcal{H}) = \bigcup_{i\in[m]} V_i$ and the associated blow-up $\widehat{\mathcal{G}} = \mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$. If two sets $T \subseteq [m]$ and $S \subset V$ have the properties*

(a) $|V_j| \geqslant (|S|+1)|T|\eta^{1/3}n$ for all $j \in T$,

(b) $|\mathcal{H}[V_{j_1}, V_{j_2}, V_{j_3}]| \geqslant |\widehat{\mathcal{G}}[V_{j_1}, V_{j_2}, V_{j_3}]| - \eta n^3$ for all $\{j_1, j_2, j_3\} \in \binom{T}{3}$, and

(c) $|L_{\mathcal{H}}(v)[V_{j_1}, V_{j_2}]| \geqslant |L_{\widehat{\mathcal{G}}}(v)[V_{j_1}, V_{j_2}]| - \eta n^2$ for all $v \in S$ and $\{j_1, j_2\} \in \binom{T}{2}$,

then there exists a selection of vertices $u_j \in V_j \setminus S$ for all $j \in [T]$ such that $U = \{u_j : j \in T\}$ satisfies $\widehat{\mathcal{G}}[U] \subseteq \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] \subseteq L_{\mathcal{H}}(v)[U]$ for all $v \in S$. In particular, if $\mathcal{H} \subseteq \widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}[U] = \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] = L_{\mathcal{H}}(v)[U]$ for all $v \in S$.

Lemma 26. *There exist $\zeta > 0$ and N_0 such that every $\mathcal{M}$-free 3-graph $\mathcal{H}$ on $n > N_0$ vertices satisfying the minimum degree $\delta(\mathcal{H}) > (3/8 - \zeta)\, n^2$ and possessing a vertex v such that $\mathcal{H}' := \mathcal{H} - v$ is $\mathcal{G}^2_{12}$-colorable is $\mathcal{G}^2_{12}$-colorable itself.*

Proof. Let $\epsilon > 0$ be a sufficiently small constant and N_0 be a sufficiently large integer. Let $\mathcal{H}$ be an $\mathcal{M}$-free 3-graph on $n > N_0$ vertices with $\delta(\mathcal{H}) \geqslant (3/8 - \zeta)n^2$, where $\zeta = \epsilon/2$. Suppose that there exists a vertex $v \in V(\mathcal{H})$ such that $\mathcal{H}' = \mathcal{H} - v$ is $\mathcal{G}^2_{12}$-colorable. Then we have

$$\delta(\mathcal{H}') \geqslant \delta(\mathcal{H}) - n \geqslant (3/8 - \epsilon)\, n^2. \tag{3}$$

Since $\mathcal{H}'$ is $\mathcal{G}^2_{12}$-colorable, let

$$\mathcal{P} = \{V_1, \ldots, V_{12}\}$$

be the partition of $V(\mathcal{H}')$ such that every edge in $\mathcal{H}'$ intersects at most one vertex in V_j for every $j \in [12]$, and there is no edges between $V_{3i-2}V_{3i-1}V_{3i}$ for $i \in [4]$. Define $\widehat{\mathcal{G}}^2_{12} = \mathcal{G}^2_{12}[V_1, \ldots, V_{12}]$ to be the associated blowup of $\mathcal{G}^2_{12}$. We say edges in $\widehat{\mathcal{G}}^2_{12} \setminus \mathcal{H}'$ are *missing edges* of $\mathcal{H}'$. For $u \in V(\mathcal{H}')$, edges in $L_{\widehat{\mathcal{G}}^2_{12}}(u) \setminus L_{\mathcal{H}'}(u)$ are also called *missing edges* of $L_{\mathcal{H}'}(u)$.

Claim 27. *We have the following hold:*

(a) $||V_i| - n/12| < 4\epsilon^{1/2}n$ *for every* $i \in [12]$.

(b) If $i \in [12]$ *and* $u \in V(\mathcal{H}') \setminus V_i$, *then* $|V_i \setminus N_{\mathcal{H}}(u)| \leqslant 1000\epsilon^{1/2}n$.

(c) For every $u \in V(\mathcal{H}')$ *the number of missing edges of* $L_{\mathcal{H}'}(u)$ *is at most* $100\epsilon^{1/2}n^2$.

