Vertex and edge orbits in nut graphs

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Abstract

A *nut graph* is a simple graph for which the adjacency matrix has a single zero eigenvalue such that all nonzero kernel eigenvectors have no zero entry. If the isolated vertex is excluded as trivial, nut graphs have seven or more vertices; they are connected, non-bipartite, and have no leaves. It is shown that a nut graph G always has at least one more edge orbit than it has vertex orbits: $o_e(G) \geqslant o_v(G) + 1$, with the obvious corollary that edge-transitive nut graphs do not exist. We give infinite families of vertex-transitive nut graphs with two orbits of edges, and infinite families of nut graphs with two orbits of vertices and three of edges. Several constructions for nut graphs from smaller starting graphs are known: double subdivision of a bridge, four-fold subdivision of an edge, a construction for extrusion of a vertex with preservation of the degree sequence. To these we add multiplier constructions that yield nut graphs from regular (not necessarily nut graph) parents. In general, constructions can have different effects on the automorphism group and counts of vertex and edge orbits, but in the case where the automorphism group is 'preserved', they can be used in a predictable way to control vertex and edge orbit numbers.

Mathematics Subject Classifications: 05C50, 05C25, 05C75, 05C92

1 Introduction

The main goal of the present paper is to find limitations on the numbers of orbits of vertices and edges of nut graphs under the action of the full automorphism group, and in particular to show that every nut graph has more than one orbit of edges. To substantiate this claim, we require some standard definitions. All graphs considered in this paper are simple and connected. By $\delta(G)$, $d(G)$ and $\Delta(G)$ we denote the minimum, average and maximum degrees of a vertex in graph G (see [20, Section 1.2]). The adjacency matrix of graph G is $\mathbf{A}(G)$ and the dimension of the nullspace of $\mathbf{A}(G)$ is the *nullity*, $\eta(G)$. Let $\Phi(M; \lambda)$

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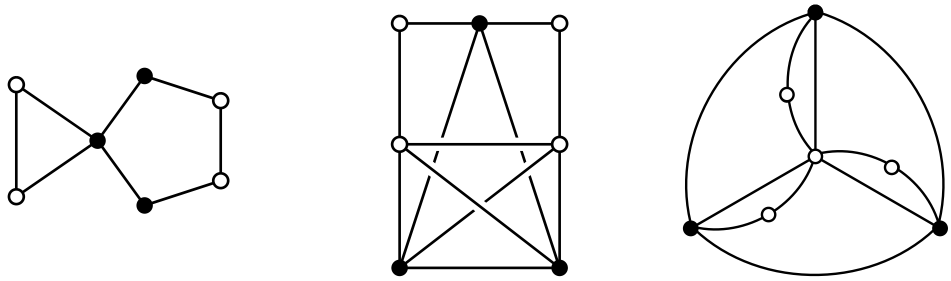
denote the characteristic polynomial of square matrix M. The characteristic polynomial of graph G, denoted $\Phi(G;\lambda)$, is the characteristic polynomial of its adjacency matrix, i.e., $\Phi(G;\lambda) = \Phi(\mathbf{A}(G);\lambda) = \det(\mathbf{A}(G) - \lambda\mathbf{I})$. The spectrum of graph G will be denoted $\sigma(G)$. For a graph G of order n, we take $V(G) = \{1, 2, \ldots, n\}$. The neighbourhood of a vertex v in graph G is denoted $N_G(v)$; where the graph G is clear from the context then we can simply write $N(v)$. For other standard definitions we refer the reader to one of the many comprehensive treatments of graph spectra and related concepts (e.g., [5, 8, 13, 14, 15]).

Nut graphs [44] are graphs that have a one-dimensional nullspace (i.e., $\eta(G) = 1$), where the nontrivial kernel eigenvector $\mathbf{x} = [x_1 \ \ldots \ x_n]^\intercal \in \ker \mathbf{A}(G)$ is full (i.e., $|x_i| > 0$ for all $i = 1, \ldots, n$). Nut graphs are connected, non-bipartite and have no leaves (i.e., $\delta(G) \geqslant 2$ for every nut graph G) [44]. As the defining paper considered the isolated vertex to be a trivial case [44], the *nontrivial* nut graphs have seven or more vertices. Nut graphs of small order have been enumerated (see, e.g., [4, 11] and [12]). If G is a *regular* nut graph, then $\delta(G) = d(G) = \Delta(G) \geqslant 3$. Note that there are no nut graphs with $\Delta(G) = 2$, as it is known that cycles are not nut graphs. The case of $\Delta(G) = 3$ is of interest in chemical applications of graph theory, as a *chemical graph* is a connected graph with maximum degree at most three. (This definition is motivated by applications of the Hückel model to carbon π-systems and is widely used in mathematical chemistry [23]. A maximum degree of four is useful in considering saturated systems such as alkanes, but a carbon atom without a spare fourth valence cannot participate directly in a π-system.) Chemical aspects of nut graphs are treated in [42]. The nut graph is a special case of the *core graph*: a core graph is a graph with $\eta(G) \geqslant 1$ for which it is possible to construct a kernel eigenvector in which all vertices of G carry a nonzero entry. Hence, a nut graph is a core graph of nullity one. Again, K_1 is presumably a trival core graph in the standard definition. Notice that a core graph may be bipartite or not, whereas a nut graph is not bipartite.

Let G and H be simple graphs. The Cartesian product of G and H, denoted $G \,\square\, H$, is the graph with the vertex set $\{(u, v) \mid u \in V(G) \text{ and } v \in V(H)\}$ and the edge set $\{(u, v)(u', v') \mid (uu' \in E(G) \text{ and } v = v') \text{ or } (u = u' \text{ and } vv' \in E(H))\}$. For further details on graph products see [26].

An *automorphism* α of a graph G is a permutation $\alpha\colon V(G) \to V(G)$ of the vertices of G that maps edges to edges and non-edges to non-edges. The set of all automorphisms of a graph G forms a group, the *(full) automorphism group* of G, denoted by $\mathrm{Aut}(G)$. The image of a vertex $v \in V(G)$ under automorphism α will be denoted v^α. Let $u, v \in V(G)$. If there is an automorphism $\alpha \in \mathrm{Aut}(G)$, such that $u^\alpha = v$, vertices u and v belong to the same vertex orbit. This relation partitions the vertex set $V(G)$ into $o_v(G)$ vertex orbits. Let $\{u_1, u_2\}, \{v_1, v_2\} \in E(G)$. If there is an automorphism $\alpha \in \mathrm{Aut}(G)$, such that $\{u_1^\alpha, u_2^\alpha\} = \{v_1, v_2\}$, then edges u_1u_2 and v_1v_2 belong to the same edge orbit. This relation partitions the edge set $E(G)$ into $o_e(G)$ edge orbits. See Figure 1 for examples. If $o_v(G) = 1$ (i.e., all vertices belong to the same vertex orbit) then the graph G is said to be *vertex-transitive*. Likewise, if $o_e(G) = 1$, then the graph G is said to be *edge-transitive*. A well-known class of vertex-transitive graphs is the *circulant graphs* [25, Section 1.5]. By $\mathrm{Circ}(\mathbb{Z}_n, S)$, where $S \subseteq \mathbb{Z}_n$, we denote the graph on the vertex set $V(G) = \mathbb{Z}_n$, where vertices $u, v \in V(G)$ are adjacent if and only if $|u - v| \in S$. Let $\mathcal{G}$ be a subgroup of $\mathrm{Aut}(G)$. The stabiliser of a vertex v in G, denoted $\mathcal{G}_v$, is the subgroup of $\mathcal{G}$ that contains all elements α such that $v^\alpha = v$. For other standard definitions from algebraic graph theory, we refer the reader to textbooks, e.g., [3, 21, 25].

Example 1. There are three non-isomorphic nut graphs on 7 vertices. We denote them S_1, S_2 and S_3 and call them the Sciriha graphs. For each Sciriha graph the numbers of vertex and edge orbits and the order of the (full) automorphism group are given; see Figure 1. $\diamondsuit$



(a) $\Omega(S_1) = (4, 5, 4)$ (b) $\Omega(S_2) = (4, 6, 4)$ (c) $\Omega(S_3) = (3, 4, 6)$

Figure 1: Vertex and edge orbits in the Sciriha graphs (i.e., the three nut graphs on 7 vertices). For brevity, we denote the triple $(o_v(G), o_e(G), |\mathrm{Aut}(G)|)$ by $\Omega(G)$. Vertices are coloured white or black to indicate equal and opposite entries in the kernel eigenvector. Note that for the Sciriha graphs, the kernel eigenvector is totally symmetric in each case (i.e., its trace is 1 for every automorphism).

1.1 Natural questions about nut graphs

Nut graphs are found within several well-known graph classes, such as *fullerenes*, *cubic polyhedra* [43] and more general *regular graphs* [2, 22]. Nut graphs within these classes tend to have low symmetry, but attention has also been paid to finding nut graphs with high symmetry (in the sense of having a small number of vertex orbits). Recently, those pairs (n, d) for which a d-regular nut circulant of order n exists have been characterised in a series of papers [2, 16, 17, 18, 19]. It is known that there are infinitely many vertex-transitive nut graphs [22]. It seems natural, therefore, to consider the possibility of edge-transitive nut graphs. (Recall that if G is edge-transitive, this does *not* imply that G is vertex-transitive, *nor* does it imply that G is regular. However, an edge-transitive connected graph has at most two vertex orbits.)

To give some background for our question, a preliminary computer search based on the census of connected edge-transitive graphs on orders $n \leqslant 47$ [9, 10] was conducted. It found no examples of nut graphs. The census contains 1894 graphs in total. Of these, 335 graphs are non-singular and 2 graphs have nullity 1 (these graphs are K_1 and P_3). In the census, there is at least one graph for every admissible nullity (i.e., for each $0 \leqslant k \leqslant 45$, there exists a graph G with $\eta(G) = k$). There are 1312 core graphs in the census (not counting K_1). Amongst these, there are 1098 bipartite graphs (945 non-regular, 25 regular non-vertex-transitive, and 128 vertex-transitive graphs). The remaining 214 non-bipartite edge-transitive core graphs are necessarily vertex-transitive, but none of these are nut graphs. On this basis, it seems plausible to question whether edge-transitive nut graphs exist. This prompted us to look for the general relationship between the numbers of vertex and edge orbits in nut graphs that is proved in the next section. In later sections, special attention is paid to nut graphs with one and two vertex orbits and the minimum number of edge orbits (respectively, two and three). Finally, the implications of constructions for the symmetry properties of nut graphs are investigated; a useful byproduct is a simple proof that there exist infinitely many nut graphs for each even number of vertex orbits and any number of edge orbits allowed by the main theorem.

2 A relation between numbers of vertex and edge orbits

The main result is embodied in the following theorem.

Theorem 2. *Let G be a nut graph. Then $o_e(G) \geqslant o_v(G) + 1$.*

This theorem immediately implies the following corollary.

Corollary 3. *Let G be a nut graph. Then G is not edge-transitive.*

It is, however, relatively easy to find infinite families of vertex-transitive nut graphs with *few* edge orbits. For example, in Sections 3.1 and 4.1, we provide infinite families of nut graphs for $(o_v, o_e) = (1, 2)$ and $(o_v, o_e) = (2, 3)$. Moreover, as we saw from our examination of the census, many core graphs are edge-transitive.

To prepare for the proof of Theorem 3, we recall some established results.

Lemma 4 ([25, Lemma 3.2.1])**.** *Let G be an edge-transitive graph with no isolated vertices. If G is not vertex-transitive, then* $\mathrm{Aut}(G)$ *has exactly two orbits, and these two orbits are a bipartition of G.*

A similar statement appears in [3] as Proposition 15.1. A theorem from a previous investigation specifies necessary conditions relating order and degree of a vertex-transitive nut graph:

Theorem 5 ([22, Theorem 10])**.** *Let G be a vertex-transitive nut graph on n vertices, of degree d. Then n and d satisfy the following conditions. Either $d \equiv 0 \pmod 4$, and $n \equiv 0 \pmod 2$ and $n \geqslant d + 4$; or $d \equiv 2 \pmod 4$, and $n \equiv 0 \pmod 4$ and $n \geqslant d + 6$.*

Lemma 6. *Let G be a vertex-transitive nut graph and let $\mathbf{x} = [x_1 \ \dots \ x_n]^\intercal \in \ker \mathbf{A}(G)$. Then the following statements hold:*

(a) $\mathbf{x} = \pm\mathbf{x}^\alpha$ *for every* $\alpha \in \mathrm{Aut}(G)$*;*

(b) $|x_i| = |x_j|$ *for all i and j;*

(c) we can take the entries to be $x_i \in \{+1, -1\}$*;*

(d) $d(G)$ *and* n *are both even.*

Proof. As G is vertex-transitive and $n \geqslant 7$, G has a nontrivial automorphism group $\mathrm{Aut}(G)$, i.e., $|\mathrm{Aut}(G)| > 1$. As G is a nut graph, the kernel eigenvector $\mathbf{x}$ belongs to a one-dimensional eigenspace, and hence spans a one-dimensional irreducible representation of $\mathrm{Aut}(G)$. As the graph is vertex-transitive, each element $\alpha \in \mathrm{Aut}(G)$ sends vertex v to an image vertex v^α (v^α may be v). The action of α on the vertices of G can be extended to $\mathbf{x}$ by defining $\mathbf{x}^\alpha = [x_{1^\alpha} \ \ldots \ x_{n^\alpha}]$. As a one-dimensional irreducible representation has trace either $+1$ or -1 under any particular automorphism $\alpha \in \mathrm{Aut}(G)$, it follows that $\mathbf{x} = \pm\mathbf{x}^\alpha$. This proves claim (a). In particular, $|x_1| = |x_{1^\alpha}|$. Since for every vertex v there exists an $\alpha \in \mathrm{Aut}(G)$ such that $v = 1^\alpha$, the claim (b) follows. To show that the entries in the kernel eigenvector can be drawn from the set $\{+1, -1\}$, it is enough to normalise the vector to $x_1 = 1$, verifying claim (c). To establish claim (d), note that the local condition for entries of the vector $\mathbf{x} \in \ker \mathbf{A}(H)$ is

$$\sum_{u \in N(v)} x_u = 0 \quad \text{for } v = 1, \ldots, n. \tag{1}$$

As all entries of $\mathbf{x}$ are in $\{+1, -1\}$, every vertex v must be of *even degree*. Since G is a regular graph, the entries of the Perron vector $\mathbf{y}$ of G (i.e., the eigenvector that corresponds to the largest eigenvalue λ_1) are all equal to $+1$. As $\mathbf{x}$ is orthogonal to $\mathbf{y}$, i.e., $\sum_{i=1}^n x_i = 0$, there are equal numbers of $+1$ and -1 entries in $\mathbf{x}$, and G must have even order n, completing claim (d). ☐

We note that the arguments used in claims (a) to (c) in the proof of Lemma 6 can be applied orbit-wise for graphs that are not vertex-transitive. Lemma 6 thus generalises naturally to the following:

Lemma 7. *Let* G *be a nut graph and let* $\mathbf{x} = [x_1 \ \ldots \ x_n]^\intercal \in \ker \mathbf{A}(G)$*. Then the following statements hold:*

(a) $\mathbf{x} = \pm\mathbf{x}^\alpha$ *for every* $\alpha \in \mathrm{Aut}(G)$*;*

(b) $|x_i| = |x_j|$ *if* i *and* j *belong to the same vertex orbit;*

(c) we can take the entries to be $x_i \in \{+a_j, -a_j\}$ *if* $i \in \mathcal{V}_j$*, where* a_j *is a nonzero constant for orbit* $\mathcal{V}_j$*.*

By Lemma 7(a), α acts on $\mathbf{x}$ in one of two ways: $\mathbf{x}^\alpha = \mathbf{x}$ or $\mathbf{x}^\alpha = -\mathbf{x}$. In that first case, the automorphism α is called *sign-preserving*, and in the second *sign-reversing*. This lemma will be used in the proof of Theorem 2. Before the proof, we introduce some definitions. The first of these deal with edges. Let G be a nut graph with k vertex orbits $V(G) = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \cdots \sqcup \mathcal{V}_k$. There are several *types* of edge (as indicated schematically in Figure 2 for the case $k = 3$):

(a) *intra-orbit* edge types, where both endvertices of an edge are in the same orbit $\mathcal{V}_i$, denoted e_i;

(b) *inter-orbit* edge types, where endvertices of an edge are in two different orbits $\mathcal{V}_i$ and $\mathcal{V}_j$, $i \neq j$, denoted $e_{ij} = e_{ji}$.

