Spectral Turán-Type Problems on Cancellative Hypergraphs

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Abstract

Let G be a cancellative 3-uniform hypergraph in which the symmetric difference of any two edges is not contained in a third one. Equivalently, a 3-uniform hypergraph G is cancellative if and only if G is $\{F_4, F_5\}$-free, where $F_4 = \{abc, abd, bcd\}$ and $F_5 = \{abc, abd, cde\}$. A classical result in extremal combinatorics states that the maximum size of a cancellative hypergraph is achieved by the balanced complete tripartite 3-uniform hypergraph, which was firstly proved by Bollobás and later by Keevash and Mubayi. In this paper, we consider spectral extremal problems for cancellative hypergraphs. More precisely, we determine the maximum p-spectral radius of cancellative 3-uniform hypergraphs, and characterize the extremal hypergraph. As a by-product, we give an alternative proof of Bollobás' result from spectral viewpoint.

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1 Introduction

Consider an r-uniform hypergraph (or r-graph for brevity) G and a family of r-graphs $\mathcal{F}$. We say G is *$\mathcal{F}$-free* if G does not contain any member of $\mathcal{F}$ as a subhypergraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges of an $\mathcal{F}$-free hypergraph on n vertices. Determining Turán numbers of graphs and hypergraphs is one of the central problems in extremal combinatorics. For graphs, the problem was asymptotically solved for all non-bipartite graphs by the celebrated Erdős-Stone-Simonovits Theorem [5, 6]. By contrast with the graph case, there is comparatively little understanding of the hypergraph Turán number. We refer the reader to the surveys [8, 11, 14].

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In this paper we consider spectral analogues of Turán-type problems for r-graphs. For $r = 2$, the picture is relatively complete, due in large part to a longstanding project of Nikiforov, see e.g., [15] for details. However, for $r \geqslant 3$ there are very few known results. In [12], Keevash, Lenz and Mubayi determined the maximum p-spectral radius of any 3-graph on n vertices not containing the Fano plane when n is sufficiently large. They also obtained a p-spectral version of the Erdős-Ko-Rado theorem on t-intersecting r-graphs. Recently, Ellingham, Lu and Wang [4] showed that the n-vertex outerplanar 3-graph of maximum spectral radius is the unique 3-graph whose shadow graph is the join of an isolated vertex and the path P_{n-1}. Gao, Chang and Hou [9] studied the spectral extremal problem for K_{r+1}^{+}-free r-graphs among linear hypergraphs, where K_{r+1}^{+} is the r-expansion of the complete graph K_{r+1}, i.e., K_{r+1}^{+} is obtained from K_{r+1} by enlarging each edge of K_{r+1} with $(r-2)$ new vertices disjoint from $V(K_{r+1})$ such that distinct edges of K_{r+1} are enlarged by distinct vertices. Generalizing Gao-Chang-Hou's result, She, Fan, Kang and Hou [18] considered the linear spectral Turán type problems for the expansion of a color critical graph.

To state our results precisely, we need some basic definitions and notations. A 3-graph is *tripartite* or 3-*partite* if it has a vertex partition into three parts such that every edge has exactly one vertex in each part. Let $T_3(n)$ be the complete 3-partite 3-graph on n vertices with part sizes $\lfloor n/3 \rfloor$, $\lfloor (n+1)/3 \rfloor$, $\lfloor (n+2)/3 \rfloor$, and $t_3(n)$ be the number of edges of $T_3(n)$. That is,

$$t_3(n) = \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n+1}{3} \right\rfloor \cdot \left\lfloor \frac{n+2}{3} \right\rfloor.$$

We call an r-graph G *cancellative* if G has the property that for any edges A, B, C whenever $A \cup B = A \cup C$, we have $B = C$. Equivalently, G is cancellative if G has no three distinct triples A, B, C satisfying $B \triangle C \subset A$, where $\triangle$ is the symmetric difference. For graphs, the condition is equivalent to saying that G is triangle-free. Moving on to 3-graphs, we observe that $B \triangle C \subset A$ can only occur when $|B \cap C| = 2$ for $B \neq C$. This leads us to identify the two non-isomorphic configurations that are forbidden in a cancellative 3-graph: $F_4 = \{abc, abd, bcd\}$ and $F_5 = \{abc, abd, cde\}$.

It is well-known that the study of Turán numbers dates back to Mantel's theorem, which states that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. As an extension of the problem to hypergraphs, Katona conjectured (c.f. [1]), and Bollobás proved the following result.

Theorem 1 ([1]). *A cancellative 3-graph on n vertices has at most $t_3(n)$ edges, with equality only for $T_3(n)$.*

In [10], Keevash and Mubayi presented a new proof of Bollobás' result, and further proved a stability theorem for cancellative hypergraphs.