Proof of Claim 27. Let $x_i = |V_i|/(n-1)$ for $i \in [12]$ and $x = (x_1, \ldots, x_{12})$. By symmetry it suffices to prove $|x_1 - 1/12| \leqslant 4\epsilon^{1/2}$. Notice that $|\mathcal{H}'| \leqslant p_{\mathcal{G}^2_{12}}(x)(n-1)^3$ and $|\mathcal{H}'| \geqslant (1/8 - \epsilon)n^2(n-1)$ by (3). Thus,

$$p_{\mathcal{G}^2_{12}}(x) \geqslant 1/8 - \epsilon. \tag{4}$$

On the other hand, let $x_{3i-2} + x_{3i-1} + x_{3i} = a_i$ for $i \in [4]$ and recall from the proof of Lemma 11 that

$$\begin{aligned} p_{\mathcal{G}^2_{12}}(x) &= \sum_{\{i,j,k\} \subset [12]} x_i x_j x_k - (x_1x_2x_3 + x_4x_5x_6 + x_7x_8x_9 + x_{10}x_{11}x_{12}) \\ &\leqslant \sum_{i=1}^{4} \left(\prod_{j \in [4] \setminus \{i\}} a_j + 3\left(\frac{a_i}{3}\right)^2 \sum_{k \in [4] \setminus \{i\}} a_k \right) \\ &= \frac{1}{3} \sum_{\{j,k\} \subset [4]} a_j a_k. \end{aligned} \tag{5}$$

Thus, by (4), we have

$$3 - 8 \sum_{\{j,k\} \subset [4]} a_j a_k \leqslant 24\epsilon,$$

which together with $\sum_{i\in[4]} a_i = 1$ yields that

$$3 - 8\sum_{\{j,k\}\subset[4]} a_j a_k = 3\left(\sum_{i\in[4]} a_i\right)^2 - 8\sum_{\{j,k\}\subset[4]} a_j a_k = \sum_{1\leqslant i<j\leqslant 4} (a_i - a_j)^2 \leqslant 24\epsilon.$$

This means $|a_i - a_j| < 5\epsilon^{1/2}$ for $i, j \in [4]$. By the triangle inequality,

$$4\left|a_1 - \frac{1}{4}\right| \leqslant \left|\sum_{i\in[4]} a_i - 1\right| + |3a_1 - a_2 - a_3 - a_4| \leqslant \sum_{i\in[2,4]} |a_1 - a_i| < 16\epsilon^{1/2}.$$

Therefore, $|a_1 - 1/4| < 4\epsilon^{1/2}$ and then $a_2 + a_3 + a_4 > 1/2$. By (5), we have

$$\begin{aligned}
p_{\mathcal{G}^2_{12}}(x) \leqslant& \sum_{i=1}^{4}\left(\prod_{j\in[4]\setminus\{i\}} a_j + (x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i})\sum_{k\in[4]\setminus\{i\}} a_k\right)\\
=&\sum_{i=1}^{4}\left(\prod_{j\in[4]\setminus\{i\}} a_j + (x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i})\sum_{k\in[4]\setminus\{i\}} a_k\right)\\
&+\sum_{i=1}^{4}\left(3\left(\frac{a_i}{3}\right)^2\sum_{k\in[4]\setminus\{i\}} a_k - 3\left(\frac{a_i}{3}\right)^2\sum_{k\in[4]\setminus\{i\}} a_k\right)\\
\leqslant&\frac{1}{3}\sum_{\{j,k\}\subset[4]} a_j a_k + \sum_{i=1}^{4}\left(\left(x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i} - 3\left(\frac{a_i}{3}\right)^2\right)\sum_{k\in[4]\setminus\{i\}} a_k\right)\\
\leqslant&\frac{1}{8} + \sum_{i=1}^{4}\left(\left(x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i} - 3\left(\frac{a_i}{3}\right)^2\right)\sum_{k\in[4]\setminus\{i\}} a_k\right).
\end{aligned}$$

This together with (4) shows that

$$\sum_{i=1}^{4}\left(\left(3\left(\frac{a_i}{3}\right)^2 - x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i}\right)\sum_{k\in[4]\setminus\{i\}} a_k\right) \leqslant \epsilon.$$

Since

$$3\left(\frac{a_i}{3}\right)^2 - x_{3i-2}x_{3i-1} + x_{3i-1}x_{3i} + x_{3i-2}x_{3i} \geqslant 0$$

for each $i \in [4]$, we have

$$\left(3\left(\frac{a_1}{3}\right)^2 - x_1x_2 + x_2x_3 + x_1x_3\right)(a_2 + a_3 + a_4) \leqslant \epsilon.$$

Thus,

$$2a_1^2 - 6(x_1x_2 + x_2x_3 + x_1x_3) \leqslant \frac{6\epsilon}{a_2 + a_3 + a_4} < 12\epsilon. \tag{6}$$

Note that

$$2a_1^2 - 6(x_1x_2 + x_2x_3 + x_1x_3) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_1 - x_3)^2.$$

We have $|x_i - x_j| < 4\epsilon^{1/2}$ for $i, j \in [3]$ by (6) and then $|x_1 - 1/12| < 4\epsilon^{1/2}$ by the triangle inequality again. This completes the proof of (a).

