Spectral Extremal Results on Trees

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Abstract

Let $\mathrm{spex}(n, F)$ be the maximum spectral radius over all F-free graphs of order n, and $\mathrm{SPEX}(n, F)$ be the family of F-free graphs of order n with spectral radius equal to $\mathrm{spex}(n, F)$. Given integers n, k, p with $n > k > 0$ and $0 \leqslant p \leqslant \lfloor (n-k)/2 \rfloor$, let $S^p_{n,k}$ be the graph obtained from $K_k \nabla (n-k) K_1$ by embedding p independent edges within its independent set, where '∇' means the join product. For $n \geqslant \ell \geqslant 4$, let $G_{n,\ell} = S^0_{n,(\ell-2)/2}$ if ℓ is even, and $G_{n,\ell} = S^1_{n,(\ell-3)/2}$ if ℓ is odd. Cioabă, Desai and Tait [SIAM J. Discrete Math. 37 (3) (2023) 2228-2239] showed that for $\ell \geqslant 6$ and sufficiently large n, if $\rho(G) \geqslant \rho(G_{n,\ell})$, then G contains all trees of order ℓ unless $G = G_{n,\ell}$. They further posed a problem to study $\mathrm{spex}(n, F)$ for various specific trees F. Fix a tree F of order $\ell \geqslant 6$, let A and B be two partite sets of F with $|A| \leqslant |B|$, and set $q = |A| - 1$. We first show that any graph in $\mathrm{SPEX}(n, F)$ contains a spanning subgraph $K_{q,n-q}$ for $q \geqslant 1$ and sufficiently large n. Consequently, $\rho(K_{q,n-q}) \leqslant \mathrm{spex}(n, F) \leqslant \rho(G_{n,\ell})$, we further respectively characterize all trees F with these two equalities holding. Secondly, we characterize the spectral extremal graphs for some specific trees and provide asymptotic spectral extremal values of the remaining trees. In particular, we characterize the spectral extremal graphs for all spiders, surprisingly, the extremal graphs are not always the spanning subgraph of $G_{n,\ell}$.

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1 Introduction

Given a graph G, let $A(G)$ be its adjacency matrix, and $\rho(G)$ or $\rho(A(G))$ be its spectral radius (i.e., the largest eigenvalue of $A(G)$). Given a graph family $\mathcal{F}$, a graph is said to be *$\mathcal{F}$-free* if it does not contain any copy of $F \in \mathcal{F}$. For convenience, we write F-free instead of $\mathcal{F}$-free if $\mathcal{F} = \{F\}$. In 2010, Nikiforov [20] proposed the following Brualdi-Soheid-Turán type problem: What is the maximum spectral radius in any F-free graph of

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order n? The aforementioned value is called the *spectral extremal value* of F and denoted by $\mathrm{spex}(n, F)$. An F-free graph G is said to be *extremal* with respect to $\mathrm{spex}(n, F)$, if $|V(G)| = n$ and $\rho(G) = \mathrm{spex}(n, F)$. Denote by $\mathrm{SPEX}(n, F)$ the family of extremal graphs with respect to $\mathrm{spex}(n, F)$. In the past decades, the Brualdi-Soheid-Turán type problem has been studied by many researchers for many specific graphs, such as complete graphs [18, 24], odd cycles [19], even cycles [2, 18, 26, 27], paths [20] and wheels [3, 28]. For more information, we refer the reader to [4, 9, 12, 13, 14, 15, 21, 22, 23].

Fix a tree F of order $\ell \geqslant 4$, let A and B be two partite sets of F with $|A| \leqslant |B|$, and set $q = |A| - 1$. If $q = 0$, then we can see that F is a star, and the spectral extremal result is trivial. It remains the case $q \geqslant 1$. Obviously, $K_{q,n-q}$ is F-free. Then it is natural to consider the following result, which will be frequently used in the following.

Theorem 1. *For $q \geqslant 1$ and sufficiently large n, any graph in $\mathrm{SPEX}(n, F)$ contains a spanning subgraph $K_{q,n-q}$.*

Given integers n, k, p with $n > k > 0$ and $p \in \{0, 1, \dots, \lfloor (n - k)/2 \rfloor\}$, let $S^p_{n,k}$ be the graph obtained from $K_k \nabla (n - k)K_1$ by embedding p independent edges into $(n - k)K_1$, where '∇' means the join product. For $n \geqslant \ell \geqslant 4$, set $G_{n,\ell} = S^0_{n,(\ell-2)/2}$ if ℓ is even and $G_{n,\ell} = S^1_{n,(\ell-3)/2}$ otherwise. Nikiforov [20] posed the following conjecture, which is a spectral version of the well-known Erdős-Sós conjecture that any graph of average degree larger than $\ell - 2$ contains all trees of order ℓ.

Conjecture 2. ([20]) Let $\ell \geqslant 6$ and G be a graph of sufficiently large order n. If $\rho(G) \geqslant \rho(G_{n,\ell})$, then G contains all trees of order ℓ unless $G \cong G_{n,\ell}$.

The validity of Conjecture 2 for P_ℓ was proved by Nikiforov [20], for all brooms was proved by Liu, Broersma and Wang [16], for the family of all ℓ-vertex trees with diameter at most 4 was proved by Hou, Liu, Wang, Gao and Lv [10] when ℓ is even and Liu, Broersma and Wang [17] when ℓ is odd. Very recently, Cioabă, Desai and Tait [1] completely solved Conjecture 2. Thus, Conjecture 2 for the family of all ℓ-vertex trees with given diameter is true. Now we give a slightly stronger result.

Theorem 3. *Let $\ell \geqslant 6$ and $d \in \{4, 5, \dots, \ell - 1\}$, and let G be a graph of sufficiently large order n.*
(i) If at least one of ℓ and d is even, then there exists a tree F of order ℓ and diameter d such that $\mathrm{SPEX}(n, F) = \{G_{n,\ell}\}$.
(ii) If both ℓ and d are odd and $\rho(G) \geqslant \rho(S^0_{n,(\ell-3)/2})$, then G contains all trees of order ℓ and diameter d unless $G \cong S^0_{n,(\ell-3)/2}$.

It is interesting to find all trees F satisfying $\mathrm{SPEX}(n, F) = \{G_{n,\ell}\}$.

Question 4. For sufficiently large n, which tree F of order $\ell \geqslant 6$ can satisfy $\mathrm{SPEX}(n, F) = \{G_{n,\ell}\}$?

A *covering* of a graph is a set of vertices which meets all edges of the graph. Let $\beta(G)$ denote the minimum number of vertices in a covering of G. Set $\delta := \min\{d_F(x) : x \in A\}$. Inspired by the work of Cioabă, Desai and Tait, we provide an answer to Question 4.

Theorem 5. *Let n be sufficiently large, and F be a tree of order $\ell \geqslant 4$.*
(i) For even ℓ, $\mathrm{SPEX}(n,F)=\{S^0_{n,(\ell-2)/2}\}$ *if and only if $\beta(F)=\ell/2$.*
(ii) For odd ℓ, $\mathrm{SPEX}(n,F)=\{S^1_{n,(\ell-3)/2}\}$ *if and only if $\beta(F)=(\ell-1)/2$ and $\delta\geqslant 2$.*

In [1], Cioabă, Desai and Tait also proposed the following question.

Question 6. ([1]) For sufficiently large n, what is the exact value of $\mathrm{spex}(n,F)$ for a tree F of order $\ell \geqslant 6$?

Now we give partial answers to Question 6 in Theorems 7 and 8.

Theorem 7. *If $q\geqslant 1$ and $\delta \geqslant 2$, then $S^1_{n,q}$ is F-free. Moreover, for sufficiently large n,*

$$\frac{q-1}{2}+\sqrt{qn-\frac{3q^2+2q-1}{4}}<\mathrm{spex}(n,F)\leqslant \rho(J)=\sqrt{qn}+\frac{q+\delta-2}{2}+O(\frac{1}{\sqrt{n}}),$$

where $J=\begin{bmatrix} q-1 & n-q\\ q & \delta-1\end{bmatrix}$.