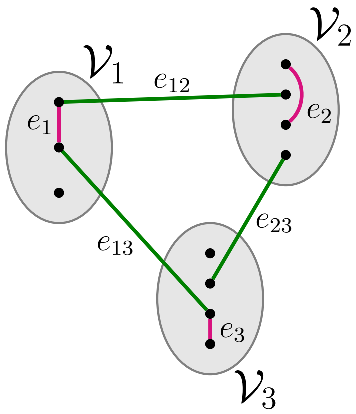


Figure 2: Schematic representation of a graph with three vertex orbits (represented by the three bags of vertices). There are six edge types, denoted e_1, e_2, e_3 (intra-orbit), e_{12}, e_{13} and e_{23} (inter-orbit).

It is clear that edges of different types cannot belong to the same edge orbit. Moreover, a single edge type may comprise several edge orbits.

An additional definition will also be useful for the proof. Let the *vertex-orbit graph* of G, denoted $\mathfrak{G}(G)$, be the graph whose vertices are vertex orbits of G and vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ are adjacent in $\mathfrak{G}(G)$ if there exists at least one edge of type e_{ij} in G. Note that in our case $\mathfrak{G}(G)$ contains k vertices. Moreover, the graph $\mathfrak{G}(G)$ is a simple graph, not to be confused with the orbit graph as defined in [32], which is typically a pregraph.

Lemma 8. *Let G be a nut graph with k vertex orbits $V(G) = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \cdots \sqcup \mathcal{V}_k$ and let* $\mathbf{x} = [x_1 \ \ldots \ x_n]^\intercal \in \ker \mathbf{A}(G)$. *Suppose there exists a vertex orbit $\mathcal{V}_\ell$ in G, such that*

(i) $\mathcal{V}_\ell$ is a leaf in $\mathfrak{G}(G)$, and

(ii) vertices $\mathcal{V}_\ell$ form an independent set in G.

Then for every orbit $\mathcal{V}_i$ of G it holds that

$$\sum_{j \in \mathcal{V}_i} x_j = 0. \tag{2}$$

Moreover, for each orbit $\mathcal{V}_i$ it holds that $|\{j \in \mathcal{V}_i : x_j > 0\}| = |\{j \in \mathcal{V}_i : x_j < 0\}|$ and the size of the orbit $\mathcal{V}_i$ is even.

Proof. Let $\mathcal{V}_{\ell'}$ be the neighbour of $\mathcal{V}_\ell$ in $\mathfrak{G}(G)$. Let d_{ij} be the number of neighbours of a vertex $v \in \mathcal{V}_i$ that reside in $\mathcal{V}_j$. The local condition says that

$$\sum_{u \in N(v)} x_u = 0 \quad \text{for } v \in \mathcal{V}_\ell. \tag{3}$$

Therefore,

$$\sum_{v \in \mathcal{V}_\ell} \sum_{u \in N(v)} x_u = \sum_{u \in \mathcal{V}_{\ell'}} d_{\ell'\ell} x_u = 0. \tag{4}$$

This implies that

$$\sum_{u \in \mathcal{V}_{\ell'}} x_u = 0.$$

Hence, the orbit $\mathcal{V}_\ell'$ must contain at least one vertex v with $x_v > 0$ and at least one vertex w with $x_w < 0$. This implies that there exists a sign-reversing $\alpha \in \mathrm{Aut}(G)$. Within each orbit, α maps vertices with positive entries in the kernel eigenvector to vertices with negative entries, and vice-versa. Therefore, the cardinalities of these two sets of vertices are equal. Equation (2) follows by Lemma 7(b), and the claim about the parity is evident. □

We can now proceed to the proof of the theorem.

Proof of Theorem 2. Let G be a nut graph with k vertex orbits $V(G) = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \cdots \sqcup \mathcal{V}_k$.

If the graph G is connected then $\mathfrak{G}(G)$ is also connected. The connectedness of $\mathfrak{G}(G)$ implies that $o_e(G) \geqslant o_v(G) - 1$ [6], since a connected graph on k vertices has at least $k-1$ edges and each edge of $\mathfrak{G}(G)$ gives rise to at least one edge orbit of G. Suppose there are no intra-orbit edges in G. In this case G is bipartite if and only if $\mathfrak{G}(G)$ is bipartite. But a bipartite graph is not a nut graph. Hence we have $o_e(G) \geqslant o_v(G)$. To avoid bipartiteness we can do one of two things:

(I) We may add intra-orbit edges to one or more vertex orbits. We need only consider addition of one such edge type, as addition of two or more would already imply $o_e(G) \geqslant o_v(G) + 1$.

(II) We may add another type of inter-orbit edge to make an odd cycle in $\mathfrak{G}(G)$. Note that $\mathfrak{G}(G)$ becomes a unicyclic graph. Again, we do not need to consider addition of more than one edge type.

First, we deal with the case $k = 1$, i.e., G is a vertex-transitive graph. By Lemma 6(c), the entries of $\mathbf{x}$ are from the set $\{+1, -1\}$. This justifies the following classification of edges of a vertex-transitive nut graph: An edge $uv \in E(G)$ is a *like* edge if $x_u x_v > 0$. An edge $uv \in E(G)$ is an *unlike* edge if $x_u x_v < 0$. Notice that every vertex of G is incident with $d/2$ like and $d/2$ unlike edges, where d is the vertex degree in G. Consider the action of $\mathrm{Aut}(G)$ on the edges of G. If $\{u, v\}$ is a like edge, then $\{u^\alpha, v^\alpha\}$ is a like edge for any choice of $\alpha \in \mathrm{Aut}(G)$. Similarly, an automorphism maps an unlike edge to an unlike edge. Therefore, G has at least two distinct edge orbits, thus $o_e(G) \geqslant o_v(G) + 1$ holds.

Now, we deal with the case $k \geqslant 2$. Suppose that $\mathfrak{G}(G)$ contains a leaf $\mathcal{V}_\ell$ that is an independent set in G (in other words, there are no intra-orbit edges in $\mathcal{V}_\ell$). Let $\mathcal{V}'_\ell$ be the neighbour of $\mathcal{V}_\ell$ in $\mathfrak{G}(G)$. By Lemma 8, the numbers of positive and negative entries in the kernel eigenvector are equal within any given orbit. This implies the existence of a sign-reversing automorphism $\alpha \in \mathrm{Aut}(G)$. Each edge of type $e_{\ell\ell'}$ can be assigned one of four

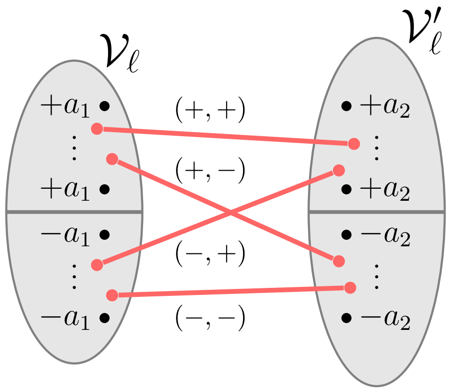


Figure 3: Vertex orbits $\mathcal{V}_\ell$ and $\mathcal{V}'_\ell$ as defined in the proof of Theorem 2, showing the four possible signatures for edges of type $e_{\ell\ell'}$.

signatures according to signs of the kernel eigenvector entries for its endvertices (shown schematically in Figure 3). We now consider the action of the automorphism α on edges of each signature. Since α is sign-reversing, edge signatures are swapped: $(+,+) \leftrightarrow (-,-)$ and $(+,-) \leftrightarrow (-,+)$. Hence, edges of type $e_{\ell\ell'}$ fall into at least two orbits, determined by relative sign of endvertex entries. Note that a like edge is of signature $(+,+)$ or $(-,-)$, while an unlike edge is of signature $(+,-)$ or $(-,+)$. By the local condition at a vertex of $\mathcal{V}_\ell$, the presence of a $(+,+)$ edge implies the presence of a $(+,-)$ edge and vice-versa, hence the two corresponding edge orbits are both nonempty. As there is at least one edge orbit included within $e_{\ell\ell'}$, the number of edge orbits in G is greater than the number of its edge types.

This proves case (I) and also case (II) where $\mathfrak{G}(G)$ is a unicyclic graph but not a cycle. If $\mathfrak{G}(G)$ is a cycle (necessarily odd) and there are no inter-orbit edges, a different argument is needed. Recall that no automorphism maps a like to an unlike edge (or vice versa), so they cannot be in the same edge orbit. If $e_{i,i+1}$ contains both like and unlike edges, this immediately implies $o_e(G) \geqslant o_v(G) + 1$. So, for every i, we can assume that $e_{i,i+1}$ contains only like or only unlike edges. Take any vertex $u \in \mathcal{V}_i$. Since there are no intra-edges it has to be connected to neighbours in $\mathcal{V}_{i-1}$ via like and neighbours in $\mathcal{V}_{i+1}$ via unlike edges or vice versa. Therefore, the edges of $\mathfrak{G}(G)$ can be properly coloured with colours 'like' and 'unlike'. But $\mathfrak{G}(G)$ is an odd cycle, so no such edge colouring exists. Hence, at least one type $e_{i,i+1}$ contains edges of both kinds. □

Lemma 8 implies that a sign-reversing automorphism exists in a nut graph if at least one vertex orbit $\mathcal{V}_\ell$ is a leaf in $\mathfrak{G}(G)$ and $\mathcal{V}_\ell$ has no intra-orbit edges. A similar structural result can also be obtained if $\mathfrak{G}(G)$ is an odd cycle and G has no intra-orbit edges.

Proposition 9. *Let G be a nut graph with k vertex orbits $V(G) = \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \cdots \sqcup \mathcal{V}_k$ and let $\mathbf{x} = [x_1 \ \ldots \ x_n]^\intercal \in \ker \mathbf{A}(G)$. Suppose that every $\mathcal{V}_\ell$ forms an independent set in G and $\mathfrak{G}(G)$ is an odd cycle. Then for every orbit $\mathcal{V}_i$ of G it holds that*

$$\sum_{j \in \mathcal{V}_i} x_j = 0. \tag{5}$$

Moreover, for each orbit $\mathcal{V}_i$ it holds that $|\{j \in \mathcal{V}_i : x_j > 0\}| = |\{j \in \mathcal{V}_i : x_j < 0\}|$ and the size of the orbit $\mathcal{V}_i$ is even.

Lemma 10. *Let n be an odd integer and let $\mathbf{A} = [a_{i,j}]_{1 \leqslant i,j \leqslant n}$ be a $n \times n$ matrix such that $a_{i,j} = 0$ unless $(i,j) \in \{(1,n),(n,1)\}$ or $|i-j| = 1$. Then $\det \mathbf{A} = a_{2,1}a_{3,2}\cdots a_{n,n-1}a_{1,n} + a_{1,2}a_{2,3}\cdots a_{n-1,n}a_{n,1}$.*

Proof. Recall that by definition

$$\det \mathbf{A} = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right), \tag{6}$$

where S_n is the set of all permutations of length n. Note that the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ necessarily contains a zero factor, unless $\sigma \in \{(1\ 2\ 3\ \ldots\ n), (1\ n\ n-1\ \ldots\ 2)\}$. □

Proof of Proposition 9. If necessary, relabel the orbits $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$, so that $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$ are neighbours in the cycle $\mathfrak{G}(G)$. Let d_{ij} be the number of neighbours of a vertex $v \in \mathcal{V}_i$ that reside in $\mathcal{V}_j$. The local condition gives us one equation for each orbit, namely

$$\sum_{v \in \mathcal{V}_i} \sum_{u \in N(v)} x_u = \sum_{u \in \mathcal{V}_{i-1}} d_{i-1,i} x_u + \sum_{u \in \mathcal{V}_{i+1}} d_{i+1,i} x_u = 0 \qquad (1 \leqslant i \leqslant k), \tag{7}$$

where we consider indices modulo k. Let us define $s_i = \sum_{u \in \mathcal{V}_i} x_u$ for $i = 1, \ldots, k$. We have the matrix equation

$$\begin{bmatrix} 0 & d_{2,1} & 0 & \ldots & 0 & d_{k,1} \\ d_{1,2} & 0 & d_{3,2} & \ddots & \vdots & 0 \\ 0 & d_{2,3} & 0 & d_{4,3} & 0 & \vdots \\ \vdots & 0 & d_{3,4} & 0 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & d_{k,k-1} \\ d_{1,k} & 0 & \ldots & 0 & d_{k-1,k} & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ \vdots \\ \vdots \\ s_k \end{bmatrix} = \mathbf{0}_{k \times 1}. \tag{8}$$

By Lemma 10, the determinant of the square matrix in Equation (8) is

$$d_{2,1}d_{3,2}\cdots d_{k,k-1}d_{1,k} + d_{1,2}d_{2,3}\cdots d_{k-1,k}d_{k,1} > 0,$$

since $d_{2,1}, d_{3,2}, \ldots$ are all positive. Hence, $s_1 = s_2 = \cdots = s_k = 0$. This already implies the existence of a sign-reversing automorphism and the fact that $|\{j \in \mathcal{V}_i : x_j > 0\}| = |\{j \in \mathcal{V}_i : x_j < 0\}|$. □

3 Vertex-transitive nut graphs

We have seen that $o_e(G) = o_v(G) = 1$ is not possible for a nut graph G. However, many other possibilities for $o_e(G)$ may exist. First, we filtered out all nut graphs from databases of small vertex-transitive graphs on up to $n \leqslant 46$ vertices [27, 39]. The counts are shown in Table 1. Recall that a vertex-transitive graph G is a nut graph if and only if $\eta(G) = 1$, so this search requires only computation of the nullity and moreover, by Theorem 5, can be limited to graphs of even order and degree. As the table shows, most of these vertex-transitive graphs are connected and a significant proportion of vertex-transitive graphs of even order are nut graphs. As a preliminary survey of symmetry aspects, we calculated the number of edge orbits for all vertex-transitive nut graphs; see Table 2, which has a number of interesting features. It has only zero entries for $o_e(G) = 1$, as demanded by Theorem 3, but there is no apparent restriction on the values of $o_e(G)$ that can occur for a large enough order of G. Note that a vertex-transitive nut graph G with a large $o_e(G)$ must have a large degree. To place these results in context, we also calculated the number of edge orbits of connected vertex-transitive graphs of even order. See Table 3. We see some intriguing gaps in Table 1 for particular pairs (n, o_e), e.g., $(n, o_e) \in \{(22, 3), (22, 5), (22, 7), (22, 10), (22, 11)\}$, even though the numbers of vertex-transitive graphs for these pairs of parameters are $37, 115, 138, 50$ and 23, respectively.