The aim of this paper is to establish a p-spectral analogue of Theorem 1. One motivation for studying this problem is that the spectral radius of an r-graph G is an upper bound for the average degree of G, and hence any upper bound on the spectral radius also gives an upper bound on the size of G. It is important to note that we will not rely on the usage of Theorem 1 to accomplish the proof of our main result (Theorem 10), although one can simplify the proof with the help of Theorem 1. As a by-product, we give an alternative proof of Bollobás' result for the case $3 \mid n$ from spectral viewpoint. Our main results can be summarized as follows.

Theorem 2. *Let $p \geqslant 1$ and G be a cancellative 3-graph on n vertices.*

(1) *If $p \geqslant 3$, then $\lambda^{(p)}(G) \leqslant \lambda^{(p)}(T_3(n))$, with equality if and only if $G = T_3(n)$.*

(2) *If $p = 1$ and G has at least one edge, then $\lambda^{(1)}(G) = 2/9$.*

2 Preliminaries

In this section we introduce definitions and notation that will be used throughout the paper, and give some preliminary lemmas.

Given an r-graph $G = (V(G), E(G))$ and a vertex v of G. The *link* $L_G(v)$ is the $(r-1)$-graph consisting of all $S \subset V(G)$ with $|S| = r-1$ and $S \cup \{v\} \in E(G)$. The *degree* $d_G(v)$ of v is the size of $L_G(v)$. As usual, we denote by $N_G(v)$ the neighborhood of a vertex v, i.e., the set formed by all the vertices which form an edge with v. In the above mentioned notation, we will skip the index G whenever G is understood from the context.

The *shadow graph* of G, denoted by $\partial(G)$, is the graph with $V(\partial(G)) = V(G)$ and $E(\partial(G))$ consisting of all pairs of vertices that belong to an edge of G, i.e., $E(\partial(G)) = \{e : |e| = 2,\ e \subseteq f \text{ for some } f \in E(G)\}$. For more definitions and notation from hypergraph theory, see e.g., [2].

For any real number $p \geqslant 1$, the p-spectral radius was introduced by Keevash, Lenz and Mubayi [12] and subsequently studied by Nikiforov [16, 17]. Let G be an r-graph of order n, the polynomial form of G is a multi-linear function $P_G(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ as

$$P_G(\boldsymbol{x}) = r! \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The *p-spectral radius* of G is defined as

$$\lambda^{(p)}(G) := \max_{\|\boldsymbol{x}\|_p = 1} P_G(\boldsymbol{x}), \tag{1}$$

where $\|\boldsymbol{x}\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$.

For any real number $p \geqslant 1$, we denote by $\mathbb{S}_{p,+}^{n-1}$ the set of all nonnegative real vectors $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_p = 1$. If $\boldsymbol{x} \in \mathbb{R}^n$ is a vector with $\|\boldsymbol{x}\|_p = 1$ such that $\lambda^{(p)}(G) = P_G(\boldsymbol{x})$, then $\boldsymbol{x}$ is called an *eigenvector* corresponding to $\lambda^{(p)}(G)$. Note that $P_G(\boldsymbol{x})$ can always reach its maximum at some nonnegative vectors. By Lagrange's method, we have the *eigenequations* for $\lambda^{(p)}(G)$ and $\boldsymbol{x} \in \mathbb{S}_{p,+}^{n-1}$ as follows:

$$\lambda^{(p)}(G) x_i^{p-1} = (r-1)! \sum_{\{i, i_2, \ldots, i_r\} \in E(G)} x_{i_2} \cdots x_{i_r} \quad \text{for } x_i > 0. \tag{2}$$

It is worth mentioning that the p-spectral radius $\lambda^{(p)}(G)$ shows remarkable connections with some hypergraph invariants. For instance, $\lambda^{(1)}(G)/r!$ is the Lagrangian of G, $\lambda^{(r)}(G)/(r-1)!$ is the usual spectral radius introduced by Cooper and Dutle [3], and $\lambda^{(\infty)}(G)/r!$ is the number of edges of G (see [16, Proposition 2.10]).

Given two vertices u and v, we say that u and v are *equivalent* in G, in writing $u \sim v$, if transposing u and v and leaving the remaining vertices intact, we get an automorphism of G.

Lemma 3 ([16]). *Let G be a uniform hypergraph on n vertices and $u \sim v$. If $p > 1$ and $\boldsymbol{x} \in \mathbb{S}_p^{n-1}$ is an eigenvector to $\lambda^{(p)}(G)$, then $x_u = x_v$.*

3 Cancellative hypergraphs of maximum p-spectral radius

The aim of this section is to give a proof of Theorem 2. We split it into Theorem 10 – Theorem 18, which deal with $p = 3$, $p > 3$ and $p = 1$, respectively.

3.1 General properties on cancellative hypergraphs

We start this subsection with a basic fact.