Obviously, $\beta(F)\leqslant |A|=q+1$. If $\beta(F)=q+1$, then let $\mathcal{A}=\{K_{q+1}\}$ and otherwise,

$$\mathcal{A}=\{F[S]\mid S \text{ is a covering of } F \text{ with } |S|\leqslant q\}.$$

Denote by $\mathrm{ex}(n,\mathcal{A})$ the maximum size in any $\mathcal{A}$-free graph of order n, and $\mathrm{EX}(n,\mathcal{A})$ the family of n-vertex $\mathcal{A}$-free graphs with $\mathrm{ex}(n,\mathcal{A})$ edges. Now we give the characterization of the spectral extremal graphs with respect to $\mathrm{spex}(n,F)$ when $\delta=1$.

Theorem 8. *For $q\geqslant 1$ and sufficiently large n,* $\mathrm{SPEX}(n,F)\subseteq \mathcal{H}(n,q,\mathcal{A})$ *if and only if $\delta=1$, where $\mathcal{H}(n,q,\mathcal{A})=\{Q_{\mathcal{A}}\nabla(n-q)K_1\mid Q_{\mathcal{A}}\in \mathrm{EX}(q,\mathcal{A})\}$. Furthermore,*
(i) $\mathrm{SPEX}(n,F)=\{K_{q,n-q}\}$ *if and only if $\delta=1$ and* $\mathrm{EX}(q,\mathcal{A})=\{qK_1\}$*;*
(ii) $\mathrm{SPEX}(n,F)=\{S^0_{n,q}\}$ *if and only if $\delta=1$ and $\beta(F)=q+1$.*

Particularly, we shall show that $\mathrm{SPEX}(n,S_{a+1,b+1})=\{K_{a,n-a}\}$ for sufficiently large n, where $1\leqslant a\leqslant b$ and $S_{a+1,b+1}$ is obtained from $K_{1,a}$ and $K_{1,b}$ by joining the centers with a new edge. If $F=S_{a+1,b+1}$, then $q=a$, $\delta=1$ and $\beta(F)=2$. By the definition of $\mathcal{A}$, we can see that $\mathrm{EX}(a,\mathcal{A})=\{aK_1\}$. By Theorem 8 (i), $\mathrm{SPEX}(n,F)=\{K_{a,n-a}\}$.

However, it seems difficult to determine $\mathrm{SPEX}(n,F)$ when $\delta\geqslant 2$, and so we leave this as a problem. In the following, we provide asymptotic spectral extremal values of all trees. Note that $\rho(K_{q,n-q})=\sqrt{q(n-q)}$. From [20] we know $\rho(S^0_{n,q})=\frac{q-1}{2}+\sqrt{qn-\frac{3q^2+2q-1}{4}}$. Combining these with Theorems 7 and 8, we have

$$\mathrm{spex}(n,F)=\sqrt{qn}+O(1). \tag{1}$$

A tree of order $\ell\geqslant 4$ is said to be a *spider* if it contains at most one vertex of degree at least 3. The vertex of degree at least 3 is called the center of the spider (if any vertex is of degree 1 or 2, then the spider is a path and any vertex of degree two can be taken to be the center). A leg of a spider is a path from the center to a leaf, and the length of a leg is the number of its edges. Let $k \geqslant 2$ and let F be a spider of order $2k+3$ with r legs of odd length and s legs of length 1. If $r \geqslant 3$ and $s \geqslant 1$, then $q = |A| - 1 = \frac{1}{2}((2k+3) - (r+s) - 1) \leqslant k-1$. By (1), we get

$$\text{spex}(n, F) = \sqrt{qn} + O(1) < \sqrt{kn} + O(1) = \rho(S^0_{n,k})$$

for sufficiently large n. This means that every graph G of order n with $\rho(G) \geqslant \rho(S^0_{n,k})$ contains F as a subgraph. Then we can derive the following result on spiders, which was originally proved by Liu, Broersma and Wang [16].

Corollary 9. ([16]) *Let $k \geqslant 2$ and let F be a spider of order $2k+3$ with r legs of odd length and s legs of length 1. If $r \geqslant 3$, $2s - r \geqslant 2$ and n is sufficiently large, then every graph G of order n with $\rho(G) \geqslant \rho(S^0_{n,k})$ contains F as a subgraph.*

The Erdős-Sós conjecture has been confirmed for some special families of spiders (see [5, 6, 7, 25]). Recently, Fan, Hong and Liu [8] has resolved this conjecture for all spiders. The spectral Erdős-Sós conjecture has also been confirmed for several classes of spiders (see [16]). In this paper, we completely characterize SPEX(n, F) for all spiders F with $q \geqslant 1$.

Theorem 10. *Let r_1, r_2, r_3, r, s and ℓ be non-negative integers with $r = r_1 + r_2 + r_3$ and $\ell \geqslant 4$, and let F be a spider of order ℓ with r_1 legs of odd length at least 5, r_2 legs of length 3, r_3 legs of length 1 and s legs of even length. Let n be sufficiently large. Then*

$$\text{SPEX}(n, F) = \begin{cases} \{S^0_{n,(\ell-r-1)/2}\} & \text{if } s \geqslant 1 \text{ and } r \geqslant 1, \\ \{S^1_{n,(\ell-3)/2}\} & \text{if } s \geqslant 1 \text{ and } r = 0, \\ \{S^1_{n,(\ell-r-1)/2}\} & \text{if } s = 0 \text{ and } r_1 \geqslant 1, \\ \{S^{r-1}_{n,(\ell-r-1)/2}\} & \text{if } s = 0,\ r_1 = 0,\ r_2 \geqslant 1 \text{ and } r_3 \in \{0, 1\}, \\ \{S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}\} & \text{if } s = 0,\ r_1 = 0,\ r_2 \geqslant 1 \text{ and } r_3 \geqslant 2. \end{cases}$$

2 Proof of Theorem 1

Before beginning our proof, we first give some notations not defined previously. Let G be a simple graph. We use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|V(G)|$ the number of vertices, $e(G)$ the number of edges, $\nu(G)$ the maximum number of independent edges, respectively. Given a vertex $v \in V(G)$ and two disjoint subsets $S, T \subseteq V(G)$. Denote by $N_G(v)$ the set of neighbors of v in G, and let $N_S(v) = N_G(v) \cap S$, $d_S(v) = |N_S(v)|$. Let $G[S]$ (resp. $G - S$) be the subgraph of G induced by S (resp. $V(G) \setminus S$). Denote by $G[S, T]$ the bipartite subgraph of G with vertex set $S \cup T$ that consists of all edges with one endpoint in S and the other endpoint in T, and let $e(S) = |E(G[S])|$ and $e(S, T) = |E(G[S, T])|$. Since $|A| + |B| = \ell$ and $|A| \leqslant |B|$, we have $|A| \leqslant \ell/2$. Moreover,

$$\beta(F) \leqslant |A| \leqslant \frac{\ell}{2}. \tag{2}$$

By the definition of q, we obtain

$$F \nsubseteq K_{q,n-q} \text{ and } F \subseteq K_{q+1,\ell}. \tag{3}$$

A standard graph theory exercise shows that for any tree F with $\ell \geqslant 2$ vertices,

$$\frac{1}{2}(\ell-2)n \leqslant \mathrm{ex}(n,F) \leqslant (\ell-2)n. \tag{4}$$

In this section, we always assume that n is sufficiently large and $G^\star$ is an extremal graph with respect to $\mathrm{spex}(n,F)$, and let $\rho^\star$ denote its spectral radius. By the Perron-Frobenius theorem, there exists a non-negative eigenvector $X=(x_1,\ldots,x_n)^{\mathrm{T}}$ corresponding to $\rho^\star$. Choose a vertex $u^\star \in V(G^\star)$ with $x_{u^\star}=\max\{x_i \mid i=1,2,\ldots,n\}=1$. We also choose a positive constant ε and a positive integer ϕ satisfying

$$\varepsilon < \frac{1}{(2\ell)^4} \quad \text{and} \quad \frac{3}{(2\ell)^{\phi-1}} < \min\left\{q\varepsilon, \frac{\varepsilon}{4\ell}\right\}, \tag{5}$$

which will be frequently used later. First, we give a rough estimation on $\rho^\star$.

