Improved Bounds Concerning the Maximum Degree of Intersecting Hypergraphs

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Abstract

For positive integers $n > k > t$ let $\binom{[n]}{k}$ denote the collection of all k-subsets of the standard n-element set $[n] = \{1, \dots, n\}$. Subsets of $\binom{[n]}{k}$ are called k-graphs. A k-graph $\mathcal{F}$ is called t-intersecting if $|F \cap F'| \geqslant t$ for all $F, F' \in \mathcal{F}$. One of the central results of extremal set theory is the Erdős-Ko-Rado Theorem which states that for $n \geqslant (k-t+1)(t+1)$ no t-intersecting k-graph has more than $\binom{n-t}{k-t}$ edges. For n greater than this threshold the t-star (all k-sets containing a fixed t-set) is the only family attaining this bound. Define $\mathcal{F}(i) = \{F \setminus \{i\} \colon i \in F \in \mathcal{F}\}$. The quantity $\varrho(\mathcal{F}) = \max_{1 \leqslant i \leqslant n} |\mathcal{F}(i)|/|\mathcal{F}|$ measures how close a k-graph is to a star. The main result (Theorem 1.3) shows that $\varrho(\mathcal{F}) > 1/d$ holds if $\mathcal{F}$ is 1-intersecting, $|\mathcal{F}| > 2^d d^{2d+1}\binom{n-d-1}{k-d-1}$ and $n \geqslant 4(d-1)dk$. Such a statement can be deduced from earlier results, however only for much larger values of n/k and/or n. The proof is purely combinatorial, it is based on a new method: shifting ad extremis. The same method is applied to obtain a nearly optimal bound in the case of $t \geqslant 2$ (Theorem 1.4).

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1 Introduction

For positive integers $n \geqslant k$, let $[n] = \{1, \dots, n\}$ be the standard n-element set and $\binom{[n]}{k}$ the collection of its k-subsets. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *t-intersecting* if $|F \cap F'| \geqslant t$ for all $F, F' \in \mathcal{F}$ and t a positive integer. In the case $t = 1$ we usually omit t and speak of intersecting families. Let us recall one of the fundamental results of extremal set theory.

Theorem 1 (Exact Erdős-Ko-Rado Theorem ([2], [4], [18])). *Let $k \geqslant t > 0$, $n \geqslant$*

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$n_0(k,t) = (k-t+1)(t+1)$. *Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *is* t*-intersecting. Then*

$$|\mathcal{F}| \leqslant \binom{n-t}{k-t}. \tag{1}$$

Let us note that $|\mathcal{S}(n,k,t)| = \binom{n-t}{k-t}$ holds for the *full star*

$$\mathcal{S}(n,k,t) = \left\{ S \in \binom{[n]}{k} : [t] \subset S \right\}$$

and for $n > n_0(k,t)$ up to isomorphism $\mathcal{S}(n,k,t)$ is the only family to achieve equality in (1). The exact bound $n_0(k,t) = (k-t+1)(t+1)$ is due to Erdős, Ko and Rado in the case $t = 1$. For $t \geqslant 15$ it was established in [4]. Wilson [18] closed the gap $2 \leqslant t \leqslant 14$ by a proof valid for all $t \geqslant 1$.

Let us recall some standard notation. Set $\cap\mathcal{F} = \cap\{F \colon F \in \mathcal{F}\}$. If $|\cap\mathcal{F}| \geqslant t$ then $\mathcal{F}$ is called a t*-star*, for $t = 1$ we usually omit the 1. If $\cap\mathcal{F} = \emptyset$ then we call $\mathcal{F}$ a *non-trivial* family.

For a subset $E \subset [n]$ and a family $\mathcal{F} \subset \binom{[n]}{k}$, define

$$\mathcal{F}(E) = \{F \setminus E \colon E \subset F \in \mathcal{F}\},\ \mathcal{F}(\overline{E}) = \{F \in \mathcal{F} \colon F \cap E = \emptyset\}.$$

In the case $E = \{i\}$ we simply use $\mathcal{F}(i)$ and $\mathcal{F}(\bar{i})$ to denote $\mathcal{F}(\{i\})$ and $\mathcal{F}(\overline{\{i\}})$, respectively. In analogy,

$$\mathcal{F}(u,v,\bar{w}) := \{F \setminus \{u,v\} \colon F \in \mathcal{F}, F \cap \{u,v,w\} = \{u,v\}\}.$$

Let us define the quantity

$$\varrho(\mathcal{F}) = \max\left\{ \frac{|\mathcal{F}(i)|}{|\mathcal{F}|} \colon 1 \leqslant i \leqslant n \right\}.$$

Since $\varrho(\mathcal{F}) = 1$ if and only if $\mathcal{F}$ is a star, in a way $\varrho(\mathcal{F})$ measures how far a family is from a star.

A set T is called a t*-transversal* of $\mathcal{F}$ if $|T \cap F| \geqslant t$ for all $F \in \mathcal{F}$. If $\mathcal{F}$ is t-intersecting then each $F \in \mathcal{F}$ is a t-transversal. Define

$$\tau_t(\mathcal{F}) = \min\{|T| \colon T \text{ is a } t\text{-transversal of } \mathcal{F}\}.$$

For $t = 1$ we usually omit the 1.

Proposition 2. *If* $\mathcal{F}$ *is* t*-intersecting, then*

$$\varrho(\mathcal{F}) \geqslant \frac{t}{\tau_t(\mathcal{F})}. \tag{2}$$

Proof. Fix a t-transversal T of $\mathcal{F}$ with $|T| = \tau_t(\mathcal{F})$. Then

$$t|\mathcal{F}| \leqslant \sum_{i \in T} |\mathcal{F}(i)| \leqslant |T| \cdot \max\{|\mathcal{F}(i)| \colon i \in T\},$$

implying (2). □

Obviously, $\tau_t(\mathcal{F}) = t$ if and only if $\mathcal{F}$ is a t-star.

Example 3. For $n > k > t > 0$ define

$$\mathcal{A}(n,k,t) = \left\{ A \in \binom{[n]}{k} : |A \cap [t+2]| \geqslant t+1 \right\}.$$

Clearly, $\mathcal{A} = \mathcal{A}(n,k,t)$ is t-intersecting, $\varrho(\mathcal{A}) = \frac{t+1+o(1)}{t+2}$, $\tau_t(\mathcal{F}) = t+1$. We should note that for $2k-t < n < (k-t+1)(t+1)$, $|\mathcal{A}| > \binom{n-t}{k-t}$ with equality for $n = (k-t+1)(t+1)$.

In [3] it was shown that for any positive ε and $n > n_1(k,t,\varepsilon)$, $\varrho(\mathcal{F}) < 1-\varepsilon$ implies $|\mathcal{F}| \leqslant |\mathcal{A}|$ for any t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$. The value of $n_1(k,t,\varepsilon)$ is implicit in [3]. With careful calculation (cf. e.g. [10]) for fixed $\varepsilon > 0$ one can prove a bound quadratic in k. Dinur and Friedgut [1] introduced the so-called junta-method that leads to strong results for $n > ck$, however the value of the constant is large and it is further dependent on the particular problem (the same is true for the recent advances of Keller and Lifschitz [16]).

The aim of the present paper is to prove some similar results concerning $\varrho(\mathcal{F})$ for t-intersecting families for $n > ck$ with relatively small constants c. Let us state here our main result for the case $t = 1$.

Theorem 4. *Let n, k, d be integers, $k > d \geqslant 2$, $n \geqslant 4(d-1)dk$. If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $|\mathcal{F}| \geqslant 2^d d^{2d+1} \binom{n-d-1}{k-d-1}$, then $\varrho(\mathcal{F}) > \frac{1}{d}$.*

Let us stress once more that $\varrho(\mathcal{F}) > \frac{1}{d}$ follows from the results of [3] and [1] however only for much larger value of n.

For $t \geqslant 2$, we obtain the following result.

Theorem 5. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a t-intersecting family with $t \geqslant 2$. If $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$ and $n \geqslant 2t(t+2)k$, then $\varrho(\mathcal{F}) > \frac{t}{t+1}$.*

2 Preliminaries

In this section, we recall some useful results that are needed in our proofs.