Before proving (b) and (c), we claim that $\Delta(\mathcal{H}') \leqslant \Delta(\widehat{\mathcal{G}}_{12}^2) \leqslant (3/8 + 50\epsilon^{1/2})n^2$. Indeed, by symmetry let $v \in V_1$. Then

$$\begin{aligned} d_{\mathcal{H}'}(v) \leqslant d_{\widehat{\mathcal{G}}_{12}^2}(v) &= (|V_2| + |V_3|) \sum_{i=4}^{12} |V_i| + \sum_{4 \leqslant i < j \leqslant 12} |V_i||V_j| \\ &\leqslant (|V_2| + |V_3|) \sum_{i=4}^{12} |V_i| + \left(\frac{\sum_{i=4}^{12} |V_i|}{9} \right)^2 \binom{9}{2}. \end{aligned} \tag{7}$$

Set $x = |V_2| + |V_3|$. Then $|x - n/6| \leqslant 8\epsilon^{1/2}n$ and

$$\sum_{i=4}^{12} |V_i| \leqslant n - |V_1| - x \leqslant \frac{11}{12}n + 4\epsilon^{1/2}n - x.$$

by (a). Treating x as a new variable, by (7), we have

$$\begin{aligned} d_{\mathcal{H}'}(v) &\leqslant x\left(\frac{11}{12}n + 4\epsilon^{1/2}n - x\right) + \left(\frac{\frac{11}{12}n + 4\epsilon^{1/2}n - x}{9}\right)^2 \binom{9}{2} \\ &\leqslant (3/8 + 50\epsilon^{1/2})n^2. \end{aligned} \tag{8}$$

Now we prove (b). If there exists some $i \in [12]$ and a vertex $u \in V(\mathcal{H}') \setminus V_i$, such that $|V_i \setminus N_{\mathcal{H}'}(u)| > 1000\epsilon^{1/2}n$. By (8), we have

$$d_{\mathcal{H}'}(u) \leqslant \Delta(\mathcal{H}') - 1000\varepsilon^{1/2}n(n/12 - 4\epsilon^{1/2}n) < (3/8 - 10\epsilon^{1/2})n^2,$$

a contradiction to (3).

For (c), the number of missing edges of $L_{\mathcal{H}'}(u)$ for each $u \in V(\mathcal{H}')$ is at most

$$\Delta(\widehat{\mathcal{G}}_{12}^2) - \delta(\mathcal{H}') \leqslant 50\epsilon^{1/2}n^2 + 2\epsilon n^2 < 100\epsilon^{1/2}n^2$$

by (3) and (8). □

Recall that $\mathcal{H}'$ is 12-partite with the vertex partition $\mathcal{P}$. The following claim shows that $\mathcal{H}$ does not contain an edge E with $v \in E$ and $|E \cap V_i| = 2$ for some $i \in [12]$.

Claim 28. *$|E \cap V_i| \leqslant 1$ for all $E \in \mathcal{H}$ and $i \in [12]$.*

Proof of Claim 28. By symmetry it suffices to show $|E \cap V_1| \leqslant 1$ for every $E \in \mathcal{H}$ containing vertex v. Assume for the sake of contradiction that there exist distinct vertices $u_1, u_1' \in V_1$ such that $E = \{v, u_1, u_1'\}$. By Claim 27 (a) and (b) for every $i \in [2, 12]$ the set $V_i' = (N_{\mathcal{H}'}(u_1) \cap N_{\mathcal{H}'}(u_1')) \cap V_i$ has size at least $n/24$. Applying Lemma 25 with $S = \{u_1, u_1'\}$, $T = [2, 12]$ and $\eta = 50\epsilon^{1/4}$ we can obtain $u_j \in V_j'$ for $j \in [2, 12]$ such that $\mathcal{H}[U \cup \{u_1\}] \cong \mathcal{H}[U \cup \{u_1'\}] \cong \mathcal{G}_{12}^2$, where $U = \{u_j : j \in [2, 12]\}$. This implies that $\mathcal{H}[U \cup \{u_1\} \cup \{u_1'\} \cup \{v\}]$ is a copy of a member of $\mathcal{K}_{13}^3$ with 14 vertices. On the other hand, $\mathcal{G}_{12}^2$ is $\mathcal{K}_{13}^3$-hom-free and not bipartite hypergraph. This means $\mathcal{H}[U \cup \{u_1\} \cup \{u_1'\} \cup \{v\}] \in \mathcal{M}$, a contradiction. □

Claim 29. *There are no two distinct index $i, j \in [12]$ such that both $|N_{\mathcal{H}}(v) \cap V_i|$ and $|N_{\mathcal{H}}(v) \cap V_j|$ are at most $n/36$.*

