Infinite Families of (q^2+1)-Tight Sets of Quadrics with an Automorphism Group $\mathrm{PSp}(n,q)$, $n=4,6$

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Abstract

We present new infinite families of (q^2+1)–tight sets of quadrics admitting the symplectic group $\mathrm{PSp}(n,q)$, $n=4,6$, as an automorphism group. Our constructions rely on the geometry of Veronese and Grassmann varieties.

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1 Introduction

Let $\mathcal{P}^r$ be a finite classical polar space of rank $r \geqslant 2$. A *tight* set is a subset $\mathcal{M}$ of points of $\mathcal{P}^r$ such that for all points P of $\mathcal{P}^r$, the intersection size $|P^\perp \cap \mathcal{M}|$ assumes exactly two values in terms of a certain parameter $i \geqslant 1$, according as $P \in \mathcal{M}$ or not, where $P^\perp$ denotes the subset of points of $\mathcal{P}^r$ that are collinear with P. Together with the notion of tight set is that of m–ovoid generalizing the classical definition of ovoid of $\mathcal{P}^r$. An m–ovoid is a subset of $\mathcal{P}^r$ met in m points by every maximal. In the literature tight sets and m–ovoids are named *intriguing sets* and they are closely connected with other combinatorial objects, such as strongly regular graphs, partial difference sets, Boolean degree one functions and the very well studied Cameron–Liebler line classes. Many examples of tight sets exist in finite polar spaces having low rank and generally tight sets with large parameters have an interesting underlying geometry and sometimes seem to be related to certain irreducible or absolutely irreducible representations of finite classical groups. For more details and results on intriguing sets, see [2]. In this paper, we construct new infinite families of (q^2+1)–tight sets in orthogonal polar spaces with an automorphism group isomorphic to the symplectic group $\mathrm{PSp}(n,q)$, $n=4,6$. As far as we know, no non trivial examples with the same parameter were previously known. Our construction relies on the geometry of the Veronese embedding of degree 2 of the finite projective space $\mathrm{PG}(3,q)$ and on the Grassmannian of lines of $\mathrm{PG}(5,q)$.

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2 Intriguing sets

Let V be a n-dimensional vector space over the finite field $\mathrm{GF}(q)$ equipped with a non–degenerate sesquilinear form or quadratic form f and let $\mathcal{P}^r$ be the associated polar space. A point of $\mathcal{P}^r$ is a 1-dimensional totally isotropic or singular subspace of V. A maximal totally isotropic or singular subspace of $\mathcal{P}^r$ is named a *generator* of $\mathcal{P}^r$ and all generators have the same dimension, say r, called the rank of the polar space. An *ovoid* of $\mathcal{P}^r$ is a set of points that meets each generator in exactly one point and its size is usually denoted by θ_r. Assuming $r \geqslant 2$, a subset $\mathcal{M}$ of $\mathcal{P}_r$ is said to be an *intriguing set* if there exist constants $h_1 \neq h_2$ such that $|P^\perp \cap \mathcal{M}| = h_1$ or h_2, according as $P \in \mathcal{M}$ or not, where P ranges over the points of $\mathcal{P}_r$. There are exactly two types of intriguing sets:
1- i–tight sets: in this case $|\mathcal{M}| = i(q^r - 1)/(q-1)$, $h_1 = q^{r-1} + i((q^{r-1}-1)/(q-1)$ and $h_2 = i((q^{r-1}-1)/(q-1)$;
2. m–ovoids: $|\mathcal{M}| = m\theta_r$, $h_1 = (m-1)\theta_{r-1} + 1$ and $h_2 = m\theta_{r-1}$

Notice that, if H is a subgroup of a finite classical group having exactly two orbits on points of $\mathcal{P}_r$ then both orbits are always intriguing sets (and necessarily of the same type).

Usually, to prove that a subset $\mathcal{M}$ of $\mathcal{P}_r$ is an intriguing set one needs to prove that $\mathcal{M}$ is a two intersection set with respect to the perp of singular points. Interestingly, from [8, Lemma 2.1], if $\mathcal{M}$ is a subset of $\mathcal{P}_r$ of size $i(q^r-1)/(q-1)$ or $m\theta_r$, for some i or r and h_1 and h_2 are the parameters determined by $\mathcal{M}$ as above, then the condition for $\mathcal{M}$ to be an intriguing set is equivalent to each of the conditions:

$$|P^\perp \cap \mathcal{M}| = h_1 \text{ for all } P \in \mathcal{M}$$
$$|P^\perp \cap \mathcal{M}| = h_2 \text{ for all } P \in \mathcal{P}_r \setminus \mathcal{M}.$$

3 The quadric Veronesean $\mathcal{V}_3$ of $PG(9,q)$

Let q be a fixed prime power. For any integer k, denote by $\mathrm{PG}(k,q)$ the k–dimensional projective space over the Galois field $\mathrm{GF}(q)$. Let $n \geqslant 1$ be an arbitrary integer. We choose homogeneous projective coordinates in $\mathrm{PG}(n,q)$ and in $\mathrm{PG}(n(n+3)/2,q)$. The *Veronesean map of degree* 2, say ν, sends a point of $\mathrm{PG}(n,q)$ with coordinates $(x_0,\ldots,x_n)$ onto the points of $\mathrm{PG}(n(n+3)/2,q)$ with coordinates

$$(x_0^2, x_1^2, \ldots, x_n^2, x_0x_1, \ldots, x_0x_n, \ldots, x_{n-1}x_n).$$

The *quadric Veronesean* or the *Veronese variety* $\mathcal{V}_n^{2^n}$, or, for short $\mathcal{V}_n$, is the image of the Veronesean map. It turns out that the Veronesean map is a bijection between points of $\mathrm{PG}(n,q)$ and points of $\mathcal{V}_n$.