Define the *lexicographic order* $A <_L B$ for $A, B \in \binom{[n]}{k}$ by $A <_L B$ if and only if $\min\{i : i \in A \setminus B\} < \min\{i : i \in B \setminus A\}$. E.g., $(1,2,9) <_L (1,3,4)$. For $n > k > 0$ and $\binom{n}{k} \geqslant m > 0$ let $\mathcal{L}(n,k,m)$ denote the first m sets $A \in \binom{[n]}{k}$ in the lexicographic order. For $X \subset [n]$ with $|X| > k > 0$ and $\binom{|X|}{k} \geqslant m > 0$, we also use $\mathcal{L}(X,k,m)$ to denote the first m sets $A \in \binom{X}{k}$ in the lexicographic order.

For $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$, we say that $\mathcal{A}, \mathcal{B}$ are *cross t-intersecting* if $|A \cap B| \geqslant t$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A powerful tool is the Kruskal-Katona Theorem ([17, 15]), especially its reformulation due to Hilton [12].

Hilton's Lemma ([12]). Let n, a, b be positive integers, $n \geqslant a+b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then $\mathcal{L}(n,a,|\mathcal{A}|)$ and $\mathcal{L}(n,b,|\mathcal{B}|)$ are cross-intersecting as well.

For $\mathcal{F} \subset \binom{[n]}{k}$ define the ℓth shadow of $\mathcal{F}$,

$$\partial^{\ell}\mathcal{F} = \{G\colon |G| = k - \ell,\ \exists F \in \mathcal{F} \text{ such that } G \subset F\}.$$

For $\ell = 1$ we often omit the superscript.

The following statement goes back to Katona [15]. Let us include the very short proof.

Proposition 6. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family. Then*

$$\partial\mathcal{F}(\bar{1}) \subset \mathcal{F}(1). \tag{3}$$

Proof. Indeed, if $E \subset F \in \mathcal{F}(\bar{1})$ and $E = F \setminus \{j\}$. Then by initiality $E \cup \{1\} \in \mathcal{F}$, i.e., $E \in \mathcal{F}(1)$. □

The Katona Intersecting Shadow Theorem gives an inequality concerning the sizes of a t-intersecting family and its shadow.

Katona Intersecting Shadow Theorem ([14]). Suppose that $n \geqslant 2k - t$, $t \geqslant \ell \geqslant 1$. Let $\emptyset \neq \mathcal{A} \subset \binom{[n]}{k}$ be a t-intersecting family. Then

$$|\partial^{\ell}\mathcal{A}| \geqslant |\mathcal{A}|\frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}} \tag{4}$$

with equality holding if and only if $\mathcal{F}$ is isomorphic to $\binom{[2k-t]}{k}$.

Let us recall an important operation called shifting introduced by Erdős, Ko and Rado [2]. For $\mathcal{F} \subset \binom{[n]}{k}$ and $1 \leqslant i < j \leqslant n$, define

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F)\colon F \in \mathcal{F}\},$$

where

$$S_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & j \in F, i \notin F \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

It is well known (cf. [5]) that shifting preserves the t-intersecting property.

Let $(x_1, \ldots, x_k)$ denote the set $\{x_1, \ldots, x_k\}$ where we know or want to stress that $x_1 < \ldots < x_k$. Let us define the *shifting partial order* $\prec$ where $P \prec Q$ for $P = (x_1, \ldots, x_k)$ and $Q = (y_1, \ldots, y_k)$ if and only if $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant k$. This partial order can be traced back to [2]. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *initial* if $F \prec G$ and $G \in \mathcal{F}$ always imply $F \in \mathcal{F}$. Note that an initial family $\mathcal{F}$ satisfies $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leqslant i < j \leqslant n$. By repeated shifting one can transform an arbitrary k-graph into a shifted k-graph with the same number of edges. Note also that $|\mathcal{F}(1)| \geqslant |\mathcal{F}(2)| \geqslant \ldots \geqslant |\mathcal{F}(n)|$ for an initial family.

We need the following property of initial families.

Proposition 7. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is initial and t-intersecting. Let $r \leqslant s < k - t$ and let $R \subset [s]$ with $|R| = r$. Then $\mathcal{F}(\overline{[s]}), \mathcal{F}(R, [s])$ are cross $(t + s - r)$-intersecting.*

Proof. Suppose for contradiction that there exist $F \in \mathcal{F}(\overline{[s]})$, $F' \in \mathcal{F}(R,[s])$ such that $|F \cap F'| = t+j \leqslant t-1+s-r$. Let $E \subset F \cap F'$ and $T \subset [s] \setminus R$ with $|E| = |T| = j+1$. Then $F'' := F \cup T \setminus E$ satisfies $F'' \prec F$ whence $F'' \in \mathcal{F}$. However $|F' \cap F''| = |F \cap F'| - |E| = t-1$, the desired contradiction. □

We need a notion called pseudo t-intersecting, which was introduced in [7]. A family $\mathcal{F} \subset \binom{[n]}{k}$ is said to be *pseudo t-intersecting* if for every $F \in \mathcal{F}$ there exists $0 \leqslant i \leqslant k-t$ such that $|F \cap [2i+t]| \geqslant i+t$.

Fact 8. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family. If $[t-1] \cup \{t+1, t+3, \ldots, 2k-t+1\} \notin \mathcal{F}$, then $\mathcal{F}$ is pseudo t-intersecting.*

Proof. Indeed, otherwise if $\mathcal{F}$ is pseudo t-intersecting then there exists $F \in \mathcal{F}$ such that $|F \cap [2i+t]| < i+t$ holds for all $i = 0, 1, \ldots, k-t$. By initiality it follows that

$$[t-1] \cup \{t+1, t+3, \ldots, 2k-t+1\} \in \mathcal{A},$$

a contradiction. □

Theorem 9 ([4])**.** *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family with $0 \leqslant t < k$. If $\mathcal{F}$ is pseudo t-intersecting, then*

$$|\mathcal{F}| \leqslant \binom{n}{k-t}. \tag{5}$$

The following property is proved in [7]. Let us include a proof as well.

Proposition 10 ([7])**.** *Let $n > \max\{2a-t, 2b-t\}$. If $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ are cross t-intersecting and both initial, then either both $\mathcal{A}$ and $\mathcal{B}$ are pseudo t-intersecting, or one of them is pseudo $(t+1)$-intersecting.*

Proof. If $\mathcal{A}$ is not pseudo t-intersecting, then there exists $A \in \mathcal{A}$ such that $|A \cap [2i+t]| < i+t$ holds for all $i = 0, 1, \ldots, a-t$. By initiality it follows that

$$A_0 := [t-1] \cup \{t+1, t+3, \ldots, t+2(a-t+1)-1\} \in \mathcal{A}.$$

Similarly, if $\mathcal{B}$ is not pseudo $(t+1)$-intersecting then

$$B_0 := [t] \cup \{t+2, t+4, \ldots, 2b-t\} \in \mathcal{B}.$$

Note that $|A_0 \cap B_0| = t-1$. By the cross t-intersecting property, we infer that if $\mathcal{B}$ is not pseudo $(t+1)$-intersecting then $\mathcal{A}$ is pseudo t-intersecting. Similarly, if $\mathcal{A}$ is not pseudo $(t+1)$-intersecting then $\mathcal{B}$ is pseudo t-intersecting. Thus the proposition follows. □

The following inequalities for cross t-intersecting families can be deduced from Proposition 10.

Corollary 11 ([4])**.** *Suppose that* $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ *are cross* t*-intersecting,* $|\mathcal{A}| \leqslant |\mathcal{B}|$*. Then either*

$$|\mathcal{B}| \leqslant \binom{n}{k-t} \text{ or} \tag{6}$$

$$|\mathcal{A}| \leqslant \binom{n}{k-t-1}. \tag{7}$$

We need the following inequalities concerning binomial coefficients.