Lemma 11. *For sufficiently large n, we have $\rho^\star \geqslant \sqrt{q(n-q)}$.*

Proof. By (3), $K_{q,n-q}$ is F-free. Hence, $\rho^\star \geqslant \rho(K_{q,n-q})=\sqrt{q(n-q)}$ as $G^\star$ is an extremal graph with respect to $\mathrm{spex}(n,F)$, as desired. □

Set $L^\eta=\{u\in V(G^\star) \mid x_u \geqslant (2\ell)^{-\eta}\}$ for some positive integer η. We shall constantly give an upper bound of $|L^\eta|$ and a lower bound for degrees of vertices in L^η (see Lemmas 12–14).

Lemma 12. *For every positive integer μ, we have $|L^\mu| \leqslant (2\ell)^{\mu+2}$.*

Proof. By Lemma 11, for some positive integer η, we get

$$\frac{\sqrt{q(n-q)}}{(2\ell)^\eta} \leqslant \rho^\star x_u = \sum_{v\in N_{G^\star}(u)} x_v \leqslant d_{G^\star}(u)$$

for each $u \in L^\eta$. Summing this inequality over all vertices in L^η, we obtain

$$|L^\eta|\sqrt{q(n-q)}\cdot\frac{1}{(2\ell)^\eta} \leqslant \sum_{u\in V(G^\star)} d_{G^\star}(u) \leqslant 2\mathrm{ex}(n,F) \leqslant 2(\ell-2)n.$$

Consequently, $|L^\eta| \leqslant n^{0.6}$ for sufficiently large n.

Given an arbitrary vertex $u\in V(G^\star)$ and a positive integer i, let $N_i(u)$ denote the set of vertices at distance i from u in $G^\star$. For simplicity, we use N_i, L_i^η and $\overline{L_i^\eta}$ instead of $N_i(u)$, $N_i(u)\cap L^\eta$ and $N_i(u)\setminus L^\eta$, respectively. By Lemma 11, we have

$$q(n-q)x_u \leqslant (\rho^\star)^2 x_u = d_{G^\star}(u)x_u + \sum_{v\in N_1}\sum_{w\in N_1(v)\setminus\{u\}} x_w. \tag{6}$$

Since $N_1(v)\backslash\{u\} \subseteq N_1 \cup N_2$, we get $(N_1(v)\backslash\{u\}) \cap L^\eta \subseteq L_1^\eta \cup L_2^\eta$ and $(N_1(v)\backslash\{u\}) \cap \overline{L^\eta} \subseteq \overline{L_1^\eta} \cup \overline{L_2^\eta}$. Now we divide $\sum_{v\in N_1}\sum_{w\in N_1(v)\backslash\{u\}} x_w$ into two cases $w \in L_1^\eta \cup L_2^\eta$ or $w \in \overline{L_1^\eta} \cup \overline{L_2^\eta}$. Clearly, $N_1 = L_1^\eta \cup \overline{L_1^\eta}$. In the case $w \in L_1^\eta \cup L_2^\eta$,

$$\sum_{v\in N_1}\sum_{w\in(L_1^\eta\cup L_2^\eta)} x_w \leqslant \big(2e(L_1^\eta) + e(L_1^\eta, L_2^\eta)\big) + \sum_{v\in\overline{L_1^\eta}}\sum_{w\in(L_1^\eta\cup L_2^\eta)} x_w. \tag{7}$$

By $|L^\eta| \leqslant n^{0.6}$ and (4), we have

$$2e(L_1^\eta) + e(L_1^\eta, L_2^\eta) \leqslant 2e(L^\eta) \leqslant 2\ell|L^\eta| \leqslant 2\ell n^{0.6}. \tag{8}$$

Now we deal with the case $w \in \overline{L_1^\eta} \cup \overline{L_2^\eta}$. Recall that $x_w \leqslant \frac{1}{(2\ell)^\eta}$ for $w \in \overline{L_1^\eta} \cup \overline{L_2^\eta}$. Then

$$\sum_{v\in N_1}\sum_{w\in\overline{L_1^\eta}\cup\overline{L_2^\eta}} x_w \leqslant \Big(e(L_1^\eta, \overline{L_1^\eta}\cup\overline{L_2^\eta}) + 2e(\overline{L_1^\eta}) + e(\overline{L_1^\eta}, \overline{L_2^\eta})\Big)\frac{1}{(2\ell)^\eta} \leqslant \frac{n}{(2\ell)^{\eta-1}}, \tag{9}$$

where $e(L_1^\eta, \overline{L_1^\eta}\cup\overline{L_2^\eta}) + 2e(\overline{L_1^\eta}) + e(\overline{L_1^\eta}, \overline{L_2^\eta}) \leqslant 2e(G^\star) \leqslant 2\mathrm{ex}(n,F) \leqslant 2\ell n$ by (4).

Combining (6)-(9), we obtain

$$\begin{aligned} qnx_u &< q^2x_u + d_{G^\star}(u) + 2\ell n^{0.6} + \sum_{v\in\overline{L_1^\eta}}\sum_{w\in(L_1^\eta\cup L_2^\eta)} x_w + \frac{n}{(2\ell)^{\eta-1}} \\ &< d_{G^\star}(u) + \sum_{v\in\overline{L_1^\eta}}\sum_{w\in(L_1^\eta\cup L_2^\eta)} x_w + \frac{2n}{(2\ell)^{\eta-1}}. \end{aligned} \tag{10}$$

Now we show that $d_{G^\star}(u) \geqslant \frac{n}{(2\ell)^{\mu+1}}$ for any $u \in L^\mu$. By (4), we have

$$e\big(\overline{L_1^\eta}, L_1^\eta \cup L_2^\eta\big) \leqslant \ell\big(|\overline{L_1^\eta}| + |L_1^\eta \cup L_2^\eta|\big) \leqslant \ell d_{G^\star}(u) + \frac{n}{(2\ell)^{\eta-1}}, \tag{11}$$

where the last inequality holds as $|\overline{L_1^\eta}| \leqslant d_{G^\star}(u)$, $|L^\eta| \leqslant n^{0.6}$ and n is sufficiently large. Combining (10) and (11), we obtain $qnx_u < (\ell+1)d_{G^\star}(u) + \frac{3n}{(2\ell)^{\eta-1}}$. Clearly, $x_u \geqslant \frac{1}{(2\ell)^\mu}$ as $u \in L^\mu$. Combining these with $\eta = \mu + 2$ we obtain

$$\frac{(\ell+4)n}{(2\ell)^{\mu+1}} \leqslant \frac{qn}{(2\ell)^\mu} \leqslant (\ell+1)d_{G^\star}(u) + \frac{3n}{(2\ell)^{\mu+1}},$$

where the first inequality holds as $q \geqslant 1$ and $\ell \geqslant 4$. Consequently, $d_{G^\star}(u) \geqslant \frac{n}{(2\ell)^{\mu+1}}$. Summing this inequality over all vertices in L^μ, we obtain

$$|L^\mu|\frac{n}{(2\ell)^{\mu+1}} \leqslant \sum_{u\in L^\mu} d_{G^\star}(u) \leqslant 2e(G^\star) \leqslant 2\mathrm{ex}(n,F) \leqslant 2\ell n,$$

which leads to $|L^\mu| \leqslant (2\ell)^{\mu+2}$, completing the proof. □

Lemma 13. *For every positive integer μ and every $u \in L^{\mu}$, we have $d_{G^\star}(u) \geqslant (x_u - \varepsilon)\, n$.*