Consider the quadric Veronesean $\mathcal{V}_3$ in $\mathrm{PG}(9,q)$ and the correspondiing Veronesean map. The image of an arbitrary plane of $\mathrm{PG}(3,q)$ under the Veronesean map is a quadric Veronesean $\mathcal{V}_2$ and the the subspace of $\mathrm{PG}(9,q)$ generated by it has dimension 5. Such a subspace is called a $\mathcal{V}_2$*–subspace* of $PG(9,q)$. The image of a line of $\mathrm{PG}(3,q)$ is a conic and the plane generated by it is called a *conic plane*. There are $(q^2+1)(q^2+q+1)$ conic planes and two conic planes are either disjoint or meet in a point of $\mathcal{V}_3$. For any q, the conic planes are contained in the chord variety $\mathcal{S}(\mathcal{V}_3)$ of $\mathcal{V}_3$ that is a hypersurface of degree 4 and has $(q^2+1)(q^4+q^3+q^2+q+1)$ points. As shown in [6], $\mathcal{S}(\mathcal{V}_3)$ is a two–character set with respect to hyperplanes. We also recall that $\mathcal{V}_3$ is a cap of $\mathrm{PG}(9,q)$ [9, Theorem 25.1.8].

The automorphism group M of $\mathcal{V}_3$ contains $\mathrm{PGL}(4,q)$ and acts transitively on conic planes of $\mathcal{V}_3$.

Assume q is odd. The stabilizer in M of a conic $\mathcal{C}$ of $\mathcal{V}_3$ induces a group N isomorphic to $\mathrm{PGL}(2,q)$ and has three orbits on the relevant conic plane, i.e., the conic $\mathcal{C}$, the orbit of internal points and the orbit of external points to $\mathcal{C}$. It follows that M has, apart from $\mathcal{V}_3$, two orbits of sizes $(q^2+1)(q^2+q+1)q(q-1)/2$ and $(q^2+1)(q^2+q+1)q(q+1)/2$ partitioning the pointset of $\mathcal{S}(\mathcal{V}_3)$. It is known that the hyperplane sections of the veronese variety $\mathcal{V}_n$ correspond to quadrics of $\mathrm{PG}(n,q)$ [9, Theorem 25.1.3]:

Proposition 1. *Let H be a hyperplane of $\mathrm{PG}(9,q)$. Then $|H\cap\mathcal{V}_3|\in\{q+1,q^2+1,q^2+q+1,(q+1)^2,2q^2+q+1\}$.*

From [6] we recall the following result.

Proposition 2. *$\mathcal{S}(\mathcal{V}_3)$ is a two–character set with respect to hyperplanes with characters $q^5+2q^4+2q^3+2q^2+q+1$ and $q^5+q^4+2q^3+2q^2+q+1$.*

4 Grassmannian

Let $\mathcal{G}(1,n,q)$ denote the set of two–dimensional subspaces of the vector space $V := \mathrm{GF}(q)^{n+1}$. A two–dimensional subspace of V corresponds to a line of the associated projective space $\mathrm{PG}(n,q)$ so that $\mathcal{G}(1,n,q)$ can be viewed as the set of such lines. Denote by $(\ldots,X_{ij},\ldots)$, with $0\leqslant i<j\leqslant n$, the homogeneous coordinates of $\mathrm{PG}(N,q) = PG(\Lambda^2(V)$, $N=n(n+1)/2-1$, where $\Lambda^2(V)$ denotes the exterior square of V. They are called the *Plücker coordinates* on $\mathcal{G}(1,n,q)$. If ℓ is a line of $\mathrm{PG}(n,q)$ defined by two points, say $P_1=P(X_1)$ and $P_2=P(X_2)$, we can associate with ℓ the point $\rho(\ell)=P(X_1\wedge X_2)\in \mathrm{PG}(N,q)$. The map of sets given by

$$\rho:\mathcal{G}(1,n,q)\to PG(N,q),$$

$$\ell\mapsto\rho(\ell)$$

is a well–defined map called the *Plücker embedding* of $\mathcal{G}(1,n,q)$. The subset $\mathcal{G}(1,n,q)$ is an algebraic variety called the *Grassmannian of lines* of $\mathrm{PG}(n,q)$ and it is intersection of quadrics [9, Theorem 24.1.6, Theorem 24.1.7]. Also from [9, Theorem 24.2.9], $\mathcal{G}(1,n,q)$ contains two systems of maximal subspaces. The first system consists of the $(n-1)$–spaces Π_{n-1} with $\rho^{-1}(\Pi_{n-1}$ the set of all lines through a common point; this system is called *Latin system* and its elements are called *Lains spaces*. The second system consists of planes Π_2 with $\rho^{-1}(\Pi_2)$ the set of lines contained in a common plane; this system is called the *Greek system* and its elements are called *Greek planes*. For more details, see [9].