3.1 Families with $(o_v, o_e) = (1, 2)$

From the line for $o_e = 2$ in Table 2 it appears likely that vertex-transitive nut graphs with two edge orbits exist for all feasible orders. This is confirmed by the next theorem.

Theorem 11. *For every even $n \geqslant 8$, there exists a nut graph G with $o_v(G) = 1$ and $o_e(G) = 2$.*

To prove this, we provide three families of quartic vertex-transitive graphs, which together cover all feasible orders and are described in Propositions 12 to 14. For the first family, let A_ℓ, where $\ell \geqslant 3$, be the antiprism on 2ℓ vertices. Gauci et al. [24] proved the following proposition.

Proposition 12 ([24])**.** *The antiprism graph A_ℓ of order 2ℓ is a nut graph if and only if $2\ell \not\equiv 0 \pmod 6$.*

The next family is composed of Cartesian products.

Proposition 13. *The graph $C_3 \,\square\, C_\ell$ of order 3ℓ is a nut graph for even $\ell \geqslant 4$ such that $\ell \not\equiv 0 \pmod 6$.*

Proof. It is known that $\sigma(G \,\square\, H) = \{\lambda + \mu \mid \lambda \in \sigma(G) \text{ and } \mu \in \sigma(H)\}$ (see [5, Section 1.4.6]). Moreover, let $\mathbf{x}_G$ be an eigenvector for an eigenvalue $\lambda \in \sigma(G)$ and let $\mathbf{x}_H$ be an eigenvector for an eigenvalue $\mu \in \sigma(H)$. Then $\mathbf{x}_{G \square H}$, defined as $\mathbf{x}_{G \square H}((u, v)) = \mathbf{x}_G(u)\mathbf{x}_H(v)$, is an eigenvector for the eigenvalue $\lambda + \mu$. It is also well known that $\sigma(C_\ell) = \{2\cos(2\pi j/\ell) \mid 0 \leqslant j < \ell\}$ (see [5, Section 1.4.3]). In particular, $\sigma(C_3) = \{2, -1, -1\}$ and

n	All VT	Connected VT	VT nut graphs	Proportion
8	14	10	1	10.00%
10	22	18	1	5.56%
12	74	64	4	6.25%
14	56	51	5	9.80%
16	286	272	20	7.35%
18	380	365	23	6.30%
20	1214	1190	150	12.61%
22	816	807	101	12.52%
24	15506	15422	1121	7.27%
26	4236	4221	508	12.04%
28	25850	25792	4793	18.58%
30	46308	46236	3146	6.80%
32	677402	677116	47770	7.05%
34	132580	132543	14565	10.99%
36	1963202	1962756	214391	10.92%
38	814216	814155	85234	10.47%
40	13104170	13102946	1815064	13.85%
42	9462226	9461929	693416	7.33%
44	39134640	39133822	7376081	18.85%
46	34333800	34333611	3281206	9.56%

Table 1: The number of nut graphs among vertex-transitive (VT) graphs on even orders $8 \leqslant n \leqslant 46$. The final column is the ratio between the number of VT nut graphs and the number of connected VT graphs on a given order, expressed as a percentage.

when ℓ is even and $\ell \not\equiv 0 \pmod 6$, it is clear that $\sigma(C_\ell)$ contains -2 with multiplicity 1, but not 1. Therefore, $C_3 \,\square\, C_\ell$ contains a 0 eigenvalue with multiplicity 1. As the eigenvector of C_ℓ for the eigenvalue -2 is full and so is the eigenvector of C_3 for the eigenvalue 2, it immediately follows that $C_3 \,\square\, C_\ell$ is a nut graph. $\square$

For the third family, a variation on the Cartesian product is used. Suppose that vertices of C_ℓ are labeled $0, 1, \ldots \ell-1$ such that i and $i+1$ are adjacent (indices modulo ℓ). Then the *twisted product* of C_k and C_ℓ, denoted $C_k \,\tau\, C_\ell$, has the vertex set $V(C_k \,\tau\, C_\ell) = V(C_k \,\square\, C_\ell)$ and the edge set

$$E(C_k \,\tau\, C_\ell) = E(C_k \,\square\, C_\ell) \setminus \{(i,0)(i,1) \mid 0 \leqslant i < k\} \cup \{(i,0)((i+1) \bmod k, 1) \mid 0 \leqslant i < k\}.$$

In other words, the construction $C_k \,\tau\, C_\ell$ is similar to the Cartesian product of C_k and C_ℓ, but with a twist introduced between the first two C_k layers.

Proposition 14. *The graph $C_3 \,\tau\, C_\ell$ of order 3ℓ is a nut graph for even $\ell \geqslant 6$ such that $\ell \equiv 0 \pmod 6$.*

o_e \ n	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	2	2	3	4	6	4	6	8	8	7	10	11	15	10	16	14	12	10
3	0	0	0	0	3	1	3	0	12	7	7	11	39	14	46	5	58	39	21	0
4	–	0	0	2	1	7	9	18	51	36	31	73	118	100	209	142	407	280	186	270
5	–	–	2	0	5	1	20	0	93	0	79	47	332	0	376	0	1079	341	349	0
6	–	–	–	1	6	6	32	38	164	119	277	258	1175	632	1604	1116	4349	2244	3944	3285
7	–	–	–	–	2	0	30	0	181	0	306	98	1457	0	2414	0	6854	1747	4512	0
8	–	–	–	–	0	0	21	4	131	34	312	171	2250	600	4181	1750	14674	6410	12993	9870
9	–	–	–	–	0	4	16	32	222	186	756	1078	4788	2363	10659	6270	37144	24419	44984	31680
10	–	–	–	–	–	0	9	0	97	5	505	70	5205	385	10743	1750	54653	11690	55182	19362
11	–	–	–	–	–	–	3	0	100	0	924	23	8242	0	26197	0	110092	6215	198328	0
12	–	–	–	–	–	–	1	5	41	105	755	1013	8438	6042	33238	27737	171053	130227	373849	334181
13	–	–	–	–	–	–	–	0	20	0	476	1	6536	0	32259	0	209056	4405	509682	0
14	–	–	–	–	–	–	–	–	3	0	197	0	3716	7	25059	140	179657	5573	507008	9870
15	–	–	–	–	–	–	–	–	0	8	110	284	3249	3807	28390	33151	298457	258510	1071473	1034877
16	–	–	–	–	–	–	–	–	–	0	39	0	1238	0	14434	8	184646	1503	599584	2460
17	–	–	–	–	–	–	–	–	–	–	9	0	682	0	13486	0	213377	739	1213286	0
18	–	–	–	–	–	–	–	–	–	–	2	12	241	588	6936	11982	150605	198368	987503	1225073
19	–	–	–	–	–	–	–	–	–	–	–	0	52	0	2790	0	93113	101	712011	0
20	–	–	–	–	–	–	–	–	–	–	–	–	2	0	883	0	44117	3	451854	10
21	–	–	–	–	–	–	–	–	–	–	–	–	0	16	379	1152	27428	38386	345284	528000
22	–	–	–	–	–	–	–	–	–	–	–	–	0	0	70	0	9106	0	125961	0
23	–	–	–	–	–	–	–	–	–	–	–	–	–	–	21	0	4024	0	102696	0
24	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	21	845	2171	37414	78705
25	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0	225	0	12892	0
26	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	28	0	3862	0
27	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1	31	972	3520
28	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0	206	0
29	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	29	0
30	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4	33
31	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0
Σ	1	1	4	5	20	23	150	101	1121	508	4793	3146	47770	14565	214391	85234	1815064	693416	7376081	3281206

Table 2: The number of vertex-transitive nut graphs of the given order n and number of edge orbits o_e.

o_e \ n	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
1	5	8	11	8	15	14	22	8	34	13	26	41	42	10	69	10	71	56	16	7
2	4	6	24	11	34	53	79	15	249	38	101	263	334	42	585	37	645	398	104	27
3	1	3	18	12	60	72	123	37	629	85	208	598	1386	146	2263	146	2588	1340	428	174
4	–	1	7	11	59	68	163	75	1086	177	422	1147	4165	441	5731	515	7726	4008	1718	878
5	–	–	4	7	44	65	185	115	1604	305	809	1956	9684	1093	12238	1559	19585	10969	5950	3508
6	–	–	–	2	32	47	184	139	2084	439	1354	2971	18878	2283	23383	3957	44631	26864	17803	11462
7	–	–	–	–	20	27	169	138	2352	549	2041	4079	32496	4153	41461	8666	93741	58552	47038	31930
8	–	–	–	–	7	13	132	116	2320	614	2828	5055	50069	6753	69326	16742	181948	114599	112933	78261
9	–	–	–	–	1	5	82	83	2000	609	3537	5746	69333	10036	109094	29248	326634	203239	250167	173735
10	–	–	–	–	–	1	38	50	1473	539	3917	6020	86177	13705	159508	46871	541098	331751	514525	355837
11	–	–	–	–	–	–	12	23	911	410	3746	5777	95517	17019	213153	68842	825026	503981	976094	676704
12	–	–	–	–	–	–	1	7	460	257	3025	4932	93632	18890	256311	91670	1151013	713859	1691134	1189545
13	–	–	–	–	–	–	–	1	171	127	2031	3642	80492	18413	273869	109238	1460133	936712	2648865	1914675
14	–	–	–	–	–	–	–	–	44	47	1101	2256	60195	15560	257721	115210	1673574	1126073	3725223	2795661
15	–	–	–	–	–	–	–	–	5	11	471	1140	38790	11276	211956	106655	1723949	1226341	4683075	3677457
16	–	–	–	–	–	–	–	–	–	1	144	453	21253	6940	151375	86183	1588363	1198385	5250045	4339178
17	–	–	–	–	–	–	–	–	–	–	29	133	9722	3583	93176	60473	1302895	1043286	5240659	4580946
18	–	–	–	–	–	–	–	–	–	–	2	25	3634	1524	49001	36624	946942	804481	4652068	4319948
19	–	–	–	–	–	–	–	–	–	–	–	2	1064	518	21711	18997	606265	546641	3666813	3633698
20	–	–	–	–	–	–	–	–	–	–	–	–	223	133	7941	8344	339719	325520	2560541	2721551
21	–	–	–	–	–	–	–	–	–	–	–	–	28	23	2309	3043	165034	168737	1579317	1810729
22	–	–	–	–	–	–	–	–	–	–	–	–	2	2	501	893	68723	75450	856413	1066672
23	–	–	–	–	–	–	–	–	–	–	–	–	–	–	70	200	24022	28725	405904	553828
24	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4	30	6862	9119	166550	251863
25	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	2	1507	2332	58439	99445
26	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	234	451	17128	33657
27	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	18	57	4061	9574
28	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	3	721	2216
29	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	86	394
30	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	4	48
31	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	3
Σ	10	18	64	51	272	365	1190	807	15422	4221	25792	46236	677116	132543	1962756	814155	13102946	9461929	39133822	34333611

Table 3: The number of connected vertex-transitive graphs of the given order n, $8 \leqslant n \leqslant 46$ even, and number of edge orbits o_e.

Proof. The *twisted product* $C_3 \,\tau\, C_\ell$ is an example of a *graph bundle* [33, 34]. Kwak et al. [7, 28] studied characteristic polynomials of some specific graph bundles [7, 28]. Here, we apply their Theorem 8 from [28]; in the language of [28, Theorem 8], our $C_3 \,\tau\, C_\ell$ is in fact $C_\ell \times^\phi C_n$, where ϕ is an $\mathrm{Aut}(C_n)$-voltage assignment and $n = 3$. $\mathrm{Aut}(C_3)$ contains $\mathbb{Z}_3$ as a subgroup. In our case, ϕ maps every directed edge of $\vec{C}_\ell$ to 0 of $\mathbb{Z}_3$, except for the directed edges $(0, 1)$ and $(1, 0)$ which are mapped to 1 and its inverse 2, respectively.

Define an $\ell \times \ell$ matrix M_z, where $z \in \mathbb{C}$, as follows

$$(M_z)_{i,j} = \begin{cases} z, & i = 0 \text{ and } j = 1; \\ \bar{z}, & i = 1 \text{ and } j = 0; \\ 1, & (i,j) \notin \{(1,0),(0,1)\} \text{ and } i - j \equiv \pm 1 \pmod{\ell}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that M_1 is the adjacency matrix of C_ℓ. Let $\omega = (-1 + \sqrt{3}i)/2$. Theorem 8 from [28] gives

$$\Phi(C_3 \,\tau\, C_\ell; \lambda) = \Phi(M_1; \lambda - 2) \cdot \Phi(M_\omega; \lambda + 1) \cdot \Phi(M_{\bar{\omega}}; \lambda + 1) = \Phi(M_1; \lambda - 2) \cdot \Phi(M_\omega; \lambda + 1)^2.$$

We show that the nullity of $C_3 \,\tau\, C_\ell$ is 1 for $\ell \equiv 0 \pmod 6$. First, $\Phi(M_1; \lambda - 2)$ contributes one factor λ, as 2 is an eigenvalue of C_ℓ with multiplicity 1. To show that $\Phi(M_\omega; \lambda + 1)$ does not contribute additional factors λ, we show that $M_\omega + I_{\ell\times\ell}$ is of full rank.