Lemma 4. *Let G be a cancellative hypergraph, and u, v be adjacent vertices. Then $L(u)$ and $L(v)$ are edge-disjoint graphs.*

Proof. Assume by contradiction that $e \in E(L(u)) \cap E(L(v))$. Since u and v are adjacent in G, we have $\{u, v\} \subset e_1 \in E(G)$ for some edge e_1. Hence, $e_2 = e \cup \{u\}$, $e_3 = e \cup \{v\}$ and e_1 are three edges of G such that $e_2 \triangle e_3 \subset e_1$, a contradiction. □

Let G be a 3-graph and $v \in V(G)$. We denote by $E_v(G)$ the collection of edges of G containing v, i.e., $E_v(G) = \{e : v \in e \in E(G)\}$. For a pair of vertices u and v in G, we denote by $T_v^u(G)$ a new 3-graph with $V(T_v^u(G)) = V(G)$ and

$$E(T_v^u(G)) = \big(E(G) \setminus E_v(G)\big) \cup \{(e \setminus \{u\}) \cup \{v\} : e \in E_u(G) \setminus E_v(G)\}.$$

Lemma 5. *Let G be a cancellative 3-graph. Then $T_v^u(G)$ is also cancellative for any $u, v \in V(G)$.*

Proof. Suppose to the contrary that there exist three edges $e_1, e_2, e_3 \in T_v^u(G)$ such that $e_1 \triangle e_2 \subset e_3$. Recalling the definition of $T_v^u(G)$, we deduce that u, v are non-adjacent in $T_v^u(G)$, and $(e \cup \{u\}) \setminus \{v\} \in E(G)$ for any $e \in E_v(T_v^u(G))$. On the other hand, since G is cancellative, we have $v \in e_1 \cup e_2 \cup e_3$. Denote by α the number of edges e_1, e_2, e_3 containing v. It suffices to consider the following three cases.

Case 1. $\alpha = 3$. We have $v \in e_1 \cap e_2 \cap e_3$. Hence, $e_1' = (e_1 \cup \{u\}) \setminus \{v\}$, $e_2' = (e_2 \cup \{u\}) \setminus \{v\}$ and $e_3' = (e_3 \cup \{u\}) \setminus \{v\}$ are three edges in G with $e_1' \triangle e_2' \subset e_3'$. This contradicts the fact that G is cancellative.

Case 2. $\alpha = 2$. Without loss of generality, we assume $v \in (e_1 \cap e_2) \setminus e_3$ or $v \in (e_1 \cap e_3) \setminus e_2$. If $v \in (e_1 \cap e_2) \setminus e_3$, then $e_3 \in E(G)$. It follows that $e_1' = (e_1 \cup \{u\}) \setminus \{v\}$,

$e_2' = (e_2 \cup \{u\}) \setminus \{v\}$ and e_3 are three edges of G with $e_1' \triangle e_2' \subset e_3$, which is a contradiction. If $v \in (e_1 \cap e_3) \setminus e_2$, then $e_2 \in E(G)$. It follows that $e_1' = (e_1 \cup \{u\}) \setminus \{v\}$, e_2 and $e_3' = (e_3 \cup \{u\}) \setminus \{v\}$ are three edges of G with $e_1' \triangle e_2 \subset e_3'$, a contradiction.

Case 3. $\alpha = 1$. Without loss of generality, we assume $v \in e_3 \setminus (e_1 \cup e_2)$. Then $e_1 \in E(G)$ and $e_2 \in E(G)$. We immediately obtain that e_1, e_2 and $e_3' = (e_3 \cup \{u\}) \setminus \{v\}$ are three edges of G with $e_1 \triangle e_2 \subset e_3'$. This is a contradiction and proves Lemma 5. $\square$

The following result can be found in [16, Proposition 7.2].

Lemma 6 ([16])**.** *Let $p > 1$ and G be a complete 3-partite 3-graph. Then*

$$\lambda^{(p)}(G) = \frac{6}{\sqrt[p]{27}} \cdot (|E(G)|)^{1-1/p}.$$

Proof. Assume that V_1, V_2 and V_3 are the vertex classes of G with $n_i := |V_i|$ and $n_1 \geqslant n_2 \geqslant n_3$. Let $\boldsymbol{x} \in \mathbb{S}_{p,+}^{n-1}$ be an eigenvector corresponding to $\lambda^{(p)}(G)$. By Lemma 3, for $i = 1, 2, 3$ we denote $a_i := x_v$ for $v \in V_i$, and set $\lambda := \lambda^{(p)}(G)$ for short. In light of eigenequation (2), we find that

$$\begin{cases} \lambda a_1^{p-1} = 2n_2 n_3 a_2 a_3, \\ \lambda a_2^{p-1} = 2n_1 n_3 a_1 a_3, \\ \lambda a_3^{p-1} = 2n_1 n_2 a_1 a_2, \end{cases}$$

from which we obtain that $a_i = (3n_i)^{-1/p}$, $i = 1, 2, 3$. Therefore,

$$\lambda = \frac{2 \cdot (27 \cdot n_1 n_2 n_3)^{1-1/p}}{9} = \frac{6}{\sqrt[p]{27}} \cdot (|E(G)|)^{1-1/p}.$$