A *linear complex* $\mathcal{L}$ of $\mathrm{PG}(n,q)$ is a set of lines whose Plučker coordinates satisfy a linear equation $\sum_{i<j} a_{ij}X_{ij} = 0$. There exists a one-to-one correspondence between linear complexes of $\mathrm{PG}(n,q)$ and hyperplane sections of $\mathcal{G}(1,n,q)$. Let $H : \sum_{i<j} a_{ij}X_{ij} = 0$ be a hyperplane of $\mathrm{PG}(N,q)$. We associate with H the $(n+1)\times(n+1)$ skew-symmetric matrix

$$A_H = \begin{pmatrix} 0 & a_{01} & \dots & a_{0n} \\ -a_{01} & 0 & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{0n} & -a_{1n} & \dots & 0 \end{pmatrix}$$

up to a non–zero scalar in $\mathrm{GF}(q)\setminus\{0\}$. The linear complex $\mathcal{L}$ is said to be of *type* r if the corresponding matrix A_H has rank r. A linear complex of type $n+1$ is said to be *non-singular*. It is known that the rank of a skew–symmetric matrix is an even number. Hence, in $\mathrm{PG}(n,q)$, n even, all linear complexes are singular.

Lemma 3. *Let $\ell = \langle P_0, P_1\rangle$ be a line of $\mathrm{PG}(n,q)$, with $P_0 = P(X)$ and $P_1 = P(Y)$ two distinct points of $\mathrm{PG}(n,q)$. The point $\rho(\ell)$ lies on the hyperplane H if and only if $XA_HY^t = 0$.*

Proof. Let $H : \sum_{i<j} a_{ij}X_{ij} = 0$, $X = (X_i)_{0\leqslant i\leqslant n}$, $Y = (Y_i)_{0\leqslant i\leqslant n}$ and $\rho(\ell) = (p_{ij})_{0\leqslant i<j\leqslant n}$, where $p_ij = X_iY_j - X_jY_i$. Since

$$\sum_{i<j} a_{ij}p_{ij} = \sum_{i<j} a_{ij}x_iy_j - \sum_{i<j} a_{ij}x_jy_i = 0,$$

we see that $XA_HY^t = 0$ if and only if $\rho(\ell) \in H$. □

Let $P = \rho(\ell)$ be a point on $\mathcal{G}(1,n,q)$ and consider the subspace $T_P(\mathcal{G}(1,n,q))$ given by the intersection of all tangent hyperplanes at P to all quadrics defining $\mathcal{G}(1,n,q)$. The subspace $T_P(\mathcal{G}(1,n,q))$ is called the *tangent space* to $\mathcal{G}(1,n,q)$ at the point P. Being $\mathcal{G}(1,n,q)$ non–singular, $\dim(T_P(\mathcal{G}(1,n,q))) = \dim\mathcal{G}(1,n,q)$ for all $P \in \mathcal{G}(1,n,q)$. From [7, Proposition 2.4], if Γ_ℓ denotes the set of all lines of $\mathrm{PG}(n,q)$ intersecting ℓ, then $T_P(\mathcal{G}(1,n,q))$ is generated by $\rho(\Gamma_\ell)$. In particular, [7, Prop. 2.5], if H is a hyperplane of $\mathrm{PG}(N,q)$ containing $T_P(\mathcal{G}(1,n,q))$ then ℓ is contained in $\mathrm{NullSpace}(A_H)$.

From now on we assume that $n = 4$ and hence $N = 9$.

Lemma 4. *[9, Lemma 24.2.15] The automorphism group K of $\mathcal{G}(1,4,q)$ in $\mathrm{PGL}(10,q)$ is isomorphic to $\mathrm{PGL}(5,q)$.*

Lemma 5. *[10, Lemma 2.5] The group K has two obits on points of $\mathrm{PG}(9,q)$ of size $(q^2+1)(q^4+q^3+q^2+q+1)$ and $(q^4+q^3+q^2+q+1)(q^3-1)q^2$, i.e., $\mathcal{G}(1,n,q)$ and its complement in $\mathrm{PG}(9,q)$.*

There are two possibilities for a hyperplane section of $\mathcal{G}(1,4,q)$ since a skew-symmetric matrix of order 5 has type 2 or 4. The corresponding singular linear complexes correspond to:

- (*type* 2) lines of $\mathrm{PG}(4,q)$ incident to or contained in a plane. In this case $\mathcal{L}$ consists of $(q^2+q+1)(q^3+q^2+1)$ lines
- (*type* 4) lines of $\mathrm{PG}(4,q)$ incident to or contained in a parabolic quadric $Q(4,q)$. In this case $\mathcal{L}$ consists of $(q+1)^2(q+1)$ lines.

Theorem 6. *The Grassmannian of lines of* $\mathrm{PG}(4,q)$ *is a two–character set of* $\mathrm{PG}(9,q)$ *with respect to hyperplanes. It has size* $(q^2+1)(q^4+q^3+q^2+q+1)$ *and the constants are* $(q^2+q+1)(q^3+q^2+1)$ *and* $(q+1)^2(q+1)$.

Theorem 7. *The Veronese variety* $\mathcal{V}_3$ *is a subset of* $\mathcal{G}(1,4,q)$.