Proposition 12 ([11])**.** *Let* n, k, i *be positive integers. Then*

$$\binom{n-i}{k} \geqslant \frac{n-ik}{n}\binom{n}{k}, \text{ for } n > ik. \tag{8}$$

Corollary 13. *Let* n, k, t *be positive integers. If* $n \geqslant 2(t-1)(k-t)$ *and* $k > t \geqslant 2$*, then*

$$\binom{n-t-2}{k-t-2} \geqslant \frac{1}{2}\binom{n-3}{k-t-2}. \tag{9}$$

Proof. Note that

$$n \geqslant 2(t-1)(k-t) = 2(t-1)(k-t-2) + 4(t-1) > 2(t-1)(k-t-2) + 3.$$

By (8) we have

$$\binom{n-t-2}{k-t-2} \geqslant \frac{(n-3)-(t-1)(k-t-2)}{n-3}\binom{n-3}{k-t-2} \geqslant \frac{1}{2}\binom{n-3}{k-t-2}. \qquad \square$$

3 Shifting ad extremis and the proof of Theorem 4

Note that for initial families one can deduce Theorem 4 under much milder constraints (cf. [8]). The problem is that one cannot transform a general family into an initial family without increasing $\varrho(\mathcal{F})$. To circumvent this difficulty we are going to apply the recently developed method of shifting ad extremis.

Let us define formally the notion of *shifting ad extremis* developed recently (cf. [6]). It can be applied to one, two or several families. For notational convenience we explain it for the case of two families in detail.

Let $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ be two families and suppose that we are concerned, as usual in extremal set theory, to obtain upper bounds for $|\mathcal{F}|+|\mathcal{G}|$, $|\mathcal{F}||\mathcal{G}|$ or some other function f of $|\mathcal{F}|$ and $|\mathcal{G}|$. For this we suppose that $\mathcal{F}$ and $\mathcal{G}$ have certain properties (e.g., cross-intersecting and non-trivial). Since $|S_{ij}(\mathcal{H})| = |\mathcal{H}|$ for all families $\mathcal{H}$, it is convenient to apply S_{ij} simultaneously to $\mathcal{F}$ and $\mathcal{G}$. Certain properties, e.g., t-intersecting, cross-intersecting or $\nu(\mathcal{F}) \leqslant r$ are known to be maintained by S_{ij}. However, some other properties may be destroyed, e.g., non-triviality, $\varrho(\mathcal{G}) \leqslant c$, etc. Let $\mathcal{P}$ be the collection of the latter properties that we want to maintain.

For any family $\mathcal{H}$, define the quantity

$$w(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{i \in H} i.$$

Obviously $w(S_{ij}(\mathcal{H})) \leqslant w(\mathcal{H})$ for $1 \leqslant i < j \leqslant n$ with strict inequality unless $S_{ij}(\mathcal{H}) = \mathcal{H}$.

Definition 14. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{\ell}$ are families having property $\mathcal{P}$. We say that $\mathcal{F}$ and $\mathcal{G}$ have been *shifted ad extremis* with respect to $\mathcal{P}$ if $S_{ij}(\mathcal{F}) = \mathcal{F}$ and $S_{ij}(\mathcal{G}) = \mathcal{G}$ for every pair $1 \leqslant i < j \leqslant n$ whenever $S_{ij}(\mathcal{F})$ and $S_{ij}(\mathcal{G})$ also have property $\mathcal{P}$.

Let us show that we can obtain shifted ad extremis families by the following shifting ad extremis process. Let $\mathcal{F}$, $\mathcal{G}$ be cross-intersecting families with property $\mathcal{P}$. Apply the shifting operation S_{ij}, $1 \leqslant i < j \leqslant n$, to $\mathcal{F}, \mathcal{G}$ simultaneously and continue as long as the property $\mathcal{P}$ is maintained. By abuse of notation, we keep denoting the current families by $\mathcal{F}$ and $\mathcal{G}$ during the shifting process. If $S_{ij}(\mathcal{F})$ or $S_{ij}(\mathcal{G})$ does not have property $\mathcal{P}$, then we do not apply S_{ij} and choose a different pair (i', j'). However we keep returning to previously failed pairs (i, j), because it might happen that at a later stage in the process S_{ij} does not destroy property $\mathcal{P}$ any longer. Note that the quantity $w(\mathcal{F}) + w(\mathcal{G})$ is a positive integer and it decreases strictly in each step. This guarantees that eventually we shall arrive at families that are shifted ad extremis with respect to $\mathcal{P}$.

Let $\mathcal{F}$, $\mathcal{G}$ be shifted ad extremis families. A pair (i, j) is called *shift-resistant* if either $S_{ij}(\mathcal{F}) \neq \mathcal{F}$ or $S_{ij}(\mathcal{G}) \neq \mathcal{G}$.

In the case of several families, $\mathcal{F}_i \subset \binom{[n]}{k_i}$, $1 \leqslant i \leqslant r$. It is essentially the same. One important property that is maintained by simultaneous shifting is *overlapping*, namely the non-existence of pairwise disjoint edges $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$ (cf. [13]).

Proof of Theorem 4. Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting, $|\mathcal{F}| \geqslant 2^d d^{2d+1} \binom{n-d-1}{k-d-1}$ and $\varrho(\mathcal{F}) \leqslant \frac{1}{d}$. Without loss of generality, we may assume that $\mathcal{F}$ is shifted ad extremis for $\varrho(\mathcal{F}) \leqslant \frac{1}{d}$. Then $S_{ij}(\mathcal{F}) \neq \mathcal{F}$ implies $\varrho(S_{ij}(\mathcal{F})) > \frac{1}{d}$. Thus, if a pair (i, j) is shift-resistant then $|\mathcal{F}(i)| + |\mathcal{F}(j)| > |\mathcal{F}|/d$.

Let $P_1, \ldots, P_s$ be a maximal collection of pairwise disjoint shift-resistant pairs, $P_i = (x_i, y_i)$, $1 \leqslant i \leqslant s$. Clearly,

$$\sum_{1 \leqslant i \leqslant s} (|\mathcal{F}(x_i)| + |\mathcal{F}(y_i)|) \geqslant \frac{s}{d} |\mathcal{F}|. \tag{10}$$

For a pair of subsets $E_0 \subset E$, let us use the notation

$$\mathcal{F}(E_0, E) = \{F \setminus E \colon F \in \mathcal{F},\ F \cap E = E_0\}.$$

Note that $\mathcal{F}(E, E) = \mathcal{F}(E)$ and $\mathcal{F}(\emptyset, E) = \mathcal{F}(\overline{E})$.

Claim 15. *For all $D \in \binom{[n]}{d}$,*

$$|\mathcal{F}(\overline{D})| \geqslant (d-1)|\mathcal{F}(D)|. \tag{11}$$

Proof. For any subset $E \subset [n]$ note the identity

$$\sum_{x\in E}|\mathcal{F}(x)| = \sum_{1\leqslant j\leqslant |E|}\sum_{E_j\in\binom{E}{j}} j|\mathcal{F}(E_j,E)| \geqslant \sum_{1\leqslant j\leqslant |E|-1}\sum_{E_j\in\binom{E}{j}}|\mathcal{F}(E_j,E)| + |E||\mathcal{F}(E,E)|$$

$$\geqslant \sum_{E'\subset E, |E'|\geqslant 1}|\mathcal{F}(E',E)| + (|E|-1)|\mathcal{F}(E)|.$$

By $\sum_{E'\subset E}|\mathcal{F}(E',E)| = |\mathcal{F}|$, we infer that

$$\sum_{x\in E}|\mathcal{F}(x)| \geqslant |\mathcal{F}| - |\mathcal{F}(\overline{E})| + (|E|-1)|\mathcal{F}(E)|. \tag{12}$$

If $|E| = d$, then $\varrho(\mathcal{F}) \leqslant \frac{1}{d}$ implies that the left hand side of (12) is less than $|\mathcal{F}|$. Comparing with the right hand side yields (11). □

Claim 16. *For all* $D \in \binom{[n]}{d}$,

$$|\mathcal{F}(D)| < d\binom{n-d-1}{k-d-1}. \tag{13}$$

Proof. For convenience assume that $D = [n-d+1, n]$. Then $\mathcal{F}(D) \subset \binom{[n-d]}{k-d}$, $\mathcal{F}(\overline{D}) \subset \binom{[n-d]}{k}$ and $\mathcal{F}(D), \mathcal{F}(\overline{D})$ are cross-intersecting. If

$$|\mathcal{F}(D)| \geqslant d\binom{n-d-1}{k-d-1} > \sum_{1\leqslant j\leqslant d}\binom{n-d-j}{k-d-1} + \binom{n-2d-2}{k-d-2},$$

then $\mathcal{L}(n-d, k-d, |\mathcal{F}(D)|)$ contains

$$\left\{A \in \binom{[n-d]}{k-d} : A\cap[d] \neq \emptyset\right\} \bigcup \left\{A \in \binom{[d+1, n-d]}{k-d} : \{d+1, d+2\} \subset A\right\}.$$