Proof. Let $\overline{L_1^{\eta}}'$ be the subset of $\overline{L_1^{\eta}}$ in which each vertex has at least q neighbors in $L_1^{\eta} \cup L_2^{\eta}$. We first claim that $|\overline{L_1^{\eta}}'| \leqslant \ell\binom{|L_1^{\eta} \cup L_2^{\eta}|}{q}$. If $|L_1^{\eta} \cup L_2^{\eta}| \leqslant q-1$, then $\overline{L_1^{\eta}}'$ is empty, as desired. Now we deal with the case $|L_1^{\eta} \cup L_2^{\eta}| \geqslant q$. Suppose to the contrary that $|\overline{L_1^{\eta}}'| > \ell\binom{|L_1^{\eta} \cup L_2^{\eta}|}{q}$. For every vertex $v \in \overline{L_1^{\eta}}'$, we can select a q-subset L_v such that $L_v \subseteq (L_1^{\eta} \cup L_2^{\eta}) \cap N_{G^\star}(v)$. Clearly, there are exactly $\binom{|L_1^{\eta} \cup L_2^{\eta}|}{q}$ q-subsets in $L_1^{\eta} \cup L_2^{\eta}$. By $\lfloor |\overline{L_1^{\eta}}'| / \binom{|L_1^{\eta} \cup L_2^{\eta}|}{q} \rfloor \geqslant \ell$ and the pigeonhole principle, there exist ℓ vertices $v_1, v_2, \ldots, v_\ell$ in $\overline{L_1^{\eta}}'$ such that $L_{v_1} = L_{v_2} = \cdots = L_{v_\ell}$. It is not hard to check that $G^\star[\{u\} \cup L_{v_1}, \{v_1, \ldots, v_\ell\}] \cong K_{q+1,\ell}$. Hence, $G^\star$ contains a copy of $K_{q+1,\ell}$, and so contains a copy of F by (3), which gives a contradiction. The claim holds. Thus,

$$e(\overline{L_1^{\eta}}, L_1^{\eta} \cup L_2^{\eta}) \leqslant (q-1)|\overline{L_1^{\eta}} \setminus \overline{L_1^{\eta}}'| + |L_1^{\eta} \cup L_2^{\eta}||\overline{L_1^{\eta}}'| \leqslant (q-1)d_{G^\star}(u) + \frac{n}{(2\ell)^{\eta-1}}, \tag{12}$$

where the last inequality holds because both $|L_1^{\eta} \cup L_2^{\eta}| \leqslant |L^{\eta}|$ and $|\overline{L_1^{\eta}}'| \leqslant \ell\binom{|L_1^{\eta} \cup L_2^{\eta}|}{q}$ are constants. Combining (10) and (12), we have $qnx_u \leqslant qd_{G^\star}(u) + \frac{3n}{(2\ell)^{\eta-1}}$. Setting $\eta = \phi$, by (5) we get $d_{G^\star}(u) \geqslant (x_u - \varepsilon)n$. $\square$

Lemma 14. *For every $u \in L^1$, $x_u \geqslant 1 - \varepsilon$ and $|N_1(u)| \geqslant (1-2\varepsilon)n$. Moreover, $|L^1| = q$.*

Proof. We first show the lower bounds of x_u and $|N_1(u)|$ for any $u \in L^1$. Suppose to the contrary that there exists a vertex $u_0 \in L^1$ with $x_{u_0} < 1 - \varepsilon$. Since $u_0 \in L^1$, we have $x_{u_0} \geqslant \frac{1}{2\ell}$. By Lemma 13, we get

$$|N_1(u^\star)| \geqslant (1-\varepsilon)\, n \quad \text{and} \quad |N_1(u_0)| \geqslant \left(\frac{1}{2\ell} - \varepsilon\right) n.$$

For convenience, we set $L_i^{\phi} = N_i(u^\star) \cap L^{\phi}$ and $\overline{L_i^{\phi}} = N_i(u^\star) \setminus L^{\phi}$. By Lemma 12, $|L^{\phi}| \leqslant (2\ell)^{\phi+2}$. Hence, $|\overline{L_1^{\phi}}| \geqslant |N_1(u^\star)| - |L^{\phi}| \geqslant \big(1 - 2\varepsilon\big)n$. Consequently, by (5)

$$\big|\overline{L_1^{\phi}} \cap N_1(u_0)\big| \geqslant \big|\overline{L_1^{\phi}}\big| + \big|N_1(u_0)\big| - n \geqslant \Big(\frac{1}{2\ell} - 3\varepsilon\Big)n > \frac{n}{4\ell}. \tag{13}$$

From (13) we can see that u_0 has a neighbor in $\overline{L_1^{\phi}}$, which is also a neighbor of $u^\star$. Thus, $u_0 \in N_1(u^\star) \cup N_2(u^\star)$. Note that $u_0 \in L^1 \subseteq L^{\phi}$. Thus, $u_0 \in L_1^{\phi} \cup L_2^{\phi}$. Now, applying $u = u^\star$ and $\eta = \phi$ to (10) gives

$$\begin{aligned} qn &\leqslant |N_1(u^\star)| + \frac{2n}{(2\ell)^{\eta-1}} + e\big(\overline{L_1^{\phi}}, (L_1^{\phi} \cup L_2^{\phi}) \setminus \{u_0\}\big) + e\big(\overline{L_1^{\phi}}, \{u_0\}\big)x_{u_0} \\ &\leqslant |N_1(u^\star)| + \frac{2n}{(2\ell)^{\eta-1}} + e\big(\overline{L_1^{\phi}}, L_1^{\phi} \cup L_2^{\phi}\big) + e\big(\overline{L_1^{\phi}}, \{u_0\}\big)\big(x_{u_0} - 1\big), \end{aligned}$$

where $x_{u_0} - 1 < -\varepsilon$ by the previous assumption. Combining this with (12) and setting $\eta = \phi$, we have

$$qn \leqslant q|N_1(u^\star)| + \frac{3n}{(2\ell)^{\phi-1}} - \varepsilon e\big(\overline{L_1^{\phi}}, \{u_0\}\big),$$

which yields that $\left|\overline{L_1^\phi} \cap N_1(u_0)\right| = e\left(\overline{L_1^\phi}, \{u_0\}\right) < \frac{n}{4\ell}$ by (5), contradicting (13). Therefore, $x_u \geqslant 1 - \varepsilon$ for each $u \in L^1$. Furthermore, it follows from Lemma 13 that for each $u \in L^1$, $|N_1(u)| \geqslant (1 - 2\varepsilon)n$.

Finally, we prove that $|L^1| = q$. We first suppose $|L^1| \geqslant q + 1$. Note that every vertex $u \in L^1$ has at most $2\varepsilon n$ non-neighbors. By (5), we can see that any $q + 1$ vertices in L^1 have at least $n - 2(q+1)\varepsilon n \geqslant \frac{n}{2}$ common neighbors. Hence, $G^\star$ contains a copy of $K_{q+1,\ell}$, and so contains a copy of F by (3), which gives a contradiction. Thus, $|L^1| \leqslant q$.

Next, suppose that $|L^1| \leqslant q - 1$. Choose an arbitrary integer $\eta \geqslant 2$. Since $u^\star \in L^1 \setminus (L_1^\eta \cup L_2^\eta)$, we have $|(L_1^\eta \cup L_2^\eta) \cap L^1| \leqslant q - 2$. We can further obtain that

$$e\left(\overline{L_1^\eta}, (L_1^\eta \cup L_2^\eta) \cap L^1\right) \leqslant |\overline{L_1^\eta}| \cdot |(L_1^\eta \cup L_2^\eta) \cap L^1| \leqslant (q-2)n.$$

By (4), we have $e\left(\overline{L_1^\eta}, (L_1^\eta \cup L_2^\eta) \setminus L^1\right) \leqslant e(G^\star) < \ell n$. Furthermore, by the definition of L^1, we know that $x_w < \frac{1}{2\ell}$ for each $w \in (L_1^\eta \cup L_2^\eta) \setminus L^1$. Applying $u = u^\star$ to (10) gives

$$\begin{aligned} qn &\leqslant d_{G^\star}(u^\star) + \sum_{v \in \overline{L_1^\eta}} \sum_{w \in (L_1^\eta \cup L_2^\eta)} x_w + \frac{2n}{(2\ell)^{\eta-1}} \\ &\leqslant \left(|N_1(u^\star)| + \frac{2n}{(2\ell)^{\eta-1}} + e\left(\overline{L_1^\eta}, (L_1^\eta \cup L_2^\eta) \cap L^1\right)\right) + e\left(\overline{L_1^\eta}, (L_1^\eta \cup L_2^\eta) \setminus L^1\right) \frac{1}{2\ell} \\ &\leqslant \left(n + \frac{2n}{2\ell}\right) + (q-2)n + \ell n \cdot \frac{1}{2\ell} \\ &\leqslant \left(q - \frac{1}{4}\right) n \qquad\qquad (\text{as } \ell \geqslant 4), \end{aligned}$$

which gives a contradiction. Therefore, $|L^1| = q$. □

For convenience, we use L, L_i and $\overline{L_i}$ instead of L^1, $N_i(u) \cap L^1$ and $N_i(u) \setminus L^1$, respectively. Now, let R_1 be the subset of $V(G^\star) \setminus L$ in which every vertex is a non-neighbor of some vertex in L and $R = V(G^\star) \setminus (L \cup R_1)$. Thus, $|R_1| \leqslant 2\varepsilon n|L| \leqslant \frac{n}{(2\ell)^3}$ by (5), and so $|R| = n - |L| - |R_1| \geqslant \frac{n}{2}$. Now, we prove that the eigenvector entries of vertices in $R \cup R_1$ are small.