Proof. The lines of $\mathrm{PG}(4,q)$ spanned by the rows of the matrix

$$\begin{pmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \end{pmatrix}.$$

represents the Veronese embedding of $\mathrm{PG}(3,q) \subset \mathrm{PG}(9,q)$ by means of its maximal minors, namely

$$(a,b,c,d) \mapsto (a^2, ab, ac, ad, b^2-ac, bc-ad, bd, c^2-bd, cd, d^2).$$

This is the variety $\mathcal{V}_3$ up to projectivities. □

5 $\mathcal{G}(1,4,q)$ and $\mathcal{S}(\mathcal{V}_3)$: the link

As already observed, $\mathcal{S}(\mathcal{V}_3)$ has size $(q^2+1)(q^4+q^3+q^2+q+1)$ that is exactly the number of points of $\mathcal{G}(1,4,q)$. Also, as a two–character set, $\mathcal{S}(\mathcal{V}_3)$ has the same characters of $\mathcal{G}(1,4,q)$ [6]. The connection between these two varieties is the well–known isomorphism $Sp(4,q) \simeq O(5,q)$, q any prime power, see [13], involving the classical generalized quadrangles $\mathcal{W}(3,q)$ and $Q(4,q)$. Here is the picture. Assume that q is odd. The projective space associated to the symmetric square of the natural 4-dimensional symplectic module, V_4, say $\mathrm{PG}(S^2(V_4))$, is where the variety $\mathcal{V}_3$ lives. In virtue of the isomorphism $Sp(4,q) \simeq O(5,q)$, considering the exterior square of the natural $O(5,q)$–module, say V_5, we see that also $\mathcal{G}(1,4,q)$ lives in $\mathrm{PG}(S^2(V_4))$ as well. This geometric context describes an action of $Sp(4,q)$ on the 10-dimensional module $\Lambda^2(V_5)$. In the paper [5] the authors actually showed that $\Lambda^2(V_5)$ is isomorphic to $S^2(V_4)$ as $Sp(4,q)$- modules. For more details on modules, see [3]. In other terms, the Plücker image of the singular lines of the parabolic quadric Q stabilized by $O(5,q)$ coincides with the pointset of the Veronese variety $\mathcal{V}_3$. In this correspondence, conics of $\mathcal{V}_3$ arising from totally isotropic lines of $\mathrm{PG}(V_4)$ are quadratic cones on Q with vertex a point P on Q. The other tangent lines to Q at P, not contained in Q, correspond to the set of internal points and external points of the associated conic on $\mathcal{V}_3$. We call $\mathcal{V}_3$ together with its conics corresponding to totally isotropic lines of $\mathrm{PG}(V_4)$ the *Symplectic Veronese variety* $\mathcal{V}_3^s$, and $\mathcal{V}_3$ together with conics corresponding to non isotropic lines of $\mathrm{PG}(V_4)$ the *non isotropic Veronese variety* $\mathcal{V}_3^n$. From our discussion above we have the following result.

Proposition 8. $\mathcal{S}(\mathcal{V}_3^s) = \mathcal{G}(1,4,q) \cap \mathcal{S}(\mathcal{V}_3)$

In other terms, to get $\mathcal{S}(\mathcal{V}_3)$ from $\mathcal{S}(\mathcal{V}_3^s)$ we need to add points on conic planes of $\mathcal{V}_3^n$. Their number is q^4+q^2. On the other hand to get $\mathcal{G}(1,4,q)$ we have to add points of $\mathrm{PG}(9,q)$ corresponding, under the Plücker map, to secant lines to Q and external lines to Q.

In the next section we show that $\mathcal{S}(\mathcal{V}_3^s)$ is also contained in a non–degenerate quadric of $\mathrm{PG}(9,q)$.

6 $\mathbf{PSp(4,q) \leqslant P\Omega^{\pm}(10,q)}$, q odd

Let $V_4 = V(4,q)$, $q=p^h$, p odd prime, $h \geqslant 1$, be a 4–dimensional vector space over the finite field $\mathrm{GF}(q)$. Let f be a non–degenerate symplectic form on V_4. We denote by $\mathrm{Sp}(4,q)$ the group of all isometries of the symplectic space (V_4, f) and by $PSp(4,q)$ its projective image, i.e. the group $Sp(4,q)/\langle \pm I \rangle$ acting faithfully on the projective space $\mathrm{PG}(3,q)$ associated to V_4. By choosing a suitable basis for V_4 we may assume that f is represented by the matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

and we may identify $\mathrm{Sp}(4,q)$ with the group of all matrices $A \in \mathrm{GL}(4,q)$ such that $A^T X A = X$.