Proof of Claim 29. Let $V_i' = N_{\mathcal{H}}(v) \cap V_i$ for $i \in [12]$. By Claim 28, $L_{\mathcal{H}}(v)$ is a 12-partite graph with the vertex partition $\mathcal{P}' = \{V_1', \ldots, V_{12}'\}$. Suppose to the contrary that there are at least two sets in $\mathcal{P}'$ that have size at most $n/36$. Then, by Claim 27 (a),

$$|L_{\mathcal{H}}(v)| \leqslant \binom{10}{2} \left(\frac{1}{12} + 4\epsilon^{\frac{1}{2}}\right)^2 n^2 + \left(\frac{n}{36}\right)^2 + 2 \times 10 \times \frac{n}{36} \left(\frac{1}{12} + 4\epsilon^{\frac{1}{2}}\right) n < \left(\frac{3}{8} - \epsilon\right) n^2,$$

a contradiction. □

Claim 30. *There is no index $i \in [12]$ such that $N_{\mathcal{H}}(v) \cap V_i \neq \emptyset$ and $|N_{\mathcal{H}}(v) \cap V_j| \geqslant n/36$ for all $j \in [12] \setminus \{i\}$.*

Proof of Claim 30. By symmetry, assume for the sake of contradiction that there exists a vertex $u_1 \in N_{\mathcal{H}}(v) \cap V_1$ and $|N_{\mathcal{H}}(v) \cap V_j| \geqslant n/36$ for all $j \in [2, 12]$. By Claim 27 (b) the set $V_j' = (N_{\mathcal{H}}(v) \cap N_{\mathcal{H}}(u_1)) \cap V_j$ has at least the size $|V_j'| \geqslant n/36 - 1000\epsilon^{1/2} n > n/50$ for $j \in [2, 12]$. Applying Lemma 25 with $S = \{u_1\}$, $T = [2, 12]$ and $\eta = 50\epsilon^{1/4}$, we can get a set $U = \{u_j : j \in [2, 12]\}$ with $u_j \in V_j'$ such that $\mathcal{H}[U \cup \{u_1\}] \cong \mathcal{G}_{12}^2$. For every $j \in [12]$ let $e_j \in \mathcal{H}$ be an edge containing both u_i and v. We have $\mathcal{H}[U \cup \{u_1\}] \cup \{e_j : j \in [12]\} \in \mathcal{K}_{13}^3$ with at most 25 vertices. On the other hand, $\mathcal{G}_{12}^2$ is $\mathcal{K}_{13}^3$-hom-free and not bipartite hypergraph. This means $\mathcal{H}[U \cup \{u_1\}] \cup \{e_j : j \in [12]\} \in \mathcal{M}$, a contradiction. □

Combining Claims 29 and 30, we may assume that $N_{\mathcal{H}}(v) \cap V_1 = \emptyset$. Recall that $\widehat{\mathcal{G}}_{12}^2$ is a blowup of $\mathcal{G}_{12}^2$. We have $L_{\widehat{\mathcal{G}}_{12}^2}(x) = L_{\widehat{\mathcal{G}}_{12}^2}(y)$ for $x, y \in V_i$ with $i \in [12]$. Choose a vertex $u \in V_1$, let $B_v = L_{\mathcal{H}}(v) \backslash L_{\widehat{\mathcal{G}}_{12}^2}(u)$ be the set of bad edges in $L_{\mathcal{H}}(v)$ and $M_v = L_{\widehat{\mathcal{G}}_{12}^2}(u) \backslash L_{\mathcal{H}}(v)$ be the set of missing edges in $L_{\mathcal{H}}(v)$. The aim is to show $B_v = \emptyset$. First, we show $|B_v|$ is small.

Claim 31. $|B_v| < 100\epsilon^{1/12} n^2$.

Proof of Claim 31. By contradiction, $|B_v| \geqslant 100\epsilon^{1/12}n^2$. Notice that each edge in B_v has one vertex in V_2 and the other vertex in V_3. Combining Claim 27 (a) and an easy averaging argument show that there exists a vertex $u_2 \in V_2$ such that

$$|N_{B_v}(u_2) \cap V_3| \geqslant \frac{|B_v|}{|V_2|} > \frac{100\epsilon^{1/12}n^2}{n/10} > 500\epsilon^{1/12}n.$$