By Hilton's Lemma, we have

$$\mathcal{L}(n-d, k, |\mathcal{F}(\overline{D})|) \subset \left\{B \in \binom{[n-d]}{k} : [d] \subset B \text{ and } B\cap\{d+1, d+2\} \neq \emptyset\right\}.$$

It follows that

$$|\mathcal{F}(\overline{D})| \leqslant \binom{n-2d-1}{k-d-1} + \binom{n-2d-2}{k-d-1} < |\mathcal{F}(D)|,$$

contradicting (11). □

Claim 17.

$$s \leqslant d^2 - d. \tag{14}$$

Proof. Assume that $s \geqslant d^2 - d + 1$. Define $E = P_1 \cup \ldots \cup P_{d^2-d+1}$ and

$$\mathcal{F}_j = \{F \in \mathcal{F} \colon |F \cap E| = j\}.$$

Clearly $|E| = 2(d^2 - d + 1)$ and

$$|\mathcal{F}_j| = \sum_{E_j \in \binom{E}{j}} |\mathcal{F}(E_j, E)|. \tag{15}$$

By (13) we have

$$\sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| < \binom{2(d^2 - d + 1)}{d} d \binom{n-d-1}{k-d-1}.$$

Note that for any set $F \in \mathcal{F}$ with $F \cap E = E_j$ and $d \leqslant j \leqslant |E|$, F is counted $\binom{j}{d}$ times in $\sum_{D \in \binom{E}{d}} |\mathcal{F}(D)|$. By (15) and $\binom{j}{d} \geqslant j$ for $j > d$, it follows that

$$\sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| = \sum_{d \leqslant j \leqslant |E|} \sum_{E_j \in \binom{E}{j}} \binom{j}{d} |\mathcal{F}(E_j, E)| \geqslant |\mathcal{F}_d| + \sum_{d < j \leqslant |E|} j|\mathcal{F}_j|.$$

By (13) we obtain that

$$|\mathcal{F}_d| + \sum_{d < j \leqslant |E|} j|\mathcal{F}_j| \leqslant \sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| < \binom{2(d^2 - d + 1)}{d} d \binom{n-d-1}{k-d-1}. \tag{16}$$

Applying (10) with $s = d^2 - d + 1$,

$$\begin{aligned}
\frac{d^2 - d + 1}{d} |\mathcal{F}| \leqslant \sum_{x \in E} |\mathcal{F}(x)| &= \sum_{1 \leqslant j \leqslant |E|} j|\mathcal{F}_j| \\
&< (d-1) \sum_{1 \leqslant j \leqslant d} |\mathcal{F}_j| + |\mathcal{F}_d| + \sum_{d < j \leqslant |E|} j|\mathcal{F}_j| \\
&\overset{(16)}{<} (d-1)|\mathcal{F}| + d \binom{2(d^2 - d + 1)}{d} \binom{n-d-1}{k-d-1}.
\end{aligned}$$

It follows that

$$|\mathcal{F}| < d^2 \binom{2(d^2 - d + 1)}{d} \binom{n-d-1}{k-d-1}.$$

Let $c(d) = d^2 \binom{2(d^2-d+1)}{d}$. For $d \geqslant 4$, since $e^d < 4^{d-1} \leqslant d^{d-1}$, using $\binom{n}{k} < \left(\frac{en}{k}\right)^k$ we have

$$c(d) < 2^d e^d d^{d+2} < 2^d d^{2d+1},$$

contradicting our assumption $|\mathcal{F}| \geqslant 2^d d^{2d+1} \binom{n-d-1}{k-d-1}$. For $d = 2, 3$, it can be checked directly that $c(d) < 2^d d^{2d+1}$, contradicting our assumption as well. $\square$

Fix $X \subset [n]$ with $|X| = 2d^2 - 2d$ and $P_1 \cup \cdots \cup P_s \subset X$. Define

$$\mathcal{T} = \left\{T \subset [n] \colon |T| \leqslant d,\ |\mathcal{F}(T)| > (2d^2)^{-|T|}|\mathcal{F}|\right\}.$$

By (10), there exists $x \in X$ such that

$$|\mathcal{F}(x)| \geqslant \frac{1}{2d}|\mathcal{F}| > \frac{1}{2d^2}|\mathcal{F}|,$$

implying $\mathcal{T} \neq \emptyset$. By (13) and $|\mathcal{F}| \geqslant 2^d d^{2d+1}\binom{n-d-1}{k-d-1}$, we know that for every $D \in \binom{[n]}{d}$,

$$|\mathcal{F}(D)| < d\binom{n-d-1}{k-d-1} \leqslant (2d^2)^{-d}|\mathcal{F}|.$$

Thus $|T| \leqslant d-1$ for each $T \in \mathcal{T}$.

Now choose $T \in \mathcal{T}$ such that $|T| = t$ is maximum. Clearly $t \geqslant 1$. Note that the maximality of t implies that for every $Z \subset [n]$ with $t < |Z| \leqslant d$

$$|\mathcal{F}(Z)| \leqslant (2d^2)^{-|Z|}|\mathcal{F}|. \tag{17}$$

Set $\mathcal{A} = \mathcal{F}(T, X \cup T)$ and $U = [n] \setminus (X \cup T)$. Assume that

$$U = \{u_1, u_2, \ldots, u_m\} \text{ with } u_1 < u_2 < \cdots < u_m.$$

Let $Q = \{u_1, u_2, \ldots, u_{2d-t}\}$. Note that $\mathcal{A}(\overline{Q}) = \mathcal{F}(T, X \cup T \cup Q)$. By (17) we have

$$\begin{aligned}|\mathcal{A}(\overline{Q})| &\geqslant |\mathcal{F}(T)| - \sum_{x \in (X\setminus T)\cup Q} |\mathcal{F}(T \cup \{x\})| \\ &> (2d^2)^{-t}|\mathcal{F}| - (2d^2 - 2d + 2d - t)(2d^2)^{-(t+1)}|\mathcal{F}| \\ &= \frac{t}{(2d^2)^{t+1}}|\mathcal{F}|.\end{aligned}$$

Then by $|\mathcal{F}| \geqslant 2^d d^{2d+1}\binom{n-d-1}{k-d-1}$ we infer that

$$|\mathcal{A}(\overline{Q})| > \binom{n-d-1}{k-d-1} = \binom{n-d-1}{(k-t)-(d+1-t)}. \tag{18}$$

Claim 18. *For every $S \subset X \setminus T$,*

$$|\mathcal{F}(S, X \cup T)| \leqslant 2^{2d-1}\binom{n-d-1-|S|}{k-d-1-|S|}. \tag{19}$$

Proof. Let $\mathcal{B} = \mathcal{F}(S, X \cup T)$. Recall that $P_1, P_2, \ldots, P_s$ is a maximal collection of pairwise disjoint shift-resistant pairs and $P_1 \cup P_2 \cup \cdots \cup P_s \subset X$. Then $\mathcal{F}$ is initial on $[n] \setminus X$. It follows that $\mathcal{A} \subset \binom{[n]\setminus(X\cup T)}{k-t}$, $\mathcal{B} \subset \binom{[n]\setminus(X\cup T)}{k-|S|}$ are initial and cross-intersecting. For any $R \subset Q$ with $|R| = r \leqslant d$, we have $\mathcal{A}(\overline{Q}) \subset \binom{[n]\setminus(X\cup T\cup Q)}{k-t}$ and $\mathcal{B}(R, Q) \subset \binom{[n]\setminus(X\cup T\cup Q)}{k-|S|-r}$.