The $\mathrm{Sp}(4,q)$–module $V_4 \otimes V_4$ has dimension 16. The group $\mathrm{Sp}(4,q)$ acts on this via: $(u\otimes v)g = (ug \otimes vg)$ and the form on this 16-dimensional module is $f^2(u\otimes v, w \otimes z) = f(u,w)f(v,z)$. With this action, the tensor square is reducible. From [3, Def. 5.2.1] one submodule is the *symmetric square* of V_4, $S^2(V_4)$. From [3, Proposition 5.2.4] the symmetric square of the matrix X (after rescaling the basis of V_4) is

$$S^2(X) = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&2 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&-4&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&-4&0&0&0&0 \\ 0&0&-1&0&0&0&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 2&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The matrix $S^2(X)$ is the Gram matrix of the symmetric form F^2 on the 10–dimensional vector space $S^2(V_4)$ [3, Prop. 5.2.4]. From a projective point of view, the set of singular vectors with respect to F^2 in $\mathrm{PG}(9,q) = \mathrm{PG}(S^2(V_4))$ is the non–degenerate quadric $\mathcal{Q}$

with equation $2X_1X_{10} + X_2X_9 - X_3X_7 + 2X_5X_8 - 2X_4^2 - 2X_6^2 = 0$, where $X_1, \dots, X_{10}$ are projective homogeneous coordinates of $\mathrm{PG}(9, q)$. Since the determinant of the matrix $S^2(X)$ is 16^2, the quadric $\mathcal{Q}$ is elliptic if $q \equiv 3 \bmod 4$ and hyperbolic if $q \equiv 1 \bmod 4$. Notice that $-I_4 \in \mathrm{Sp}(4, q)$ acts trivially on $S^2(V_4)$ hence the image of $\mathrm{Sp}(4, q)$ in this representation is $G = \mathrm{PSp}(4, q)$.

Let $\mathcal{V}_3$ be the Veronese variety of $\mathrm{PG}(9, q)$ with parametric equations:

$$(a^2, ab, ac, ad, b^2, bc, bd, c^2, cd, d^2),$$

where a, b, c, d vary in $\mathrm{GF}(q)$ and $(a, b, c, d) \neq (0, 0, 0, 0)$. An easy check show that $\mathcal{V}_3$ is a subset of $\mathcal{Q}$. The group $\mathrm{PSp}(4, q)$ is transitive on totally isotropic points, totally isotropic lines and non-isotropic lines of $\mathrm{PG}(3, q)$. Without loss of generality consider the line $\ell : (0, 0, \alpha, \beta)$, $\alpha, \beta \in \mathrm{GF}(q)$, $(\alpha, \beta) \neq (0, 0)$. Then ℓ is a totally isotropic line and its image under ν is the conic, $\mathcal{C}$, with parametric equations $(0, 0, 0, 0, 0, 0, 0, c^2, cd, d^2)$ in the plane $\pi : X_1 = \cdots = X_7 = 0$. Also π is totally singular for $\mathcal{Q}$. Of course $\mathcal{C} = \mathcal{V}_3 \cap \pi$. Since $\mathrm{PSp}(4, q)$ is transitive on totally isotropic lines, we get $q^3 + q^2 + q + 1$ planes of $\mathcal{Q}$ each intersecting $\mathcal{V}_3$ in a conic and two such conics either are disjoint or meet in a point necessarily belonging to $\mathcal{V}_3$. These are the conic planes of $\mathcal{V}_3$ corresponding to totally isotropic lines of $\mathrm{PG}(3, q)$. Considering the pointset of $\mathcal{Q}$ covered by these conic planes we get a subset of $\mathcal{Q}$ consisting of $q^3 + q^2 + q + 1 + q^2(q^3 + q^2 + q + 1) = (q^2 + 1)(q^3 + q^2 + q + 1)$ points. Actually, this pointset is the chord variety $\mathcal{S}(\mathcal{V}_3^s)$ arising from the Symplectic Veronese variety introduced in the previous section.

Theorem 9. *The set $\mathcal{S}(\mathcal{V}_3^s)$ is a $(q^2 + 1)$–tight set of $\mathcal{Q}^-(9, q)$, $q \equiv 3$ mod (4).*

Proof. Let P be a point of $\mathcal{V}_3$. By transitivity, we may choose $P = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. Then $P^\perp : X_{10} = 0$ and $P^\perp$ meets $\mathcal{V}_3$ in the set $(a^2, ab, ac, b^2, bc, c^2, 0, 0, 0, 0)$, $a, b, c \in GF(q)$ and $(a, b, c) \neq (0, 0, 0)$. This is a Veronese surface $\mathcal{V}_2$ lying in a projective 5-subspace T of $\mathrm{PG}(9, q)$ and it arises from the Veronese embedding of a plane σ of $\mathrm{PG}(3, q)$. A plane of $\mathrm{PG}(3, q)$ contains $q + 1$ totally isotropic lines and any other totally isotropic line of $\mathrm{PG}(3, q)$ meets σ in a point. It follows that $P^\perp$ meets $\mathcal{S}(\mathcal{V}_3^s)$ in $q^2 + q + 1 + (q + 1)q^2 + (q^3 + q^2)q = q^4 + 2q^3 + 2q^2 + q + 1 = q^3 + (q^2 + 1)(q^2 + q + 1)$ points. The stabilizer of the conic $\mathcal{C}$ in G has three orbits on π: the conic $\mathcal{C}$, the set of internal points to $\mathcal{C}$ of size $q(q - 1)/2$ and the set of external points to $\mathcal{C}$ of size $q(q + 1)/2$. It follows that $\mathcal{S}(\mathcal{V}_3^s)$ is the union of three G–orbits of sizes $q^3 + q^2 + q + 1$, $q(q - 1)(q^3 + q^2 + q + 1)/2$ and $q(q + 1)(q^3 + q^2 + q + 1)/2$. Call the last two orbits $\mathcal{I}$ and $\mathcal{E}$, respectively.