Let $B = M_\omega + I_{\ell\times\ell}$. Let B_i denote the i-th column of B. From B we can obtain an equivalent matrix C by defining $C_i = B_i - B_{i+1}$ for $i \geqslant 1$ (indices modulo ℓ) and $C_0 = B_0 - \omega^2 B_1$. Note that

$$C_{ij} = \begin{cases} \omega, & i = 0 \text{ and } j = 1; \\ 1, & (i,j) \neq (0,1) \text{ and } i + 1 \equiv j \pmod{\ell}; \\ -\omega^2, & (i,j) \in \{(2,0),(1,\ell-1)\}; \\ -1, & (i,j) \notin \{(2,0),(1,\ell-1)\} \text{ and } i - 2 \equiv j \pmod{\ell}; \\ 0, & \text{otherwise.} \end{cases}$$

From matrix C we can obtain an equivalent matrix D by permuting columns, namely

$$D = [C_1\ C_4\ C_7 \ldots C_{\ell-2} \mid C_2\ C_5\ C_8 \ldots C_{\ell-1} \mid C_0\ C_3\ C_6 \ldots C_{\ell-3}].$$

Note that matrix D is composed of three blocks of size $\ell \times (\ell/3)$. Block i, $0 \leqslant i \leqslant 2$, contains nonzero entries only in rows j, $0 \leqslant j < \ell$, such that $j \equiv i \pmod 3$. Now, we can define a matrix E that is equivalent to matrix D by defining, for $0 \leqslant i < \ell/3$,

$$E_i = \sum_{j=0}^{\ell/3-i-1} D_j + \sum_{j=\ell/3-i}^{\ell/3-1} \omega D_j;$$

$$E_{i+\ell/3} = \sum_{j=0}^{i-1} \omega^2 D_{j+\ell/3} + \sum_{j=i}^{\ell/3-1} D_{j+\ell/3};$$

$$E_{i+2\ell/3} = \sum_{j=0}^{\ell/3-i-1} D_{(j+1) \bmod (\ell/3)+2\ell/3} + \sum_{j=\ell/3-i}^{\ell/3-1} \omega D_{(j+1) \bmod (\ell/3)+2\ell/3}.$$

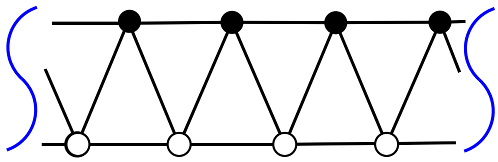
Matrix E has a single nonzero entry in each row and each column; $2\ell/3-1$ of these entries are $\omega - 1$ and $\ell/3+1$ of these entries are $\omega + 2$, and the determinant is

$$(-1)^{\ell/6}(\omega-1)^{2\ell/3-1}(\omega+2)^{\ell/3+1} = 3^{\ell/2}\omega^{\ell/6+2} \neq 0.$$

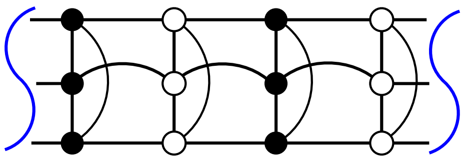
But matrix E is equivalent to B which is therefore of full rank. Hence the nullity of B is 1 and therefore $C_3 \,\tau\, C_\ell$ is a nut graph. □

As a referee has observed, there is an alternative shorter proof of Proposition 14, that exploits the observation $C_k \,\tau\, C_\ell \cong \mathrm{Circ}(k\ell, \{1, \ell\})$. Armed with this observation, Corollary 7 from [16] can be applied to prove the proposition. Similarly, $C_3 \,\square\, C_\ell \cong \mathrm{Circ}(3\ell, \{3, \ell\})$ under the requirements of Proposition 13.

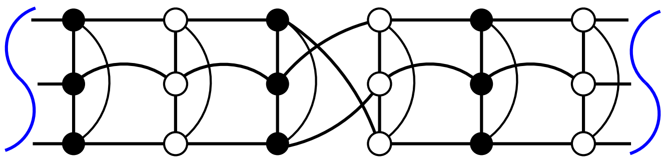
Note that our proof does not require explicit construction of the kernel eigenvector. However, it is easily obtained. Define $\mathbf{x}\colon V(C_3 \,\tau\, C_\ell) \to \mathbb{R}$ by $\mathbf{x}((i,j)) = (-1)^j$. Observe that $\mathbf{x} \in \ker \mathbf{A}(C_3 \,\tau\, C_\ell)$ and is a full vector.



(a) A_4



(b) $C_3 \,\square\, C_4$

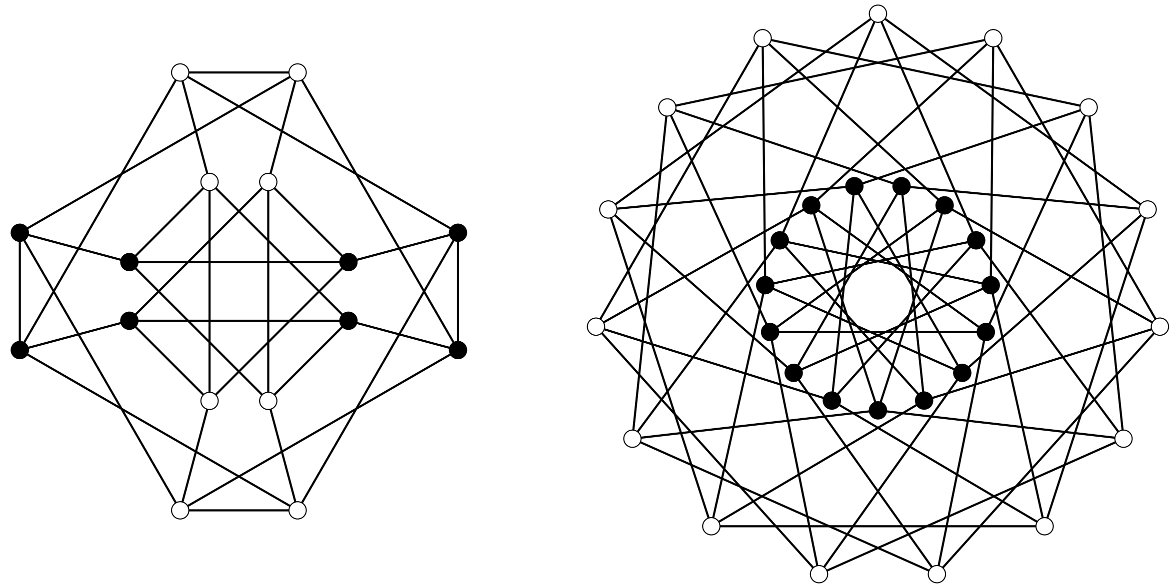


(c) $C_3 \,\tau\, C_6$

Figure 4: The smallest examples of each of the families described in Propositions 12 to 14. They are shown embedded on a circular strip; the end blue curves are to be identified. Entries in the kernel eigenvector in each graph are all of equal magnitude and represented by circles colour-coded for sign.

Proof of Theorem 11. By Theorem 5, orders and degrees of vertex-transitive nut graphs are even. In fact, our families are all quartic. The family A_ℓ, where $\ell \geqslant 4$ even, described in Proposition 12, covers orders $\{n \geqslant 8 \mid n \text{ even and } n \not\equiv 0 \pmod 6\}$. The family $C_3 \,\square\, C_\ell$, where $\ell \geqslant 4$ even, described in Proposition 13, covers orders $\{n \geqslant 12 \mid n \equiv 0 \pmod 6 \text{ and } n \not\equiv 0 \pmod{18}\}$. Finally, the family $C_3 \,\tau\, C_\ell$, where $\ell \geqslant 6$ even, described in Proposition 14, covers orders $\{n \geqslant 18 \mid n \equiv 0 \pmod{18}\}$. □

There exist vertex-transitive nut graphs that are not Cayley graphs. The three minimal examples of non-Cayley nut graphs have order 16, with invariants $(d(G), o_e(G), |\mathrm{Aut}(G)|)$ of $(4, 3, 32)$, $(6, 4, 32)$ and $(10, 5, 32)$, respectively. The quartic example is shown in Figure 5(a); it is a tetracirculant with vertex set $\{u_i, v_i, w_i, z_i \mid i \in \mathbb{Z}_4\}$ and edge set $\{u_iv_i, v_iw_i, u_iz_i, u_iu_{i+1}, v_iv_{i+1}, z_iw_{i+1}, z_iw_{i+2}, z_iw_{i+3} \mid i \in \mathbb{Z}_4\}$. The second smallest quartic example is shown in Figure 5(b) and is one of 14 non-Cayley nut graphs of order 30; it has 2 edge orbits and its automorphism group is of order 120. This is a generalisation of Rose Window graphs; its vertex set is $\{u_i, v_i \mid i \in \mathbb{Z}_{15}\}$ and its edge set is $\{u_iv_i, u_iv_{i+5}, v_iv_{i+3}, u_iu_{i+6} \mid i \in \mathbb{Z}_{15}\}$.



(a) (b)

Figure 5: The two smallest 4-valent non-Cayley vertex-transitive nut graphs. Entries in the kernel eigenvector in each graph are all of equal magnitude and represented by circles colour-coded for sign.

4 Nut graph with two vertex orbits

Data are available for graphs with two vertex orbits [40] and Table 4 shows our analysis for small graphs of this class. We observe that $(o_v(G), o_e(G)) = (2, 1)$ and $(o_v(G), o_e(G)) = (2, 2)$ do not occur in the table. This observation is, of course, consistent with Theorem 2 from Section 2. We also observe that the number of edge orbits can be large. Here, we provide infinite families of nut graphs with two vertex orbits and three edge orbits.

o_e \ n	9	10	12	14	15	16	18	20	21	22	24	25	26	27
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	4	7	6	7	10	20	10	19	33	13	26	19
4	0	3	6	2	16	12	16	72	62	6	169	46	19	124
5	0	0	12	1	5	24	78	133	40	20	665	66	44	90
6	–	0	6	3	6	31	99	134	48	122	1460	160	327	227
7	–	–	5	1	3	31	133	171	77	94	3418	191	348	445
8	–	–	4	1	0	78	102	310	77	110	7031	234	552	671
9	–	–	1	0	0	53	136	264	40	184	12081	429	1118	777
10	–	–	0	–	1	80	71	381	88	45	19694	599	283	1984
11	–	–	–	–	0	73	82	392	193	14	28013	156	340	5192
12	–	–	–	–	–	49	18	366	4	154	36902	574	2258	797
13	–	–	–	–	–	17	20	165	49	0	41123	267	77	3996
14	–	–	–	–	–	13	2	147	0	0	44395	8	4	292
15	–	–	–	–	–	3	0	238	0	0	39101	1	15	261
16	–	–	–	–	–	0	–	52	0	10	36325	0	735	420
17	–	–	–	–	–	–	–	9	0	0	24477	0	0	1239
18	–	–	–	–	–	–	–	18	–	–	19068	2	0	136
19	–	–	–	–	–	–	–	1	–	–	8568	2	0	171
20	–	–	–	–	–	–	–	0	–	–	5638	–	20	0
21	–	–	–	–	–	–	–	–	–	–	2173	–	0	0
22	–	–	–	–	–	–	–	–	–	–	838	–	–	0
23	–	–	–	–	–	–	–	–	–	–	140	–	–	0
24	–	–	–	–	–	–	–	–	–	–	63	–	–	–
25	–	–	–	–	–	–	–	–	–	–	7	–	–	–
26	–	–	–	–	–	–	–	–	–	–	0	–	–	–
Σ	1	4	38	15	37	471	767	2873	688	778	331382	2748	6166	16841

Table 4: The number of nut graphs with precisely two vertex orbits of the given order n and number of edge orbits o_e.

4.1 Families with $(o_v, o_e) = (2, 3)$

From the line for $o_e = 3$ in Table 4 it appears that nut graphs with two vertex orbits and three edge orbits exist for all orders $n \geqslant 9$ such that n is not a prime; see Conjecture 24. Here, we provide two families of such nut graphs; one that covers orders that are multiples of three, and one that covers orders that are multiples of two but not multiples of three.

Proposition 15. *Let $\mathcal{T}_n$ be an n-cycle with a triangle fused to every vertex (see Figure 6(a) for an example). The graph $\mathcal{T}_n$ is a nut graph for every $n \geqslant 3$.*

Proof. Let the vertices of the n-cycle be labeled $0, 1, \ldots, n-1$. Let $a_0, a_1, \ldots, a_{n-1}$ denote

o_e \ n	9	10	12	14	15	16	18	20	21	22	24	25	26	27
1	5	5	8	8	15	11	14	21	24	16	31	23	18	30
2	29	43	98	103	151	190	285	420	341	315	869	433	449	628
3	34	74	270	305	402	718	1341	2117	1332	1624	6279	1961	2968	3743
4	12	52	331	363	514	1352	2903	5318	2573	3621	22524	4379	8593	10968
5	4	17	284	258	469	1738	4359	9211	3725	5488	56544	7739	16838	23380
6	–	1	183	129	345	1879	5130	12453	4459	6462	112054	11823	26114	41411
7	–	–	110	50	251	1831	5496	14313	4999	6614	190905	16078	35084	66769
8	–	–	53	13	152	1787	5305	14885	5255	6056	292831	19694	42043	99346
9	–	–	22	2	87	1627	4714	14377	5316	4993	416618	22044	45677	139401
10	–	–	3	–	30	1427	3597	13039	4845	3683	555666	22452	45324	182434
11	–	–	–	–	8	1086	2365	11191	3992	2411	694869	20502	40992	221082
12	–	–	–	–	–	734	1213	9054	2763	1406	809588	16446	33697	242741
13	–	–	–	–	–	392	499	6826	1626	721	872753	11346	25145	239567
14	–	–	–	–	–	169	128	4666	747	318	863949	6566	16956	208850
15	–	–	–	–	–	49	21	2832	277	117	780210	3118	10260	160119
16	–	–	–	–	–	9	–	1457	66	30	639027	1179	5482	106253
17	–	–	–	–	–	–	–	624	12	5	471486	340	2524	60839
18	–	–	–	–	–	–	–	204	–	–	311318	69	959	29385
19	–	–	–	–	–	–	–	48	–	–	182116	8	288	11915
20	–	–	–	–	–	–	–	6	–	–	93435	–	60	3857
21	–	–	–	–	–	–	–	–	–	–	41330	–	8	993
22	–	–	–	–	–	–	–	–	–	–	15463	–	–	173
23	–	–	–	–	–	–	–	–	–	–	4716	–	–	21
24	–	–	–	–	–	–	–	–	–	–	1120	–	–	–
25	–	–	–	–	–	–	–	–	–	–	180	–	–	–
26	–	–	–	–	–	–	–	–	–	–	14	–	–	–
Σ	84	192	1362	1231	2424	14999	37370	123062	42352	43880	7435895	166200	359479	1853905

Table 5: The number of connected graphs with precisely two vertex orbits of the given order n and number of edge orbits o_e. Only orders where nut graphs with two vertex orbits exist are included.

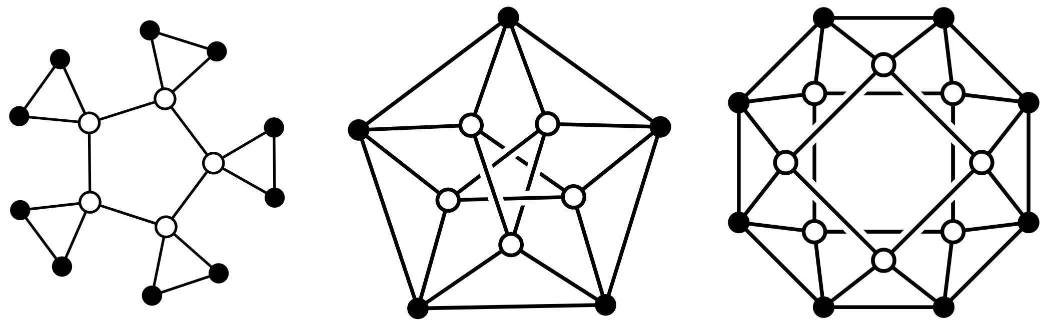
the entries on the vertices of the n-cycle in a kernel eigenvector of $\mathcal{T}_n$. It is easy to see that both vertices of the triangle fused to vertex i of the cycle must then carry entry $-a_i$. The local condition at vertices of the cycle is

$$a_{i-1} - 2a_i + a_{i+1} = 0 \qquad \text{for } i = 0, \dots, n-1, \tag{9}$$

where indices are modulo n. Equation (9) in matrix form is

$$A(C_n)\mathbf{x} = 2\mathbf{x}, \tag{10}$$

where $A(C_n)$ is the adjacency matrix of the n-cycle and $\mathbf{x} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}$. The cycle C_n is a 2-regular connected graph, and thus has a unique eigenvalue 2 in its spectrum, with $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \end{bmatrix}$ and the solution to Equation (9) is $a_0 = a_1 = \cdots = a_{n-1} = 1$. □



(a) $\mathcal{T}_5$ (b) $R_5(1,2)$ (c) $R_8(1,2)$

Figure 6: Small examples of each of the families described in Propositions 15 and 16. Entries in the kernel eigenvector in each graph are all of equal magnitude and represented by circles colour-coded for sign.