Let $V'_1 = V_1$, $V'_3 = N_{B_v}(u_2) \cap V_3$, and $V'_j = N_{\mathcal{H}}(v) \cap V_j$ for $j \in [4, 12]$. Then $|V'_j| > n/36$ for $j \in [4, 12]$ by Claim 29. Applying Lemma 25 with $S = \{u_2\}$, $T = [12] \setminus \{2\}$, and $\eta = 50\epsilon^{1/4}$ we can obtain $u_j \in V'_j$ for $j \in [12] \setminus \{2\}$ such that the induced subgraph of $\mathcal{H}[\{u_1, \ldots, u_{12}\}] \cong \mathcal{G}^2_{12}$. For $j \in [4, 12]$ let e_j be an edge containing both v and u_j. Let $F = \mathcal{H}[\{u_1, \ldots, u_{12}\}] \cup \{e_j : j \in [4, 12]\} \cup \{vu_2u_3\}$. Then $F \in \mathcal{K}^3_{12}$ with the core $\{v, u_2, \ldots, u_{12}\}$. Note that F is $\mathcal{G}^2_{12}$-colorable, as F is not bipartite. We can find a map $\psi : V(F) \to V(\mathcal{G}^2_{12})$ such that $\psi(e) \in \mathcal{G}^2_{12}$ for all $e \in F$. Since both $\{u_1, \ldots, u_{12}\}$ and $\{v, u_2, \ldots, u_{12}\}$ are 2-covered in F, the restrictions of ψ on $\{u_1, \ldots, u_{12}\}$ and $\{v, u_2, \ldots, u_{12}\}$ are both injective for all $e \in F$, and $\psi(v) = \psi(u_1)$. Notice that $L_F(u_1)[\{u_2, \ldots, u_{12}\}]$ has exactly 54 edges and $\{u_2u_3\} \notin L_F(u_1)[\{u_2, \ldots, u_{12}\}]$. This means the degree of $\psi(v)$ in $\mathcal{G}^2_{12}$ should be at least $54 + 1 = 55$, which contradicts the fact that the maximum degree of $\mathcal{G}^2_{12}$ is 54. $\square$

A consequence of Claim 31 is that the size of M_v satisfies

$$\begin{aligned}
|M_v| = |L_{\widehat{\mathcal{G}}^2_{12}}(u) \setminus L_{\mathcal{H}}(v)| &= |L_{\widehat{\mathcal{G}}^2_{12}}(u)| - |L_{\widehat{\mathcal{G}}^2_{12}}(u) \cap L_{\mathcal{H}}(v)| \\
&= |L_{\widehat{\mathcal{G}}^2_{12}}(u)| - (|L_{\mathcal{H}}(v)| - |B_v|) \\
&\leqslant 54 \times (n/12 + 4\epsilon^{1/2}n)^2 - ((3/8 - \epsilon)n^2 - |B_v|) \\
&< 200\epsilon^{1/12}n^2. \qquad (9)
\end{aligned}$$

Claim 32. $B_v = \emptyset$.

Proof of Claim 32. By contradiction, there is an edge $u_2u_3 \in B_v$ with $u_2 \in V_2$ and $u_3 \in V_3$. Let $V'_j = V_j$ for $j \in [3]$ and $V'_j = V_j \cap N_{\mathcal{H}}(v) \cap N_{\mathcal{H}}(u_2) \cap N_{\mathcal{H}}(u_3)$ for $j \in [4, 12]$. By (9) and Claim 27, we have $|V'_j| \geqslant |V_j|/2 > n/25$. Applying Lemma 25 with $S = \{v, u_2, u_3\}, T = [12]$ and $\eta = 200\epsilon^{1/36}$ we can obtain $w_j \in V'_j$ for $j \in [12]$ such that

(a) $\mathcal{H}[\{w_1, \ldots, w_{12}\}] \cong \mathcal{G}^2_{12}$,

(b) $L_{\mathcal{H}}(v)[\{w_2, \ldots, w_{12}\}] = L_{\widehat{\mathcal{G}}^2_{12}}(w_1)[\{w_2, \ldots, w_{12}\}]$,

(c) $L_{\mathcal{H}}(u_2)[\{w_1, w_3, \ldots, w_{12}\}] = L_{\widehat{\mathcal{G}}^2_{12}}(w_2)[\{w_1, w_3, \ldots, w_{12}\}]$, and

(d) $L_{\mathcal{H}}(u_3)[\{w_1, w_2, w_4, \ldots, w_{12}\}] = L_{\widehat{\mathcal{G}}^2_{12}}(w_3)[\{w_1, w_2, w_4, \ldots, w_{12}\}]$.

Clearly, $\mathcal{H}[\{v, u_2, u_3, w_4, \ldots, w_{12}\}] \in \mathcal{K}^3_{12}$ with the core $\{v, u_2, u_3, w_4, \ldots, w_{12}\}$. Let $F = \mathcal{H}[\{v, u_2, u_3, w_1, w_2, \ldots, w_{12}\}]$. We can find a map $\psi : V(F) \to V(\mathcal{G}^2_{12})$ such that $\psi(e) \in \mathcal{G}^2_{12}$ for all $e \in F$. Notice that both $\mathcal{H}[\{w_1, \ldots, w_{12}\}]$ and $\mathcal{H}[\{v, u_2, u_3, w_4, \ldots, w_{12}\}]$ are 2-covered in F, so the restrictions of ψ on sets $\{w_1, \ldots, w_{12}\}$ and $\{v, u_2, u_3, w_4, \ldots, w_{12}\}$