Take a point $P \in \mathcal{I}$. Then P is internal to a conic of $\mathcal{V}_3$ in a conic plane. Without loss of generality, we can assume that $P \in \pi$. It is easy to see that the set of internal points to $\mathcal{C}$ is the set $\{(a^2, a, 1 - \theta^2\lambda) : a \in \mathrm{GF}(q)^*\} \cup \{(1, 0, -\theta^2\lambda)\}$, where λ is a non square in $\mathrm{GF}(q)$. Then P can always be chosen as the point $(0, 0, 0, 0, 0, 0, 0, 1, 0, -\mu)$, μ a non square in $\mathrm{GF}(q)$. It follows that $P^\perp : X_5 = \mu X_1$. There are $q^2 + q$ plane conics meeting $\mathcal{C}$ in a point. Of course $P^\perp$ contains π and hence the set of internal points I to $\mathcal{C}$. We have that $P^\perp$ meets $\mathcal{V}_3$ in the set where $b^2 = \lambda a^2$. This means necessarily that $a = b = 0$ and hence $P^\perp$ meets $\mathcal{V}_3$ just in $\mathcal{C}$. $P^\perp$ meets the $q^2 + q$ planes above in a line that is tangent to $\mathcal{C}$ and so does not meet the set of internal points of the relevant conic and the remaining q^3 conic planes in an external line to the relevant conic and hence containing $(q+1)/2$ internal points. Then $P^\perp$ meets $\mathcal{I}$ in $q(q-1)/2+q^3(q+1)/2$ points and $\mathcal{E}$ in $q(q+1)/2+q(q^2+q)+q^3(q+1)/2$.

We have that

$$q+1+q(q-1)/2+q^3(q+1)/2+q(q+1)/2+q(q^2+q)+q^3(q+1)/2 = q^3+(q^2+1)(q^2+q+1).$$

Take a point $P \in \mathcal{E}$. Again, without loss of generality we may choose P external to the conic $\mathcal{C}$ in the plane π, say $P = (0,0,0,0,0,0,0,0,1,0)$. Then $P^\perp : X_2 = 0$. There are q^2+q plane conics meeting $\mathcal{C}$ in a point. Of course $P^\perp$ contains π and hence the set of external points E to $\mathcal{C}$. We have that $P^\perp$ meets $\mathcal{V}_3$ in $2q^2+q+1$ points (correspondent to two planes in $\mathrm{PG}(3,q)$ sharing a totally isotropic line). Indeed, $P^\perp$ meets $\mathcal{V}_3$ in the subset of $\mathcal{V}_3$ for which $ab = 0$. This means that $P^\perp$ contains $2q+1$ conic planes, π included. Also $P^\perp$ meets q^2-q planes in a line that is tangent to $\mathcal{C}$ and so contains q points of the subset of external points of the relevant conic and the remaining q^3 conic planes in a secant line and hence containing $(q-1)/2$ external points. Then $P^\perp$ meets $\mathcal{E}$ in $(2q+1)q(q+1)/2+q(q^2-q)+q^3(q-1)/2$ points. It follows that $P^\perp$ meets $\mathcal{I}$ in $(2q+1)q(q-1)/2+q^3(q-1)/2$ points. We have that

$$2q^2+q+1+(2q+1)q(q-1)/2+q^3(q-1)/2+(2q+1)q(q+1)/2+q(q^2-q)+q^3(q-1)/2 =$$

$$= q^3+(q^2+1)(q^2+q+1).$$

We have proved that for any $P \in \mathcal{S}(\mathcal{V}_3^s)$, $P^\perp$ meets $\mathcal{S}(\mathcal{V}_3^s)$ in $q^3+(q^2+1)(q^2+q+1)$ points. The result now follows from [8, Lemma 2.1]. □

Corollary 10. *The full stabilizer H of $\mathcal{S}(\mathcal{V}_3^s)$ in $\mathrm{P}\Omega^\pm(10,q)$ is $2 \times \mathrm{PSp}(4,q)$.*

Proof. Clearly $PSp(4,q) \leqslant H$. Let $A = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where ω is a primitive element of $\mathrm{GF}(q)$. Then $A^T X A = \omega X$ and hence $A \in CSp(4,q)$, the conformal symplectic group. The symmetric square of A multiplied by $\omega^{-1} I_{10}$ is the matrix $\mathrm{diag}(\omega,1,\omega,1,\omega^{-1},1,\omega^{-1},\omega,1,\omega^{-1})$ that belongs to $SO^\pm(10,q)$. It is easy to check that its spinor norm is a non–square in $\mathrm{GF}(q)$ and hence the diagonal automorphism δ' of $\Omega^\pm(10,q)$ induces the involutorial diagonal automorphism δ of $\mathrm{Sp}(4,q)$. For more details, see [3, Prop. 5.5.2]. Also, H is a maximal subgroup of $\mathrm{P}\Omega^\pm(10,q)$, see [3, Table 8.67, Table 8.69]. □

Proposition 11. *$\mathcal{S}(\mathcal{V}_3^s) \subset \mathcal{Q}^-(9,q)$ cannot be the union of q^2+1 mutually disjoint three–dimensional subspaces*