This completes the proof of Lemma 6. $\square$

3.2 Extremal *p*-spectral radius of cancellative hypergraphs

Let $\mathrm{SPEX}_p(n, \{F_4, F_5\})$ be the set of all 3-graphs attaining the maximum p-spectral radius among cancellative hypergraphs on n vertices. If $p = 3$, we will denote it by $\mathrm{SPEX}(n, \{F_4, F_5\})$ for short. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$ and a set $S \subset [n] := \{1, 2, \ldots, n\}$, we write $\boldsymbol{x}(S) := \prod_{i \in S} x_i$ for short. The *support set* S of a vector $\boldsymbol{x}$ is the set of the indices of non-zero elements in $\boldsymbol{x}$, i.e., $S = \{i \in [n] : x_i \neq 0\}$. Also, we denote by $x_{\min} := \min\{|x_i| : i \in [n]\}$ and $x_{\max} := \max\{|x_i| : i \in [n]\}$.

Lemma 7. *Let $p > 1$, $G \in \mathrm{SPEX}_p(n, \{F_4, F_5\})$, and $\boldsymbol{x} \in \mathbb{S}_{p,+}^{n-1}$ be an eigenvector corresponding to $\lambda^{(p)}(G)$. If u, v are two non-adjacent vertices, then $x_u = x_v$.*

Proof. Assume u and v are two non-adjacent vertices in G. Since G is a cancellative 3-graph, we have $T_u^v(G)$ is also cancellative by Lemma 5. It follows from (1) and (2) that

$$\begin{aligned} \lambda^{(p)}(T_u^v(G)) &\geqslant 6 \sum_{e \in E(G)} \boldsymbol{x}(e) - 6 \sum_{e \in E_u(G)} \boldsymbol{x}(e) + 6 \sum_{e \in E_v(G)} \boldsymbol{x}(e \setminus \{v\}) \cdot x_u \\ &= \lambda^{(p)}(G) - 3\lambda^{(p)}(G) x_u^p + 3\lambda^{(p)}(G) x_v^{p-1} x_u \\ &= \lambda^{(p)}(G) + 3\lambda^{(p)}(G)(x_v^{p-1} - x_u^{p-1}) \cdot x_u, \end{aligned}$$

which yields that $x_u \geqslant x_v$. Likewise, we also have $x_v \geqslant x_u$ by considering $T_v^u(G)$. Hence, $x_u = x_v$, completing the proof of Lemma 7. $\square$

Lemma 8. *Let $p > 1$, $G \in \mathrm{SPEX}_p(n, \{F_4, F_5\})$, and u, v be two non-adjacent vertices. Then there exists a cancellative 3-graph H such that*

$$L_H(u) = L_H(v), \quad \lambda^{(p)}(H) = \lambda^{(p)}(G), \ \textit{and} \ d_H(w) \leqslant d_G(w), \quad w \in V(G). \tag{3}$$

Proof. Assume that $\boldsymbol{x} \in \mathbb{S}_{p,+}^{n-1}$ is an eigenvector corresponding to $\lambda^{(p)}(G)$. By Lemma 7, $x_u = x_v$. Without loss of generality, we assume $d_G(u) \geqslant d_G(v)$. In view of (1) and (2), we have

$$\begin{aligned}
\lambda^{(p)}(T_u^v(G)) &\geqslant 6 \sum_{e \in E(G)} \boldsymbol{x}(e) - 6 \sum_{e \in E_u(G)} \boldsymbol{x}(e) + 6 \sum_{e \in E_v(G)} \boldsymbol{x}(e \setminus \{v\}) \cdot x_u \\
&= \lambda^{(p)}(G) + 3\lambda^{(p)}(G)(x_v^{p-1} - x_u^{p-1}) \cdot x_u \\
&= \lambda^{(p)}(G).
\end{aligned}$$

Observe that $T_u^v(G)$ is a cancellative 3-graph and $G \in \mathrm{SPEX}_p(n, \{F_4, F_5\})$. We immediately obtain that $\lambda^{(p)}(T_u^v(G)) = \lambda^{(p)}(G)$. It is straightforward to check that $H := T_u^v(G)$ is a cancellative 3-graph satisfying (3), as desired. $\square$

Next, we give an estimation on the entries of eigenvectors corresponding to $\lambda^{(p)}(G)$.