In 2008, the family of *Rose Window* graphs was introduced [45]. A Rose Window graph, denoted $R_n(a,r)$, is defined by

$$V(R_n(a,r)) = \{v_0, v_1, \dots, v_{n-1}\} \cup \{u_0, u_1, \dots, u_{n-1}\} \text{ and}$$
$$E(R_n(a,r)) = \{v_i v_{i+1}, u_i u_{i+r} \mid i = 0, \dots, n-1\} \cup \{u_i v_i, u_i v_{i+a} \mid i = 0, \dots, n-1\},$$

where all indices are modulo n. We will consider the subset with $a = 1$ and $r = 2$ (see Figures 6(b) and 6(c) for examples).

Proposition 16. *Let $n \geqslant 5$. The graph $R_n(1,2)$ is a core graph for all $n \geqslant 5$. The graph $R_n(1,2)$ is a nut graph if and only if $n \not\equiv 0 \pmod 3$.*

Proof. Let $\mathbf{x} \in \ker \mathbf{A}(R_n(1,2))$ and let $a_0 = \mathbf{x}(v_0), a_1 = \mathbf{x}(v_1), b_{-2} = \mathbf{x}(u_{n-2}), b_{-1} = \mathbf{x}(u_{n-1}), b_0 = \mathbf{x}(u_0)$ and $b_1 = \mathbf{x}(u_1)$. See Figure 7 for an illustration. Using the local condition (1) at vertices $v_1, \dots, v_{n-2}$ and $u_0, \dots, u_{n-3}$ the entries in $\mathbf{x}$ of vertices $v_2, \dots, v_{n-1}$

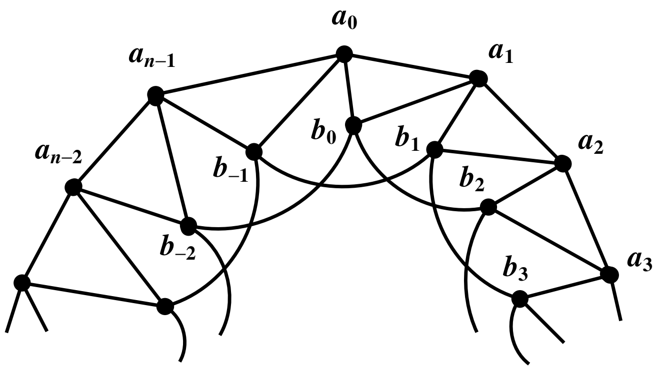


Figure 7: Labelling scheme for the Rose Window graph $R_n(1,2)$. Entries in the candidate kernel eigenvector are a_i on vertices v_i and b_i on vertices u_i, all indices taken modulo n.

and $u_2,\dots,u_{n-1}$ can be expressed as linear combinations of $a_0, a_1, b_{-2}, b_{-1}, b_0$ and b_1. Namely,

$$\begin{aligned} a_i &= -a_{i-2} - b_{i-2} - b_{i-1} \qquad (2 \leqslant i < n), \\ b_i &= -a_{i-2} - a_{i-1} - b_{i-4} \qquad (2 \leqslant i < n), \end{aligned} \tag{11}$$

where $a_i = \mathbf{x}(v_i)$ and $b_i = \mathbf{x}(u_i)$. Every entry $\mathbf{x}(v)$, $v \in V(R_n(1,2))$, can be assigned a row vector $\boldsymbol{\xi}(v) \in \mathbb{R}^6$, acting as proxy for $\mathbf{x}(v) = \boldsymbol{\xi}(v) \cdot [a_0\ a_1\ b_{-2}\ b_{-1}\ b_0\ b_1]$. Solving the linear recurrence relations (11) we obtain

$$a_k = \begin{cases} \begin{bmatrix} 1 & 0 & \frac{k}{3} & 0 & 0 & -\frac{k}{3} \end{bmatrix}, & k \equiv 0 \pmod 3; \\ \begin{bmatrix} \frac{k-1}{3} & \frac{k+2}{3} & \frac{k-1}{3} & \frac{k-1}{3} & 0 & 0 \end{bmatrix}, & k \equiv 1 \pmod 3; \\ \begin{bmatrix} -\frac{k+1}{3} & \frac{k-2}{3} & 0 & \frac{k-2}{3} & -1 & -\frac{k+1}{3} \end{bmatrix}, & k \equiv 2 \pmod 3; \end{cases} \tag{12}$$

and

$$b_k = \begin{cases} \begin{bmatrix} \frac{k}{3} & -\frac{k}{3} & 0 & -\frac{k}{3} & 1 & \frac{k}{3} \end{bmatrix}, & k \equiv 0 \pmod 3; \\ \begin{bmatrix} 0 & 0 & -\frac{k-1}{3} & 0 & 0 & \frac{k+2}{3} \end{bmatrix}, & k \equiv 1 \pmod 3; \\ \begin{bmatrix} -\frac{k+1}{3} & -\frac{k+1}{3} & -\frac{k+1}{3} & -\frac{k-2}{3} & 0 & 0 \end{bmatrix}, & k \equiv 2 \pmod 3. \end{cases} \tag{13}$$

By using the local condition (1) at vertices $v_0, v_{n-1}, u_{n-2}, u_{n-1}$ we obtain the four linear equations

$$\begin{aligned} a_{n-1} + b_{-1} + b_0 + a_1 &= 0, \\ a_{n-2} + b_{-2} + b_{-1} + a_0 &= 0, \\ a_{n-1} + a_0 + b_{n-3} + b_1 &= 0, \\ a_{n-2} + a_{n-1} + b_{n-4} + b_0 &= 0, \end{aligned} \tag{14}$$

relating $a_0, a_1, b_{-2}, b_{-1}, b_0$ and b_1 to each other. Two more equations can be obtained from the fact that $b_{n-2} = \boldsymbol{\xi}(u_{n-2}) = b_{-2}$ and $b_{n-1} = \boldsymbol{\xi}(u_{n-1}) = b_{-1}$:

$$\begin{aligned} b_{n-2} - b_{-2} &= 0, \\ b_{n-1} - b_{-1} &= 0, \end{aligned} \tag{15}$$

There are three cases to consider.

Case $n \equiv 0 \pmod 3$: The equations (14) and (15) can be written in matrix form

$$\begin{bmatrix} -\mu & \mu & 0 & \mu & 0 & -\mu \\ \mu & \mu & \mu & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu & \mu & 0 & \mu & 0 & -\mu \\ 0 & 0 & -\mu & 0 & 0 & \mu \\ -\mu & -\mu & -\mu & -\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_{-2} \\ b_{-1} \\ b_0 \\ b_1 \end{bmatrix} = \mathbf{0}_{6\times 1}, \tag{16}$$

where $\mu = \frac{n}{3}$. It is easy to see that the matrix in (16) is of rank 3. This implies that $R_n(1,2)$ has nullity 3.

Case $n \equiv 1 \pmod 3$: The equations (14) and (15) can be written in matrix form

$$\begin{bmatrix} 1 & 1 & \mu & 1 & 1 & -\mu \\ 1-\mu & \mu-1 & 1 & \mu & -1 & -\mu \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \mu & 0 & 1 & -\mu-1 \\ -\mu & -\mu & -\mu-1 & 1-\mu & 0 & 0 \\ \mu & -\mu & 0 & -\mu-1 & 1 & \mu \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_{-2} \\ b_{-1} \\ b_0 \\ b_1 \end{bmatrix} = \mathbf{0}_{6\times 1}, \tag{17}$$

where $\mu = \frac{n-1}{3}$. Note that $\mu \geqslant 2$ as $n \geqslant 5$. Using elementary linear algebra, the matrix in (17) can be reduced to its echelon form, from which it can be seen that it is of rank 5. This implies that $R_n(1,2)$ has nullity 1.

Case $n \equiv 2 \pmod 3$: The equations (14) and (15) can be written in matrix form

$$\begin{bmatrix} \mu & \mu+2 & \mu & \mu+1 & 1 & 0 \\ 2 & 0 & \mu+1 & 1 & 0 & -\mu \\ 1 & 1 & 0 & 1 & 0 & 1 \\ \mu+1 & \mu+1 & \mu+1 & \mu & 1 & 0 \\ \mu & -\mu & -1 & -\mu & 1 & \mu \\ 0 & 0 & -\mu & -1 & 0 & \mu+1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_{-2} \\ b_{-1} \\ b_0 \\ b_1 \end{bmatrix} = \mathbf{0}_{6\times 1}, \tag{18}$$

where $\mu = \frac{n-2}{3}$. Note that $\mu \geqslant 1$ as $n \geqslant 5$. As before, the matrix in (18) can be reduced to its echelon form, from which it can be seen that it is of rank 5. This implies that $R_n(1,2)$ has nullity 1 also in the present case.

It is easily seen that there exists a full vector in $\ker \mathbf{A}(R_n(1,2))$ in all three cases. Simply take $a_i = 1$ and $b_i = -1$ for all i, hence $R_n(1,2)$ is a nut graph if $n \not\equiv 0 \pmod 3$ and merely a core graph if $n \equiv 0 \pmod 3$. □

The graph $R_n(1,2)$, for $n \equiv 0 \pmod 3$, has nullity 3. Possible choices of basis for the nullspace are depicted in Figure 8.

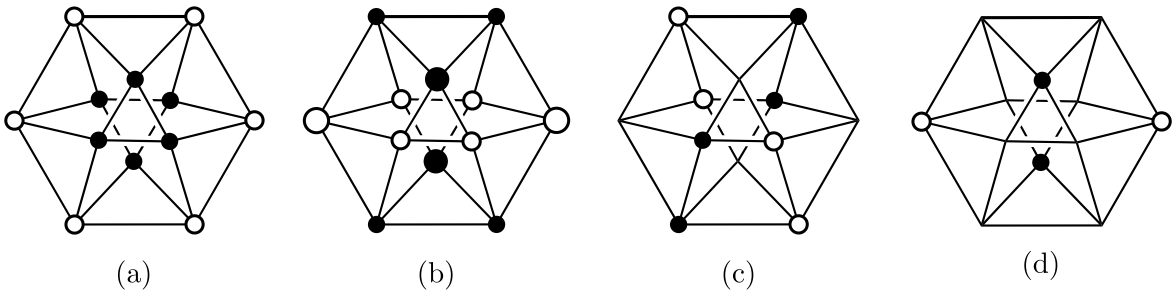


Figure 8: Kernel eigenvectors for the graph $R_6(1,2)$. Vectors (a) to (c) form an orthogonal basis that includes the rotationally symmetric vector that is present in the nullspace of $R_n(1,2)$ for every $n \geqslant 5$. An alternative basis consists of vector (d) and its rotations by $\pm 60°$. Signs of eigenvector entries are indicated by colour and relative magnitudes by area of the circles, where the possible magnitudes are 0, 1 and 2.

4.2 Multiplier constructions

The family $\mathcal{T}_n$ can in fact be substantially generalised by defining the *triangle-multiplier construction.* Unlike some constructions in the literature [24], the triangle-multiplier applies to parent graphs that are not necessarily nut graphs.

Proposition 17. *Let G be a connected $(2t)$-regular graph, where $t \geqslant 1$. Let $\mathcal{M}_3(G)$ be the graph obtained from G by fusing a bouquet of t triangles to every vertex of G. Then $\mathcal{M}_3(G)$ is a nut graph.*

Proof. We follow the pattern of the proof of Proposition 15, where Equation (9) is replaced by

$$\sum_{u \in N(v)} \mathbf{x}(u) - 2t\,\mathbf{x}(v) = 0 \qquad \text{for } v \in V(G), \tag{19}$$

which in matrix form is $A(G)\mathbf{x} = 2t\,\mathbf{x}$. So, kernel eigenvectors of $\mathcal{M}_3(G)$ are precisely eigenvectors of G for the eigenvalue $2t$. But the graph G is $2t$-regular, so the solution of Equation (19) is unique and $\mathbf{x}$, i.e., the Perron eigenvector, is full. □

The choice of name for the construction is justified by the fact that $|V(\mathcal{M}_3(G))| = (2t+1)|V(G)|$. As Proposition 31 in Section 5 will show, the triangle-multiplier construction adds one vertex orbit and two edge orbits to the graph G, irrespective of the value t. We can define a *pentagon-multiplier construction* as follows. As in the case of the triangle-multiplier, this construction applies to graphs that are not necessarily nut graphs.

Proposition 18. *Let G be a bipartite connected $(2p)$-regular graph, where $p \geqslant 1$. Let $\mathcal{M}_5(G)$ be the graph obtained from G by fusing a bouquet of p pentagons (i.e., 5-cycles) to every vertex of G. Then $\mathcal{M}_5(G)$ is a nut graph.*

Proof. We follow the pattern of the proof of Proposition 17. Consider a pentagon fused at a vertex $v \in V(G)$. The vertices of the pentagon that are adjacent to v both carry entry $+\mathbf{x}(v)$, while the remaining two vertices carry entry $-\mathbf{x}(v)$.

Equation (19) is replaced by

$$\sum_{u\in N(v)} \mathbf{x}(u) + 2p\,\mathbf{x}(v) = 0 \qquad \text{for } v \in V(G), \tag{20}$$

which in matrix form is $A(G)\mathbf{x} = -2p\,\mathbf{x}$. Since G is connected, bipartite and $(2p)$-regular, it has a unique eigenvalue $-2p$ in its spectrum, and the corresponding eigenvector is full. ☐

We note that in Proposition 17, any fused triangle could be replaced by a $(4q+3)$-cycle for any $q \geqslant 0$. Likewise, in Proposition 18, any fused pentagon may be replaced by a $(4q+5)$-cycle for any $q \geqslant 0$. In fact, these changes are just repeated applications of the subdivision construction on the triangles (resp. pentagons) of the graph $\mathcal{M}_3(G)$ (resp. $\mathcal{M}_5(G)$).