By Proposition 7, we infer that $\mathcal{A}(\overline{Q})$, $\mathcal{B}(R,Q)$ are cross $(2d-t-r+1)$-intersecting. Since $r \leqslant d$ implies $2d-t-r+1 \geqslant d+1-t$, by (18) and (5) we see that $\mathcal{A}(\overline{Q})$ is not pseudo $(2d-t-r+1)$-intersecting. By Proposition 10, it follows that $\mathcal{B}(R,Q)$ is pseudo $(2d-t-r+2)$-intersecting. Thus by (5) we have

$$|\mathcal{B}(R,Q)| \leqslant \binom{n-|X\cup T\cup Q|}{k-|S|-r-(2d-t-r+2)} = \binom{n-|X\cup T\cup Q|}{k-2d-2+t-|S|}.$$

Since $t \leqslant d-1$, $|X\cup T\cup Q| \geqslant |S|+2d-t \geqslant |S|+d+1$ and

$$\frac{n-d-1-|S|}{2} \geqslant k-d-1-|S| > k-2d-2+t-|S|,$$

we infer that

$$|\mathcal{B}(R,Q)| \leqslant \binom{n-d-1-|S|}{k-2d-2+t-|S|} < \binom{n-d-1-|S|}{k-d-1-|S|}.$$

Moreover, $|\mathcal{B}(R)| \leqslant \binom{n-d-1-|S|}{k-d-1-|S|}$ for $|R|=d+1$. Thus,

$$\begin{aligned}
|\mathcal{B}| = \sum_{R\subset Q} |\mathcal{B}(R,Q)| &= \sum_{R\subset Q,|R|\leqslant d} |\mathcal{B}(R,Q)| + \sum_{R\subset Q,|R|\geqslant d+1} |\mathcal{B}(R,Q)| \\
&\leqslant \sum_{R\subset Q,|R|\leqslant d} |\mathcal{B}(R,Q)| + \sum_{R\subset Q,|R|=d+1} |\mathcal{B}(R)| \\
&< \sum_{0\leqslant i\leqslant d} \binom{2d-t}{i}\binom{n-d-1-|S|}{k-d-1-|S|} + \binom{2d-t}{d+1}\binom{n-d-1-|S|}{k-d-1-|S|} \\
&\leqslant \binom{n-d-1-|S|}{k-d-1-|S|} \sum_{0\leqslant i\leqslant d+1} \binom{2d-1}{i} \\
&\leqslant 2^{2d-1}\binom{n-d-1-|S|}{k-d-1-|S|}.
\end{aligned}$$

□

By (19),

$$\begin{aligned}
|\mathcal{F}(\overline{T})| = \sum_{S\subset X\setminus T} |\mathcal{F}(S, X\cup T)| &< \sum_{0\leqslant j\leqslant |X\setminus T|} \binom{|X\setminus T|}{j} 2^{2d-1}\binom{n-d-1-j}{k-d-1-j} \\
&< \sum_{0\leqslant j\leqslant |X\setminus T|} \binom{2d^2-2d}{j} 2^{2d-1}\binom{n-d-1-j}{k-d-1-j}.
\end{aligned}$$

Note that $n \geqslant 4d(d-1)k$ implies

$$\frac{\binom{2d^2-2d}{j+1}\binom{n-d-2-j}{k-d-2-j}}{\binom{2d^2-2d}{j}\binom{n-d-1-j}{k-d-1-j}} = \frac{(2d^2-2d-j)(k-d-1-j)}{(j+1)(n-d-1-j)} < \frac{(2d^2-2d)k}{n} \leqslant \frac{1}{2}.$$

It follows that

$$\sum_{0\leqslant j\leqslant |X\setminus T|}\binom{2d^2-2d}{j}\binom{n-d-1-j}{k-d-1-j}<\binom{n-d-1}{k-d-1}\sum_{0\leqslant i\leqslant\infty}2^{-i}=2\binom{n-d-1}{k-d-1}.$$

Thus,

$$|\mathcal{F}(\overline{T})|<2^{2d}\binom{n-d-1}{k-d-1}<\frac{1}{d}|\mathcal{F}|$$

and therefore

$$\sum_{x\in T}|\mathcal{F}(x)|\geqslant|\mathcal{F}|-|\mathcal{F}(\overline{T})|>\frac{d-1}{d}|\mathcal{F}|.$$

Since $|T|=t\leqslant d-1$, there exists some $x\in T$ with $|\mathcal{F}(x)|>\frac{1}{d}|\mathcal{F}|$, contradicting $\varrho(\mathcal{F})\leqslant\frac{1}{d}$. Thus the theorem holds. □

4 Proof of Theorem 5

In this section we consider the maximum degree ratio problem for t-intersecting families.

Let us recall the t-covering number $\tau_t(\mathcal{F})$:

$$\tau_t(\mathcal{F})=\min\{|T|\colon |T\cap F|\geqslant t \text{ for all } F\in\mathcal{F}\}.$$

It should be clear that $\tau_t(\mathcal{F})=t$ if and only if $\mathcal{F}$ is a t-star. Proposition 2 yields

$$\varrho(\mathcal{F})\geqslant\frac{t}{\tau_t(\mathcal{F})}\tag{20}$$

for any t-intersecting family $\mathcal{F}\subset\binom{[n]}{k}$.

We say that a t-intersecting family $\mathcal{F}$ is *saturated* if any addition of an extra k-set to $\mathcal{F}$ would destroy the t-intersecting property.

In the case $\tau_t(\mathcal{F})=t+1$ one can improve on (20).

Proposition 19. *Suppose that $\mathcal{F}\subset\binom{[n]}{k}$ is t-intersecting, $n\geqslant 2k$, $\tau_t(\mathcal{F})\leqslant t+1$ and $\mathcal{F}$ is saturated. Then $\varrho(\mathcal{F})>\frac{t+1}{t+2}$.*

Proof. Without loss of generality let $[t+1]$ be a t-transversal of $\mathcal{F}$, i.e., $|F\cap[t+1]|\geqslant t$ for all $F\in\mathcal{F}$. Define

$$\mathcal{F}_i=\{F\setminus[t+1]\colon F\in\mathcal{F}, F\cap[t+1]=[t+1]\setminus\{i\}\}$$

and $\mathcal{F}_0=\mathcal{F}([t+1])$. By saturatedness $\mathcal{F}_0=\binom{[t+2,n]}{k-t-1}$. Obviously, $\mathcal{F}_i,\mathcal{F}_j$ are cross-intersecting for $1\leqslant i<j\leqslant t+1$. By Hilton's Lemma, $\min\{|\mathcal{F}_i|,|\mathcal{F}_j|\}\leqslant\binom{n-t-2}{k-t-1}$. Assume by symmetry $|\mathcal{F}_1|\leqslant|\mathcal{F}_2|\leqslant\ldots\leqslant|\mathcal{F}_{t+1}|$. Then

$$|\mathcal{F}_1|\leqslant\binom{n-t-2}{k-t-1}<\binom{n-t-1}{k-t-1}=|\mathcal{F}_0|.$$

Note that

$$|\mathcal{F}(1)| = |\mathcal{F}_2| + \cdots + |\mathcal{F}_{t+1}| + |\mathcal{F}_0| > (t+1)|\mathcal{F}_1|,\ |\mathcal{F}(\bar{1})| = |\mathcal{F}_1|.$$

Thus

$$\varrho(\mathcal{F}) \geqslant \frac{|\mathcal{F}(1)|}{|\mathcal{F}(1)| + |\mathcal{F}(\bar{1})|} > \frac{t+1}{t+2}.$$

□

Remark 20. Considering all k-subsets of $[2k-t]$ shows that without some conditions on $|\mathcal{F}|$ one cannot hope to prove better than $\varrho(\mathcal{F}) \geqslant \frac{k}{2k-t}$.

In the case of cross t-intersecting families, $t \geqslant 2$, we cannot apply Hilton's Lemma. To circumvent this difficulty we prove a similar albeit somewhat weaker inequality.