are both injective, and moreover, $\psi(v) = \psi(w_1)$ (due to (b), v is adjacent to all vertices in $\{w_2, \ldots, w_{12}\}$, so $\psi(v)$ is distinct from $\{\psi(w_2), \ldots, \psi(w_{12})\}$), $\psi(u_2) = \psi(w_2)$ (due to (c) and a similar reason), and $\psi(u_3) = \psi(w_3)$ (due to (d) and a similar reason). Let $w = \psi(v) = \psi(w_1)$. Notice that the induced subgraph of $|L_F(w_1)[\{w_2, \ldots, w_{12}\}]|$ has size 54 and $u_2u_3 \in L_F(v) \setminus L_F(w_1)$. The degree of w in $\mathcal{G}^2_{12}$ is at least $54 + 1 = 55$, which contradicts the fact that the maximum degree of $\mathcal{G}^2_{12}$ is 54. □

Define

$$V_j^* = \begin{cases} V_1 \cup \{v\}, & \text{if } j = 1, \\ V_j, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}^* = \{V_1^*, \ldots, V_{12}^*\}$ is a vertex partition of $\mathcal{H}$. By Claim 32, $L_{\mathcal{H}}(v) \subseteq L_{\widehat{\mathcal{G}}^2_{12}}(x)$ for $x \in V_1$. Therefore, $\mathcal{H}$ is $\mathcal{G}^2_{12}$-colorable with $\mathcal{P}^*$. This completes the proof of Lemma 26. □

5 Feasible Region of $\mathcal{M}$

In this section we consider the feasible region $\Omega(\mathcal{M})$ of $\mathcal{M}$ and prove Theorem 5. First, from Lemma 4.2 in [22], we have the following simple corollary of Theorem 4.

Corollary 33. *There exist constants $\epsilon_0 \in (0, 1)$ and N_0 such that the following statement holds for all $\epsilon \leqslant \epsilon_0$ and $n \geqslant N_0$. Suppose that $\mathcal{H}$ is an $\mathcal{M}$-free 3-graph on n vertices with at least $(1/8 - \epsilon)n^3$ edges. Then there exists a set $Z_\epsilon \subseteq V(\mathcal{H})$ of size at most $\epsilon^{1/2}n$ such that the subgraph $\widetilde{\mathcal{H}} = \mathcal{H} - Z_\epsilon$ is either bipartite or $\mathcal{G}^2_{12}$-colorable, $\delta(\widetilde{\mathcal{H}}) \geqslant (3/8 - 3\epsilon^{1/2})n^2$ and $|\widetilde{\mathcal{H}}| \geqslant (1/8 - 2\epsilon^{1/2})n^3$.*

Using Corollary 33, we prove

Lemma 34. *Let $\epsilon > 0$ be sufficiently small and N_0 be sufficiently large. Suppose that $\mathcal{H}$ is an $\mathcal{M}$-free 3-graph on $n \geqslant N_0$ vertices with at least $(1/8 - \epsilon)n^3$ edges. Then,*

$$\text{either } \left| |\partial\mathcal{H}| - \frac{11}{24}n^2 \right| < 100\epsilon^{1/4}n^2 \text{ or } \frac{1}{2}n^2 - 12\epsilon^{1/2}n^2 < |\partial\mathcal{H}| \leqslant \binom{n}{2}.$$

Proof. Let $\mathcal{H}$ be an $\mathcal{M}$-free 3-graph on $n \geqslant N_0$ vertices with at least $(1/8-\epsilon)n^3$ edges. By Corollary 33, there exists a set $Z_\epsilon \subseteq V(\mathcal{H})$ of size at most $\epsilon^{1/2}n$ such that the subgraph $\widetilde{\mathcal{H}} = \mathcal{H} - Z_\epsilon$ is either bipartite or $\mathcal{G}^2_{12}$-colorable, $\delta(\widetilde{\mathcal{H}}) \geqslant (3/8 - 3\epsilon^{1/2})n^2$ and $|\widetilde{\mathcal{H}}| \geqslant (1/8 - 2\epsilon^{1/2})n^3$. Let $\widetilde{n} = |V(\widetilde{\mathcal{H}})|$. Then

$$\widetilde{n} = n - |Z_\epsilon| \geqslant n - \epsilon^{1/2}n. \tag{10}$$

Suppose that $\widetilde{\mathcal{H}}$ is bipartite with two parts A_1 and A_2. Note from Claim 21 that for $i = 1, 2$,

$$\frac{n}{2} - 6\epsilon^{1/2}n < |A_i| < \frac{n}{2} + 6\epsilon^{1/2}n. \tag{11}$$