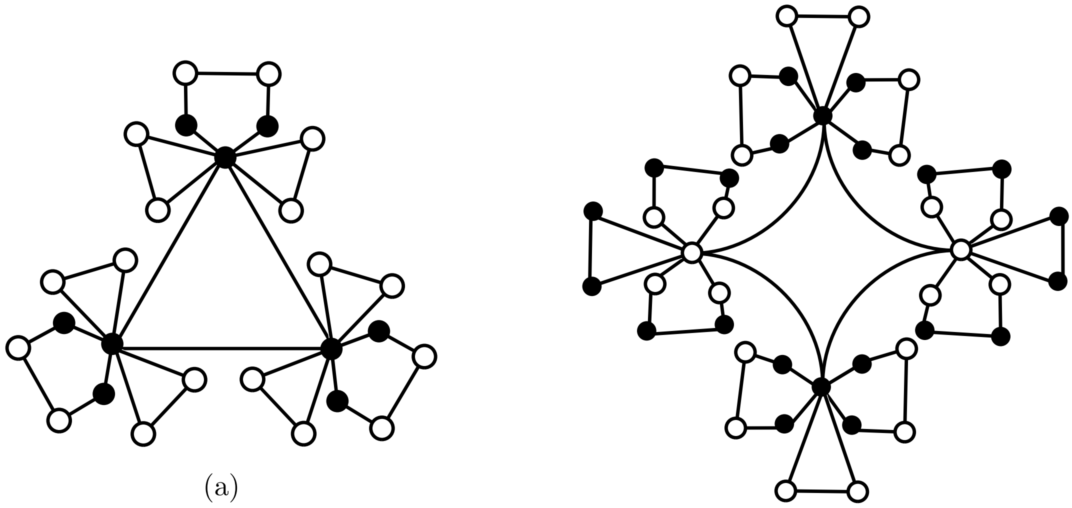
It seems natural to ask, what would happen if we fuse a mixture of triangles and pentagons to some vertices of a graph? Consider the case where we fuse a triangle and a pentagon to a vertex in a graph.

Proposition 19. *Let G be a nut graph and let $v \in V(G)$ be a vertex. Let $\mathcal{P}(G, v)$ be the graph obtained from G by fusing a triangle and a pentagon to vertex v. Then $\mathcal{P}(G, v)$ is a nut graph.*

Proof. Let $\mathbf{x}$ be a kernel eigenvector of $\mathcal{P}(G, v)$. Consider the two vertices on the fused triangle that are adjacent to v. Their entries in $\mathbf{x}$ are $-\mathbf{x}(v)$. Now consider the two vertices of the fused pentagon that are adjacent to v. Their entries in $\mathbf{x}$ are $\mathbf{x}(v)$; the entries of the remaining two vertices are $-\mathbf{x}(v)$. The local condition at vertex v is simply $\sum_{u\in N(v)} \mathbf{x}(u) = 0$. This means that $\eta(\mathcal{P}(G, v)) = \eta(G)$. Thus, $\mathcal{P}(G, v)$ is a nut graph if and only if G is a nut graph. ☐

This is a special case of the coalescence construction devised by Sciriha [41]. Corollary 21 in [41] is equivalent to the statement that coalescence of any two nut graphs G_1 and G_2 at any pair of vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ produces a nut graph. The fusion of a triangle and a pentagon is one of the three Sciriha graphs; see Figure 1(a). Note that the above construction is used on a single vertex of a nut graph G. Had we used it iteratively on all vertices of G, that would give us yet another multiplier construction. In fact, various sorts of mixed objects can be envisaged. After the initial application of $\mathcal{M}_3$ or $\mathcal{M}_5$ on an appropriate parent graph, which gives rise to a nut graph, the way lies open to application of the coalescence construction, locally or globally. See Figure 9 for examples.

The triangle-multiplier and pentagon-multiplier constructions may be generalised to a k-multiplier construction: Let G be a $(2r)$-regular graph and let $k \geqslant 3$. Let $\mathcal{M}_k(G)$ be the graph obtained from G by fusing a bouquet of r k-cycles to every vertex of G. In fact, Propositions 17 and 18 have natural generalisations to every $\mathcal{M}_k$, where $\{k \geqslant 3$ and $k \equiv 3 \pmod 4\}$ and $\{k \geqslant 5$ and $k \equiv 1 \pmod 4\}$, respectively. These generalisations follow immediately by the subdivision construction (see Section 5).



(a)

(b)

Figure 9: These two nut graphs were obtained from (a) C_3 and (b) C_4 by an application of the $\mathcal{M}_3$ resp. $\mathcal{M}_5$, followed by repeated application of Proposition 19. Entries in the kernel eigenvector in each graph are all of equal magnitude and represented by circles colour-coded for sign.

4.3 Characterisation of orders for nut graphs with 2 vertex orbits

Observe that columns for prime values of n are absent from Table 4. This is because the search did not reveal any examples in the range. As the next theorem shows, this is no coincidence.

Theorem 20. *Let G be a nut graph of order n with precisely two vertex orbits. Then n is not a prime number.*

The next proposition will be useful in the proof of the above theorem.

Proposition 21. *Let G be a nut graph and let $\mathbf{x} = [x_1 \ \ldots \ x_n]^\intercal \in \ker \mathbf{A}(G)$. If there exists a sign-reversing automorphism $\alpha \in \mathrm{Aut}(G)$ then all orbits are of even size. Moreover, for every j, half of the entries $\{x_i \mid i \in \mathcal{V}_j\}$ are positive, and the other half are negative.*

Proof. Think of the automorphism α as a product of disjoint cycles. Note that elements of any given cycle of α are contained in the same vertex orbit. Since α is a sign-reversing automorphism, every vertex i is mapped to a vertex i^α carrying the opposite sign (i.e., $x_i \cdot x_{i^\alpha} < 0$). Therefore, every cycle of α is of even length and contains a perfect matching whose edges join vertices with entries of opposite sign. Hence, $\{x_i \mid i \in \mathcal{V}_j\}$ contains the same number of positive and negative elements. □

Corollary 22. *Let G be a nut graph of order n. If n is odd then* $\mathrm{Aut}(G)$ *does not contain any sign-reversing automorphism. Moreover, kernel eigenvector entries are constant within a given orbit.*

Proof of Theorem 20. Let $n_1 = |\mathcal{V}_1|$ and $n_2 = |\mathcal{V}_2|$ with $n = n_1 + n_2$ and $n_1, n_2 \geqslant 1$. Let d_{ij} be the number of neighbours of a vertex $v \in \mathcal{V}_i$ that reside in $\mathcal{V}_j$, where $1 \leqslant i, j \leqslant 2$. Since G is simple and connected, $0 \leqslant d_{ii} < n_i$ and $1 \leqslant d_{ji} \leqslant n_i$. The number of inter-orbit edges is

$$d_{12}n_1 = d_{21}n_2. \tag{21}$$

The proof proceeds by contradiction. Suppose that n is a prime. The case $n = 2$ is trivial, since there are no nut graphs on 2 vertices. As n is odd, kernel eigenvector entries within each orbit are constant by Corollary 22. Let a_i be the entry in $\mathcal{V}_i$. The local conditions are

$$\begin{aligned} d_{11}a_1 + d_{12}a_2 &= 0, \\ d_{21}a_1 + d_{22}a_2 &= 0. \end{aligned} \tag{22}$$

First, we note that Equation (21) has a unique solution and implies

$$d_{12}(n_1 + n_2) = (d_{21} + d_{12})n_2.$$

Since $n_1 + n_2$ is a prime factor, it divides either $d_{21} + d_{12}$ or n_2. As it clearly cannot divide n_2, it divides $d_{21} + d_{12}$. But $d_{21} + d_{12} \leqslant n_1 + n_2$. Divisibility is possible only in the case where $d_{21} = n_1$ and $d_{12} = n_2$. For this case, Equation (22) can be expressed in matrix form as

$$\begin{bmatrix} n_1 & d_{22} \\ d_{11} & n_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}$$

The determinant of the matrix is $n_1 n_2 - d_{11}d_{22} > 0$ and therefore $a_1 = a_2 = 0$. This contradicts the fact that G is a nut graph. Therefore, n cannot be prime. ☐

We have justified the claim that nut graphs with two vertex orbits cannot be of prime order n. We add to the picture by showing that a nut graph with two vertex orbits exists for all composite orders $n \geqslant 9$.

Theorem 23. *Let $n \geqslant 9$ such that n is not a prime. Then there exists a nut graph G of order n with $o_v(G) = 2$.*

Proof. Let $n = p_1 p_2 \cdots p_k$ be the decomposition of n into prime factors. Without loss of generality, we may assume that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$.

Case 1: Suppose that $p_1 = 2$. If $n \not\equiv 0 \pmod 3$ then Proposition 16 guarantees a solution. If $n \equiv 0 \pmod 3$ then Proposition 15 guarantees a solution.

Case 2: Now suppose that $p_1 > 2$. Clearly, p_1 is an odd integer. The strategy is to find a vertex-transitive graph H of order $\widetilde{n} = n/p_1$ and degree $\widetilde{d} = p_1 - 1$. Then use the triangle-multiplier construction to obtain $\mathcal{M}_3(H)$. Since n is not a prime $n/p_1 \geqslant p_1$ and so $\widetilde{n} > \widetilde{d}$. Note that $\widetilde{d}$ is an even number. The circulant graph $H = \mathrm{Circ}(\widetilde{n}, \{1, 2, \ldots, \widetilde{d}/2\})$ has the prescribed order and degree and it is of course vertex-transitive as required. The graph $\mathcal{M}_3(H)$ is of order n. By Proposition 31, this graph has two vertex orbits. ☐

Theorem 23 shows that there is at least one nonzero entry in every column of Table 4. However, we believe that row $o_e = 3$ by itself consists of nonzero entries.

Conjecture 24. Let $n \geqslant 9$ such that n is not a prime. Then there exists a nut graph G of order n with $o_v(G) = 2$ *and* $o_e(G) = 3$.

Conjecture 24 holds for all even numbers (covered by Case 1 of the proof), all multiples of 3 (covered by Proposition 15) and all perfect squares (in that case graph H in the proof is a complete graph). The conjecture can also be validated for those values of n, such that there exists an edge-transitive graph H of order $\widetilde{n}$ and degree $\widetilde{d}$ in the proof of Theorem 23. Table 6 shows the orders up to 300 that are not resolved by anything mentioned thus far. For some of these orders we were able to provide graph H (see proof of Theorem 23). N/A in the table indicates that no such graph H exists (based on the census by Conder and Verret [9, 10]). Order 35, for example, cannot be resolved in this way, because there is only one vertex-transitive graph of order 7 and degree 4, namely $\mathrm{Circ}(7, \{1, 2\})$, but it is not edge-transitive, and so a completely different approach is required. For order 295, H would have to be a 4-regular edge-transitive graph of order 59, but no such graph is known (see [35, 36, 37, 38]). The classes of edge-transitive circulants provided in [31] could be used to resolve some orders beyond Table 6. All graphs H provided in Table 6 are circulants. However, for some orders there are non-circulant alternative possibilities, e.g., the graph $C_5 \,\square\, C_5$ could be used for $n = 125$ and $\mathrm{Cay}(\mathbb{Z}_5 \times \mathbb{Z}_5, \{(0, 1), (1, 0), (1, 1)\})$ for order $n = 175$.

Order	Graph H
$35 = 5 \cdot 7$	N/A
$55 = 5 \cdot 11$	N/A
$65 = 5 \cdot 13$	$\mathrm{Circ}(13, \{1, 5\})$
$77 = 7 \cdot 11$	N/A
$85 = 5 \cdot 17$	$\mathrm{Circ}(17, \{1, 4\})$
$91 = 7 \cdot 13$	$\mathrm{Circ}(13, \{1, 3, 4\})$
$95 = 5 \cdot 19$	N/A
$115 = 5 \cdot 23$	N/A
$119 = 7 \cdot 17$	N/A
$125 = 5^3$	$\mathrm{Circ}(25, \{1, 7]\}$
$133 = 7 \cdot 19$	$\mathrm{Circ}(19, \{1, 7, 8\})$
$143 = 11 \cdot 13$	N/A
$145 = 5 \cdot 29$	$\mathrm{Circ}(29, \{1, 12\})$
$155 = 5 \cdot 31$	N/A
$161 = 7 \cdot 23$	N/A
$175 = 5^2 \cdot 7$	$\mathrm{Circ}(35, \{1, 6\})$
$185 = 5 \cdot 37$	$\mathrm{Circ}(37,\ \{1,\ 6\})$

Order	Graph H
$187 = 11 \cdot 17$	N/A
$203 = 7 \cdot 29$	N/A
$205 = 5 \cdot 41$	$\mathrm{Circ}((41, \{1, 9\})$
$209 = 11 \cdot 19$	N/A
$215 = 5 \cdot 43$	N/A
$217 = 7 \cdot 31$	$\mathrm{Circ}(31, \{1, 5, 6\})$
$221 = 13 \cdot 17$	N/A
$235 = 5 \cdot 47$	N/A
$245 = 5 \cdot 7^2$	$\mathrm{Circ}(35, \{1, 11, 16\})$
$247 = 13 \cdot 19$	N/A
$253 = 11 \cdot 23$	N/A
$259 = 7 \cdot 37$	$\mathrm{Circ}(37, \{1, 10, 11\})$
$265 = 5 \cdot 53$	$\mathrm{Circ}(53, \{1, 23\})$
$275 = 5^2 \cdot 11$	$\mathrm{Circ}(25, \{1, 4, 6, 9, 11\})$
$287 = 7 \cdot 41$	N/A
$295 = 5 \cdot 59$	Unknown
$299 = 13 \cdot 23$	N/A

Table 6: List of all integers $9 \leqslant n \leqslant 300$ that are not prime, not even, not multiples of three and not perfect squares. Where a graph H is listed, it proves Conjecture 24 for the particular order. N/A indicates that no graph H with the desired properties exists. For order 295 it is not known whether such a graph exists.

Question 25. Find a nut graph G with 2 vertex orbits and 3 edge orbits for orders $n = 35, 55, 77, 95, \ldots$ and other non-resolved orders.

5 How constructions influence symmetry

Several constructions have been described for producing a larger nut graph when applied to a smaller nut graph G; literature examples include the bridge construction (insertion of two vertices on a bridge) [44], the subdivision construction (insertion of four vertices on an edge) [44] and the so-called Fowler construction [24], which has the net result of introducing $2d$ new vertices in the proximity of a vertex v of degree d. In Section 4.2 we have given examples of constructions that do not require the parent G to be a nut graph. Here, we are interested in the implications of the various constructions for numbers of vertex and edge orbits of the constructed nut graph.

Proposition 26. *Let G be a nut graph and let $e = uv \in E(G)$ be a bridge in G. Let $\mathcal{E}$ be the orbit of the bridge e under* $\mathrm{Aut}(G)$*. The graph obtained from G by applying the bridge construction on every edge from $\mathcal{E}$, denoted $B(G, \mathcal{E})$, is a nut graph and* $\mathrm{Aut}(G) \leqslant \mathrm{Aut}(B(G, \mathcal{E}))$.

If, in addition, $\mathrm{Aut}(G) \cong \mathrm{Aut}(B(G, \mathcal{E}))$*, then the following statements hold.*

(i) If there exists an element $\varphi \in$ $\mathrm{Aut}(G)$ such that $u^\varphi = v$ and $v^\varphi = u$, then $o_v(B(G, \mathcal{E})) = o_v(G) + 1$ and $o_e(B(G, \mathcal{E})) = o_e(G) + 1$.

(ii) If there is no element $\varphi \in$ $\mathrm{Aut}(G)$ such that $u^\varphi = v$ and $v^\varphi = u$, then $o_v(B(G, \mathcal{E})) = o_v(G) + 2$ and $o_e(B(G, \mathcal{E})) = o_e(G) + 2$.