Proposition 21. *Let n, k, ℓ, t, s be integers, $s > t \geqslant 2$, $k, \ell > s$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{\ell}$ are cross t-intersecting. Assume that $|\mathcal{G}| > \binom{n}{\ell-s}$. Then*

$$|\mathcal{F}| < \binom{s-1}{t}\binom{n-s-1}{k-t} + 2^s\binom{n-t-1}{k-t-1}. \tag{21}$$

Moreover, if $n \geqslant s(k-t)$, then

$$|\mathcal{F}| < \binom{s-1}{t}\binom{n-s}{k-t} + \left(\frac{2}{s-1}\binom{s}{t-1} + \binom{s-1}{t-1} + 2\binom{s}{t+1}\right)\binom{n-s}{k-t-1}. \tag{22}$$

Proof. Assume the contrary. Without loss of generality, we can suppose that $\mathcal{F}$ and $\mathcal{G}$ are initial (S_{ij} does not change $|\mathcal{F}|, |\mathcal{G}|$). Since $|\mathcal{G}| > \binom{n}{\ell-s}$, by Theorem 9 we infer that $\mathcal{G}$ is not pseudo s-intersecting. That is, there exists $G \in \mathcal{G}$ such that $|G \cap [2i+s]| < i+s$ for all $i = 0, 1, \ldots, \ell-s$. It follows that

$$(1, 2, \ldots, s-1, s+1, s+3, \ldots) =: G_0 \in \mathcal{G}.$$

Let $T_0 \in \binom{[s-1]}{t-1}$. Then by the cross t-intersecting property

$$T_0 \cup (s, s+2, s+4, \ldots) \notin \mathcal{F}.$$

Define $T = T_0 \cup \{s\}$. Then $\mathcal{F}(T, [s]) \subset \binom{[s+1,n]}{k-t}$ and

$$E_0 := (s+2, s+4, \ldots, s+2(k-t)) \notin \mathcal{F}(T, [s]).$$

By Fact 8, $\mathcal{F}(T, [s])$ is pseudo intersecting. Thus,

$$|\mathcal{F}(T, [s])| \leqslant \binom{n-s}{k-t-1}. \tag{23}$$

For $R \subset [s]$,

$$|\mathcal{F}(R, [s])| \leqslant \binom{n-s}{k-|R|}. \tag{24}$$

We shall use (24) for R with $|R| > t$ and $|R| = t$ but $s \notin R$.

Claim 22. *For $R \subset [s]$ with $|R| = t-i$ and $i \geqslant 1$, $\mathcal{F}(R, [s])$ is pseudo $(2i+1)$-intersecting.*

Proof. Let

$$\tilde{R} := (s+1, s+2, \dots, s+2i-1, s+2i, s+2i+2, \dots, s+2(k-t)).$$

Set $Q = [t-1] \cup (s, s+2, \dots, s+2(k-t))$ and note $|Q \cap G_0| = t-1$ whence $Q \notin \mathcal{F}$. Since $Q \prec R \cup \tilde{R}$, $R \cup \tilde{R} \notin \mathcal{F}$, i.e., $\tilde{R} \notin \mathcal{F}(R, [s])$. By Fact 8, we infer that $\mathcal{F}(R, [s])$ is pseudo $(2i+1)$-intersecting. □

For $|R| = t - i$ with $1 \leqslant i \leqslant t$, by Claim 22

$$|\mathcal{F}(R, [s])| \leqslant \binom{n-s}{k-(t-i)-2i-1} = \binom{n-s}{k-t-i-1}.$$

Now

$$\begin{aligned}
|\mathcal{F}| &= \sum_{R \subset [s]} |\mathcal{F}(R, [s])| \\
&= \sum_{R \subset [s], |R| \leqslant t-1} |\mathcal{F}(R, [s])| + \sum_{R \in \binom{[s]}{t}} |\mathcal{F}(R, [s])| + \sum_{R \subset [s], |R| \geqslant t+1} |\mathcal{F}(R, [s])| \\
&\leqslant \sum_{0 \leqslant i \leqslant t-1} \binom{s}{i}\binom{n-s}{k-2t+i-1} + \binom{s-1}{t}\binom{n-s}{k-t} + \binom{s-1}{t-1}\binom{n-s}{k-t-1} \\
&\quad + \sum_{t+1 \leqslant i \leqslant s} \binom{s}{i}\binom{n-s}{k-i}. \qquad (25)
\end{aligned}$$

Using $\binom{n-s}{k-2t+i-1} < \binom{n-s}{k-t-1}$ for $i \leqslant t-1$ and $\binom{n-s}{k-i} \leqslant \binom{n-s}{k-t-1}$ for $i \geqslant t+1$, we conclude that

$$\begin{aligned}
|\mathcal{F}| &< \binom{s-1}{t}\binom{n-s}{k-t} + \sum_{0 \leqslant i \leqslant s} \binom{s}{i}\binom{n-s}{k-t-1} - \binom{s-1}{t}\binom{n-s}{k-t-1} \\
&\leqslant \binom{s-1}{t}\binom{n-s-1}{k-t} + 2^s\binom{n-s}{k-t-1}.
\end{aligned}$$

This proves (21).

If $n \geqslant s(k-t)$ then for $1 \leqslant i \leqslant t-1$

$$\begin{aligned}
\frac{\binom{s}{i}\binom{n-s}{k-2t+i-1}}{\binom{s}{i-1}\binom{n-s}{k-2t+i-2}} &= \frac{(s-i+1)(n-s-k+2t-i+2)}{i(k-2t+i-1)} \\
&\geqslant \frac{(s-t+2)(n-s-k+t+3)}{(t-1)(k-t-2)} \\
&> \frac{(s-t+2)(s-1)(k-t-1)}{(t-1)(k-t-2)} \\
&> 2
\end{aligned}$$

and

$$\frac{\binom{n-s}{k-t-2}}{\binom{n-s}{k-t-1}} = \frac{k-t-1}{n-s-k+t+2} < \frac{k-t-1}{(s-1)(k-t-1)} \leqslant \frac{1}{s-1}.$$

It follows that

$$\begin{aligned}\sum_{0\leqslant i\leqslant t-1}\binom{s}{i}\binom{n-s}{k-2t+i-1} &< \binom{s}{t-1}\binom{n-s}{k-t-2}\sum_{i=0}^{\infty}2^{-i}\\ &= 2\binom{s}{t-1}\binom{n-s}{k-t-2}\\ &< \frac{2}{s-1}\binom{s}{t-1}\binom{n-s}{k-t-1}. \end{aligned} \tag{26}$$

For $t+1 \leqslant i \leqslant s-1$,

$$\begin{aligned}\frac{\binom{s}{i+1}\binom{n-s}{k-i-1}}{\binom{s}{i}\binom{n-s}{k-i}} &= \frac{(s-i)(k-i)}{(i+1)(n-s-k+i+1)}\\ &\leqslant \frac{(s-t-1)(k-t-1)}{(t+2)(n-s-k+t+2)}\\ &< \frac{(s-t-1)(k-t-1)}{(t+2)(s-1)(k-t-1)}\\ &< \frac{1}{2}.\end{aligned}$$

It follows that

$$\sum_{t+1\leqslant i\leqslant s}\binom{s}{i}\binom{n-s}{k-i} < \binom{s}{t+1}\binom{n-s}{k-t-1}\sum_{i=0}^{\infty}2^{-i} = 2\binom{s}{t+1}\binom{n-s}{k-t-1}. \tag{27}$$

Combining (25), (26) and (27), we conclude that

$$\begin{aligned}|\mathcal{F}| &< \frac{2}{s-1}\binom{s}{t-1}\binom{n-s}{k-t-1} + \binom{s-1}{t}\binom{n-s}{k-t} + \binom{s-1}{t-1}\binom{n-s}{k-t-1}\\ &\quad + 2\binom{s}{t+1}\binom{n-s}{k-t-1}\\ &= \binom{s-1}{t}\binom{n-s}{k-t} + \left(\frac{2}{s-1}\binom{s}{t-1} + \binom{s-1}{t-1} + 2\binom{s}{t+1}\right)\binom{n-s}{k-t-1}. \qquad \square\end{aligned}$$

Consider the obvious construction:

$$\mathcal{G} = \left\{G \in \binom{[n]}{\ell} : [s] \subset G\right\},\ \mathcal{F} = \left\{F \in \binom{[n]}{k} : |F \cap [s]| \geqslant t\right\}.$$

Then $\mathcal{F}, \mathcal{G}$ are cross t-intersecting and

$$|\mathcal{G}| = \binom{n-s}{\ell-s}, \ |\mathcal{F}| = \binom{s}{t}\binom{n-s}{k-t} + \sum_{t<j\leqslant s}\binom{s}{j}\binom{n-s}{k-j},$$

showing that (21) does not hold for $|\mathcal{G}| \leqslant \binom{n-s}{\ell-s}$.