First we prove a lower bound for $|\partial\mathcal{H}|$. Let a (or (b)) denote the number of edges of $\widetilde{\mathcal{H}}$ which intersect A_1 (or A_2) exactly two vertices. Since $\widetilde{\mathcal{H}}$ is bipartite, the following inequalities hold:

$$\begin{aligned} a+b=|\widetilde{\mathcal{H}}| &\geqslant (1/8-2\epsilon^{1/2})n^3 \\ |A_2||(\partial\widetilde{\mathcal{H}})[A_1]| &\geqslant a \\ |A_1||(\partial\widetilde{\mathcal{H}})[A_2]| &\geqslant b \\ |A_1||(\partial\widetilde{\mathcal{H}})[A_1,A_2]| &\geqslant 2a \\ |A_2||(\partial\widetilde{\mathcal{H}})[A_1,A_2]| &\geqslant 2b. \end{aligned}$$

This together with (11) yields

$$\begin{aligned} |\partial\widetilde{\mathcal{H}}| &= |(\partial\widetilde{\mathcal{H}})[A_1]| + |(\partial\widetilde{\mathcal{H}})[A_2]| + |(\partial\widetilde{\mathcal{H}})[A_1,A_2]| \\ &\geqslant \frac{a}{|A_2|}+\frac{b}{|A_1|}+\frac{2a+2b}{|A_1|+|A_2|} \\ &\geqslant \frac{a}{n/2+6\epsilon^{1/2}n}+\frac{b}{n/2+6\epsilon^{1/2}n}+\frac{2a+2b}{n} \\ &\geqslant \frac{(1/8-2\epsilon^{1/2})n^3}{n/2+6\epsilon^{1/2}n}+\frac{2(1/8-2\epsilon^{1/2})n^3}{n} \\ &> \left(\frac{1}{2}-12\epsilon^{1/2}\right)n^2. \end{aligned}$$

Therefore,

$$\frac{1}{2}n^2-12\epsilon^{1/2}n^2 < |\partial\widetilde{\mathcal{H}}| \leqslant |\partial\mathcal{H}| \leqslant \binom{n}{2}.$$

Suppose that $\widetilde{\mathcal{H}}$ is $\mathcal{G}^2_{12}$-colorable and let $\mathcal{P}=\{V_1,\dots,V_{12}\}$ be the vertex partition of $\widetilde{\mathcal{H}}$ such that there is no edge between $V_{3i-2}V_{3i-1}V_{3i}$ for $i\in[4]$. Notice from Claim 27 that for every $i\in[12]$,

$$\left||V_i|-\frac{n}{12}\right| \leqslant 8\epsilon^{1/4}n. \tag{12}$$

First we prove a lower bound for $|\partial\mathcal{H}|$. Note that $\partial\widetilde{\mathcal{H}}$ is 12-partite with the vertex partition $\mathcal{P}$. Let $\mathcal{G}$ denote the blow up of $\mathcal{G}^2_{12}$ with $\mathcal{P}$. By (12), for each $e\in\partial\mathcal{G}\setminus\partial\widetilde{\mathcal{H}}$, there are at least $9(n/12-8\epsilon^{1/4}n)$ sets $E\in\mathcal{G}\setminus\widetilde{\mathcal{H}}$ such that $e\subset E$. Thus,

$$|\partial\mathcal{G}\setminus\partial\widetilde{\mathcal{H}}| \leqslant \frac{3|\mathcal{G}\setminus\widetilde{\mathcal{H}}|}{9(n/12-8\epsilon^{1/4}n)} \leqslant \frac{3\times 2\epsilon^{1/2}n^3}{9(n/12-8\epsilon^{1/4}n)} < 20\epsilon^{1/2}n^2,$$

and it follows that

$$|\partial\widetilde{\mathcal{H}}| > |\partial\mathcal{G}|-20\epsilon^{1/2}n^2 \geqslant \binom{12}{2}\left(\frac{n}{12}-8\epsilon^{1/4}n\right)^2-20\epsilon^{1/2}n^2 > \frac{11}{24}n^2-100\epsilon^{1/4}n^2.$$

Therefore,

$$|\partial\mathcal{H}| \geqslant |\partial\widetilde{\mathcal{H}}| > 11n^2/24 - 100\epsilon^{1/4}n^2. \tag{13}$$

Next, we give an upper bound for $|\partial\mathcal{H}|$. As in the proof of Claim 28, we can show $L_{\mathcal{H}}(v)[V_i] = \emptyset$ for $i \in [12]$ and every $v \in Z_\epsilon$. Thus,

$$|\partial\mathcal{H}| \leqslant \binom{12}{2}\left(\frac{\widetilde{n}}{12}\right)^2 + \widetilde{n}|Z_\epsilon| + \binom{|Z_\epsilon|}{2} < \frac{11}{24}n^2 + 100\epsilon^{1/4}n^2. \tag{14}$$