Proof. It is clear that $B(G, \mathcal{E})$ is a nut graph [44]. Every element $\alpha \in \mathrm{Aut}(G)$ can be extended in a natural way to an element $\widehat{\alpha} \in \mathrm{Aut}(B(G, \mathcal{E}))$. More precisely, since the arc (u, v) was subdivided, so was its image (u^α, v^α). Let the new vertices on (u, v) be labeled w_1 and w_2, where w_1 is adjacent to u. And let the new vertices on (u^α, v^α) be labeled w_1' and w_2' where w_1' is adjacent to u^α. Then $w_1^{\widehat{\alpha}} = w_1'$ and $w_2^{\widehat{\alpha}} = w_2'$. This immediately implies that $\mathrm{Aut}(G) \leqslant \mathrm{Aut}(B(G, \mathcal{E}))$.

Note that $\mathrm{Aut}(B(G, \mathcal{E}))$ may include additional automorphisms that were not induced by $\mathrm{Aut}(G)$. These may cause merging of vertex orbits and merging of edge orbits. If $\mathrm{Aut}(G) \cong \mathrm{Aut}(B(G, \mathcal{E}))$, then we know that there are no such additional automorphisms. Note that graph $B(G, \mathcal{E})$ has at most two new vertex orbits, namely, the orbit of w_1 and the orbit of w_2. The edge uv was substituted by the three edges uw_1, w_1w_2 and w_2v. If there exists an element $\varphi \in \mathrm{Aut}(G)$ such that $u^\varphi = v$ and $v^\varphi = u$, then using its extension $\widetilde{\varphi}$ we get $w_1^{\widetilde{\varphi}} = w_2$. This means that vertices w_1 and w_2 are in the same vertex orbit. Similarly, edges uw_1 and w_2v are in the same edge orbit. The claim follows. ☐

Note that since $\mathrm{Aut}(G) \leqslant \mathrm{Aut}(B(G, \mathcal{E}))$, the condition $|\mathrm{Aut}(G)| = |\mathrm{Aut}(B(G, \mathcal{E}))|$ automatically implies $\mathrm{Aut}(G) \cong \mathrm{Aut}(B(G, \mathcal{E}))$.

The proof of the following proposition is analogous to that of Proposition 26 and is skipped here.

Proposition 27. *Let G be a nut graph and let $e = uv \in E(G)$ be an edge in G. Let $\mathcal{E}$ be the orbit of the edge e under $\mathrm{Aut}(G)$. The graph obtained from G by applying the subdivision construction on every edge from $\mathcal{E}$, denoted $S(G, \mathcal{E})$, is a nut graph and $\mathrm{Aut}(G) \leqslant \mathrm{Aut}(S(G, \mathcal{E}))$.*

If, in addition, $\mathrm{Aut}(G) \cong \mathrm{Aut}(S(G, \mathcal{E}))$, then the following statements hold.

(i) If there exists an element $\varphi \in \mathrm{Aut}(G)$ such that $u^\varphi = v$ and $v^\varphi = u$, then $o_v(S(G, \mathcal{E})) = o_v(G) + 2$ and $o_e(S(G, \mathcal{E})) = o_e(G) + 2$.

(ii) If there is no element $\varphi \in \mathrm{Aut}(G)$ such that $u^\varphi = v$ and $v^\varphi = u$, then $o_v(S(G, \mathcal{E})) = o_v(G) + 4$ and $o_e(S(G, \mathcal{E})) = o_e(G) + 4$.

Definition 28. Let G be a nut graph and let $v \in V(G)$ be a vertex of degree d in G. Let $N(v) = \{u_1, \ldots, u_d\}$. The graph $F(G, v)$ is obtained from G in the following way: (a) edges incident to v are deleted; (b) let $w_1, \ldots, w_d$ and $x_1, \ldots, x_d$ denote $2d$ newly added vertices; (c) new edges are added such that $x_i \sim u_i$ for $i = 1, \ldots, d$; and $x_i \sim w_j$ for $i \neq j$, $1 \leqslant i, j \leqslant d$; and $w_i \sim v$ for all $i = 1, \ldots, d$. The construction $F(G, v)$ has been called 'the Fowler construction' in the nut-graph literature [24].

Figure 10 illustrates the definition. Note that $u_1, \ldots, u_d$ are at distance 3 from v in $F(G, v)$. Moreover, the degrees of all the newly added vertices are d.

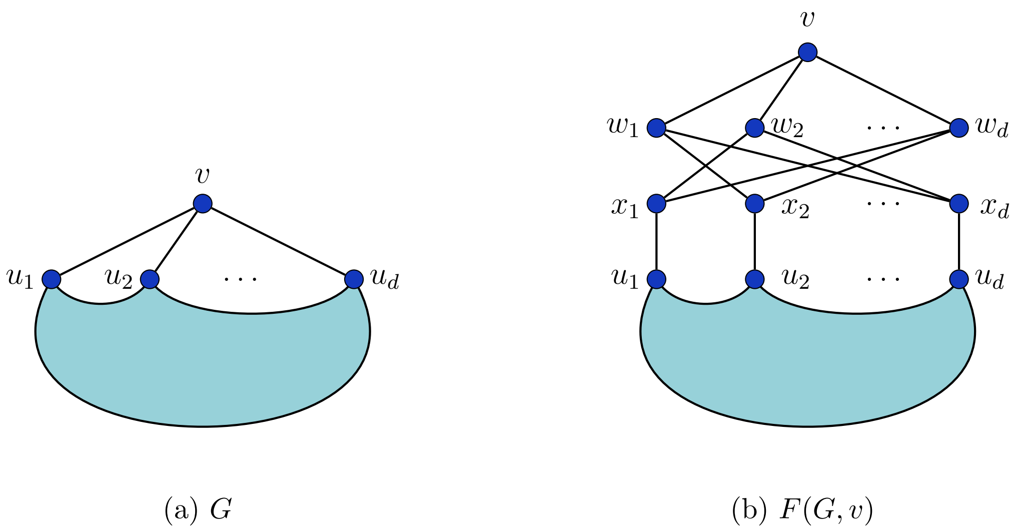


Figure 10: A construction for expansion of a nut graph G about vertex v of degree d, to give $F(G, v)$. Panel (a) shows the neighbourhood of vertex v in G. Panel (b) shows additional vertices and edges in $F(G, v)$.

Proposition 29. *Let G be a nut graph and let $v \in V(G)$ be a vertex in G. Let $\mathcal{V}$ be the orbit of the vertex v under $\mathrm{Aut}(G)$. The graph obtained from G by applying the Fowler construction on every vertex from $\mathcal{V}$, denoted $F(G, \mathcal{V})$, is a nut graph and $\mathrm{Aut}(G) \leqslant \mathrm{Aut}(F(G, \mathcal{V}))$.*

Suppose that, in addition, $\mathrm{Aut}(G) \cong \mathrm{Aut}(F(G, \mathcal{V}))$. *Let the vertices in the neighbourhood of* v *in* G *and in the first, second and third neighbourhood of* v *in* $F(G, \mathcal{V})$ *be labeled as in Definition 28. The stabiliser* $\mathrm{Aut}(G)_v$ *fixes* $N(v)$ *set-wise in the graph* G *and partitions* $N(v)$ *into* t *orbits. Let* $\mathcal{S} = \{(w_i, x_j) \mid i \neq j; 1 \leqslant i, j \leqslant d\}$. $\mathrm{Aut}(F(G, \mathcal{V}))_v$ *partitions* $\mathcal{S}$ *into* τ *orbits. Then* $o_v(F(G, \mathcal{V})) = o_v(G) + 2t$ *and* $o_e(F(G, \mathcal{V})) = o_e(G) + t + \tau$.

Note that we define the action of $\mathrm{Aut}(F(G, \mathcal{V}))_v$ on pairs (w_i, x_j) by taking $(w_i, x_j)^\alpha = (w_i^\alpha, x_j^\alpha)$, where $\alpha \in \mathrm{Aut}(F(G, \mathcal{V}))_v$. The proof of the above proposition uses the same approach as that of Proposition 26 and is skipped here.

We are interested in the growth of the number of edge orbits under the construction. Let us define $\Phi(G, v) = o_e(F(G, \mathcal{V})) - o_e(G)$, where $\mathcal{V}$ is the orbit of the vertex v. Proposition 29 has the following corollary.

Corollary 30. *Let* G *be a nut graph and let* $v \in V(G)$ *be a vertex in* G. *Let* $\mathcal{V}$ *be the orbit of the vertex* v *under* $\mathrm{Aut}(G)$. *Let* $F(G, \mathcal{V})$ *be as in Proposition 29 and also let* $\mathrm{Aut}(G) \cong \mathrm{Aut}(F(G, \mathcal{V}))$. *Then*

$$4 \leqslant \Phi(G, v) \leqslant d^2. \tag{24}$$

If $\deg(v) \geqslant 3$, *then* $\Phi(G, v) \geqslant 5$.

Proof. To get the upper bound, assume that each vertex of $N(v)$ is in its own orbit and therefore $t = d$. Similarly, each element of $\mathcal{S}$ is in its own orbit and therefore $\tau = d^2 - d$.

For parameters t and τ from Proposition 29, it holds that $t \geqslant 1$ and $\tau \geqslant 1$. Therefore, $2 \leqslant o_e(F(G, \mathcal{V})) - o_e(G)$. By Lemma 7, $\alpha \in \mathrm{Aut}(G)_v$ cannot be sign-reversing. There must be at least one vertex in $N(v)$ with a positive entry in the kernel eigenvector and at least one with a negative entry. Therefore, the group $\mathrm{Aut}(G)_v$ partitions $N(v)$ in at least 2 orbits, say $\mathcal{U}$ and $\mathcal{U}'$. Vertices $x_1, \ldots, x_d$ cannot be in the same orbit under $\mathrm{Aut}(G)_v$ as any other vertex of $F(G, v)$; see Figure 10. The same is true for $w_1, \ldots, w_d$. Since $\mathrm{Aut}(G) \cong \mathrm{Aut}(F(G, \mathcal{V}))$, $\{x_1, \ldots, x_d\}$ and $\{w_1, \ldots, w_d\}$ are partitioned into orbits under $\mathrm{Aut}(G)_v$ in the same way as $\{u_1, \ldots, u_d\}$ (i.e., x_i and x_j belong to the same orbit if and only if u_i and u_j belong to the same orbit). Orbits $\mathcal{U}$ and $\mathcal{U}'$ induce orbits $\mathcal{X}$ and $\mathcal{X}'$ on $\{x_1, \ldots, x_d\}$ and orbits $\mathcal{W}$ and $\mathcal{W}'$ on $\{w_1, \ldots, w_d\}$. There exists at least one edge connecting $\mathcal{U}$ to $\mathcal{X}$, at least one edge connecting $\mathcal{U}'$ to $\mathcal{X}'$, at least one edge connecting $\mathcal{X}$ to $\mathcal{W}'$ and at least one edge connecting $\mathcal{X}'$ to $\mathcal{W}$. Each of these four edges must be in a distinct new edge orbit, hence $4 \leqslant o_e(F(G, \mathcal{V})) - o_e(G)$.

If $\deg(v) \geqslant 3$ then either $\mathcal{U}$ or $\mathcal{U}'$ contains at least 2 vertices. Without loss of generality assume that $|\mathcal{U}| \geqslant 2$. Then there exists at least one edge connecting $\mathcal{X}$ to $\mathcal{W}$. This edge cannot share the orbit with any of the above four edges, hence $5 \leqslant o_e(F(G, \mathcal{V})) - o_e(G)$. □

The upper bound is best possible, because the equality in (24) can be attained if we take G to be any asymmetric nut graph (i.e., $|\mathrm{Aut}(G)| = 1$). This bound is attained even within the class of vertex-transitive graphs, when we take G to be a GRR [29, 30] nut graph, such as the one in Figure 11.

The lower bound is more interesting. The restriction of G to nut graphs in Corollary 30 is significant since, for example, if we take a complete graph on $n \geqslant 4$ vertices then

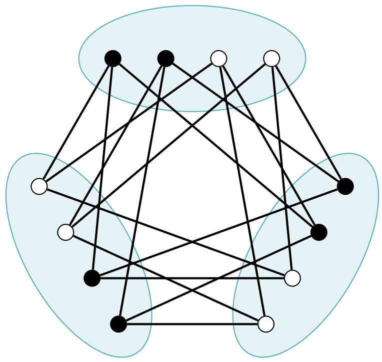
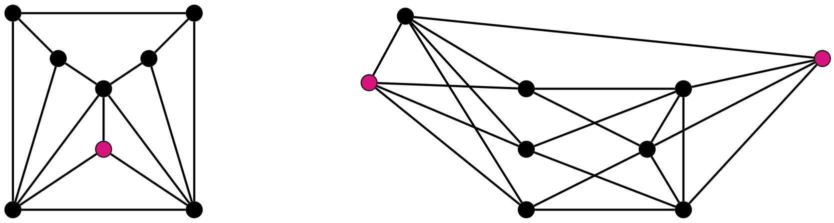


Figure 11: The smallest GRR nut graph has order 12 and degree 6. The graph contains three cliques represented by shaded regions; edges within cliques are not drawn. Entries in the kernel eigenvector are all of equal magnitude and represented by circles colour-coded for sign.