Proof. Apart from conic planes, $\mathcal{S}(\mathcal{V}_3^s)$ contains another family $\mathcal{T}$ of q^3+q^2+q+1 planes that are tangent to $\mathcal{V}_3$: it is easy to check that the plane $\tau : X_1 = X_2 = X_4 = X_5 = X_6 = X_7 = X_{10} = 0$ is tangent to $\mathcal{V}_3$ at the point $R = (0,0,0,0,0,0,0,1,0,0)$ and τ

meets the conic plane $\pi : X_1 = \cdots = X_7 = 0$ along the line $t : X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_{10} = 0$ that is tangent to the conic $\mathcal{C}$, with parametric equations $(0,0,0,0,0,0,0,c^2,cd,d^2)$. We have that τ is a singular plane of $\mathcal{Q}^{\pm}(9,q)$. Under the action of G we get the family $\mathcal{T}$. The plane τ contains the $q+1$ tangent lines to the conics of $\mathcal{V}_3$ passing through the point R and hence τ and its G–orbit is contained in $\mathcal{S}(\mathcal{V}_3^s)$. From Proposition 8 the dimension of a projective subspace of $\mathcal{S}(\mathcal{V}_3^s)$ is at most three. In particular, the three–space $\langle \tau, \pi \rangle$ is singular for $\mathcal{Q}^{\pm}(9,q)$ and it meets $\mathcal{S}(\mathcal{V}_3^s)$ in $2q^2+q+1$ points and so cannot lie in $\mathcal{S}(\mathcal{V}_3^s)$. This excludes the possibility, in the elliptic case, that $\mathcal{S}(\mathcal{V}_3^s)$ is the union of q^2+1 mutually disjoint three–dimensional subspaces, as desired. □

Remark 12. When q is even, the symmetric square representation of the symplectic group $\mathrm{PSp}(4,q)$ is reducible in $\mathrm{PGL}(10,q)$ and no non degenerate polar space is involved. For more details see [12].

7 Higher dimensions

In this section we provide other infinite families of (q^2+1)–tight sets of $\mathcal{Q}^-(13,q)$, $q=p^h$ odd, h odd, and $p \equiv 11 \bmod (12)$ or $q=2^h$, h odd, and of $\mathcal{Q}(12,3^h)$. Our construction relies on the geometry of the Grassmannian, say $\mathcal{I}$, of totally isotropic lines of $\mathrm{PG}(5,q)$, with respect to a symplectic polarity $\mathcal{N}$ with isometry group $\mathrm{PSp}(6,q)$. Notice that $|\mathcal{I}| = (q^2+1)(q^5+q^4+q^3+q^2+q+1)$.

Lemma 13. *Let ℓ be a totally isotropic line of $\mathrm{PG}(5,q)$ with respect to $\mathcal{N}$. The stabilizer S of ℓ in $\mathrm{PSp}(6,q)$ has five orbits on totally isotropic lines of $\mathcal{N}$.*

Proof. As already observed, the number of totally isotropic lines of $\mathrm{PG}(5,q)$ with respect to $\mathcal{N}$ is $(q^2+1)(q^3+1)(q^2+q+1)$. Of course $\{\ell\}$ is an orbit. There are $q+1$ totally isotropic planes on ℓ permuted by S each containing q^2+q lines distinct from ℓ and again permuted by S. We get an orbit $\mathcal{O}_1$ of size $q(q+1)^2$. On each point of $\mathrm{PG}(5,q)$ there are q^3+q^2+q+1 totally isotropic lines. Fix $P \in \ell$. There are q^3 totally isotropic lines on P not lying in totally isotropic planes on ℓ. Varying P on ℓ under S we get an orbit $\mathcal{O}_2$ of size $q^3(q+1)$. This means that there are $(q+1)(q^6+2q^4-q^3+q^2-q+1)-1$ totally isotropic lines disjoint from ℓ. On a point $P \in \ell$ there are q^3+q^2 totally isotropic planes not containing ℓ and permuted by S. If two such planes meet in a line, such a line necessarily passes through P. In each such plane there are q^2 lines disjoint from ℓ. We get an orbit $\mathcal{O}_3$ of size $q^4(q+1)^2$. The last orbit $\mathcal{O}_4$ of S has size q^7 consisting of totally isotropic lines disjoint from $\ell^{\mathcal{N}}$. Notice that $|\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3| = q^6+2q^5+2q^4+2q^3+2q^2+q+1 = q^5+(q^2+1)(q^4+q^3+q^2+q+1)$. □

Proposition 14. *The Grassmannian $\mathcal{I}$ is embedded in an elliptic quadric $\mathcal{Q}^-(13,q)$, $q=p^h$ odd, h odd, and $p \equiv 11$ mod (12) or $q=2^h$, h odd.*

Proof. From [11, Prop. 9.3.3 iv), Prop. 9.3.4], the group $\mathrm{PSp}(6,q)$ can be realized as a subgroup of $\mathrm{P}\Omega^-(14,q)$. Indeed, $Sp(6,q)$ acts on a 14–dimensional submodule of $\Lambda^2(V_6)$, where V_6 is the underlying vector space of $\mathrm{PG}(5,q)$, yielding $\mathcal{I}$ as a $\mathrm{PSp}(6,q)$–orbit on $\mathcal{Q}^-(13,q)$. In other terms, $\mathcal{I}$ is obtained intersecting $\mathcal{G}(1,5,q)$ with $\mathcal{Q}^-(13,q)$. □