Lemma 9. *Let $G \in \mathrm{SPEX}_p(n, \{F_4, F_5\})$ and $\boldsymbol{x} \in \mathbb{S}_{p,+}^{n-1}$ be an eigenvector corresponding to $\lambda^{(p)}(G)$. If $1 < p \leqslant 3$, then*

$$x_{\min} > \left(\frac{3}{4}\right)^{2/(p-1)} \cdot x_{\max}.$$

Proof. Suppose to the contrary that $x_{\min} \leqslant \left(\frac{3}{4}\right)^{2/(p-1)} \cdot x_{\max}$. Let u and v be two vertices such that $x_u = x_{\min}$ and $x_v = x_{\max} > 0$. Then we have

$$\left(1 + \frac{x_u}{x_v}\right)\left(\frac{x_u}{x_v}\right)^{p-1} \leqslant \left(1 + \left(\frac{3}{4}\right)^{2/(p-1)}\right)\left(\frac{3}{4}\right)^2 \leqslant \frac{7}{4} \cdot \frac{9}{16} < 1,$$

which implies that

$$x_v^p - x_u^p > x_u^{p-1} x_v. \tag{4}$$

On the other hand, by eigenequations we have

$$2 \sum_{e \in E_v(G) \setminus E_u(G)} \boldsymbol{x}(e) \geqslant \lambda^{(p)}(G)(x_v^p - x_u^p). \tag{5}$$

Now, we consider the cancellative 3-graph $T_u^v(G)$. In light of (1) and (5), we have

$$\begin{aligned}
\lambda^{(p)}(T_u^v(G)) &\geqslant 6 \sum_{e \in E(G)} \boldsymbol{x}(e) - 6 \sum_{e \in E_u(G)} \boldsymbol{x}(e) + 6 \sum_{e \in E_v(G) \setminus E_u(G)} \boldsymbol{x}(e \setminus \{v\}) \cdot x_u \\
&\geqslant \lambda^{(p)}(G) - 3\lambda^{(p)}(G) x_u^p + 3\lambda^p(G)(x_v^p - x_u^p) \cdot \frac{x_u}{x_v} \\
&> \lambda^{(p)}(G) + 3\lambda^{(p)}(G)\Big(-x_u^p + x_u^{p-1} x_v \cdot \frac{x_u}{x_v}\Big) \\
&= \lambda^{(p)}(G),
\end{aligned}$$

where the third inequality is due to (4). This contradicts the fact that G has maximum p-spectral radius over all cancellative hypergraphs. $\square$

Now, we are ready to give a proof of Theorem 2 for $p = 3$.

Theorem 10. *Let G be a cancellative 3-graph on n vertices. Then $\lambda^{(3)}(G) \leqslant \lambda^{(3)}(T_3(n))$ with equality if and only if $G = T_3(n)$.*

Proof. According to Lemma 8, we assume that $G^* \in \mathrm{SPEX}(n, \{F_4, F_5\})$ is a 3-graph such that $L_{G^*}(u) = L_{G^*}(v)$ for any non-adjacent vertices u and v.

Our first goal is to show $G^* = T_3(n)$ by Claim 11 – Claim 13. Assume that $\boldsymbol{x} \in \mathbb{S}^{n-1}_{3,+}$ is an eigenvector corresponding to $\lambda^{(3)}(G^*)$; u_1 is a vertex in G^* such that $x_{u_1} = x_{\max}$ and u_2 is a vertex with $x_{u_2} = \max\{x_v : v \in N_{G^*}(u_1)\}$. Let $U_1 := V(G^*) \setminus N_{G^*}(u_1)$ and $U_2 := V(G^*) \setminus N_{G^*}(u_2)$. Since $u_2 \in V(G^*) \setminus U_1$, there exists a vertex u_3 such that $\{u_1, u_2, u_3\} \in E(G^*)$. Let $U_3 = V(G^*) \setminus N_{G^*}(u_3)$. Recall that for any non-adjacent vertices u and v we have $L_{G^*}(u) = L_{G^*}(v)$. Hence, the sets U_1, U_2 and U_3 are well-defined.

Claim 11. *The following statements hold:*

(1) $d_{G^*}(u_1) > n(n-1)/9$;

(2) $d_{G^*}(u_2) > n(n-1)/12$;

(3) $d_{G^*}(v) > n(n-1)/16$, $v \in V(G^*)$.

Proof of Claim 11. Since $T_3(n)$ is a cancellative 3-graph, it follows from Lemma 6 that

$$\lambda^{(3)}(G^*) \geqslant \lambda^{(3)}(T_3(n)) = 2 \cdot (t_3(n))^{2/3},$$

which is equivalent to

$$27 \cdot \left(\frac{\lambda^{(3)}(G^*)}{2}\right)^{3/2} \geqslant 27 \cdot t_3(n) = \begin{cases} n^3, & n \equiv 0 \pmod{3}, \\ (n-1)^2(n+2), & n \equiv 1 \pmod{3}, \\ (n-2)(n+1)^2, & n \equiv 2 \pmod{3}. \end{cases}$$

By simple algebra we see

$$\lambda^{(3)}(G^*) \geqslant \frac{2 \cdot \left((n-2)(n+1)^2\right)^{2/3}}{9} > \frac{2n(n-1)}{9}. \tag{6}$$