Lemma 15. *Let $u \in R \cup R_1$. Then $x_u \leqslant \frac{1}{2\ell^2}$.*

Proof. For any vertex $u \in R \cup R_1$, we can see that

$$d_R(u) \leqslant \ell - 1. \tag{14}$$

Indeed, if $d_R(u) \geqslant \ell$, then $G^\star[N_R(u) \cup \{u\} \cup L]$ contains a copy of $K_{q+1,\ell}$, and so contains a copy of F by (3), a contradiction. By Lemma 14 and (2), $|L| = q \leqslant (\ell - 2)/2$. Then,

$$d_{G^\star}(u) = d_L(u) + d_R(u) + d_{R_1}(u) \leqslant \frac{3}{2}\ell + d_{R_1}(u).$$

Note that $|R_1| \leqslant \frac{n}{(2\ell)^3}$ and $e(R_1) \leqslant \ell|R_1|$ by (4). Thus,

$$\rho^\star \sum_{u \in R_1} x_u \leqslant \sum_{u \in R_1} d_{G^\star}(u) \leqslant \sum_{u \in R_1} \left(\frac{3}{2}\ell + d_{R_1}(u)\right) \leqslant \frac{3}{2}\ell|R_1| + 2e(R_1) \leqslant \frac{7}{2}\ell|R_1| \leqslant \frac{7n}{16\ell^2},$$

which yields $\sum_{u\in R_1} x_u \leqslant \frac{7n}{16\ell^2\rho^\star}$. Combining $|L| \leqslant (\ell-2)/2$ and (14), we obtain

$$\rho^\star x_u = \sum_{v\in N_{G^\star}(u)} x_v \leqslant \sum_{v\in N_L(u)} x_v + \sum_{v\in N_R(u)} x_v + \sum_{v\in N_{R_1}(u)} x_v \leqslant \frac{3}{2}\ell + \frac{7n}{16\ell^2\rho^\star}.$$

Note that $\rho^\star \geqslant \sqrt{q(n-q)} \geqslant \sqrt{n-1}$. Dividing both sides by $\rho^\star$, we get

$$x_u \leqslant \frac{3\ell}{2\rho^\star} + \frac{7n}{16\ell^2(\rho^\star)^2} \leqslant \frac{3\ell}{2\sqrt{n-1}} + \frac{7n}{16\ell^2(n-1)} \leqslant \frac{1}{2\ell^2},$$

where the last inequality holds as n is sufficiently large, as desired. □

Now we complete the proof of Theorem 1.

Proof of Theorem 1 . From (4) we know that $e(R_1) \leqslant \ell|R_1|$. Then there exists a vertex $v_1 \in R_1$ with $d_{R_1}(v_1) \leqslant \frac{2e(R_1)}{|R_1|} \leqslant 2\ell$. We modify the graph $G^\star$ by deleting all edges incident to v_1 and joining v_1 to all vertices in L to obtain the graph $G^{\star\star}$. We first claim that $G^{\star\star}$ is F-free. Suppose to the contrary, then $G^{\star\star}$ contains a subgraph F' isomorphic to F. From the modification, we can see that $v_1 \in V(F')$. Since $|R| \geqslant \frac{n}{2}$, we have $|R \setminus V(F')| \geqslant |R| - \ell > \ell$. Then there exists a vertex $w_1 \in R \setminus V(F')$. Clearly, $N_{G^{\star\star}}(v_1) = L \subseteq N_{G^{\star\star}}(w_1)$. This indicates that a copy of F is already present in $G^\star$, which gives a contradiction. Hence, $G^{\star\star}$ is F-free.

Now we claim that $\rho(G^{\star\star}) > \rho^\star$. By (14) and Lemma 15, we have

$$\sum_{w\in N_{L\cup R\cup R_1}(v_1)} x_w \leqslant (q-1) + \sum_{w\in N_R(v_1)} x_w + \sum_{w\in N_{R_1}(v_1)} x_w \leqslant (q-1) + 3\ell\cdot\frac{1}{2\ell^2}, \tag{15}$$

By Lemma 14, $\sum_{w\in L} x_w \geqslant q(1-\varepsilon)$. Combining this with (15) and (5), we have

$$\rho(G^{\star\star}) - \rho^\star \geqslant \frac{2}{X^{\mathrm{T}}X} x_{v_1} \left(\sum_{w\in L} x_w - \sum_{w\in N_{L\cup R\cup R_1}(v_1)} x_w \right) \geqslant 0.$$

If $\rho(G^{\star\star}) = \rho^\star$, then $x_{v_1} = 0$ and X is also a non-negative eigenvector of $G^{\star\star}$ corresponding to $\rho^\star$. This implies that $\rho(G^{\star\star})x_{v_1} = \sum_{w\in L} x_w \geqslant q(1-\varepsilon)$, and so $x_{v_1} > 0$, a contradiction. Thus, $\rho(G^{\star\star}) > \rho^\star$, contradicting that $G^\star$ is an extremal graph with respect to $\mathrm{spex}(n, F)$. Therefore, R_1 is empty, and thus $G^\star$ contains a spanning subgraph $K_{q,n-q}$, completing the proof. □

3 Proofs of the remaining theorems

In this section, we first record several technique lemmas that we will use.

Lemma 16. ([21]) *Let H_1 be a graph on n_0 vertices with maximum degree d and H_2 be a graph on $n - n_0$ vertices with maximum degree d'. H_1 and H_2 may have loops or multiple edges, where loops add 1 to the degree. Let $H = H_1 \nabla H_2$. Define*

$$J^\star = \begin{bmatrix} d & n - n_0 \\ n_0 & d' \end{bmatrix}.$$

Then $\rho(H) \leqslant \rho(J^\star)$.

The well–known König–Egerváry theorem is as follows.

Lemma 17. ([11]) *For any bipartite graph G, we have $\beta(G) = \nu(G)$.*

By the proof of Theorem 1, we can see that $G^\star = G^\star[L] \nabla G^\star[R]$. We then give three lemmas to characterize $G^\star[L]$ and $G^\star[R]$, which help us to present an approach to prove the remaining theorems.

Lemma 18. *Let n be sufficiently large and H be a graph of order q. Then $H \nabla (n-q)K_1$ is F-free if and only if H is $\mathcal{A}$-free. Furthermore, if $G^\star[L] \cong K_q$, then $\beta(F) = q + 1$.*

Proof. Suppose first that H is $\mathcal{A}$-free. Then we show that $H \nabla (n-q)K_1$ is F-free. Otherwise, embed F into $H \nabla (n-q)K_1$ and set $S = V(F) \cap V(H)$. Then S is a covering set of F. By the definition of $\mathcal{A}$, $F[S] \in \mathcal{A}$. However, $F[S] \subseteq H[S]$, which contradicts that H is $\mathcal{A}$-free. Hence, $H \nabla (n-q)K_1$ is F-free. Suppose then that H is not $\mathcal{A}$-free. By the definition of $\mathcal{A}$, there exists a covering set S of F such that $|S| \leqslant q$ and $F[S] \subseteq H$. We can further find that $H \nabla (n-q)K_1$ contains a copy of F. Therefore, $H \nabla (n-q)K_1$ is F-free if and only if H is $\mathcal{A}$-free.