Corollary 23. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be t-intersecting with $n \geqslant (t+2)(k-t)$ and $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$. If $\varrho(\mathcal{F}) < \frac{t}{t+1}$, then for every $P \in \binom{[n]}{2}$,*

$$|\mathcal{F}(P)| \leqslant (t+1)\binom{n-t-2}{k-t-2} + \frac{5t^2+19t+24}{6}\binom{n-t-3}{k-t-3}. \tag{28}$$

Proof. If there exists $\{x, y\} \subset [n]$ such that

$$\begin{aligned}|\mathcal{F}(x,y)| &> (t+1)\binom{n-t-2}{k-t-2} + \frac{5t^2+19t+24}{6}\binom{n-t-3}{k-t-3}\\ &> (t+1)\binom{n-t-4}{k-t-2} + \left(\frac{2}{t+1}\binom{t+2}{t-1} + \binom{t+1}{t-1} + 2\binom{t+2}{t+1}\right)\binom{n-t-4}{k-t-3},\end{aligned}$$

note that $\mathcal{F}(x,y) \subset \binom{[n]\setminus\{x,y\}}{k-2}$, $\mathcal{F}(\bar{x},\bar{y}) \subset \binom{[n]\setminus\{x,y\}}{k}$ are cross t-intersecting, by applying Proposition 21 with $s = t+2$ we infer

$$|\mathcal{F}(\bar{x},\bar{y})| \leqslant \binom{n-2}{k-t-2}.$$

Since $\varrho(\mathcal{F}) < \frac{t}{t+1}$ implies

$$|\mathcal{F}(\bar{x})|, |\mathcal{F}(\bar{y})| > \frac{1}{t+1}|\mathcal{F}| > \binom{n-1}{k-t-1},$$

it follows that

$$\mathcal{F}(\bar{x},y) \geqslant |\mathcal{F}(\bar{x})| - |\mathcal{F}(\bar{x},\bar{y})| > \binom{n-2}{k-t-1}$$

and

$$\mathcal{F}(x,\bar{y}) \geqslant |\mathcal{F}(\bar{y})| - |\mathcal{F}(\bar{x},\bar{y})| > \binom{n-2}{k-t-1}.$$

But $\mathcal{F}(\bar{x},y)$, $\mathcal{F}(x,\bar{y})$ are cross t-intersecting. This contradicts Corollary 11. □

Lemma 24. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be initial, t-intersecting, $n \geqslant 2(t+1)(k-t)$ and $|\mathcal{F}| \geqslant 2t(t+1)(t+2)\binom{n-t-4}{k-t-2}$ then*

$$\varrho(\mathcal{F}) > \frac{t}{t+1}.$$

Proof. Consider a subset $P \subset [t+1]$, $|P| \leqslant t-1$.

Claim 25. *$\mathcal{F}(P,[t+1])$ is $1+2(t-|P|)$-intersecting.*

Proof. Suppose for contradiction that $\bar{F}_1, \bar{F}_2 \in \mathcal{F}(P,[t+1])$ satisfy $\bar{F}_1 \cap \bar{F}_2 = D$ with $|D| \leqslant 2(t-|P|)$. Since $\mathcal{F}$ is t-intersecting, we infer $|D| \geqslant t-|P|$. If $|D| = t-|P|$ then choose $y \in D$ and $x \in [t+1] \setminus P$ and set $F_1 = \bar{F}_1 \cup P$, $F_2 = (\bar{F}_2 \cup P \cup \{x\}) \setminus \{y\}$. By initiality $F_2 \prec \bar{F}_2 \cup P$ implies $F_2 \in \mathcal{F}$. But $|F_1 \cap F_2| = |P| + |D| - 1 = t-1$, a contradiction.

If $|D| \geqslant t+1-|P|$, then choose $E \subset D$, $|E| = t+1-|P|$ and set $F_1 = \bar{F}_1 \cup P$, $F_2 = (\bar{F}_2 \cup [t+1]) \setminus E$. Then $F_1 \cap F_2 = P \cup D \setminus E$ whence

$$|F_1 \cap F_2| = |P| + |D| - |E| \leqslant |P| + 2t - 2|P| - (t+1-|P|) = t-1.$$

Now $F_1 \in \mathcal{F}$ and $\bar{F}_2 \cup P \in \mathcal{F}$ by definition and $F_2 \prec \bar{F}_2 \cup P$. Hence $F_2 \in \mathcal{F}$ contradicting the t-intersecting property. □

Define $\mathcal{F}_i = \mathcal{F}([t+1] \setminus \{i\}, [t+1])$. By initiality

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{t+1}.$$

Apply Claim 25 with $P = [t+1] \setminus \{1\}$, $\mathcal{F}_1$ is intersecting. Thus by (4) $|\partial \mathcal{F}_1| \geqslant |\mathcal{F}_1|$. By initiality $\partial \mathcal{F}_1 \subset \mathcal{F}_0 := \mathcal{F}([t+1])$. Then

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| + \cdots + |\mathcal{F}_{t+1}| \geqslant (t+2)|\mathcal{F}_1|. \tag{29}$$

For any $P \in \binom{[2,t+1]}{t-j}$, by Claim 25 we know $\mathcal{F}(P,[t+1])$ is $(2j+1)$-intersecting. Note that $n \geqslant 2(t+1)(k-t) > (2j+2)(k-t-j)$ for $j = 1, \ldots, t$. By (1), we infer

$$|\mathcal{F}(P,[t+1])| \leqslant \binom{n-t-1-2j-1}{k-(t-j)-2j-1} = \binom{n-t-2-2j}{k-t-1-j}.$$

Note that $\mathcal{F}(P,[t+1])$ is $(k-t+j)$-uniform and $j \geqslant k-t$ implies $2j+1 > k-t+j$. It follows that $|\mathcal{F}(P,[t+1])| = 0$ for $j \geqslant k-t$. Thus,

$$\begin{aligned}
|\mathcal{F}(\bar{1})| = & \sum_{P \subset [2,t+1]} |\mathcal{F}(P,[t+1])| \\
= & |\mathcal{F}_1| + \sum_{0 \leqslant i \leqslant t-1} \sum_{P \in \binom{[2,t+1]}{i}} |\mathcal{F}(P,[t+1])| \\
\leqslant & |\mathcal{F}_1| + \sum_{1 \leqslant j \leqslant \min\{t,k-t-1\}} \binom{t}{t-j}\binom{n-t-2-2j}{k-t-1-j}.
\end{aligned}$$

For $k = t+2$, we have

$$|\mathcal{F}(\bar{1})| \leqslant |\mathcal{F}_1| + t\binom{n-t-4}{k-t-2}.$$

For $2 \leqslant j \leqslant \min\{t, k-t-1\}$,

$$\frac{\binom{t}{j}\binom{n-t-2-2j}{k-t-1-j}}{\binom{t}{j-1}\binom{n-t-2j}{k-t-j}} = \frac{(t-j+1)(k-t-j)(n-k-j)}{j(n-t-2j)(n-t-2j-1)} \leqslant \frac{(t-1)(k-t-2)(n-k-2)}{2(n-3t)(n-3t-1)}.$$

Since $n \geqslant 2(t+1)(k-t)$ and $k \geqslant t+3$ implies that

$$\frac{n-k-2}{n-3t} < 2, \ \frac{(t-1)(k-t-2)}{n-3t-1} < \frac{1}{2},$$

it follows that $\binom{t}{j}\binom{n-t-2-2j}{k-t-1-j} < \frac{1}{2}\binom{t}{j-1}\binom{n-t-2j}{k-t-j}$. Thus,

$$\begin{aligned}|\mathcal{F}(\bar{1})| &\leqslant |\mathcal{F}_1| + \sum_{1 \leqslant j \leqslant t} \binom{t}{t-j}\binom{n-t-2-2j}{k-t-1-j} \\ &< |\mathcal{F}_1| + t\binom{n-t-4}{k-t-2}\sum_{i=0}^{\infty} 2^{-i} \\ &= |\mathcal{F}_1| + 2t\binom{n-t-4}{k-t-2}.\end{aligned}$$