(1). By eigenequation with respect to u_1, we have

$$\lambda^{(3)}(G^*) x_{u_1}^2 = 2 \sum_{\{u_1,i,j\} \in E(G^*)} x_i x_j \leqslant 2 d_{G^*}(u_1) x_{u_1}^2.$$

Combining with (6), we get

$$d_{G^*}(u_1) \geqslant \frac{\lambda^{(3)}(G^*)}{2} > \frac{n(n-1)}{9}. \tag{7}$$

(2). Since $L_{G^*}(u) = L_{G^*}(v)$ for any pair $u, v \in U_1$, we immediately obtain that $|(e \setminus \{u_2\}) \cap U_1| \leqslant 1$ for each $e \in E_{u_2}(G^*)$ by the definition of U_1. It follows from $x_{u_2} = \max\{x_v : v \in V(G^*) \setminus U_1\}$ that

$$\lambda^{(3)}(G^*)x_{u_2}^2 = 2 \sum_{\{u_2,i,j\} \in E(G^*)} x_i x_j \leqslant 2d_{G^*}(u_2)x_{u_1}x_{u_2},$$

which, together with Lemma 9 for $p = 3$, gives

$$\begin{aligned} d_{G^*}(u_2) &\geqslant \frac{x_{u_2}}{x_{u_1}} \cdot \frac{\lambda^{(3)}(G^*)}{2} \\ &\geqslant \frac{3}{4} \cdot \frac{\lambda^{(3)}(G^*)}{2} \\ &> \frac{1}{12} n(n-1). \end{aligned}$$

The last inequality is due to (6).

(3). Let v be an arbitrary vertex in $V(G^*)$. Then

$$\lambda^{(3)}(G^*)x_v^2 = 2 \sum_{\{v,i,j\} \in E(G^*)} x_i x_j \leqslant 2d_{G^*}(v)x_{u_1}^2.$$

Hence, by Lemma 9 and (6) we have

$$d_{G^*}(v) \geqslant \left(\frac{x_v}{x_{u_1}}\right)^2 \cdot \frac{\lambda^{(3)}(G^*)}{2} > \frac{1}{16} n(n-1),$$

as desired. □

Next, we consider the graph $H = L_{G^*}(u_1) \cup L_{G^*}(u_2) \cup L_{G^*}(u_3)$. Let $\phi : E(H) \to [3]$ be a mapping such that $\phi(f) = i$ if $f \in L_{G^*}(u_i)$, $i \in [3]$. By Lemma 4, ϕ is an edge coloring of H. For convenience, we denote $L := V(G^*) \setminus (U_1 \cup U_2 \cup U_3)$.

Claim 12. *If $L \neq \emptyset$, then there is no rainbow star $K_{1,3}$ in the induced subgraph $H[L]$ with the coloring ϕ.*

Proof of Claim 12. Suppose to the contrary that there exist $v_1, v_2, v_3, v_4 \in L$ with $\phi(v_1v_2) = 1$, $\phi(v_1v_3) = 2$ and $\phi(v_1v_4) = 3$. We first show that $\{v_1, v_2, v_3, v_4\}$ induces a clique in $\partial(G^*)$ by contradiction. Without loss of generality, we assume $v_2v_3 \notin E(\partial(G^*))$. Then $L_{G^*}(v_2) = L_{G^*}(v_3)$. Since $\phi(v_1v_2) = 1$ and $\phi(v_1v_3) = 2$, we have $\{u_1, v_1, v_2\} \in E(G^*)$ and $\{u_2, v_1, v_3\} \in E(G^*)$. This implies that $e_1 = \{u_1, u_2, u_3\}$, $e_2 = \{u_1, v_1, v_2\}$ and $e_3 = \{u_2, v_1, v_2\}$ are three edges in G^* with $e_2 \triangle e_3 \subset e_1$, which is impossible.

On the other hand, since $L = V(G^*) \setminus (U_1 \cup U_2 \cup U_3)$, we have $v_iu_j \in E(\partial(G^*))$ for any $i \in [4]$, $j \in [3]$. Therefore, every pair of vertices in $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3\}$ is contained in an edge of G^*. Consider the graph

$$H' := \left(\bigcup_{i=1}^{3} L_{G^*}(u_i)\right) \bigcup \left(\bigcup_{i=1}^{4} L_{G^*}(v_i)\right).$$