By Theorem 1, $G^\star[L] \nabla (n-q)K_1 \subseteq G^\star$. Since $G^\star$ is F-free, so does $G^\star[L] \nabla (n-q)K_1$. Thus, $G^\star[L]$ is $\mathcal{A}$-free. Assume that $G^\star[L] \cong K_q$. Now we prove that $\beta(F) \geqslant q+1$. If not, then there exists a covering set S of F with $|S| = \beta(F) \leqslant q$. Clearly, $F[S] \subseteq K_q$ and $F[S] \in \mathcal{A}$. It follows that $G^\star[L]$ contains a member of $\mathcal{A}$, contradicting that $G^\star[L]$ is $\mathcal{A}$-free. Hence, $\beta(F) \geqslant q+1$. This, together with $\beta(F) \leqslant |A| = q+1$, gives that $\beta(F) = q+1$. This completes the proof. □

Given a non-nagative integer $p \leqslant b/2$, let $K^p_{a,b}$ be the graph obtained from $aK_1 \nabla bK_1$ by embedding p independent edges into the partite set of size b.

Lemma 19. *Let n be sufficiently large and $\delta = 1$. Then $e(G^\star[R]) = 0$ and $G^\star[L] \in \mathrm{EX}(q, \mathcal{A})$.*

Proof. Since $\delta = 1$, there exists a vertex $v \in A$ of degree 1 in F. Let $A' = A \setminus \{v\}$ and $B' = B \cup \{v\}$. Obviously, $|A'| = q$ and $F[B']$ consists of an edge and some isolated vertices, which implies that $F \subseteq K^1_{q,\ell-q}$. If $e(G^\star[R]) \geqslant 1$, then $G^\star$ must contain a copy of $K^1_{q,n-q}$, and so contains a copy of F, a contradiction. Thus, $e(G^\star[R]) = 0$.

By Lemma 18, $G^\star[L]$ is $\mathcal{A}$-free, which implies that $e(G^\star[L]) \leqslant \mathrm{ex}(q, \mathcal{A})$. Now we prove that $e(G^\star[L]) = \mathrm{ex}(q, \mathcal{A})$. Suppose to the contrary, then $e(G^\star[L]) < e(Q_{\mathcal{A}})$, where $Q_{\mathcal{A}} \in \mathrm{EX}(q, \mathcal{A})$. Clearly, $e(Q_{\mathcal{A}}) \leqslant e(K_q) = \binom{q}{2}$. By Lemma 14 and (5), we have

$$
\begin{aligned}
\sum_{uv\in E(Q_{\mathcal{A}})} x_u x_v - \sum_{uv\in E(G^\star[L])} x_u x_v &\geqslant e(Q_{\mathcal{A}})(1-\varepsilon)^2 - e(G^\star[L]) \\
&> e(Q_{\mathcal{A}}) - 2\varepsilon e(Q_{\mathcal{A}}) - e(G^\star[L]) \\
&\geqslant 1 - 2\varepsilon\binom{q}{2} \\
&> 0.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\rho(Q_{\mathcal{A}}\nabla(n-q)K_1) - \rho(G^\star) &\geqslant \frac{1}{X^{\mathrm{T}}X} X^{\mathrm{T}}(A(Q_{\mathcal{A}}\nabla(n-q)K_1) - A(G^\star))X \\
&\geqslant \frac{2}{X^{\mathrm{T}}X}\left(\sum_{uv\in E(Q_{\mathcal{A}})} x_u x_v - \sum_{uv\in E(G^\star[L])} x_u x_v\right) \\
&> 0.
\end{aligned}
$$

By Lemma 18, $Q_{\mathcal{A}}\nabla(n-q)K_1$ is F-free. However, this contradicts that $G^\star$ is an extremal graph with respect to $\mathrm{spex}(n, F)$. Hence, $e(G^\star[L]) = \mathrm{ex}(q, \mathcal{A})$. From the proof in Lemma 18 we know that $G^\star[L]$ is $\mathcal{A}$-free. Therefore, $G^\star[L] \in \mathrm{EX}(q, \mathcal{A})$. □

Lemma 20. *Let n be sufficiently large and $\delta \geqslant 2$. Then $S^1_{n,q}$ is F-free and $e(G^\star[R]) \geqslant 1$.*

Proof. We first prove that $S^1_{n,q}$ is F-free, where Y_1 is the set of dominating vertices of $S^1_{n,q}$ and $Y_2 = V(S^1_{n,q}) \setminus Y_1$. Otherwise, embed F into $S^1_{n,q}$. Set $A_i = A \cap Y_i$ for each $i \in \{1, 2\}$. Then $A = A_1 \cup A_2$. Since $|A| = q+1 = |Y_1|+1 > |A_1|$, we have $A_2 \neq \varnothing$. In the graph F, let B_1 be the set of vertices in Y_1 adjacent to at least one vertex in A_2. Then, $B_1 \subseteq B$, and thus $A_1 \subseteq Y_1 \setminus B_1$ as $A_1 \subseteq A$. Obviously, $S^1_{n,q}[Y_2]$ contains exactly one edge, say e. Since $F[A_2 \cup B_1]$ is a forest, we have $e(F[A_2 \cup B_1]) \leqslant |A_2| + |B_1| - 1$. On the other hand, since $\delta \geqslant 2$, we can see that $e(F[A_2 \cup B_1]) \geqslant 2|A_2| - 1$ if there exists a vertex in A_2 incident to e, and $e(F[A_2 \cup B_1]) \geqslant 2|A_2|$ if there exists no vertex in A_2 incident to e. In both situations,

$$2|A_2| - 1 \leqslant e(F[A_2 \cup B_1]) \leqslant |A_2| + |B_1| - 1,$$

which yields that $|A_2| \leqslant |B_1|$. Combining $A_1 \subseteq Y_1 \setminus B_1$, we obtain

$$q + 1 = |A| = |A_1| + |A_2| \leqslant |Y_1 \setminus B_1| + |B_1| = |Y_1| = q,$$

a contradiction. Hence, $S^1_{n,q}$ is F-free. It follows that $\rho(G^\star) \geqslant \rho(S^1_{n,q})$.

Now we prove that $e(G^\star[R]) \geqslant 1$. Otherwise, $e(G^\star[R]) = 0$, which implies that $G^\star$ is a proper subgraph of $S^1_{n,q}$. Then, $\rho(G^\star) < \rho(S^1_{n,q})$, contradicting $\rho(G^\star) \geqslant \rho(S^1_{n,q})$. Hence, $e(G^\star[R]) \geqslant 1$. This completes the proof. □

Combining Lemmas 19 and 20, we can directly get Theorem 8. Having Lemmas 16-20, we are ready to complete the proofs of the remaining theorems.

Proof of Theorem 5. (i) Recall that $G^\star$ is an extremal graph with respect to $\mathrm{spex}(n, F)$ and $G^\star = G^\star[L]\nabla G^\star[R]$. Suppose that $\mathrm{SPEX}(n, F) = \{S^0_{n,(\ell-2)/2}\}$. Then, $G^\star[L] \cong K_{(\ell-2)/2}$ and $e(G^\star[R]) = 0$. Since $e(G^\star[R]) = 0$, we have $\delta = 1$ by Lemma 20. Since $G^\star[L] \cong K_{(\ell-2)/2}$, by Lemma 18, we have $\beta(F) = q + 1 = \ell/2$.

Conversely, if $\beta(F) = \ell/2$, then $\beta(F) = |A| = |B|$ by (2). Then, $\delta = 1$ as F is a tree. Since $\beta(F) = |A| = q + 1$, by the definition of $\mathcal{A}$ we obtain $\mathcal{A} = \{K_{q+1}\}$, and hence $\mathrm{EX}(q, \mathcal{A}) = \{K_{(\ell-2)/2}\}$. By Lemma 19, $G^\star[L] \cong K_{(\ell-2)/2}$ and $e(G^\star[R]) = 0$, that is, $\mathrm{SPEX}(n, F) = \{S^0_{n,(\ell-2)/2}\}$, as desired.

(ii) Suppose $\mathrm{SPEX}(n, F) = \{S^1_{n,(\ell-3)/2}\}$, that is, $G^\star[L] \cong K_{(\ell-3)/2}$ and $e(G^\star[R]) = 1$. Since $e(G^\star[R]) = 1$, we have $\delta \geqslant 2$ by Lemma 19. Since $G^\star[L] \cong K_{(\ell-3)/2}$, by Lemma 18, we have $\beta(F) = q + 1 = (\ell - 1)/2$.