Theorem 15. *The symplectic Grassmannian $\mathcal{I}$ is a (q^2+1)-tight set of $\mathcal{Q}^-(13,q)$ admitting* $\mathrm{PSp}(6,q)$ *as an automorphism group.*

Proof. Let P be a point of $\mathcal{I}$. Then the tangent space $T_P(\mathcal{G}(1,5,q))$ is contained in the hyperplane $P^\perp$, where $\perp$ is the orthogonal polarity of $\mathrm{PG}(13,q)$ associated to $\mathcal{Q}^-(13,q)$. We want to establish how $P^\perp$ meets $\mathcal{I}$. What we know is that a hyperplane section of $\mathcal{G}(1,5,q)$ corresponds to a non–degenerate linear complex of $\mathrm{PG}(5,q)$, hence to a symplectic polarity $\mathcal{M}$ of $\mathrm{PG}(5,q)$. It follows that $|P^\perp \cap \mathcal{I}|$ corresponds to the number η of totally isotropic lines of $\mathrm{PG}(5,q)$ which are simultaneously totally isotropic for both the polarities $\mathcal{N}$ and $\mathcal{M}$.

From [4, Section 3],

$$(q+1)\eta = q^3 N_1 + (q^6-1)(q^3-1)/(q-1)^2,$$

where $N_1 = |\{p \in \mathrm{PG}(5,q) : p^{\mathcal{N}} \subseteq p^{\mathcal{M}}\}|$. Now, since $T_P(\mathcal{G}(1,5,q)) \subset P^\perp$, we have that $N_1 = q^3+q^2+q+1$. This yields $\eta = q^5 + (q^2+1)(q^4+q^3+q^2+q+1)$. It follows that $P^\perp \cap \mathcal{I}$ consists of the points corresponding to $\rho(\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3)$. □

Of course, if $P \in \mathcal{Q}^-(13,q) \setminus \mathcal{I}$, then $P^\perp$ meets $\mathcal{I}$ in $(q^2+1)(q^4+q^3+q^2+q+1)$ points. From [4, Section 3], in this case $N_1 = q+1$ and it seems that a possible configuration in $\mathrm{PG}(5,q)$ consists of a non-isotropic line joining the center P of a star, say $\mathcal{L}$, of totally isotropic lines, with a point Q not in the subspace $P^{\mathcal{N}}$.

For completeness we also consider the case when $P \in \mathcal{G}(1,5,q)$ corresponds to a non–isotropic line.

Lemma 16. *Let ℓ be a non–isotropic line of* $\mathrm{PG}(5,q)$ *with respect to $\mathcal{N}$. The stabilizer S of ℓ in* $\mathrm{PSp}(6,q)$ *has four orbits on totally isotropic lines of $\mathcal{N}$.*

Proof. A first orbit $\mathcal{R}_1$ consists of the q^3+q^2+q+1 totally isotropic lines of ℓ^N. The lines of the form $\langle P, Q\rangle$ where $P \in \ell$ and $Q \in P^{\mathcal{N}}$ are totally isotropic and form an orbit $\mathcal{R}_2$ of size $(q+1)(q^3+q^2+q+1)$. If $P \in \ell^{\mathcal{N}}$, then there are q^3+q^2+q+1 totally isotropic lines on P of which $q+1$ are in $\ell^{\mathcal{N}}$. We get $(q^3+q^2+q+1)(q^3+q^2)$ totally isotropic lines of which $(q+1)(q^3+q^2+q+1)$ have already been counted. By difference, we get an orbit $\mathcal{R}_3$ of size $(q^3+q^2+q+1)(q^3+q^2+1)$. All the remaining totally isotropic lines are permuted in a single orbit, say $\mathcal{R}_4$. □

From the above corollary, we can say that if P is a point on the Grassmannian $\mathcal{G}(1,5,q)$ corresponding to ℓ, then $P^\perp$ meets $\mathcal{I}$ in $(q^3+q^2+q+1)(q^3+q^2+1)$ points. Indeed, from [4, Section 3], if $N_1 = q^3+q^2+2q+2$, then $\eta = (q^3+q^2+q+1)(q^3+q^2+1)$.

When $q = 3^i$, from [11, Prop. 9.3.1], $\mathrm{PSp}(6,3^i) \leqslant \Omega(13,3^i)$. This depends on the fact that in the 14–dimensional representation of $\mathrm{PSp}(6,3^i)$ the orthogonal form is not non–degenerate and there exists a point of $\mathrm{PG}(13,3^i)$ respect to which all points of $\mathrm{PG}(13,3^i)$

are conjugate. Then one can pass to the quotient geometry $\mathrm{PG}(13, 3^i)/P$ on which we have a non–degenerate orthogonal form. For further results on this 13–dimensional representation, see [1, Section 4].

Theorem 17. *Assume $q = 3^i$. The symplectic Grassmannian $\mathcal{I}$ is a (q^2+1)-tight set of $\mathcal{Q}(12, q)$ admitting $\mathrm{PSp}(6, q)$ as an automorphism group.*

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