By Claim 11, we have

$$\begin{aligned}|E(H')| &= \sum_{1\leqslant i\leqslant 3} d_{G^*}(u_i) + \sum_{1\leqslant j\leqslant 4} d_{G^*}(v_j)\\ &> \left(1+\frac{3}{4}+5\times\frac{9}{16}\right)\cdot\frac{1}{9}n(n-1)\\ &= \frac{73}{144}n(n-1)\\ &> \binom{n}{2},\end{aligned}$$

a contradiction completing the proof of Claim 12. □

Claim 13. *$L=\emptyset$.*

Proof of Claim 13. Suppose to the contrary that $L\neq\emptyset$. For $i=1,2,3$, let L_i be the set of vertices in L which is not contained in an edge with coloring i. By Claim 12, we have $L=L_1\cup L_2\cup L_3$. Without loss of generality, we assume $L_1\neq\emptyset$. Let w be a vertex in L_1. Then there exists an edge f in G^* such that $f=\{u_1,w,w'\}$, where $w'\in U_2\cup U_3$. If $w'\in U_2$, then $f'=\{u_1,u_3,w'\}\in E(G^*)$. Since G^* is cancellative, w is not a neighbor of u_3 in G^*. This implies that $w\in U_3$, a contradiction to $w\in L$. Similarly, if $w'\in U_3$, then $w\in U_2$, which is also a contradiction. □

Now, we continue our proof. By Claim 13, we immediately obtain that G^* is a complete 3-partite 3-graph with vertex classes U_1, U_2 and U_3. Hence, $G^*=T_3(n)$ by Lemma 6.

Finally, it is enough to show that $G=T_3(n)$ for any $G\in \mathrm{SPEX}(n,\{F_4,F_5\})$. According to Lemma 8 and Claim 13, we can transfer G to the complete 3-partite 3-graph $T_3(n)$ by a sequence of switchings $T_u^v(\,\cdot\,)$ that keep the spectral radius unchanged. Let $T_1,\ldots,T_s$ be such a sequence of switchings $T_u^v(\,\cdot\,)$ which turn G into $T_3(n)$. Consider the 3-graphs $G=G_0,G_1,\ldots,G_s=T_3(n)$ in which G_i is obtained from G_{i-1} by applying T_i. Let $\boldsymbol{z}\in\mathbb{S}_{3,+}^{n-1}$ be an eigenvector corresponding to $\lambda^{(3)}(G_{s-1})$ and $T_u^v(G_{s-1})=T_3(n)$, and denote

$$A:=V(G_{s-1})\setminus\big(N_{G_{s-1}}(v)\cup\{u\}\cup\{v\}\big).$$

Hence, we have $L_{G_{s-1}}(w)=L_{G_{s-1}}(v)=L_{T_3(n)}(v)$ for each $w\in A$. In what follows, we shall prove $L_{G_{s-1}}(u)=L_{G_{s-1}}(v)$, and therefore $G_{s-1}=T_3(n)$. If $L_{G_{s-1}}(u)\neq L_{G_{s-1}}(v)$, there exists an edge $e=v_1v_2\in L_{G_{s-1}}(u)\setminus L_{G_{s-1}}(v)$ since $z_u=z_v$ by Lemma 7. Let M_1 and M_2 be two subsets of $V(G_{s-1})$ such that $M_1\cup M_2=N_{G_{s-1}}(v)$ and $L_{G_{s-1}}(v)=K_{|M_1|,|M_2|}$. If $\{v_1,v_2\}\subset N_{G_{s-1}}(v)$, then $\{v_1,v_2\}\subset M_1$ or $\{v_1,v_2\}\subset M_2$. It follows that there exists a vertex $w\in N_{G_{s-1}}(v)$ such that $f_1:=\{v,w,v_1\}\in E(G_{s-1})$ and $f_2:=\{v,w,v_2\}\in E(G_{s-1})$. However, $f_1\triangle f_2\subset\{u,v_1,v_2\}\in E(G_{s-1})$, a contradiction. So we obtain $\{v_1,v_2\}\cap A\neq\emptyset$. Without loss of generality, we assume $v_1\in A$. Then $L_{G_{s-1}}(v_1)=L_{G_{s-1}}(v)$, i.e., $uv_2\in L_{G_{s-1}}(v)$. Thus, $u\in N_{G_{s-1}}(v)$, a contradiction. This implies that $G_{s-1}=T_3(n)$. Likewise, $G_{i-1}=G_i$ for each $i\in[s-1]$, and therefore $G=T_3(n)$. This completes the proof of the theorem. □

According to Theorem 10, we can give an alternative proof of Bollobás' result for $n \equiv 0 \pmod 3$.

Corollary 14. *Let G be a cancellative 3-graph on n vertices with $n \equiv 0 \pmod 3$. Then $|E(G)| \leqslant t_3(n)$ with equality if and only if $G = T_3(n)$.*

Proof. Denote by $\boldsymbol{z}$ the all-ones vector of dimension n. In view of (1), we deduce that

$$\lambda^{(3)}(G) \geqslant \frac{P_G(\boldsymbol{z})}{\|\boldsymbol{z}\|_3^3} = \frac{6|E(G)|}{n}.$$

On the other hand, by Theorem 10 we have

$$\lambda^{(3)}(G) \leqslant \lambda^{(3)}(T_3(n)) = 2 \cdot (t_3(n))^{2/3}.$$

As a consequence,

$$|E(G)| \leqslant \frac{n}{3} \cdot (t_3(n))^{2/3} = t_3(n).$$

Equality may occur only if $\lambda^{(3)}(G) = 2 \cdot (t_3(n))^{2/3} = \lambda^{(3)}(T_3(n))$, and therefore $G = T_3(n)$ by Theorem 10. □

Next, we will prove Theorem 2 for the case $p > 3$ as stated in Theorem 16.