Conversely, suppose $\beta(F) = (\ell - 1)/2$ and $\delta \geqslant 2$. Combining (2) gives $|A| = q + 1 = (\ell - 1)/2$. We first claim that $G^\star[R]$ is $2K_2$-free. Otherwise, $G^\star$ contains a copy of $K^2_{q,n-q}$. Let v_1, v_2 be two endpoints of a longest path P in F. Since F is not a star, the path P is of length at least 3, which implies that v_1, v_2 have no common neighbors. Since $\delta \geqslant 2$, we have $v_1, v_2 \in B$. Set $A' = B \setminus \{v_1, v_2\}$ and $B' = A \cup \{v_1, v_2\}$. Then A' is an independent set of F with $|A'| = (\ell - 3)/2 = q$, and $F[B']$ consists of two independent edges and some isolated vertices. This indicates that $F \subseteq K^2_{q,\ell-q}$. However, $G^\star$ contains a copy of $K^2_{q,n-q}$, and so contains a copy of F, a contradiction. Hence, $G^\star[R]$ is $2K_2$-free.

We then claim that $G^\star[R]$ is P_3-free. Since $\delta \geqslant 2$, we have

$$\ell - 1 = e(F) = \sum_{v \in A} d_A(v) \geqslant \delta \frac{\ell - 1}{2} \geqslant \ell - 1.$$

This indicates that all vertices in A are of degree 2. Choose an arbitrary vertex $v_0 \in A$. Set $A'' = A \setminus \{v_0\}$ and $B'' = B \cup \{v_0\}$. Then A'' is an independent set of F with $|A''| = (\ell - 3)/2$, and $F[B'']$ consists of a path of length 2 with center v_0 and some isolated vertices. This implies that $G^\star[R]$ is P_3-free.

Combining the above two claims, we can see that $e(G^\star[R]) \leqslant 1$, and hence $G^\star \subseteq S^1_{n,(\ell-3)/2}$ as $q = (\ell - 3)/2$. By $\delta \geqslant 2$ and Lemma 20, $S^1_{n,(\ell-3)/2}$ is F-free. Therefore, $G^\star \cong S^1_{n,(\ell-3)/2}$. The result follows. □

Proof of Theorem 3. For non-negative integers a, b, c with $a \geqslant b + 1$ and $c \geqslant 1$, let $S(a, b, c)$ be the spider with $a - b - 1$ legs of length 1, b legs of length 2 and one leg of length c. Clearly,

$$|V(S(a, b, c))| = (a - b - 1) + 2b + c + 1 = a + b + c.$$

We can find integers α and γ such that $0 \leqslant \gamma \leqslant 1$ and $\ell - d - 1 = 2\alpha + \gamma$. Then $S(\alpha + \gamma + 2, \alpha + 1, d - 2)$ is a spider of order ℓ and diameter d.

(i) Suppose first that ℓ is even. Whether d is even or not, we always obtain that $\beta(S(\alpha + \gamma + 2, \alpha + 1, d - 2)) = \ell/2$. By Theorem 5 (i), $\mathrm{SPEX}(n, S(\alpha + \gamma + 2, \alpha +$

$1, d-2)) = \{G_{n,\ell}\}$. Suppose now that ℓ is odd and d is even. It is not hard to check that $\gamma = 0$, $\delta = 2$ and $\beta(S(\alpha+\gamma+2, \alpha+1, d-2)) = (\ell-1)/2$. By Theorem 5 (ii), $\mathrm{SPEX}(n, S(\alpha+\gamma+2, \alpha+1, d-2)) = \{G_{n,\ell}\}$, as desired.

(ii) Suppose that both ℓ and d are odd. Let F be a graph of order ℓ and diameter d. Then two endpoints of a longest path in F belong to different partite sets, which implies that $\delta = 1$. On the one hand, $\beta(S(\alpha+\gamma+2, \alpha+1, d-2)) = q+1 = (\ell-1)/2$. By $\delta = 1$ and Theorem 8, $\mathrm{SPEX}(n, S(\alpha+\gamma+2, \alpha+1, d-2)) = \{S^0_{n,(\ell-3)/2}\}$. This means that $S^0_{n,(\ell-3)/2}$ does not contain a copy of $S(\alpha+\gamma+2, \alpha+1, d-2)$. On the other hand, By $\delta = 1$ and Lemma 19, $e(G^\star[R]) = 0$. Then, any graph in $\mathrm{SPEX}(n, F)$ is a subgraph of $S^0_{n,q}$, and consequently, it is also a subgraph of $S^0_{n,(\ell-3)/2}$ as $q+1 = |A| \leqslant (\ell-1)/2$. This means that $\mathrm{spex}(n, F) \leqslant \rho(S^0_{n,(\ell-3)/2})$, with equality if and only if $G^\star \cong S^0_{n,(\ell-3)/2}$. Therefore, if $\rho(G) \geqslant \rho(S^0_{n,(\ell-3)/2})$, then G contains all trees of order ℓ and diameter d unless $G \cong S^0_{n,(\ell-3)/2}$, as desired. □

Proof of Theorem 7*.* We first consider the lower bound. From [20] we know $\rho(S^0_{n,q}) = \frac{q-1}{2} + \sqrt{qn - \frac{3q^2+2q-1}{4}}$. This, together with Lemma 20, gives that

$$\rho(G^\star) \geqslant \rho(S^1_{n,q}) > \rho(S^0_{n,q}) = \frac{q-1}{2} + \sqrt{qn - \frac{3q^2+2q-1}{4}}.$$

It remains the upper bound. We shall prove that $\Delta \leqslant \delta - 1$, where Δ is the maximum degree of $G^\star[R]$. Suppose to the contrary that there exists a vertex $\widetilde{u} \in R$ with $d_R(\widetilde{u}) \geqslant \delta$. Choose a vertex $u_0 \in A$ with $d_F(u_0) = \delta$. Then we can embed F into $G^\star$ by embedding $A \setminus \{u_0\}$ into L, and embedding $B \cup \{u_0\}$ into R such that $\widetilde{u} = u_0$. This contradicts that $G^\star$ is F-free. The claim holds. Applying $d = q-1$, $n_0 = q$ and $d' = \Delta$ with Lemma 16, we have $\rho^\star \leqslant \rho(J^\star)$. By direct computation, we have

$$\rho(J^\star) = \frac{q+\Delta-1}{2} + \frac{1}{2}\sqrt{(q+\Delta-1)^2 - 4((q-1)\Delta - q(n-q))},$$

and

$$\rho(J) = \frac{q+\delta-2}{2} + \frac{1}{2}\sqrt{(q+\delta-2)^2 - 4((q-1)(\delta-1) - q(n-q))}.$$

Since n is sufficiently large and $\Delta \leqslant \delta - 1$, we obtain that

$$\rho^\star \leqslant \rho(J^\star) \leqslant \rho(J) = \sqrt{qn} + \frac{q+\delta-2}{2} + O(\frac{1}{\sqrt{n}}).$$

This completes the proof. □

Proof of Theorem 10*.* Let $v^\star$ be the center of the spider F, and let C denote the set of vertices at odd distance from $v^\star$ in F. Then $C \in \{A, B\}$. Combining Lemma 17, we can observe that

$$\delta = \begin{cases} 1 & \text{if } r \geqslant 1 \text{ and } s \geqslant 1, \\ 2 & \text{otherwise,} \end{cases} \tag{16}$$

and

$$\beta(F) = \nu(F) = |A| = \begin{cases} (\ell - r + 1)/2 & \text{if } r \geqslant 1, \\ (\ell - 1)/2 & \text{if } r = 0. \end{cases} \tag{17}$$

We first give the following claim.

Claim 21. *$G^\star[R]$ is P_3-free.*

Proof. Since F is not a star, we can select a leg of length $k \geqslant 2$, say $v^\star v_1 \cdots v_k$. Clearly, $v_i \in A$ for some $i \in \{1, 2\}$. Set $A' = A \setminus \{v_i\}$ and $B' = B \cup \{v_i\}$. Then, A' is an independent set of F with $|A'| = |A| - 1 = |L|$, and $F[B']$ consists of a path of length 2 with center v_i and some isolated vertices. Thus, $G^\star[R]$ is P_3-free. □

Now we distinguish two cases to complete the proof.

Case 1. $s \geqslant 1$.