Combining (13) and (14), if $\widetilde{\mathcal{H}}$ is $\mathcal{G}^2_{12}$-colorable, then

$$\left||\partial\mathcal{H}| - \frac{11}{24}n^2\right| < 100\epsilon^{1/4}n^2$$

This completes the proof of Lemma 34. □

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Since $\mathcal{G}^1_n$ is $\mathcal{M}$-free and $|\partial\mathcal{G}^1_n| = \binom{n}{2}$, it follows from Observation 1.5 in [20] that $\mathrm{proj}\Omega(\mathcal{M}) = [0,1]$. Recall that $g(\mathcal{M})(x)$ is the feasible region function of $\mathcal{M}$. Theorem 4 shows that $g(\mathcal{M})(x) \leqslant 3/4$ for all $x \in [0,1]$ and $g(\mathcal{M})(11/12) = g(\mathcal{M})(1) = 3/4$. Now suppose that $(\mathcal{H}_k)_{k=1}^{\infty}$ is a sequence of $\mathcal{M}$-free 3-graphs with $\lim_{k\to\infty} v(\mathcal{H}_k) = \infty$, $\lim_{k\to\infty} \rho(\partial\mathcal{H}_k) = x_0$, and $\lim_{k\to\infty} \rho(\mathcal{H}_k) = 3/4$. For any sufficiently small $\epsilon > 0$ and sufficiently large N_0, there exists k_0 such that $v(\mathcal{H}_k) \geqslant N_0$ and $|\mathcal{H}_k| > (1/8 - \epsilon)(v(\mathcal{H}_k))^3$ for all $k \geqslant k_0$. Therefore, by Lemma 34, for every $k \geqslant k_0$ either

$$\frac{11}{12} - 200\epsilon^{1/4} < \frac{|\partial\mathcal{H}_k|}{\binom{v(\mathcal{H}_k)}{2}} < \frac{11}{12} + 200\epsilon^{1/4}$$

or

$$1 - 24\epsilon^{1/2} < \frac{|\partial\mathcal{H}_k|}{\binom{v(\mathcal{H}_k)}{2}} \leqslant 1$$

Letting $\epsilon \to 0$ we obtain either $x_0 = \frac{11}{12}$ or $x_0 = 1$, and this completes the proof. □

6 Concluding remarks

In this paper, we construct a finite family of triple systems $\mathcal{M}$, determine its Turán number, and prove that there are two near-extremal $\mathcal{M}$-free constructions that are far from each other in edit-distance. We believe that families of hypergraphs having the stability number large than one is an universal phenomenon in extremal combinatorics.

The forbidden family $\mathcal{M}$ is a suitably chosen family based on $\mathcal{G}^1_n$ and $\mathcal{G}^2_n$. Now we generalize $\mathcal{G}^1_n$ and $\mathcal{G}^2_n$. A 3-graph $\mathcal{H}$ is called *weak-ℓ-partite* if $V(\mathcal{H})$ has a partition $V_1 \cup \ldots \cup V_\ell$ such that $\mathcal{H}[V_i] = \emptyset$ for every $i \in [\ell]$. For an integer $t \geqslant 2$, let $\mathcal{F}^t_{3t^2}$ be the

3-graph with vertex set $[3t^2]$ whose complement is a perfect matching. For $n > 3t^2$, let $\mathcal{H}_n^t$ be the complete balanced weak-t-partite 3-graph on n vertices and $\mathcal{F}_n^t$ be a 3-graph on n vertices which is a blowup of $\mathcal{F}_{3t^2}^t$ with the maximum number of edges. It is easy to see that $\mathcal{H}_n^2 = \mathcal{G}_n^1$ and $\mathcal{F}_n^2 = \mathcal{G}_n^2$. An obvious calculation shows

$$|\mathcal{H}_n^t| \approx \binom{n}{3} - t\binom{\frac{n}{t}}{3} \approx \frac{t^2-1}{6t^2}n^3$$

and

$$|\mathcal{F}_n^t| \approx \left(\binom{3t^2}{3} - t^2\right)\left(\frac{n}{3t^2}\right)^3 \approx \frac{t^2-1}{6t^2}n^3.$$

It is natural to ask the following problem.

Problem 35. Let $t \geqslant 2$ be an integer. Does there exist a finite family of triple systems $\mathcal{M}$ such that there are exactly two near-extremal $\mathcal{M}$-free constructions $\mathcal{H}_n^t$ and $\mathcal{F}_n^t$?

This paper only addresses the problem when $t = 2$. After the submission of this paper, Liu and Pikhurko [24] gave a more general result, which implies Theorem 5 and a weaker form of Theorem 4 and answers Problem 35. We refer the reader to [24] for more details.

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