Lemma 15 ([16])**.** *Let $p \geqslant 1$ and G be an r-graph with m edges. Then the function*

$$f_G(p) := \left(\frac{\lambda^{(p)}(G)}{r!m} \right)^p$$

is non-increasing in p.

Theorem 16. *Let $p > 3$ and G be a cancellative 3-graph on n vertices. Then $\lambda^{(p)}(G) \leqslant \lambda^{(p)}(T_3(n))$ with equality if and only if $G = T_3(n)$.*

Proof. Assume that $p > 3$ and G is a 3-graph in $\mathrm{SPEX}_p(n, \{F_4, F_5\})$ with m edges. It is enough to show that $G = T_3(n)$. By Lemma 15, we have

$$\left(\frac{\lambda^{(p)}(G)}{6m} \right)^p \leqslant \left(\frac{\lambda^{(3)}(G)}{6m} \right)^3,$$

which, together with $\lambda^{(3)}(G) \leqslant 2 \cdot (t_3(n))^{2/3}$ by Theorem 10 and Lemma 6, gives

$$\lambda^{(p)}(G) \leqslant (6m)^{1-3/p} \cdot (\lambda^{(3)}(G))^{3/p} \leqslant 2^{3/p} \cdot (6m)^{1-3/p} \cdot (t_3(n))^{2/p}.$$

On the other hand, we have

$$\lambda^{(p)}(G) \geqslant \lambda^{(p)}(T_3(n)) = \frac{6}{\sqrt[p]{27}} \cdot (t_3(n))^{1-1/p}.$$

We immediately obtain $m \geqslant t_3(n)$. The result follows from Theorem 1. □

Finally, we shall give a proof of Theorem 2 for the remaining case $p = 1$. In what follows, we always assume that $\boldsymbol{x} \in \mathbb{S}^{n-1}_{1,+}$ is an eigenvector such that $\boldsymbol{x}$ has the minimum possible number of non-zero entries among all eigenvectors corresponding to $\lambda^{(1)}(G)$. Before continuing, we need the following result.

Lemma 17 ([7]). *Let G be an r-graph and S be the support set of $\boldsymbol{x}$. Then for each pair vertices u and v in S, there is an edge in $G[S]$ containing both u and v.*

Theorem 18. *Let G be a cancellative 3-graph. Then $\lambda^{(1)}(G) = 2/9$.*

Proof. Assume that G is a cancellative 3-graph with support set S. Let $H := G[S]$. By Lemma 17, for any $u, v \in S$ there is an edge in H containing both u and v. Hence, each pair of edges of H has at most one common vertex by H being cancellative. So the shadow graph of H is the complete graph $K_{|S|}$. Since H is cancellative, the link graphs $L_H(u)$ and $L_H(v)$ are edge-disjoint graphs for any distinct vertices $u, v \in S$. It follows from (2) that

$$|S| \cdot \lambda^{(1)}(G) = 2 \sum_{uv \in E(\partial(H))} x_u x_v \leqslant 1 - \frac{1}{|S|}, \tag{8}$$

where the last inequality follows from Motzkin–Straus Theorem [13]. On the other hand, set

$$z_v = \begin{cases} 1/|S|, & v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

We immediately have

$$\lambda^{(1)}(G) \geqslant 6 \sum_{e \in E(H)} \boldsymbol{z}(e) = 2 \sum_{v \in V(H)} \left(z_v \cdot \sum_{f \in L_H(v)} \boldsymbol{z}(f) \right) = \frac{|S| - 1}{|S|^2},$$

where the last inequality follows from the fact that $d_H(v) = (|S| - 1)/2$ for $v \in V(H)$. Combining with (8) we get

$$\lambda^{(1)}(G) = \frac{|S| - 1}{|S|^2}.$$

Clearly, $(|S| - 1)/|S|^2$ attains its maximum at $|S| = 3$ when $|S| \geqslant 3$. Hence, we see $\lambda^{(1)}(G) \leqslant 2/9$. Finally, noting that $\lambda^{(1)}(G)$ is at least the Lagrangian of an edge $K_3^{(3)}$, i.e.,

$$\lambda^{(1)}(G) \geqslant \lambda^{(1)}(K_3^3) = \frac{2}{9},$$

we obtain $\lambda^{(1)}(G) = 2/9$, as desired. □

Remark 19. For an r-graph G on n vertices, it is well-known that $\lambda^{(1)}(G)/r!$ is the Lagrangian of G. In [19], Yan and Peng present a tight upper bound on $\lambda^{(1)}(G)$ for F_5-free 3-graphs, see [19] for details.

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