Suppose first that $r \geqslant 1$. By (16) and (17), we have $\delta = 1$ and $\beta(F) = q + 1 = (\ell - r + 1)/2$. Combining Theorem 8, we have $\mathrm{SPEX}(n, F) = \{S^0_{n,(\ell-r-1)/2}\}$, as desired. Suppose then that $r = 0$. By (17), $|L| = q = (\ell - 3)/2$. Since $s = d_F(v^\star) \geqslant 2$, we can select two legs of even length, say $v^\star v_1 v_2 \cdots v_{2k_1-1} v_{2k_1}$ and $v^\star w_1 w_2 \cdots w_{2k_2-1} w_{2k_2}$. Obviously, $v_{2k_1}, w_{2k_2} \in B$. Set $A' = B \setminus \{v_{2k_1}, w_{2k_2}\}$ and $B' = A \cup \{v_{2k_1}, w_{2k_2}\}$. Then, A' is an independent set of F with $|A'| = |B| - 2 = (\ell - 3)/2$, and $F[B']$ consists of two independent edges and some isolated vertices. This implies that $G^\star[R]$ is $2K_2$-free. Combining Claim 21, we have $e(G^\star[R]) \leqslant 1$, and thus $G^\star \subseteq S^1_{n,(\ell-3)/2}$. On the other hand, from (16) we get $\delta = 2$. By Lemma 20, $S^1_{n,(\ell-3)/2}$ is F-free. Thus, $G^\star \cong S^1_{n,(\ell-3)/2}$, as desired.

Case 2. $s = 0$.

Obviously, $r \geqslant 2$. Since F is not a star, we have $r_1 + r_2 \geqslant 1$. Now, we divide the proof into the following three subcases.

Subcase 2.1. $r_1 \geqslant 1$.

Then, there exists a leg of length $2k+1 \geqslant 5$, say $v^\star v_1 \ldots v_{2k+1}$. Clearly, $v^\star, v_2, \ldots, v_{2k} \in A$. Set $A' = (A \setminus \{v_2, v_4\}) \cup \{v_3\}$ and $B' = V(F) \setminus A'$. Then, A' is an independent set of F with $|A'| = |A| - 1 = (\ell - r - 1)/2$ by (17), and $F[B']$ consists of two independent edges v_1v_2, v_4v_5 and some isolated vertices. This indicates that $G^\star[R]$ is $2K_2$-free. Combining Claim 21, we have $e(G^\star[R]) \leqslant 1$, and thus $G^\star \subseteq S^1_{n,(\ell-r-1)/2}$. On the other hand, since $\delta = 2$ by (16), by Lemma 20, $S^1_{n,(\ell-r-1)/2}$ is F-free. Thus, $G^\star \cong S^1_{n,(\ell-r-1)/2}$, as desired.

Subcase 2.2. $r_1 = 0$, $r_2 \geqslant 1$ and $r_3 \in \{0, 1\}$.

By (17), $|A| = (\ell - r + 1)/2$ and $|B| = (\ell + r - 1)/2$. Moreover, there are exactly r leaves in F, say $v_1, v_2, \ldots, v_r$, which contains no common neighbors as $r_3 \in \{0, 1\}$. Obviously, $v_1, v_2, \ldots, v_r \in B$. Set $A' = B \setminus \{v_1, v_2, \ldots, v_r\}$ and $B' = A \cup \{v_1, v_2, \ldots, v_r\}$. Then A' is an independent set of F, and $F[B']$ consists of r independent edges and some isolated vertices. Since $|L| = |A'|$, we can observe that $G^\star[R]$ is rK_2-free, and $S^r_{n,(\ell-r-1)/2}$ contains a copy of F. Combining Claim 21, we can see that $G^\star[R]$ consists of at most $r - 1$ independent edges and some isolated vertices, and thus $G^\star \subseteq S^{r-1}_{n,(\ell-r-1)/2}$.

Note that $S^{r}_{n,(\ell-r-1)/2}$ contains a copy of F. Then $r' \leqslant r$, where r' is the minimum integer such that $S^{r'}_{n,(\ell-r-1)/2}$ contains a copy of F. By (16), $\delta \geqslant 2$. Then, from Lemma 20 we know that $S^{1}_{n,(\ell-r-1)/2}$ is F-free, which implies that $r' \geqslant 2$. Now we shall prove $r' = r$. Otherwise, $r' < r$. Embed F into $S^{r'}_{n,(\ell-r-1)/2}$, where Y_1 is the set of dominating vertices of $S^{r'}_{n,(\ell-r-1)/2}$ and $Y_2 = V(S^{r'}_{n,(\ell-r-1)/2}) \setminus Y_1$. Set $V(F) \cap V(Y_1) = A'$ and $V(F) \cap V(Y_2) = B'$. By the definition of r', $F[B']$ contains exactly r' independent edges, say $e_1, e_2, \ldots, e_{r'}$, and some isolated vertices. Contracting e_i as a vertex for each $i \in \{1, \ldots, r'\}$ in F and $S^{r'}_{n,(\ell-r-1)/2}$, we obtain a corresponding spider F' and a corresponding graph $S^{0}_{n-r',(\ell-r-1)/2}$. Then, $F' \subseteq S^{0}_{n-r',(\ell-r-1)/2}$ as $F \subseteq S^{r'}_{n,(\ell-r-1)/2}$. Now we shall prove that $S^{0}_{n-r',(\ell-r-1)/2}$ is F'-free, which gives a contradiction. By Claim 21, any leg of F has at most one of these independent edges. If $r_3 = 0$, then F' has exactly $r - r' \geqslant 1$ legs of length 3 and r' legs of length 2. If $r_3 = 1$, then either F' has exactly $r - 1 - r'$ legs of length 3, r' legs of length 2 and one leg of length 1, or F' has exactly $r - r' \geqslant 1$ legs of length 3 and $r' - 1$ legs of length 2. Let A' and B' be two partite sets of F' with $|A'| \leqslant |B'|$. In all situations, we can see that $|V(F')| = \ell - r'$, F' has exactly $r - r' \geqslant 1$ legs of odd length and at least $r' - 1 \geqslant 1$ legs of length 2. Hence, $\min\{d_{F'}(x) \,:\, x \in A'\} = 1$ and

$$\beta(F') = \nu(F') = |A'| = ((\ell - r') - (r - r') + 1)/2 = (\ell - r + 1)/2.$$

By Theorem 8 (ii), we know that $S^{0}_{n-r',(\ell-r-1)/2}$ is F'-free, a contradiction. Hence, $r' = r$. By the definition of r', $S^{r-1}_{n,(\ell-r-1)/2}$ is F-free. Recall that $G^\star \subseteq S^{r-1}_{n,(\ell-r-1)/2}$. Then by the definition of $G^\star$, we have $G^\star \cong S^{r-1}_{n,(\ell-r-1)/2}$, as desired.

Subcase 2.3. $r_1 = 0$, $r_2 \geqslant 1$ and $r_3 \geqslant 2$.

Clearly, $\ell = 1 + 3r_2 + r_3 = 1 + r + 2r_2$, and consequently $r_2 = (\ell - r - 1)/2$. By Claim 21, $G^\star[R]$ is P_3-free. By (17), $q = (\ell - r - 1)/2$. Combining these with Theorem 1, $G^\star \subseteq S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$. It suffices to show that $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$ is F-free. Suppose to the contrary that $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$ contains a copy of F. Then embed F into $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$, where Y_1 is the set of dominating vertices of $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$ and $Y_2 = V(S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}) \setminus Y_1$. Set $V(F) \cap Y_1 = A'$ and $V(F) \cap Y_2 = B'$. Clearly, $F - \{v^\star\}$ consists of r_2 paths of length 2, say $P^1, P^2, \ldots, P^{r_2}$, and r_3 isolated vertices, say $w_1, w_2, \ldots, w_{r_3}$. Since $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}[Y_2]$ is P_3-free, at least one vertex of P^i belongs to A' for each $i \in \{1, 2, \ldots, r_2\}$, and at least one vertex of $\{v^\star, w_1, w_2\}$ belongs to A'. It follows that $|A'| \geqslant r_2 + 1 = (\ell - r + 1)/2$, which contradicts that $|A'| \leqslant |Y_1| = (\ell - r - 1)/2$. Hence, $S^{\lfloor (2n-\ell+r+1)/4 \rfloor}_{n,(\ell-r-1)/2}$ is F-free.

This completes the proof of Theorem 10. □

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