$o_e(F(K_n, \mathcal{V})) - o_e(K_n) = 2$. For vertices of degree 2, Corollary 30 implies $o_e(F(K_n, \mathcal{V})) - o_e(K_n) = 4$. Small graphs with vertices of degree $d \in \{3, 4\}$ furnish examples where $o_e(F(K_n, \mathcal{V})) - o_e(K_n) = 5$; see Figure 12. The search on nut graphs of orders $n \leqslant 12$



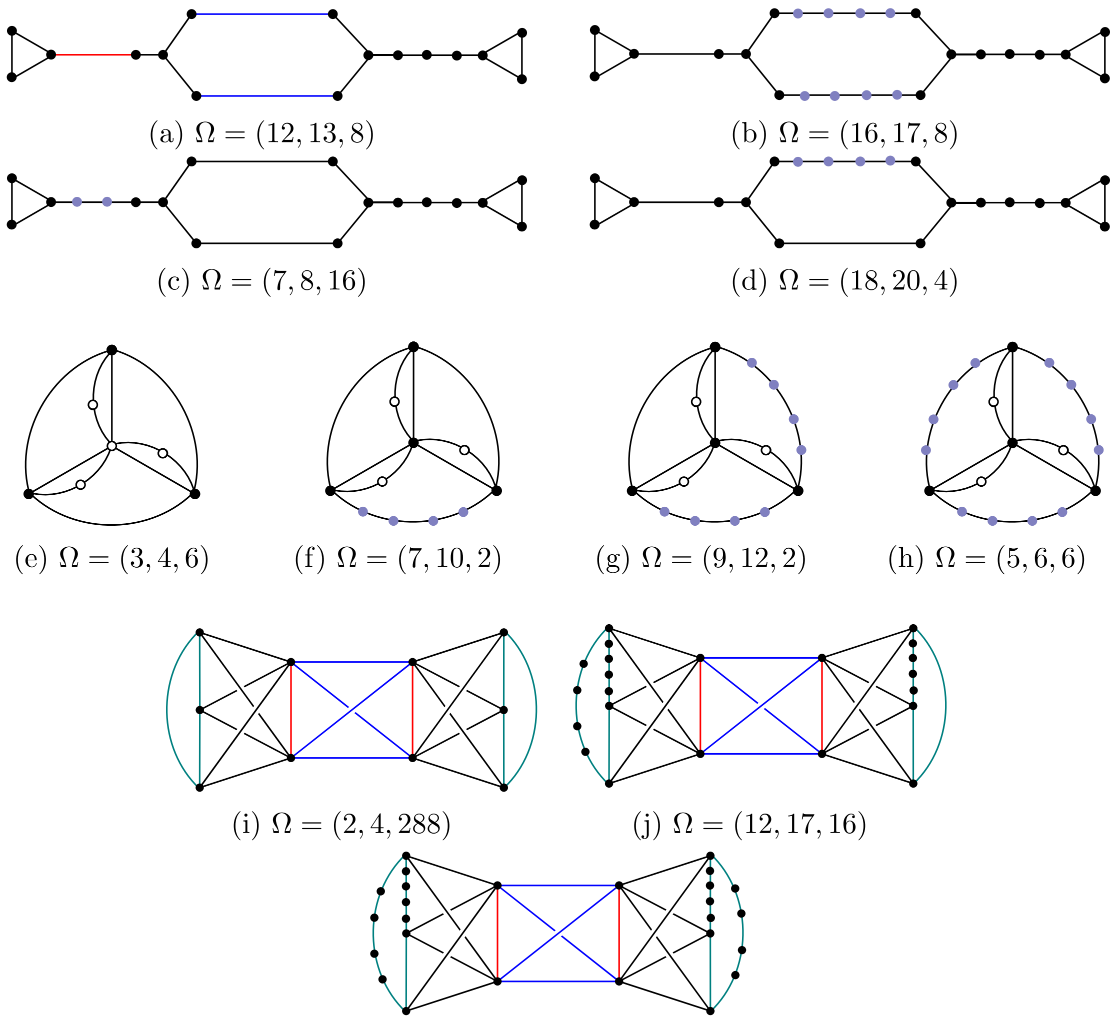
(a) (b)

Figure 12: The smallest graphs where $\Phi(G, v) = 5$. (a) For $d = 3$ the smallest example is unique and of order 8. (b) For $d = 4$ one of the two smallest examples of order 9 is shown. Vertices for which the bound is met are coloured magenta.

yields examples for $d \in \{5, 6, 7\}$ for which $\Phi(G, v) = 6$; see [1]. Examples of graphs with small $\Phi(G, v)$ can also be found in the class of regular graphs; see [1]. Moreover, examples with small $\Phi(G, v)$ can be found in the class of vertex-transitive graphs. For example, $K_{3,3} \,\square\, K_4$ is a sextic vertex-transitive nut graph with $\Phi(K_{3,3} \,\square\, K_4, v) = 6$.

It is important to note that in Propositions 26, 27 and 29 the respective construction was applied to all edges/vertices within a given edge/vertex orbit. This ensures that the graph obtained by the construction inherits all the symmetries of the original graph G. The requirement that the order of the automorphism group of the graph does not increase upon applying the given construction, i.e., $\mathrm{Aut}(G) \cong \mathrm{Aut}(F(G, \mathcal{V}))$, is also crucial to the propositions. Figure 13 illustrates some of the complications that can arise.

Proposition 31. *Let G be a connected $(2t)$-regular graph, where $t \geqslant 1$. Then $o_v(\mathcal{M}_3(G)) = 2o_v(G)$ and $o_e(\mathcal{M}_3(G)) = o_e(G) + 2o_v(G)$. Moreover, $|\mathrm{Aut}(\mathcal{M}_3(G))| = (2^t t!)^{|V(G)|} |\mathrm{Aut}(G)|$.*



(a) $\Omega = (12, 13, 8)$ (b) $\Omega = (16, 17, 8)$

(c) $\Omega = (7, 8, 16)$ (d) $\Omega = (18, 20, 4)$

(e) $\Omega = (3, 4, 6)$ (f) $\Omega = (7, 10, 2)$ (g) $\Omega = (9, 12, 2)$ (h) $\Omega = (5, 6, 6)$

(i) $\Omega = (2, 4, 288)$ (j) $\Omega = (12, 17, 16)$

(k) $\Omega = (7, 10, 32)$

Figure 13: Interplay of constructions, orbits and automorphism groups. Parts (a) to (d): in the starting graph (a) two orbits are marked in red and blue; in (b) subdivision on an entire orbit preserves the automorphism group; in (c) the bridge construction on an entire orbit leads to doubling in order of the automorphism group; in (d) subdivision in a part orbit leads to halving. Parts (e) to (h): the Sciriha graph S_3 shown in (e) is progressively subdivided; in (f) subdivision in a part orbit gives broken symmetry; in (g) subdivision in a part orbit of (f) preserves the order of the automorphism group; in (h) subdivision of the final edge causes merging of orbits and restoration of the original symmetry. Parts (i) to (k): the highly symmetric graph (i) has four edge orbits; in (j) the symmetry is considerably reduced by applying subdivision in a part of the green orbit of (i); the graph in (k) is obtained from (j) by applying subdivision on one of two equivalent edges but the symmetry increases; subdivision of the two remaining green edges would restore the full symmetry of (i). The signature Ω given for each graph G denotes the triple $(o_v(G), o_e(G), |\mathrm{Aut}(G)|)$ as in Figure 1.

Proof. Every element $\alpha \in \operatorname{Aut}(G)$ can be extended in a natural way to an element $\widehat{\alpha} \in \mathcal{M}_3(G)$; the element $\widehat{\alpha}$ moves the vertices of the original graph in the same way as α, and also moves the corresponding attached triangles. Therefore, $|\operatorname{Aut}(\mathcal{M}_3(G))| \geqslant |\operatorname{Aut}(G)|$. Now, let us consider action of the stabiliser within $\operatorname{Aut}(\mathcal{M}_3(G))$ that fixes the subgraph G. Consider the triangles attached to an arbitrary vertex $v \in V(G)$. Clearly, the stabiliser permutes the t triangles; this contributes $t!$ to the order of the stabiliser. In addition, there exist involutions that swap two degree-2 endvertices in any attached triangle; this contributes 2^t to the order of the stabiliser. Finally, all these operations can be done independently at every vertex $v \in V(G)$. Therefore, the order of the stabiliser is $(2^t t!)^{|V(G)|}$. Using the Orbit-Stabiliser Lemma [25, Lemma 2.2.2], we obtain $|\operatorname{Aut}(\mathcal{M}_3(G))| = (2^t t!)^{|V(G)|} |\operatorname{Aut}(G)|$. Now that the full automorphism group of $\mathcal{M}_3(G))$ is known, counting the vertex- and edge-orbits is straightforward. $\square$

Example 32. Consider graphs K_7, $\operatorname{Circ}(12, \{1, 5\})$ and the hypercube Q_6. All these graphs are vertex and edge-transitive. Let us determine the order of the automorphism group of $\mathcal{M}_3(G)$ for G from the above list. It is easy to see that $|\operatorname{Aut}(\operatorname{Circ}(12, \{1, 5\}))| = 2^8 \cdot 3$, $|\operatorname{Aut}(K_7)| = 7!$ and $|\operatorname{Aut}(Q_6)| = 2^6 \cdot 6!$. Graphs K_7 and Q_6 are 6-regular, while $\operatorname{Circ}(12, \{1, 5\})$ is 4-regular. By Proposition 31,

$$\begin{aligned} |\operatorname{Aut}(\mathcal{M}_3(\operatorname{Circ}(12, \{1, 5\}))| &= (2^2 \cdot 2!)^{12} \cdot (2^8 \cdot 3) = 52776558133248, \\ |\operatorname{Aut}(\mathcal{M}_3(K_7))| &= (2^3 \cdot 3!)^7 \cdot 7! = 2958824445050880, \\ |\operatorname{Aut}(\mathcal{M}_3(Q_6))| &= (2^3 \cdot 3!)^{64} \cdot (2^6 \cdot 6!) \approx 1.832 \cdot 10^{112}. \end{aligned}$$

Note that even though the automorphism group of $\mathcal{M}_3(G)$ might be absurdly large, the numbers of vertex and edge orbits remain small, and in determining them we can ignore the extra symmetries. $\Diamond$

Proposition 31 has a natural generalisation.

Proposition 33. *Let $k \geqslant 3$ be an odd integer and let G be a connected $(2t)$-regular graph, where $t \geqslant 1$. If $k \equiv 1 \pmod 4$ then the graph G is further required to be bipartite. Then $o_v(\mathcal{M}_k(G)) = \frac{k+1}{2} o_v(G)$ and $o_e(\mathcal{M}_k(G)) = o_e(G) + \frac{k+1}{2} o_v(G)$. Moreover, $|\operatorname{Aut}(\mathcal{M}_k(G))| = (2^t t!)^{|V(G)|} |\operatorname{Aut}(G)|$.*

Proof of Proposition 33 follows the same pattern as the proof of Proposition 31 and is left as an exercise to the reader.

Finally, Proposition 33 implies some further results on the existence of infinite sets of graphs for given pairs (o_v, o_e). In Subsections 3.1 and 4.1 we provided infinite families of graphs for which $(o_v, o_e) = (1, 2)$ and $(o_v, o_e) = (2, 3)$, respectively. Using the machinery of multiplier constructions and their effects on symmetry, we obtain the next theorem.

Theorem 34. *Let $r \geqslant 2$ be even. For every $k \geqslant r + 1$ there exist infinitely many nut graphs G with $o_v(G) = r$ and $o_e(G) = k$.*

Lemma 35. *For every $k \geqslant 1$ it holds that* $\operatorname{Aut}(\operatorname{Circ}(n, \{1, 2, \ldots, k\})) \cong \operatorname{Dih}(n)$ *for all $n \geqslant 2k + 3$.*

Proof. Let $G = \mathrm{Circ}(n, \{1, 2, \ldots, k\})$. Recall that $V(G) = \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. It is clear that $\mathrm{Dih}(n) \leqslant \mathrm{Aut}(G)$. If $k = 1$, the graph G is isomorphic to the cycle graph C_n. In this case it is clear that $\mathrm{Aut}(G) \cong \mathrm{Dih}(n)$. Hence, we can assume that $k \geqslant 2$.

Let $\mathcal{G} \leqslant \mathrm{Aut}(G)$. The Orbit-Stabiliser Lemma [25, Lemma 2.2.2] says $|\mathcal{G}| = |\mathcal{G}_v| \cdot |v^{\mathcal{G}}|$ for any $v \in V(G)$. Since G is vertex-transitive it follows that $|\mathrm{Aut}(G)| = n \cdot |\mathrm{Aut}(G)_0|$. What is the orbit of vertex 1 inside $\mathrm{Aut}(G)_0$? Note that $d_G(0, i) = 1$ for $i \in \{1, 2, \ldots, k\} \cup \{-1, -2, \ldots, -k\}$ and $d_G(0, i) = 2$ for $i \in \{k+1, k+2, -k-1, -k-2\}$, where $d_G(u, v)$ is the distance between vertices u and v in graph G. Let us define $f_0(i) = |\{j \in N_G(i) \mid d(0, j) = 2\}|$ for $v \in \{1, \ldots, k\} \cup \{-1, \ldots, -k\}$. Note that $f_0(1) = f_0(-1) = 1$ and $f_0(i) \geqslant 2$ if $i \notin \{-1, 1\}$. Since graph automorphisms preserve distances, it follows that $1^{\mathrm{Aut}(G)_0} = \{-1, 1\}$, as these are the only two vertices at distance 1 from 0 that have a single neighbour at distance 2. Therefore, $|\mathrm{Aut}(G)_0| = 2 \cdot |\mathrm{Aut}(G)_{0,1}|$, where $\mathrm{Aut}(G)_{0,1}$ is the stabiliser that fixes both 0 and 1. It only remains to show that $\mathrm{Aut}(G)_{0,1}$ is trivial. Vertices $\{2, 3, \ldots, k+1\} \cup \{0, -1, -k+1\}$ are at distance 1 from vertex 1. For these vertices we define $f_1(i) = |\{j \in N_G(i) \mid d(1, j) = 2\}|$. The only vertex ℓ for which $d_G(\ell, 1) = 1$ and $f_1(\ell) = 1$ and $d_G(\ell, 0) = 1$ is the vertex $\ell = 2$. Therefore, $\mathrm{Aut}(G)_{0,1}$ fixes vertex 2. By iteration of the argument, all vertices are fixed, so $|\mathrm{Aut}(G)_{0,1}| = 1$. □

We remark in passing that $\mathrm{Circ}(2k+1, \{1, 2, \ldots, k\}) \cong K_{2k+1}$ and its automorphism group has order $(2k+1)!$; and that $\mathrm{Circ}(2k+2, \{1, 2, \ldots, k\}) \cong K_{2k+2} - (k+1)K_2$ and its automorphism group has order $2^{k+1}(k+1)!$.

Proof of Theorem 34. For every $k \geqslant 1$ there exist infinitely many vertex-transitive graphs with precisely k edge orbits; they include the circulants $\mathrm{Circ}(n, \{1, 2, \ldots, k\})$ for $n \geqslant 2k+3$, provided by Lemma 35. By Proposition 33,

$$o_v(\mathcal{M}_{4q-1}(\mathrm{Circ}(n, \{1, 2, \ldots, k\}))) = 2q,$$
$$o_e(\mathcal{M}_{4q-1}(\mathrm{Circ}(n, \{1, 2, \ldots, k\}))) = k + 2q.$$ □

6 Future work

The present paper gives a theorem for the relationship between vertex-orbit and edge-orbit counts for nut graphs. The result (Theorem 2) that $o_e \geqslant o_v + 1$, compares to Buset's result $o_e \geqslant o_v - 1$ for all connected graphs [6, Theorem 2]. Edge-transitive nut graphs are therefore impossible objects.

We also provided a complete characterisation of the orders for which nut graphs with $(o_v, o_e) = (1, 2)$ exist. A partial answer was also found for the pair $(o_v, o_e) = (2, 3)$ (see Conjecture 24 and Question 25). It was possible to provide infinite families of nut graphs for the pairs (o_v, o_e), where o_v is an even number and $o_e > e_v$. The case where o_v is an odd number remains to be completed. The ultimate goal is, of course, the complete characterisation of orders for all (o_v, o_e) pairs.

During this work we encountered smallest examples of several interesting classes, including the non-Cayley nut graphs (see Figure 5), and GRR nut graphs (see Figure 11), which suggest directions for future explorations. The three infinite families used to prove

Theorem 11, and the Rose Window family (see Proposition 16) are all quartic, but the problem of characterising orders for (o_v, o_e) pairs is also a natural one for regular graphs, or graphs of prescribed degree. It is planned to investigate the cubic case first, because of its significance for chemical graph theory.

A substantial part of the paper was devoted to constructions of nut graphs and their effects on symmetry, which can be complicated. In some cases, the automorphism group of a constructed nut graph can be impressively large (see Example 32). The multiplier constructions (Subsection 4.2) give access to highly symmetric graphs with controlled number of vertex orbits. This prompts the question: For a given n, what is the most symmetric nut graph on that order, where by 'most symmetric' we mean in the sense of order of the automorphism group? From this perspective it is interesting that the graph with 288 automorphisms shown in Figure 13(i) is the nut graph with the largest full automorphism group amongst all nut graphs on 10 vertices and yet it is not vertex-transitive.

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