By (29) and $|\mathcal{F}| \geqslant 2t(t+1)(t+2)\binom{n-t-4}{k-t-2}$, it follows that

$$|\mathcal{F}(\bar{1})| \leqslant \frac{1}{t+2}|\mathcal{F}| + 2t\binom{n-t-4}{k-t-2} \leqslant \frac{1}{t+2}|\mathcal{F}| + \frac{1}{(t+1)(t+2)}|\mathcal{F}| = \frac{1}{t+1}|\mathcal{F}|.$$

Thus the lemma follows. □

Proof of Theorem 5. Suppose to the contrary that $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$ and $\varrho(\mathcal{F}) \leqslant \frac{t}{t+1}$. Since $n \geqslant 2t(t+2)k \geqslant 4(t+2)k$, we infer

$$|\mathcal{F}| > (t+1)\frac{n-1}{k-t-1}\binom{n-2}{k-t-2} > 4(t+1)(t+2)\binom{n-t-2}{k-t-2}. \quad (30)$$

Shift $\mathcal{F}$ ad extremis for $\varrho(\mathcal{F}) \leqslant \frac{t}{t+1}$ and let $\mathbb{H}$ be the graph formed by the shift-resistant pairs. For every $P \in \binom{[n]}{2}$, by (28) and $n \geqslant 2t(t+2)k > \frac{5t^2+19t+24}{6}k$ we infer

$$|\mathcal{F}(P)| < (t+1)\binom{n-t-2}{k-t-2} + \frac{5t^2+19t+24}{6}\binom{n-t-3}{k-t-3} < (t+2)\binom{n-t-2}{k-t-2}. \quad (31)$$

Claim 26. $\mathbb{H}$ *is intersecting.*

Proof. Suppose that there are disjoint pairs $(a_1, b_1), (a_2, b_2) \in \mathbb{H}$. Set $\mathcal{G}_i = \{F \in \mathcal{F} \colon F \cap \{a_i, b_i\} \neq \emptyset\}$, $i = 1, 2$. Since $\varrho(S_{a_i b_i}(\mathcal{F})) > \frac{t}{t+1}|\mathcal{F}|$, we infer $|\mathcal{G}_i| > \frac{t}{t+1}|\mathcal{F}|$. By (31) we have

$$|\mathcal{G}_1 \cap \mathcal{G}_2| \leqslant \sum_{i=1,2}\sum_{j=1,2} \mathcal{F}(\{a_i, b_j\}) < 4(t+2)\binom{n-t-2}{k-t-2}.$$

It follows that

$$|\mathcal{F}| \geqslant |\mathcal{G}_1| + |\mathcal{G}_2| - |\mathcal{G}_1 \cap \mathcal{G}_2| > \frac{2t}{t+1}|\mathcal{F}| - 4(t+2)\binom{n-t-2}{k-t-2}$$

$$\overset{(30)}{>} \frac{2t}{t+1}|\mathcal{F}| - \frac{1}{t+1}|\mathcal{F}| \geqslant |\mathcal{F}|,$$

a contradiction. □

Note that $n \geqslant 2t(t+2)k$ implies

$$|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1} > 2t(t+1)(t+2)\binom{n-t-4}{k-t-2}. \tag{32}$$

By Lemma 24, we may assume that $\mathbb{H} \neq \emptyset$. For convenience assume that $(n-1,n) \in \mathbb{H}$. Let

$$\mathcal{A} = \left\{A \in \binom{[n-2]}{k-1} : A \cup \{x\} \in \mathcal{F} \text{ with } \ x = n-1 \text{ or } x = n\right\},\ \mathcal{B} = \binom{[n-2]}{k} \cap \mathcal{F}.$$

Since $\varrho(S_{n-1,n}(\mathcal{F})) > \frac{t}{t+1}|\mathcal{F}|$ implies

$$|\mathcal{F}(n-1,n)| + |\mathcal{F}(\overline{n-1},n) \cup \mathcal{F}(n-1,\overline{n})| > \frac{t}{t+1}|\mathcal{F}|,$$

by (31) and $t \geqslant 2$ we infer

$$\begin{aligned}
\mathcal{A}(\bar{1},\bar{2}) &\geqslant \frac{t}{t+1}|\mathcal{F}| - |\mathcal{F}(n-1,n)| - \sum_{i\in\{1,2\},j\in\{n-1,n\}} |\mathcal{F}(i,j)| \\
&\geqslant \frac{t}{t+1}|\mathcal{F}| - 5(t+2)\binom{n-t-2}{k-t-2} \\
&\overset{(32)}{\geqslant} 4t(t+2)\binom{n-t-2}{k-t-2} - 5(t+2)\binom{n-t-2}{k-t-2} \\
&\overset{(9)}{\geqslant} 3(t+2)\cdot\frac{1}{2}\binom{n-3}{k-t-2} \\
&> \binom{n-4}{k-t-2}.
\end{aligned} \tag{33}$$

Fix $R \subset [2]$ with $|R| \leqslant 1$. Since $\mathcal{A}, \mathcal{B}$ are initial and cross t-intersecting, by Proposition 7 we infer that $\mathcal{A}(\bar{1},\bar{2})$ and $\mathcal{B}(R,[2])$ are cross $(t+2-|R|)$-intersecting. By (33) we know that $\mathcal{A}(\bar{1},\bar{2})$ is not pseudo $(t+2-|R|)$-intersecting. By Proposition 10 we infer that $\mathcal{B}(R,[2])$ is pseudo $(t+3-|R|)$-intersecting. Therefore,

$$|\mathcal{B}(R,[2])| \leqslant \binom{n-4}{k-|R|-(t+3-|R|)} = \binom{n-4}{k-t-3}.$$

Note that (31) implies $\mathcal{B}([2]) < (t+2)\binom{n-t-2}{k-t-2}$. Thus,

$$\begin{aligned}
|\mathcal{B}| &= \sum_{R\subset[2]} |\mathcal{B}(R,[2])| \\
&< 3\binom{n-4}{k-t-3} + (t+2)\binom{n-t-2}{k-t-2} \\
&= \frac{3(k-t-2)}{n-3}\binom{n-3}{k-t-2} + (t+2)\binom{n-t-2}{k-t-2} \\
&\overset{(9)}{\leqslant} \frac{6(k-t-2)}{n-3}\binom{n-t-2}{k-t-2} + (t+2)\binom{n-t-2}{k-t-2} \\
&< (t+3)\binom{n-t-2}{k-t-2}.
\end{aligned}$$

Then $\varrho(\mathcal{F}) \leqslant \frac{t}{t+1}$ implies

$$|\mathcal{F}(\overline{n-1},n)| = |\mathcal{F}(\overline{n-1})| - |\mathcal{B}| > \frac{1}{t+1}|\mathcal{F}| - (t+3)\binom{n-t-2}{k-t-2} \overset{(30)}{>} (3t+5)\binom{n-t-2}{k-t-2}$$

and

$$|\mathcal{F}(n-1,\overline{n})| = |\mathcal{F}(\bar{n})| - |\mathcal{B}| > \frac{1}{t+1}|\mathcal{F}| - (t+3)\binom{n-t-2}{k-t-2} \overset{(30)}{>} (3t+5)\binom{n-t-2}{k-t-2}.$$

Now by (31)

$$|\mathcal{F}(\overline{\{1,n-1\}},n)| \geqslant |\mathcal{F}(\overline{n-1},n)| - |\mathcal{F}(1,n)| > (2t+3)\binom{n-t-2}{k-t-2}$$

and

$$|\mathcal{F}(\overline{\{1,n\}},n-1)| \geqslant |\mathcal{F}(n-1,\overline{n})| - |\mathcal{F}(1,n-1)| > (2t+3)\binom{n-t-2}{k-t-2}.$$

By (9),

$$(2t+3)\binom{n-t-2}{k-t-2} \geqslant \frac{(2t+3)}{2}\binom{n-3}{k-t-2} > \binom{n-3}{k-t-2},$$

this contradicts the fact that $\mathcal{F}(\overline{\{1,n-1\}},n), \mathcal{F}(\overline{\{1,n\}},n-1) \subset \binom{[2,n-2]}{k-1}$ are cross $(t+1)$-intersecting. Thus the theorem holds. □

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